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A Novel Method for Decision Making by Double-Quantitative Rough Sets in Hesitant Fuzzy Systems

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Abstract: In some complex decision-making issues such as economy, management, and social development, decision makers are often hesitant to reach a consensus on the decision-making results due to different goals. How to reduce the influence of decision makers' subjective arbitrariness on decision results is an inevitable task in decision analysis. Following the principle of improving the fault-tolerance capability, this paper firstly proposes the graded and the variable precision rough set approaches from a single-quantitative decision-making view in a hesitant fuzzy environment (HFEn). Moreover, in order to improve the excessive overlap caused by the high concentration of single quantization, we propose two kinds of double-quantitative decision-making methods by cross-considering relative quantitative information and absolute quantitative information. The proposal of this method not only solves the fuzzy system problem of people's hesitation in the decision-making process, but also greatly enhances the fault-tolerant ability of the model in application. Finally, we further compare the approximation process and decision results of the single-quantitative models and the double-quantitative models, and explore some basic properties and corresponding decision rules of these models. Meanwhile, we introduce a practical example of housing purchase to expound and verify these theories, which shows that the application value of these theories is impressive.

Keywords: double-quantitative method; graded rough set; hesitant fuzzy environment; variable precision rough set

MSC: 03E72; 68T30; 68U35

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1. Introduction

With the complexity of information and data, people tend to be hesitant to make decisions, which affects both the efficiency of decision making and reduces the quality of decision results. The hesitant fuzzy set (HFS) was first proposed by Torra [1] in 2009, as an extension of the fuzzy set, and it can accurately reflect situations of different preference degrees of a decision-making group. Unlike a fuzzy set [2–4], the membership degree of an HFS [5] is not a deterministic value and does not obey a certain distribution, but is a set of possible values. Thus, the HFS contains more complete and specific information about the object while dealing with some complex decision-making problems. The introduction of the HFS not only depicts the indecision of decision makers when making decisions, but also reflects various decision information of decision-making groups.

The information system (IS) is the most intuitive and effective way to express knowledge. Nevertheless, with the complexity and diversification of data in practical applications, a great mass of data are not suitable for the classical information system or the fuzzy information system. Therefore, the combination and application of IS and HFS has attracted the attention of scholars in various decision-making fields [6] at home and abroad, and has been widely applied in many fields, such as supply chain management, pattern recognition, and medical diagnosis. Xu et al. [7] defined the distance, similarity, and correlation degree in hesitant fuzzy systems. Then the formulas of hesitation fuzzy entropy and cross entropy were discussed and applied to the hesitant fuzzy multiple attribute decision-making problems [8]. Peng et al. [9] investigated multi-criteria decision-making problems in hesitant fuzzy environments. Moreover, some researchers have devoted themselves to the study of hesitant fuzzy information measurement methods, which can be summarized as the following two aspects: (1) a hesitant fuzzy information measurement method based on the traditional distance; (2) a hesitant fuzzy information measurement method based on the perspective of information theory. In terms of hesitant fuzzy aggregation operators, some scholars have applied the aggregation operators in HFEn and achieved remarkable results. Xia et al. [10] defined a hesitant fuzzy weighted average operator, geometric average operator, mixed average operator, and generalized hesitant fuzzy average operator. Xia et al. [11] proposed some hesitant fuzzy quasi-operators based on the quasi-arithmetic mean. Furthermore, Xia et al. [12] put forward the reliability-induced hesitant fuzzy aggregation operator by considering the reliability level of the aggregation information. Based on the Einstein rule, Yu et al. [13] proposed a method of hesitant fuzzy information aggregation. However, the above operators only considered the importance of attributes themselves while ignoring the relationship among these various attributes. In order to repair this shortcoming, Zhu et al. [14,15] and Yu et al. [16] proposed a hesitant fuzzy Bonferroni operator and generalized hesitant fuzzy Bonferroni operator, respectively, and applied them to multi-attribute decision-making problems. Liao et al. [17] and Peng et al. [18] put forward hesitant fuzzy dynamic aggregation operators to process the multi-stage multi-attribute decision-making problems. With the development of the theories of HFSs, various information fusion methods and models are used to deal with the decision-making problems with hesitant fuzzy information. Zhang et al. [19] and Liao et al. [20]. studied the VIKOR method in HFEn. Zhu et al. [21], Liao et al. [22] and Xia et al. [23] defined the hesitant fuzzy preference relation and discussed its application in multi-attribute group decision making. In HFEn, Liao et al. [24] established an optimization model based on scheme satisfaction and discussed the interactive group decision-making method. Farhadinia [25,26] proposed a new comparison method of HFSs for the shortcomings of the HFS comparison function. To realize the comparison of HFSs, Zhang et al. [27] presented a new hesitant fuzzy measure function. For one side, the hesitation process of humans is synthesized by the method of aggregation operators. For the other side, the method of utilizing aggregation operators comprehensively analyzes the relationship between attributes, extending and optimizing the solution to the hesitant fuzzy decision-making problem. Accordingly, the research background of our work is a hesitant fuzzy information system (HFIS). Consistent with these scholars, we intend to use the aggregation operator of HFSs to investigate decision problems.

