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The Dual Characterization of Structured and Skewed Structured Singular Values

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Abstract: The structured singular values and skewed structured singular values are the well-known mathematical quantities and bridge the gap between linear algebra and system theory. It is well-known fact that an exact computation of these quantities is NP-hard. The NP-hard nature of structured singular values and skewed structured singular values allow us to provide an estimations of lower and upper bounds which guarantee the stability and instability of feedback systems in control. In this paper, we present new results on the dual characterization of structured singular values and skewed structured singular values. The results on the estimation of upper bounds for these two quantities are also computed.

Keywords: eigenvalues; singular values; structured singular values; skewed structured singular values

MSC: 15A18; 15A03; 05B20; 15A23

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1. Introduction

The structured singular value (SSV) was first introduced by J. C. Doyel [1] and Safonov [2] as a mathematical tool, which is widely used to investigate the robustness, performance and stability of linear feedback systems in control. In control system analysis, the problem associated with the determination of stability and robustness in the presence of uncertainties is among the most fundamental issue in control and it has attracted a reasonable amount of researchers in the last three decades. Much of the research work has been done in robustness analysis for two different class of problems from system theory which involves the uncertainties. For this purpose, two different kinds of approaches has been developed.

One of the approach is based upon the frequency-based robust stability conditions in the form of the small gain condition. The small gain condition is most useful for analyzing those problems from system theory which are associated with the norm bounded unstructured or complex structured uncertainties. An example of such an approach is based upon the structured singular values introduced in [1,3].

An another approach is largely inspired by Kharitonov work [4]. The main aim of the work by Kharitonov is to determine the stability robustness with a finite number of conditions. This approach also aims to study the problems in control when real parametric uncertainties consisting of real-valued uncertain parameters are involved, for a more details see, e.g., [5–7].

In principle, both types of methodologies can be modified to deal with real and complex parametric uncertainties. Indeed, the early developments in structured singular values deals with the uncertainties which are only pure complex. However, the extensions have been made so that the real-valued uncertainties can also be considered [8,9]. The Kharitonov type approaches deal with the μ-value problems while considering the real-valued uncertainties [10–12].

The exact computation of SSV is impenetrable which makes it NP-hard [13]. The NP-hard nature of SSV broach to foster methods for approximation of its lower and upper bounds. However, the SSV lower bounds are computed using the generalization of power method [14,15]. Moreover, the balanced AMI technique developed by [16] is the utilization of bounds introduced by [17] and is particularized in a preeminent style in [16]. The given matrix, under consideration, is first balanced while imploging a variation of Osborne's [18] generalized to crank the repeated real/complex scalars and the number of full blocks. Further, Perron approach is a determination for balancing the given matrix. The Perron eigenvector methodology is established on the idea apt by Safonov [2].

The D-Scaling upper bound presented in [1] is the most extensively used paper for the approximation of the upper bound of structured singular values. The D-Scaling for complex structures acquiring full complex blocks is close to the original SSV. For more details, we suggest the reader to consult [19] and the reference therein. Meanwhile, for non-trivial complex structures, the D-Scaling upper bound turns out to be more flexible [1,19].

The SSV theory for mixed real/complex cases is an extension of SSV that acquiesce the structure to consist of real and complex parts. The computation of upper bounds for the mixed SSV presented by [17] is also known as (D, q)-Scaling upper bound of skewed structured singular values ν, and is quite apart from actual mixed SSV [20].

The investigation for the non-fragile asynchronous H_∞ control while considering the stochastic memory systems with Bernoulli distribution has been recently studied by [21]. An efficient algorithm for the computation of budget allocation procedure for the selection of top candidate solution for objective performance measure has been extensively studied by [22].

In this article, we give an analytical treatment for the dual characterization of structured singular values and skewed structured singular values. We present some new results for the computation of an upper bounds of these quantities.

The rest of the paper is organized as: In Section 2, we provide the preliminaries of our article. In particular, we give the Definitions of the block diagonal structure, structured singular values and skewed structured singular values for a set of block diagonal matrices and subset of positive definite matrices. In Section 3, we present new results on the computation of the dual characterization of structured singular values and skewed structured singular values. The computation of upper bounds of skewed structured singular values are also presented in Section 3 of our article. Finally, Section 4 is about the conclusion of our presented work.