Rough set theory (RST) was proposed by Pawlak [28] in 1982. As an important tool of data to process imprecision and uncertainty information, its basic idea is to approximate some inaccurate and uncertain concepts with known knowledge. The biggest advantage of the rough set model is that it does not require any prior knowledge about the data. However, the approximations of a classical rough set are defined on the basis of the qualitative relationship between concepts, without considering the intersection degree of concepts, thus it is not applicable for the treatment of many practical problems. Moreover, RST also defines three regions [29] by using the set-inclusion relation and the set non-empty overlapping condition. In some ways, these three regions can be understood as three-way decisions [30,31] including three decisions of acceptance, rejection, and abstention. In view of the shortcomings of the stringency and the lack of fault-tolerant competence of the Pawlak rough set model, Yao et al. [32] proposed the decision-theoretic rough set (DTRS) model in 1990. The main starting point of the DTRS is to define the degree of intersection of concepts with conditional probability, and to define the probability upper and lower approximation sets by two thresholds, α and β. In 1993, Ziarko [33] proposed a variable precision rough set (VPRS) model from the perspective of the degree of set inclusion, which is a special case of DTRSs [34]. Subsequently, a parameterized rough set model was brought up by Skowron in [35], and a Bayesian rough set model was presented by Ślezak et al. [36] in 2005. In combination with probability theory, the DTRS model includes both a comprehensive quantitative description of rough set theory and a complete semantic model based on the Bayesian decision process. Meanwhile, the DTRS model also gives a practical and effective method for computing these thresholds. These works lay a solid theoretical foundation for further research on the theory of rough sets, and provide a new idea for us to study uncertain knowledge. Liang and Xu et al. [37] have made a deep study of the risk decision in HFEn on the basis of decision theory. Liang and Xu et al. [38–44] studied three-way decision problems based on the decision-theoretic rough set model in different systems.

In terms of information quantification, the VPRS model primarily describes the approximate spaces from the perspective of the relative quantitative information, while the graded rough set (GRS) model [45] depicts the approximate spaces from the view of absolute quantitative information. Relative quantitative information and absolute quantitative information are two distinct objective aspects of describing approximate space, each with its application environment and limitations. However, it is worth mentioning that some composite models based on DTRSs with double fault-tolerance capabilities are proposed by Li and Xu [46], which thoroughly describe the approximation space through the double-quantitative indicators with a complementary relationship. Yu et al. [47] introduced multi-granulation methods to Dq-DTRS models. Xu et al. [48] further explored generalized multi-granulation Dq-DTRS models. In addition, Fan et al. [49] discussed logical double-quantitative rough fuzzy sets and local logical disjunction double-quantitative rough sets.

Neither the classical decision-theoretic rough set theory nor the fuzzy decision-theoretic rough set that scholars have already studied can be used to describe the indecisive attitude of decision makers in the decision-making process. However, there exist many decision-making problems in the current stage of social and economic management, such as house purchases, the stock market, medical diagnosis, etc. The problem of improving the quality of decision making and enhancing the effectiveness of decision making is an urgent issue to be solved. In view of the superiority of HFSs and the high tolerance of DTRSs, how to utilize these two models to reduce the impact of decision makers' subjective arbitrariness on decision results is a motivation of this study.

With the diversification of decision information and the hesitancy of people to make a decision, this paper integrates the GRS model and the VPRS model to construct two kinds of composite models with double fault tolerance, which comprehensively and in detail describe the approximate space. They further explore the uncertainty and imprecise knowledge in data and provide decision makers with scientific and effective decisions. As shown in Figure 1, the contributions of our work are as follows:

(1) A novel hesitant fuzzy aggregation operator that both reflects the hesitant attitude of decision makers and synthesizes hesitant information is proposed. On this basis, we introduce the hesitant fuzzy aggregation operator to the VPRS model and GRS model.
(2) Two HFEn-based double-quantitative decision-making models are presented by integrating the VPRS-HEFn model and the GRS-HEFn model. Furthermore, we discuss the properties of two models and make a comparison from the internal relationship between the two models.
(3) In order to further illustrate the superiority of two types of MARS models, we design a corresponding algorithm and case analysis. The comparison results of single-quantitative models and double-quantitative models in the case verify the effectiveness of the proposed method.

This paper is organized as follows. In Section 2, some basic concepts of HFSs, the original GRS model, and the VPRS model derived from a Bayesian decision procedure are reviewed. Regarding HFEn, Section 3 describes the extension of the GRS model and the VPRS model and analyzes their properties. In Section 4, considering the combination of two kinds of logical operators, two HFEn-based double-quantitative decision-making models (Models 1 and 2) are proposed and their basic internal relations are discussed. In Section 5, a case study is given to illustrate these double-quantitative decision-making methods in the HFEn. Finally, Section 6 gives a summary of this research and further elaborates the future research.

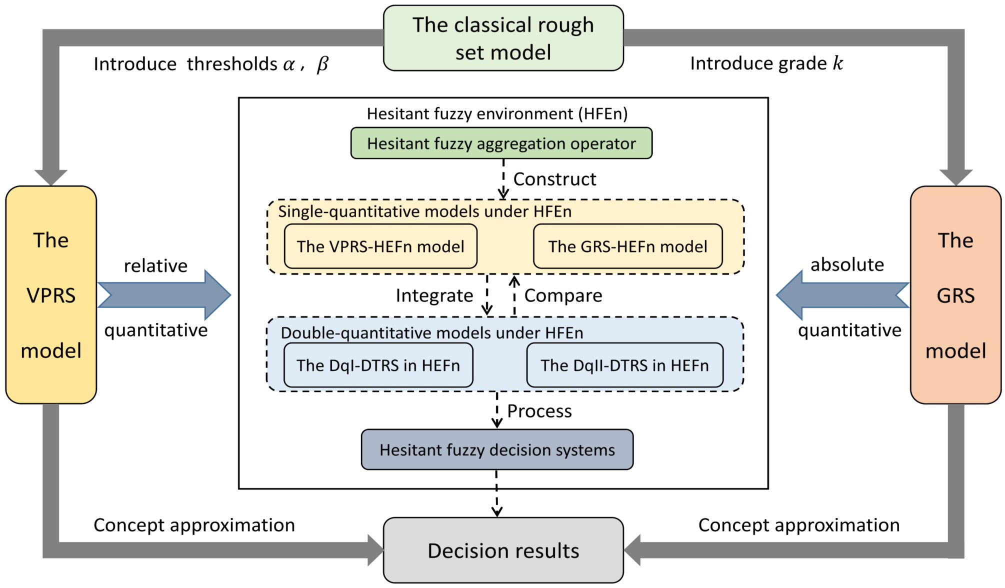

Figure 1. System diagram of our work.

2. Preliminaries

For the convenience of understanding, this section reviews some of the relevant background knowledge needed for this paper, such as the theories of hesitant fuzzy set, graded rough set, variable precision rough set, and double-quantitative decision-theoretic rough fuzzy set models.

2.1. Hesitant Fuzzy Set

In real life, it is difficult to reach a consensus on the views of all members of a group when hesitation exists. From this point of view, Torah [1] shows that the difficulty in establishing membership degrees of an element is caused by a set of possible values. Then, he proposed the concept of a hesitant fuzzy set (HFS). Subsequently, Xia and Xu [7] expressed the HFS by a mathematical symbol for more profound understanding and application. The details can be found in [45].

Definition 1. *Let U be a universe with finite elements, a hesitant fuzzy set (HFS) over U can be defined as :*

$$E = \{\langle x, h_E(x)\rangle : x \in U\},$$

where $h_E(x) \in 2^{[0,1]}$ is a set of values in $[0,1]$, denoting the possible membership degrees of the element $x \in U$ to the set E. For simplicity, $h_E(x)$, called a hesitant fuzzy element (HFE), is abbreviated as $h(x)$ or h in this paper. Suppose the set of all hesitant fuzzy sets (HFSs) on U is denoted by $H(U)$. Obviously, $\forall E \in H(U)$. The complement of E is denoted as E^C, $h_{E^C}(x) = h^C(x) = \{1 - h : \forall h \in h(x)\}$.

Considering the randomness of values in the HFS and that the number of them in different HFSs may be unequal, we stipulate that $\Im h(x)$ represents the number of the possible membership degrees in $h(x)$, $h(x) = \{h^{\sigma(1)}(x), h^{\sigma(2)}(x), \cdots, h^{\sigma(n)}(x)\}$ which implies that $h(x)$ is arranged in increasing order, where $n = \Im h(x)$ and $h^{\sigma(j)}(x)$ $(j = 1, 2, \cdots, n)$ is the j_{th} minimum value in $h(x)$. On this basis, the following criterion [7,8,34] can be obtained.

Definition 2. *For a given HFE h, $S(h) = \frac{1}{\Im h}\sum_{j=1}^{\Im h} h^{\sigma(j)}$ is called the score function of h, where $\Im h$ is the number of elements in h, and we specify that $\frac{0}{0} = 0$. For two HFEs, h_1 and h_2, we have* *the following comparison rules: if $S(h_1) > S(h_2)$, then $h_1 \succ h_2$; if $S(h_1) = S(h_2)$, then $h_1 \approx h_2$; if $S(h_1) \geq S(h_2)$, then $h_1 \succcurlyeq h_2$. Otherwise, it is the opposite. It is important to note that the " $\prec$ " and " $\succ$ " indicate preference relations here.*

Remark 1. (1) *E is referred to as an empty HFS [3] if $h(x) = \{0\}$ for all x in U, and the empty HFS is denoted by in this paper.*
(2) *E is referred to as a full HFS [3] if $h(x) = \{1\}$ for all x in U, and the empty HFS is denoted by U in this paper.*

2.2. Graded Rough Set

Let U be a reference set and R be an equivalence relation of $U \times U$. The equivalence relation R induces a partition of U, denoted by $U/R = \{[x]_R : x \in U\}$, where $[x]_R$ represents the equivalence class of x with regard to R. In a Pawlak rough set, an arbitrary subset X of U can characterized by the upper approximation $\overline{R}(X) = \{x \in U : [x]_R \cap X \neq \varnothing\} = \cup\{[x]_R : [x]_R \cap X \neq \varnothing\}$ and the lower approximation $\underline{R}(X) = \{x \in U : [x]_R \subseteq X\} = \cup\{[x]_R : [x]_R \subseteq X\}$. Furthermore, suppose k is a non-negative integer and is called grade, the grade k upper and lower approximations of X are defined as follows [36]:

$$\overline{R}_k(X) = \{x \in U : |[x]_R \cap X| > k\} = \cup\{[x]_R \in U/R : |[x]_R \cap X| > k\},$$
$$\underline{R}_k(X) = \{x \in U : |[x]_R| - |[x]_R \cap X| \leqslant k\} = \cup\{[x]_R \in U/R : |[x]_R| - |[x]_R \cap X| \leqslant k\}.$$

If $\overline{R}_k(X) = \underline{R}_k(X)$, then X is called a definable set by grade k; otherwise, X is called a rough set by grade k. $\overline{R}_k$ and $\underline{R}_k$ are called grade k upper and lower approximation operators, respectively. They have the following properties:

(1) $\overline{R}_k(\phi) = \phi, \underline{R}_k(\phi) = \{x \in U : |[x]_R| \leqslant k\}$;
(2) $\overline{R}_k(U) = \{x \in U : |[x]_R| > k\}, \underline{R}_k(U) = U$;
(3) $\overline{R}_k(X^C) = (\underline{R}_k(X))^C, \underline{R}_k(X^C) = (\overline{R}_k(X))^C$;
(4) $k \geqslant 1 \Longleftrightarrow \overline{R}_k(X) \subseteq \overline{R}_1(X), \underline{R}_k(X) \supseteq \underline{R}_1(X)$.

In particular, $\overline{R}_k(X) = \overline{R}(X)$ and $\underline{R}_k(X) = \underline{R}(X)$ when $k = 0$, and the GRS model degenerates into the classical Pawlak rough set model. We have to point out here that the inclusion relationship between $\overline{R}_k(H)$ and $\underline{R}_k(H)$ is generally unfounded.

$$POS_k(X) = \overline{R}_k(X) \cap \underline{R}_k(X); \qquad NEG_k(X) = (\overline{R}_k(X) \cup \underline{R}_k(X))^C;$$
$$UBN_k(X) = \overline{R}_k(X) - \underline{R}_k(X); \qquad LBN_k(X) = \underline{R}_k(X) - \overline{R}_k(X);$$
$$BND_k(X) = UBN_k(X) \cup LBN_k(X)$$

are called the grade k positive region, negative region, upper boundary region, lower boundary region, and boundary region with respect to X, respectively.