2. Preliminaries

Before we proceed, we give some essential definitions that will act as prerequisites for the subsequent results.

Definition 1 ([23])**.** *The set of block diagonal matrices X is defined as*

$$X \quad := \quad \left\{diag(\alpha_i I_i; \beta_j I_j; C_t) : \alpha_i \in \mathbb{R}, \beta_j \in \mathbb{C}, C_t \in \mathbb{C}^{m_t \times m_t}\right\},$$

where $\alpha_i I_i$, $\beta_j I_j$, and C_t denote the number of repeated real scalar blocks with different sizes for all $i = 1, \cdots, r$, the number of complex scalar blocks with different sizes for all $j = 1, \cdots, c$ and the number of full complex blocks with different sizes for all $t = 1, \cdots, k$, respectively.

Definition 2 ([1]). *For a given n-dimensional complex valued matrix $M \in \mathbb{C}^{n\times n}$, the structured singular value with respect to X is defined as*

$$\mu_X(M) := \begin{cases} 0, & \text{if } det(I - M\Delta) \neq 0,\ \forall\, \Delta \in X \\ \frac{1}{\min\limits_{\Delta}\{\|\Delta\|_2 : \Delta \in X,\, det(I - M\Delta) = 0\}}, & \text{else.} \end{cases} \tag{1}$$

The matrix valued function Δ is an uncertainty that occurs in the linear feedback system.

Definition 3 ([23]). *The sets D_X and G_X*

$$D_X := \Big\{diag(P_1, \ldots, P_r; P_1, \ldots, P_c; P_1, \ldots, P_t)\Big\}$$

and

$$G_X := \Big\{diag(H_1, \ldots, H_r; O_1, \ldots, O_c; O_1, \ldots, O_t\Big\}$$

contains positive definite matrices P_i for all $i = 1, \cdots, r$, P_j for all $j = 1, \cdots, c$ and P_t for all $t = 1, \cdots, k$ and Hermitian matrices H_i for all $i = 1, \cdots, r$, and repeated null complex scalar blocks O_j for all $j = 1, \cdots, c$ and number of null full complex blocks O_t for all $t = 1, \cdots, k$, respectively.

Definition 4 ([23]). *For a given n dimensional complex valued square matrix $M \in \mathbb{C}^{n\times n}$ and $\beta \in \mathbb{R}$, the matrix-value function $f_\beta(D, G)$ is defined as*

$$f_\beta(D, G) := M^H DM + i(GM - M^H G) - \beta^2 D,$$

where matrices D, G belongs to D_X and G_X, receptively.

Definition 5 ([23]). *The upper bound of $\mu_X(M)$ is denoted by $\nu_X(M)$ and is defined as*

$$\nu_X(M) := \inf_{\beta>0}\Big\{\beta : \exists\ D \in D_X \text{ and } G \in G_X \text{ s.t. } f_\beta(D, G) < 0\Big\}.$$

Let $M \in \mathbb{C}^{m\times n}$ be a given matrix and (m_r, m_c, m_C) represent an m-tuples of positive integers and let

$$K = (k_1, \ldots, k_{m_r}, k_{m_{r+1}}, \ldots, k_{m_r+m_c}, k_{m_r+m_{c+1}}, \ldots, k_{m_C}), \tag{2}$$

where $\sum_{i=1}^{m} k_i = n$.

Definition 6 ([23]). *The set of block diagonal matrices is defined as*

$$X_K := \{\Delta = diag(\delta_1 I_1, \ldots, \delta_r I_r, \delta_1 I_1, \ldots, \delta_c I_c, \Delta_1, \ldots, \Delta_t)\}.$$

In Definition 6, $\delta_i \in \mathbb{R}\ \forall i = 1, \cdots, r$, $\delta_j \in \mathbb{C}\ \forall j = 1, \cdots, c$, $\Delta_t \in \mathbb{C}^{t\times t}\ \forall t = 1, \cdots, k$. The set X_K is pure real if $\delta_j = 0$ and pure complex if $\delta_i = 0$, otherwise it is with mixed real and complex block perturbation. For $\Delta_t \in \mathbb{C}^{t\times t}$, the set X_K turns out to be a set of full complex blocks perturbation.