2.3. Variable Precision Rough Set

Inspired by [31], the loss function of a decision-theoretic rough set (DTRS) is considered in particular and then the probability approximations of the variable precision rough set (VPRS) model can be obtained according to the decision rules. As an extension model of the Pawlak rough set, the VPRS model is an effective tool to characterize the upper and lower approximations of Bayesian decision process using the conditional probability and two thresholds α and β. We will briefly introduce the concept of the VPRS which can be confirmed in [31,33]

In accordance with the Bayesian decision procedure, suppose that $\Omega = \{A, A^C\}$ and $\mathscr{A} = \{a_1, a_2, a_3\}$ are a finite set of states and a finite set of three possible actions, respectively. For convenience, A and A^C are recorded as A_1 and A_2, respectively. $P(A_j|x)(j = 1, 2)$ are the conditional probabilities of an object x being in state A given that the object is described by x. $\lambda(a_i|A_j)(i = 1, 2, 3; j = 1, 2)$ express the loss or cost for taking action a_i when the states are A_j. Moreover, the expected loss function associated with the action a_i is $R(a_i|x) = \sum_{j=1}^{2} \lambda(a_i|A_j)P(A_j|x)(i = 1, 2, 3; j = 1, 2)$, which represents the expected loss of action a_i under the state A_j. Then, the minimum risk decision rules are as follows:

$(P-0)$ If $R(a_1|[x]) \leqslant R(a_2|[x])$ and $R(a_1|[x]) \leqslant R(a_3|[x])$, decide $x \in POS(A)$;
$(N-0)$ If $R(a_1|[x]) \geqslant R(a_2|[x])$ and $R(a_2|[x]) \leqslant R(a_3|[x])$, decide $x \in NEG(A)$;
$(B-0)$ If $R(a_3|[x]) \leqslant R(a_1|[x])$ and $R(a_3|[x]) \leqslant R(a_2|[x])$, decide $x \in BND(A)$.

A special case of loss function and calculation rules of parameters α, β, and γ are considered as follows: $(\lambda_{PN}-\lambda_{BN})(\lambda_{NP}-\lambda_{BP}) \geqslant (\lambda_{BN}-\lambda_{NN})(\lambda_{BP}-\lambda_{PP})$, and $\alpha = \frac{\lambda_{PN}-\lambda_{BN}}{(\lambda_{BP}-\lambda_{BN})-(\lambda_{PP}-\lambda_{PN})}$, $\beta = \frac{\lambda_{BN}-\lambda_{NN}}{(\lambda_{NP}-\lambda_{NN})-(\lambda_{BP}-\lambda_{BN})}$, $\gamma = \frac{\lambda_{PN}-\lambda_{NN}}{(\lambda_{NP}-\lambda_{NN})-(\lambda_{PP}-\lambda_{PN})}$. Then, we can derive $\alpha \geqslant \gamma \geqslant \beta$ and the following minimum risk decision rules:
(P) If $P(A|[x]) \geqslant \alpha$, then decide $x \in POS(A)$;
(N) If $P(A|[x]) \leqslant \beta$, then decide $x \in NEG(A)$;
(B) If $\beta < P(A|[x]) < \alpha$, then decide $x \in BND(A)$.

Meanwhile, the upper and lower approximations with respect to X of the VPRS model are obtained as follows:

$$\overline{R_{(\alpha,\beta)}}(X) = \{x \in U : P(X|[x]_R) > \beta\};$$
$$\underline{R_{(\alpha,\beta)}}(X) = \{x \in U : P(X|[x]_R) \geqslant \alpha\},$$

where $P(X|[x]_R) = \frac{|[x]_R \cap X|}{|[x]_R|}$ refers to the relative number of the elements belonging to X in equivalence class $[x]_R$. $POS_{(\alpha,\beta)}(X) = \underline{R_{(\alpha,\beta)}}(X)$, $NEG_{(\alpha,\beta)}(X) = (\overline{R_{(\alpha,\beta)}}(X))^C$, and $BND_{(\alpha,\beta)}(X) = \overline{R_{(\alpha,\beta)}}(X) - \underline{R_{(\alpha,\beta)}}(X)$ are the positive region, negative region, and boundary region with respect to X, respectively.

2.4. Double-Quantitative Decision-Theoretic Rough Fuzzy Set

With a view to the method of information fusion that can improve fault-tolerance ability, Fan and Xu et al. [35] proposed two models that combine logical composite operators and logical disjunctive operators. The details mentioned below refer to [46]. It should be pointed out that $0 \leqslant \beta < \alpha < 1$, $\widetilde{X} \in \mathscr{F}(U)$ for any $x \in U$, where the $\mathscr{F}(U)$ stands for all fuzzy sets of U.

The first kind of double-quantitative decision-theoretic rough fuzzy set (Dq-DTRFS) model called the logical conjunction Dq-DTRFS model ($\wedge$-Dq-DTRFS) is denoted by $(U, \overline{R_{k\wedge(\alpha,\beta)}}(\widetilde{X}), \underline{R_{k\wedge(\alpha,\beta)}}(\widetilde{X}))$, where $\overline{R_{k\wedge(\alpha,\beta)}}$ and $\underline{R_{k\wedge(\alpha,\beta)}}$ are approximation operators. For a fuzzy set $\widetilde{X}$, the corresponding upper and lower approximations are

$$\overline{R_{k\wedge(\alpha,\beta)}}(\widetilde{X}) = \{x \in U : P(\widetilde{X}|[x]_R) > \beta,\ \Sigma_{y\in[x]_R}\widetilde{X}(y) > k\};$$
$$\underline{R_{k\wedge(\alpha,\beta)}}(\widetilde{X}) = \{x \in U : P(\widetilde{X}|[x]_R) \geqslant \alpha,\ \Sigma_{y\in[x]_R}(1-\widetilde{X}(y)) \leqslant k\},$$

and the positive region, negative region, upper and lower boundary region with respect to $\widetilde{X}$ are defined as follows:
$POS_k^{\wedge}(\widetilde{X}) = \overline{R_{k\wedge(\alpha,\beta)}}(\widetilde{X}) \cap \underline{R_{k\wedge(\alpha,\beta)}}(\widetilde{X})$; $NEG_k^{\wedge}(\widetilde{X}) = (\overline{R_{k\wedge(\alpha,\beta)}}(\widetilde{X}) \cup \underline{R_{k\wedge(\alpha,\beta)}}(\widetilde{X}))^C$;
$UBN_k^{\wedge}(\widetilde{X}) = \overline{R_{k\wedge(\alpha,\beta)}}(\widetilde{X}) - \underline{R_{k\wedge(\alpha,\beta)}}(\widetilde{X})$; $LBN_k^{\wedge}(\widetilde{X}) = \underline{R_{k\wedge(\alpha,\beta)}}(\widetilde{X}) - \overline{R_{k\wedge(\alpha,\beta)}}(\widetilde{X})$.

Naturally, we have the following decision rules for the $\wedge$-Dq-DTRFS model:
$Des([x]_R) \rightarrow Des_{P_k^{\wedge}}(\widetilde{X})$, for $x \in POS_k^{\wedge}(\widetilde{X})$;
$Des([x]_R) \rightarrow Des_{N_k^{\wedge}}(\widetilde{X})$, for $x \in NEG_k^{\wedge}(\widetilde{X})$;
$Des([x]_R) \rightarrow Des_{UB_k^{\wedge}}(\widetilde{X})$, for $x \in UBN_k^{\wedge}(\widetilde{X})$;
$Des([x]_R) \rightarrow Des_{LB_k^{\wedge}}(\widetilde{X})$, for $x \in LBN_k^{\wedge}(\widetilde{X})$.

The second kind of Dq-DTRFS model called the logical disjunction Dq-DTRFS model ($\vee$-Dq-DTRFS) is denoted by $(U, \overline{R_{k\vee(\alpha,\beta)}}(\widetilde{X}), \underline{R_{k\vee(\alpha,\beta)}}(\widetilde{X}))$, where $\overline{R_{k\vee(\alpha,\beta)}}$ and $\underline{R_{k\vee(\alpha,\beta)}}$ are approximation operators. For a fuzzy set $\widetilde{X}$, the corresponding upper and lower approximations are

$$\overline{R_{k\vee(\alpha,\beta)}}(\widetilde{X}) = \{x \in U : P(\widetilde{X}|[x]_R) > \beta \text{ or } \Sigma_{y\in[x]_R}\widetilde{X}(y) > k\};$$
$$\underline{R_{k\vee(\alpha,\beta)}}(\widetilde{X}) = \{x \in U : P(\widetilde{X}|[x]_R) \geqslant \alpha \text{ or } \Sigma_{y\in[x]_R}(1-\widetilde{X}(y)) \leqslant k\},$$

and the positive region, negative region, upper and lower boundary regions with respect to $\widetilde{X}$ are defined as follows:

$POS_k^{\vee}(\widetilde{X}) = \overline{R_{k\vee(\alpha,\beta)}}(\widetilde{X}) \cap \underline{R_{k\vee(\alpha,\beta)}}(\widetilde{X})$; $NEG_k^{\vee}(\widetilde{X}) = (\overline{R_{k\vee(\alpha,\beta)}}(\widetilde{X}) \cup \underline{R_{k\vee(\alpha,\beta)}}(\widetilde{X}))^C$;

$UBN_k^{\vee}(\widetilde{X}) = \overline{R_{k\vee(\alpha,\beta)}}(\widetilde{X}) - \underline{R_{k\vee(\alpha,\beta)}}(\widetilde{X})$; $LBN_k^{\vee}(\widetilde{X}) = \underline{R_{k\vee(\alpha,\beta)}}(\widetilde{X}) - \overline{R_{k\vee(\alpha,\beta)}}(\widetilde{X})$.

Naturally, we have the following decision rules for the $\vee$-Dq-DTRFS model:

$Des([x]_R) \to Des_{P_k^{\vee}}(\widetilde{X})$, for $x \in POS_k^{\vee}(\widetilde{X})$;

$Des([x]_R) \to Des_{N_k^{\vee}}(\widetilde{X})$, for $x \in NEG_k^{\vee}(\widetilde{X})$;

$Des([x]_R) \to Des_{UB_k^{\vee}}(\widetilde{X})$, for $x \in UBN_k^{\vee}(\widetilde{X})$;

$Des([x]_R) \to Des_{LB_k^{\vee}}(\widetilde{X})$, for $x \in LBN_k^{\vee}(\widetilde{X})$.

3. Two Single-Quantitative Models under Hesitant Fuzzy Environment

In real-life applications, the decision makers often struggle to reach agreement on final decision outcomes due to the pursuit of different goals. Apart from that, the discrepancy of the individual status of the decision makers and the different interest orientations lead to complexity in decision-making results, which further enhances the difficulty of group decision-making problems. In order to enhance the scalability and applicability of these models described above, the specific semantics of complex decision problems in the practical application must be considered. Accordingly, it is essential to establish the basic framework of the double-quantitative decision models in a hesitant fuzzy environment (HFEn) in this section.

3.1. Graded Rough Set in HFEn

This subsection establishes the GRS-HFEn model by combining the hesitant fuzzy aggregation operator with the GRS model, and discusses properties of lower and upper approximation operators of GRS-HFEn.