Definition 7 ([1]). *For given matrix $M \in \mathbb{C}^{m\times n}$ and X_K, the structured singular value; denoted by $\mu_{X_K}(M)$ and is defined as*

$$\mu_{X_K}(M) := \begin{cases} 0, & \text{if } det(I - M\Delta) \neq 0, \Delta \in X_K \\ \frac{1}{\min\limits_{\Delta\in X_K}\{\|\Delta\|_2 : det(I - M\Delta) = 0\}}, & \text{else,} \end{cases} \tag{3}$$

where $\|\Delta\|_2$ denotes the largest singular value of Δ.

Definition 8 ([23]). *The set Y_K of block diagonal structure is defined as*

$$Y_K : \{\Delta_v = diag(\delta_1 I_1, \ldots, \delta_r I_r, \delta_1 I_1, \ldots, \delta_c I_c; \delta_1 I_1, \ldots, \delta_c I_c; \Delta_1, \ldots, \Delta_t)\}.$$

In Definition 8, $\delta_i \in \mathbb{R}\ \forall i = 1, \cdots, r,\ \delta_j \in \mathbb{C}\ \forall j = 1, \cdots, c,\ \Delta_t \in \mathbb{C}^{t \times t}\ \forall t = 1, \cdots, k.$

Definition 9 ([23]). *The secondary set $Z_{\hat{K}}$ of block diagonal structure is defined as*

$$Z_{\hat{K}} : \{\Delta_v = diag(\delta_1 I_1, \ldots, \delta_r I_r, \delta_1 I_1, \ldots, \delta_c I_c; \Delta_1, \ldots, \Delta_t)\}.$$

In Definition 9, $\delta_i \in \mathbb{R}\ \forall i = 1, \cdots, r,\ \delta_j \in \mathbb{C}\ \forall j = 1, \cdots, c,\ \Delta_t \in \mathbb{C}^{t \times t}\ \forall t = 1, \cdots, k.$

Definition 10 ([23]). *The set Z_K is restricted to the unit ball and is defined as*

$$BZ_{\hat{Z}} = \left\{\Delta_f \in Z_{\hat{K}} : \|\Delta_f\|_2 \leq 1\right\}.$$

Definition 11 ([23]). *The block structure $W_{K,\hat{K}}$ is defined as*

$$W_{K,\hat{K}} = \left\{\Delta = diag(\Delta_f, \Delta_v)\right\}, \tag{4}$$

or

$$\Delta = \left(\begin{array}{c|c} \Delta_f & 0 \\ \hline 0 & \Delta_v \end{array}\right). \tag{5}$$

Definition 12 ([23]). *For given matrix $M \in \mathbb{C}^{m \times n}$ and $Z_{\hat{K}}$, the skewed structured singular value is denoted by $\mu_{Z_{\hat{K}}}(M)$ and is defined as*

$$\mu_{Z_{\hat{K}}}(M) := \begin{cases} 0, & \text{if } det(I - M\Delta) \neq 0, \Delta \in W_{K,\hat{K}} \\ \frac{1}{\min\limits_{\Delta \in W_{K,\hat{K}}} \{\|\Delta_v\|_2 : det(I - M\Delta) = 0\}}, & \text{else.} \end{cases} \tag{6}$$

3. The Main Results

In the section, we present some new results on the computation of structured singular values and skewed structured singular value.

3.1. Dual Characterization of $\mu_X(M)$ and $\nu_X(M)$

We give a dual characterization of $\mu_X(M)$ and $\nu_X(M)$. The characterization is the dual in the sense that they act as an application for duality argument in convex sets. The following result given by Boyed [24] is considered as a standard result for the separation of the hyper-planes.

Lemma 1 ([24]). *Let $P(E) \in \mathbb{C}^{m \times m}$ and P depends affinely on $E \in \mathbb{C}^{n \times n}$. Let γ be the some convex subset of $\mathbb{C}^{n \times n}$. Then, there exists no $E \in \gamma$ such that the Hermitian part of $P(E)$ becomes negative, that is,*

$$He(P(E)) < 0$$

if and only if there is some non-zero matrix W, that is, $W = W^H$ and is non-negative such that

$$Re(Tr(WP(E))) \geq 0.$$

We make use of the following assumptions to prove our main result for dual characterization of $\nu_X(M)$.

Assumption 1. *The matrix $E \in \mathbb{C}^{n \times n}$ in $D_X + G_X$ is Hermitian, that is $E = E^H$.*

Assumption 2. *The matrices* $(M - \beta I), (M^H + \beta I)$ *are Hermitian, that is,*

$$(M - \beta I) = (M + \beta I)^H$$

and

$$(M^H + \beta I) = (M^H + \beta I)^H.$$

Theorem 1 (Dual Characterization of $\nu_x(M)$)**.** *Let* $M \in \mathbb{C}^{n \times n}$ *and X be the set of block diagonal matrices, as defined above. The quantity* $\beta \in \mathbb{R}$ *is lower bound of an upper bound of* $\mu_X(M)$

$$\beta \leq \nu_X(M) \Leftrightarrow \exists\, W = W^H \geq 0$$

such that

$$\eta_i[(M - \beta I)W(M^H + \beta I)E] \geq 0, \forall E \in D_X + iG_X.$$

Proof. The matrices $(M - \beta I)$, $(M^H + \beta I)$, W and E are Hermitian. The unitary diagonalization of the matrix $((M - \beta I)W(M^H + \beta I)E)$ implies that

$$(M - \beta I)W(M^H + \beta I)E = Q\Lambda Q^*$$

or

$$\Lambda = diag(\lambda_1, \ldots, \lambda_z, \lambda_{z+1}, \ldots, \lambda_n) = Q^*((M - \beta I)W(M^H + \beta I)E)Q \tag{7}$$

We construct matrices M_0, M_+, which pack $\lambda_1, \cdots, \lambda_z$ as zero eigen-values and $\lambda_{z+1}, \cdots$, λ_n as strictly positive eigen-values.

$$M_0 = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in \mathbb{C}^{n_z \times n_z},$$

and

$$M_+ = \begin{pmatrix} \frac{1}{\lambda_{z+1}^{1/2}} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\lambda_n^{1/2}} \end{pmatrix} \in \mathbb{C}^{n_p \times n_p},$$

and then assemble all eigen-values into matrix $\mathbb{B}$ as

$$\mathbb{B} = \begin{pmatrix} M_0 & 0 \\ 0 & M_+ \end{pmatrix}.$$

A simple calculation shows that

$$(Q\mathbb{B})^*((M - \beta I)W(M^H + \beta I)E)(Q\mathbb{B}) = \mathbb{B}^*\Lambda\mathbb{B} = \begin{pmatrix} O_z & 0 \\ 0 & I_p \end{pmatrix}. \tag{8}$$

In Equation (8), O_z and I_p are of dimensions n_z and n_p.

Suppose that there exists an Hermitian matrix $\hat{H}$, which is a similar matrix to $((M - \beta I)W(M^H + \beta I)E)$ such that

$$Q_{(M-\beta I)W(M^H+\beta I)E}, \mathbb{B}_{(M-\beta I)W(M^H+\beta I)E} = (M - \beta I)W(M^H + \beta I)E, \tag{9}$$

and

$$Q_{\hat{H}}, \mathbb{B}_{\hat{H}} = \hat{H}. \tag{10}$$

In a similar manner,

$$(Q_{(M-\beta I)W(M^H+\beta I)E}\mathbb{B}_{(M-\beta I)W(M^H+\beta I)E})^*[(M-\beta I)W(M^H+\beta I)E]$$
$$(Q_{(M-\beta I)W(M^H+\beta I)E}\mathbb{B}_{(M-\beta I)W(M^H+\beta I)E}) = \begin{bmatrix} O_z & 0 \\ 0 & I_p \end{bmatrix} = (Q_{\hat{H}}\mathbb{B}_{\hat{H}})^*\hat{H}(Q_{\hat{H}}\mathbb{B}_{\hat{H}}).$$

In turn, the quantity $(M-\beta I)W(M^H+\beta I)E$ implies that,

$$(Q_{(M-\beta I)W(M^H+\beta I)E}\mathbb{B}_{(M-\beta I)W(M^H+\beta I)E})^{-1} = (Q_{\hat{H}}\mathbb{B}_{\hat{H}})^*\hat{H}(Q_{\hat{H}}\mathbb{B}_{\hat{H}})(Q_{\hat{H}}\mathbb{B}_{\hat{H}})^{-1}. \quad (11)$$