Definition 3. *Let $IT = (U, CS \cup DS, V, F)$ be an information table with hesitant fuzzy decision information, where U is a non-empty finite universe. For any $H \in H(U)$, $x \in U$, k is a non-negative integer and called "grade". In the graded rough set model in a hesitant fuzzy environment (GRS-HFEn), the definitions of the upper and lower approximations of set H based on the equivalence relation R are given as follows:*

$$\overline{R_k}(H) = \{x \in U : \sum_{j=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x) > k\} = \bigcup\{[x]_R \in U/R : \sum_{j=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x) > k\};$$
$$\underline{R_k}(H) = \{x \in U : \sum_{j=1}^{\Im\psi_S} (\psi_S^{\sigma(j)}(x))^C \le k\} = \bigcup\{[x]_R \in U/R : \sum_{j=1}^{\Im\psi_S} (\psi_S^{\sigma(j)}(x))^C \le k\},$$

where $\psi_S(x) = \{S(H(x)) : x \in [x]_R\}$. $H(U)$ denotes all hesitant fuzzy subsets of U, $\psi_S^{\sigma(j)}(x)$ indicates the j_{th} minimum value in $\psi_S(x)$, and $\Im\psi_S$ is the number of the elements of $\psi_S(x)$. If $\overline{R_k}(H) = \underline{R_k}(H)$, then H is called an accurate set, otherwise H is called a rough set. $\overline{R_k}$ and $\underline{R_k}$ are upper and lower approximation operators, respectively.

Proposition 1. *Let $IT = (U, CS \cup DS, V, F)$ be an information table with hesitant fuzzy decision information, where U represents a non-empty finite universe. For any $x \in U$, k is a non-negative integer and called "grade". The following properties are satisfied:*

(1)$\overline{R_k}(\phi) = \phi$, $\underline{R_k}(\phi) = \{x \in U : \Im\psi_S(x) \leqslant k\}$;

(2)$\overline{R_k}(U) = \{x \in U : \Im\psi_S(x) > k\}$, $\underline{R_k}(U) = U$;

(3)$\overline{R_k}(H^C) = (\underline{R_k}(H))^C, \underline{R_k}(H^C) = (\overline{R_k}(H))^C$;
(4)$k \geqslant 1 \Longleftrightarrow \overline{R_k}(H) \subseteq \overline{R_1}(H), \underline{R_k}(H) \supseteq \underline{R_1}(H)$.

Proof. It is not difficult to obtain that (1) and (2) are valid by Definition 3. The proof processes of (3) and (4) are presented as follows:

(3) By the definition of the complement set $h_{E^C}(x)$ of E in Definition 1 and the definition of score function $S(h)$ of h in Definition 2, we can know that $S(h^C(x)) = \frac{1}{\Im h}\sum_{j=1}^{\Im h}(h^{\sigma(j)}(x))^C$. Then, we can obtain that $\overline{R_k}(H^C) = \{x \in U : \sum_{j=1}^{\Im \psi_S}(\psi_S^{\sigma(j)}(x))^C > k\} = (\underline{R_k}(H))^C$ by $\psi_S(x) = \{S(H^C(x)) : x \in [x]_R\}$. Thus, the first equation in (3) has been proved. The proof of the second equation in (3) can be verified similarly.

(4) " $\Rightarrow$ " When it is clear that $k \geqslant 1$, on the one hand, we have $\sum_{j=1}^{\Im \psi_S} \psi_S^{\sigma(j)}(x) > k \geqslant 1$ if any $x \in \overline{R_k}(H)$. That is to say, $\sum_{j=1}^{\Im \psi_S} \psi_S^{\sigma(j)}(x) > 1$. Thus, we can find that $x \in \overline{R_1}(H)$ by Definition 3. On the other hand, we can obtain that $\sum_{j=1}^{\Im \psi_S}(\psi_S^{\sigma(j)}(x))^C \leq 1$ if any $x \in \underline{R_1}(H)$, then we deduce that $\sum_{j=1}^{\Im \psi_S}(\psi_S^{\sigma(j)}(x))^C \leq k$. Therefore, we derive that $x \in \underline{R_k}(H)$. With the above analysis, it shows that the sufficiency is tenable.

" $\Rightarrow$ " Analogously, the necessity can also be proved.

Analogously, the positive region, negative region, upper boundary region, lower boundary region, and boundary region with respect to hesitant fuzzy set H are as follows:

$POS_k(H) = \overline{R_k}(H) \cap \underline{R_k}(H)$;
$NEG_k(H) = (\overline{R_k}(H) \cup \underline{R_k}(H))^C$;
$UBN_k(H) = \overline{R_k}(H) - \underline{R_k}(H)$;
$LBN_k(H) = \underline{R_k}(H) - \overline{R_k}(H)$;
$BND_k(H) = UBND_k(X) \cup LBND_k(H)$.
□

Corollary 1. *Let U be a non-empty finite universe, R represents the equivalence relation on U, and, for each $H \in H(U)$, we have the following conclusion:*

H is a accurate set $\Longleftrightarrow BND_k(H) = \phi$.

Proof. It is only proof of sufficiency because definitions of the accurate set and the $BND_k(H)$ show that the necessity is clear. It is easy to verify that $BND_k(H) = \phi$ if and only if $UBN_k(H) \cup LBN_k(H) = \phi$. Therefore, we can clearly obtain that $UBN_k(H) = \phi$ and $LBN_k(H) = \phi$, that is, $UBN_k(H) = \overline{R_k}(H) - \underline{R_k}(H) = \phi$ and $LBN_k(H) = \underline{R_k}(H) - \overline{R_k}(H) = \phi$. Accordingly, $\underline{R_k}(H) = \overline{R_k}(H)$. □

Corollary 2. *The number of values of the membership degree will become 1 if H is degenerated into an original fuzzy set $\widetilde{X} \subseteq \mathscr{F}(U)$, then the GRS-HFEn model can be degenerated into the form of the GRFS model [46]. In light of the conclusions in [46], the GRFS model is extended from the original GRS model, so we can be sure that the GRS-HFEn model is an extended form of the GRS model.*

Proof. The corresponding proof refers to [46]. □

3.2. Variable Precision Rough Set in HFEn

Benefiting from the Bayesian decision theory and the original probability approximations, this subsection mainly discusses the VPRS-HFEn model, which is a generalization model of VPRSs in the practical application.