Thus, finally we obtain the following expression for the quantity $(M-\beta I)W(M^H+\beta I)E$, that is,

$$(M-\beta I)W(M^H+\beta I)E = TV.$$

Here the matrix $T = (Q_{\hat{H}}\mathbb{B}_{\hat{H}}\mathbb{B}^{-1}_{(M-\beta I)W(M^H+\beta I)E}Q^{-1}_{(M-\beta I)W(M^H+\beta I)E})^*$, and the matrix $V = \hat{H}(Q_{\hat{H}}\mathbb{B}_{\hat{H}}\mathbb{B}^{-1}_{(M-\beta I)W(M^H+\beta I)E}Q^{-1}_{(M-\beta I)W(M^H+\beta I)E})$. The result in above equation implies the existence of some invertible matrix Z such that

$$(M-\beta I)W(M^H+\beta I)E = Z^*\hat{H}Z.$$

Reduce $(M-\beta I)W(M^H+\beta I)E$ and $\hat{H}$ to the form of Equation (8) as:

$$(\Upsilon\Omega)^*((M-\beta I)W(M^H+\beta I)E)(\Upsilon\Omega) = \begin{pmatrix} O_z & 0 \\ 0 & I_p \end{pmatrix},$$

where $\Upsilon = Q_{(M-\beta I)W(M^H+\beta I)E}$ and $\Omega = \mathbb{B}_{(M-\beta I)W(M^H+\beta I)E}$ and hence, we have that,

$$(Q_{\hat{H}}\mathbb{B}_{\hat{H}})^*\hat{H}(Q_{\hat{H}}\mathbb{B}_{\hat{H}}) = \begin{bmatrix} O_{\hat{z}} & 0 \\ 0 & I_{\hat{p}} \end{bmatrix}. \quad (12)$$

Next, we intend to show that $z = \hat{z}$ and $p = \hat{p}$. However, since, $(M-\beta I)W(M^H+\beta I)E = Z^*\hat{H}Z$ and from Equation (12), we get

$$(Z\Upsilon\Omega)^*\hat{H}(Z\Upsilon\Omega) = \begin{bmatrix} O_z & 0 \\ 0 & I_p \end{bmatrix}. \quad (13)$$

Equations (12) and (13) have two similar transformations for $\hat{H}$. We write these as

$$Y^*\hat{H}Y = \begin{pmatrix} O_z & 0 \\ 0 & I_p \end{pmatrix}; \quad \hat{Y}^*\hat{H}\hat{Y} = \begin{pmatrix} O_{\hat{z}} & 0 \\ 0 & I_{\hat{p}} \end{pmatrix}$$

where matrices Y and $\hat{Y}^*$ are invertible. We show $z = \hat{z}$ and skip to show $p = \hat{p}$ to avoid redundancy in working rules.

Let

$$\mathcal{W} = \mathcal{N}(Y^*\hat{H}Y), \ \dim(\mathcal{W}) = z.$$

Take $w \in \mathcal{W}$, then we get $Y^*\hat{H}Yw = 0$. As the matrix Y is invertible, so $Y^*\hat{H}w = 0$. For $\hat{Y}^{-1}Yw \in \hat{Y}^{-1}Y\mathcal{W}$, then we have

$$\hat{Y}^*\hat{H}\hat{Y}x = \hat{Y}^*\hat{H}\hat{Y}w.$$

Furthermore, $\hat{Y}^{-1}Y\mathcal{W} \subset \mathcal{N}(\hat{Y}^*\hat{H}Y)$ and by making use of the fact that Y and $\hat{Y}$ are invertible,

$$\hat{z} = dim(\mathcal{N}(\hat{Y}^*BY)) \geq dim(\hat{Y}^{-1}Y\mathcal{W}) = dim(\mathcal{W}) = dim(\mathcal{N}(Y^*\hat{H}Y)) = z.$$

Finally, by switching the roles of $Y^*\hat{H}Y$ and $(\hat{Y}^*\hat{H}\hat{Y})$, it follows that $z \geq z^*$, so, $z = z^*$. □

Assumption 3. *For $t \in \mathbb{C}^{n\times 1}$, the matrix $tt^H = W$ with $W = W^H \geq 0$.*

Theorem 2. *(Dual Characterization of $\mu_X(M)$). Let X be any set of block diagonal matrices as defined in (8). Then, $\mu_X(M) \geq \beta$ if and only if there exists $t \in \mathbb{C}^{n\times 1}$ such that*

$$Re\ tr[(M - \beta I)tt^H(M^H + \alpha I)E] \geq 0, \forall E \in D_x + iG_x.$$

Proof. The proof is similar to that of Theorem 1 by making use of Assumption 3. □

Theorem 3. *For $W \in \mathbb{C}^{m\times n}, W = W^H \geq 0$ there exists $B \geq 0$ s.t. $W^{1/2} = B$.*

Proof. The proof is followed from spectral decomposition Theorem. Indeed, we do have $W = QDQ^*$ with $QQ^* = I = Q^*Q$ and

$$D = diag(\lambda_1, \lambda_2, \ldots, \lambda_n),\ \forall i = 1, \cdots, n.$$

For all $i \in [1, \cdots, n],\ \lambda_i(W) \geq 0$. Set $B := Q\tilde{D}Q^*$ with $\tilde{D} = diag(\lambda_1^{1/2}, \lambda_2^{1/2}, \ldots, \lambda_n^{1/2})$. In turn, this implies that

$$W^{1/2} = B,\ or\ , B^2 = W. \tag{14}$$

The proof is done. □

Lemma 2. *If $M \in \mathbb{C}^{m\times n}$ has rank one, then $\mu_X(M) = \nu_X(M)$.*

Proof. It is sufficient to show $\nu_X(M) \geq \beta$ implies that $\mu_X(M) \geq \beta$ for $\beta \in [0, \infty)$. For this purpose, consider that $\nu_x(M) \geq \beta$, then from Theorem 3.1, there is a nonzero non-negative definite matrix W such that $W = W^H \geq 0$, which satisfies matrix inequality

$$(M - \beta I)W(M^H + \beta I)E \geq 0,\ \forall E \in DX + iG_X.$$

Next, we pick the largest rank-1 piece of M, that is, $\sigma_1 u_1 \theta_1^H$

$$M = u_1 \left(\begin{array}{c|c} \sigma_1 & 0 \\ \hline 0 & 0 \end{array} \right) \theta_1^H,\ \sigma_1 > 0, u_1, \theta_1 \in \mathbb{C}^{n\times 1}$$

Factorize W as

$$W = tt^H + \hat{W}. \tag{15}$$

In Equation (15), $\hat{W}$ is chosen such that $\hat{W}\theta_1 = 0, \hat{W} = \hat{W}^H \geq 0,\ t \in \mathbb{C}^{n\times 1}$. If $B\theta_1 = 0$, then $W = \hat{W}$ for $t = 0$. The PSD-matrix B is defined in Theorem 3. If $B\theta_1 \neq 0$, then take $t = \frac{1}{(\theta_1^H W\theta_1)^{1/2}} W\theta_1,\ \hat{W} = W - \frac{1}{(\theta_1^H W\theta_1)^{1/2}} W\theta_1\theta_1^H W$. This shows that

$$\begin{aligned} (M - \beta I)tt^H(M^H + \beta I) &= (\sigma_1 u_1 \theta_1^H - \beta I)(W - \hat{W})(\theta_1 u_1^H + \beta I) \\ &= (M - \beta I)W(M^H + \beta I) + \beta^2\hat{W}. \end{aligned}$$

Since, by Theorem 3, we have that

$$(M - \beta I)W(M^H + \beta I)E \geq 0,\ \forall E \in D_X + iG_X,$$

then so does

$$(M - \beta I)tt^H(M^H + \beta I)E$$

as the factor $\beta^2\hat{W}$ is non-negative. □

3.2. Computing Upper Bound of Skewed Structured Singular Value

In this section, we present some new results on the computation of the upper bounds of structured singular values, that is, $\mu_s(\cdot)$.

Theorem 4. *For a given matrix $M \in \mathbb{C}^{m\times n}$ and block diagonal structure*

$$S = \left(\begin{array}{c|c} I_f & 0 \\ \hline 0 & \nu^I \nu \end{array}\right).$$

The inequality holds true, that is, $\mu_s(M_s(\nu)) \leq \sigma_1(M_s(\nu))$ with

$$M_s(\nu) := S^{-1}M = \left(\begin{array}{c|c} M_{11} & M_{12} \\ \hline \frac{1}{\nu}M_{21} & \frac{1}{\nu}M_{22} \end{array}\right).$$