Let U be a non-empty finite universe, R be an equivalence relation of U, and P be probabilistic measure. For any $H \in H(U)$, $P(H|[x]_R)$ can be explained as the conditional probability of the event H given the description $[x]_R$. The definition of the $P(H|[x]_R)$ is as follows:

$$P(H|[x]_R) = \frac{\sum_{j=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x)}{\Im\psi_S},$$

where $\psi_S(x) = \{S(H(x)) : x \in [x]_R\}$. $\psi_S^{\sigma(j)}(x)$ indicates the j_{th} minimum value in $\psi_S(x)$ and $\Im\psi_S$ is the number of elements of $\psi_S(x)$. It needs to be pointed out that $P(H|[x]_R) + P(H^C|[x]_R) = 1$ is always true.

It is necessary to note that $P(\widetilde{X}|[x]_R) = \frac{\sum_{y\in[x]_R} \widetilde{X}(y)}{|[x]_R|}$ when H degenerates into a traditional fuzzy set $\widetilde{X} \subseteq \mathscr{F}(U)$, and the $\mathscr{F}(U)$ means all fuzzy sets of U. It is consistent with the conditional probabilistic measure in [46], so it can be concluded that $P(\widetilde{X}|[x]_R)$ is a special case of $P(H|[x]_R)$ in this paper, in other words, $P(H|[x]_R)$ is a generalization of $P(\widetilde{X}|[x]_R)$. The literature [46] indicates that the $P(X|[x]_R)$ is a special case of the $P(\widetilde{X}|[x]_R)$ when fuzzy set $\widetilde{X}$ degenerates into a classical set X. Therefore, $P(X|[x]_R)$ is a special case of the $P(H|[x]_R)$, that is, the $P(H|[x]_R)$ is an extended form of the $P(X|[x]_R)$.

Suppose $\Omega = \{H, H^C\}$ and $\mathscr{A} = \{a_P, a_B, a_N\}$ are a finite set of states and a finite set of three possible actions, respectively. $\Omega = \{H, H^C\}$ represents that an element x, whether in object set X, a_P, a_B, and a_N, symbolizes $x \in POS(H)$, $x \in BND(H)$, and $x \in NEG(H)$ when classifying object x, respectively. A description of the risk or cost of taking different actions under different conditions is shown in Table 1.

Table 1. The hesitant fuzzy loss function.

	$\boldsymbol{H(P)}$	$\boldsymbol{H^C(N)}$
a_P	$h_{\lambda_{PP}}$	$h_{\lambda_{PN}}$
a_N	$h_{\lambda_{NP}}$	$h_{\lambda_{NN}}$
a_B	$h_{\lambda_{BP}}$	$h_{\lambda_{BN}}$

Where $h_{\lambda_{PP}}$, $h_{\lambda_{BP}}$, $h_{\lambda_{NP}}$ are loss degrees induced by actions a_P, a_B, a_N, respectively. Then, $\Im h_{\lambda_{PP}}$, $\Im h_{\lambda_{BP}}$, $\Im h_{\lambda_{NP}}$ are numbers of $h_{\lambda_{PP}}$, $h_{\lambda_{BP}}$, $h_{\lambda_{NP}}$, respectively. Similarly, $h_{\lambda_{PN}}$, $h_{\lambda_{BN}}$, $h_{\lambda_{NN}}$ are loss degrees suffered by actions a_P, a_B, a_N, respectively. Then, $\Im h_{\lambda_{PN}}$, $\Im h_{\lambda_{BN}}$ and $\Im h_{\lambda_{NN}}$ are numbers of $h_{\lambda_{PN}}$, $h_{\lambda_{BN}}$, and $h_{\lambda_{NN}}$, respectively. In addition, there is a reality that the nether expected losses can be found:

$$R(a_P|[x]_R) = h_{\lambda_{PP}} P(H|[x]_R) + h_{\lambda_{PN}} P(H^C|[x]_R);$$
$$R(a_N|[x]_R) = h_{\lambda_{NP}} P(H|[x]_R) + h_{\lambda_{NN}} P(H^C|[x]_R);$$
$$R(a_B|[x]_R) = h_{\lambda_{BP}} P(H|[x]_R) + h_{\lambda_{BN}} P(H^C|[x]_R).$$

By the statement in [34], it is easy to see $\alpha \in (0,1]$, $\beta \in [0,1)$, and $\gamma \in (0,1)$ when we make reasonable assumptions about loss functions $h_{\lambda_{PP}} \preccurlyeq h_{\lambda_{BP}} \prec h_{\lambda_{NP}}$ and $h_{\lambda_{NN}} \preccurlyeq h_{\lambda_{BN}} \prec h_{\lambda_{PN}}$ to simplify classification rules. Meanwhile, they are prerequisites for the VPRS-HFEn model in this article. Going a step further, minimum-risk hesitant fuzzy decision rules are written as:

$(P-1)$ If $P(H|[x]_R) \geqslant \alpha$ and $P(H|[x]_R) \geqslant \gamma$, then decide $x \in POS(H)$;
$(N-1)$ If $P(H|[x]_R) \leqslant \beta$ and $P(H|[x]_R) \leqslant \gamma$, then decide $x \in NEG(H)$;
$(B-1)$ If $P(H|[x]_R) \leqslant \alpha$ and $P(H|[x]_R) \geqslant \beta$, then decide $x \in BND(H)$.