Proof. For given matrix $M \in \mathbb{C}^{m\times n}$, there exists unitary matrices $U \in \mathbb{C}^{m\times n}, V \in \mathbb{C}^{m\times n}$ such that

$$M = U\left(\begin{array}{c|c} \sigma_1 & 0 \\ \hline 0 & T \end{array}\right)V^H.$$

Take σ_1 and $\theta_1 \in \mathbb{C}^{n\times 1}$ such that $\sigma_1 = \|M\theta_1\|_2 = \|M\|_2$ and $\|\theta_1\|_2 = 1$. Let $u_1 = \frac{M\theta_1}{\sigma_1}$, then $\|u_1\|_2 = \frac{\|M\theta_1\|_2}{\sigma_1} = \frac{\|M\theta_1\|_2}{\|M\|_2} = 1$. Take $U_2 \in \mathbb{C}^{m\times m-1}, V_2 \in \mathbb{C}^{n\times n-1}$ so that, U and V become $U = (u_1|U_2)$ and $V = (v_1|V_2)$ with U, V being unitary matrices. Then, the product of matrices $U^H MV$ takes the form as:

$$(u_1|U_2)M(v_1|V_2) = \left(\begin{array}{c|c} u_1^H M\theta_1 & u_1^H MV_2 \\ \hline U_2 M\theta_1 & U_2^H MV_2 \end{array}\right) = \left(\begin{array}{c|c} \sigma_1 u_1^H u1 & u_1^H MV_2 \\ \hline \sigma_1 U_2^H u_1 & U_2^H MV_2 \end{array}\right) = \left(\begin{array}{c|c} \sigma_1 & w^H \\ \hline 0 & B \end{array}\right),$$

with $u_1^H u_1 = 1, U_2^H u_1 = 0, w = V_2^H M^H u_1$, and $B = U_2^H MV_2$.

By taking $w = 0$, we have that

$$\sigma_1^2 = \|M\|_2^2 = \|U^H MV\|_2^2 = \max_{x\neq 0}\frac{\|U^H MVx\|_2^2}{\|x\|_2^2} = \max_{x\neq 0}\frac{\|\left(\begin{array}{c|c} \sigma_1 & u^H \\ \hline 0 & B \end{array}\right)x\|_2^2}{\|x\|_2^2}.$$

Replace $x \longrightarrow w$, we get

$$\sigma_1^2 \geq \frac{(\sigma_1^2 + w^H w)^2}{(\sigma_1^2 + w^H w)} = \sigma_1^2 + w^H w.$$

In turn, this implies that $w = 0$ and

$$U^H MV = \left(\begin{array}{c|c} \sigma_1 & 0 \\ \hline 0 & B \end{array}\right) \quad \text{or} \quad M = U\left(\begin{array}{c|c} \sigma_1 & 0 \\ \hline 0 & B \end{array}\right)V^H. \tag{16}$$

To see the fact that $\mu_s(M_s) \leq \sigma_1(M_s)$, we have

$$M_s(\nu) := S^{-1}M = \left(\begin{array}{c|c} M_{11} & M_{12} \\ \hline \frac{1}{\nu}M_{21} & \frac{1}{\nu}M_{22} \end{array}\right)$$

The largest singular value $\sigma_1(M_s)$ depends upon ν. Furthermore,

$$\left(\begin{array}{c|c} I & M_s(\nu) \\ \hline M_s^H(\nu) & I \end{array}\right) > 0 \Longleftrightarrow I - M_s(\nu)I^{-1}M_s^H(\nu) \geq 0.$$

It follows that

$$\lambda_i(I - M_s(\nu)M_s^H(\nu)) \geq 0, \ \forall i$$

or

$$1 - \lambda_i(M_s(\nu)M_s^H(\nu)) \geq 0, \forall i$$

or

$$\lambda_i(M_s(\nu)M_s^H(\nu)) \leq 1, \ \forall i.$$

Finally, we have

$$\sigma_1(M_s(\nu)) \leq 1.$$

The proof is done. □

4. Conclusions

We have introduced some new results for upper bounds of structured singular values and skewed structured singular values along with their dual characterizations. The characterization is defined in the sense that they act as an application for some duality argument in the given convex sets. The accomplished results on the dual characterization of structured and skewed structured singular values can be used to obtain the new direction for the computation of lower bounds of both of these quantities. The numerical treatment on the dual characterization of both structured and skewed structured singular values is our future work. The interested researchers may use this opportunity to carry out their research while making use of this contribution.

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