The calculation of parameters α, β, and γ are recommended to follow several rules:

$$\alpha = \frac{S(h_{\lambda_{PN}}) - S(h_{\lambda_{BN}})}{(S(h_{\lambda_{BP}}) - S(h_{\lambda_{BN}})) - (S(h_{\lambda_{PP}}) - S(h_{\lambda_{PN}}))};$$

$$\beta = \frac{S(h_{\lambda_{BN}}) - S(h_{\lambda_{NN}})}{(S(h_{\lambda_{NP}}) - S(h_{\lambda_{NN}})) - (S(h_{\lambda_{BP}}) - S(h_{\lambda_{BN}}))};$$

$$\gamma = \frac{S(h_{\lambda_{PN}}) - S(h_{\lambda_{NN}})}{(S(h_{\lambda_{NP}}) - S(h_{\lambda_{NN}})) - (S(h_{\lambda_{PP}}) - S(h_{\lambda_{PN}}))}.$$

If the corresponding score functions and loss functions satisfy the condition $(S(h_{\lambda_{PN}}) - S(h_{\lambda_{BN}}))(S(h_{\lambda_{NP}}) - S(h_{\lambda_{BP}})) \geqslant (S(h_{\lambda_{BN}}) - S(h_{\lambda_{NN}}))(S(h_{\lambda_{BP}}) - S(h_{\lambda_{PP}}))$, then we can obtain that $\beta \leqslant \gamma \leqslant \alpha$, even in the case of $0 \leqslant \beta < \gamma < \alpha \leqslant 1$ when $\alpha > \beta$. Three decision rules of the VPRS-HFEn model can be written as follows:

$(P-2)$ If $P(H|[x]_R) \geqslant \alpha$, decide $x \in POS(H)$;
$(N-2)$ If $P(H|[x]_R) \leqslant \beta$, decide $x \in NEG(H)$;
$(B-2)$ If $\beta < P(H|[x]_R) < \alpha$, decide $x \in BND(H)$.

On the basis of the above rules, the probability rough approximations in HFEn can be formed. The upper and lower approximations of the new VPRS-HFEn model are defined as below:

$$\overline{R_{(\alpha,\beta)}}(H) = \{x \in U : P(H|[x]_R) > \beta\};$$
$$\underline{R_{(\alpha,\beta)}}(H) = \{x \in U : P(H|[x]_R) \geqslant \alpha\}.$$

If $\overline{R_{(\alpha,\beta)}}(H) = \underline{R_{(\alpha,\beta)}}(H)$, then H is a definable set, otherwise H is rough. $\overline{R_{(\alpha,\beta)}}$ and $\underline{R_{(\alpha,\beta)}}$ are upper and lower approximation operators, respectively. In light of $\alpha > \beta$, it should be noted that the lower approximation must be included in the upper approximation in this model. The positive region, negative region, and boundary region with respect to hesitant fuzzy set H are as follows:

$POS_{(\alpha,\beta)}(H) = \overline{R_{(\alpha,\beta)}}(H) \cap \underline{R_{(\alpha,\beta)}}(H)$;
$NEG_{(\alpha,\beta)}(H) = (\overline{R_{(\alpha,\beta)}}(H) \cup \underline{R_{(\alpha,\beta)}}(H))^C$;
$BND_{(\alpha,\beta)}(H) = \overline{R_{(\alpha,\beta)}}(H) - \underline{R_{(\alpha,\beta)}}(H)$.

Theorem 1. *The previous upper and lower approximations with respect to H under equivalence relation R in the VPRS-HFEn model can be re-arranged as follows:*

$$\overline{R_{(\alpha,\beta)}}(H) = \{x \in U : \sum_{j=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x) > \beta\Im\psi_S\};$$
$$\underline{R_{(\alpha,\beta)}}(H) = \{x \in U : \sum_{j=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x) \geqslant \alpha\Im\psi_S\},$$

where $\psi_S(x) = \{S(H(x)) : x \in [x]_R\}$, $\psi_S^{\sigma(j)}(x)$ indicates the j_{th} minimum value in $\psi_S(x)$ and $\Im\psi_S$ is the number of the elements of $\psi_S(x)$.

Proof. It can be confirmed by the definitions of $P(H|[x]_R)$ and the definitions of upper and lower approximations $\overline{R_{(\alpha,\beta)}}(H)$ and $\underline{R_{(\alpha,\beta)}}(H)$. □

Proposition 2. *The following basic properties hold for the upper and lower approximations of the VPRS-HFEn model:*

(1) $\overline{R_{(\alpha,\beta)}}(\phi) = \underline{R_{(\alpha,\beta)}}(\phi) = \phi$;
(2) $\overline{R_{(\alpha,\beta)}}(U) = \underline{R_{(\alpha,\beta)}}(U) = U$;
(3) $\underline{R_{(\alpha,\beta)}}(H) \subseteq \overline{R_{(\alpha,\beta)}}(H)$.

Proof. It is easy to confirm (1) and (2) for $\alpha > \beta$ by the definition of the upper and lower approximations in the VPRS-HFEn model. The proof process of (3) is as follows: Based on $\alpha > \beta$, we deduce that $\alpha\Im\psi_S > \beta\Im\psi_S$ and, consequently, $x \in \overline{R_{(\alpha,\beta)}}(H)$ if $x \in \underline{R_{(\alpha,\beta)}}(H)$. To sum up, $\underline{R_{(\alpha,\beta)}}(H) \subseteq \overline{R_{(\alpha,\beta)}}(H)$ has been completed. □

Corollary 3. *The number of values of the membership degree will become 1 if H is degenerated into an original fuzzy set $\widetilde{X} \subseteq \mathscr{F}(U)$, the VPRS-HFEn model can be degenerated into the form of the DTRFS model [46]. On account of conclusions in [46], the DTRFS model is extended from* *the DTRS model. Therefore, we can obtain that the VPRS-HFEn model is an extended model of the VPRS model.*

4. Double-Quantitative Decision-Making Models in HFEn

In this section, according to the basic framework of the established model, we construct two double-quantitative decision-making models with logical conjunction operation and logical disjunction operation, respectively. As shown in Figure 2, from the perspective of improving fault-tolerance ability, the relative and absolute quantitative information, logical conjunction operation, and logical disjunction operation will be considered in a staggered manner. The new models effectively avoid the singleness of information selection in the previous two quantification models, but also enhance the fault-tolerance ability in the application. Meanwhile, the new model also takes into account the subjective assumptions and hesitations of decision makers. The feasibility and effectiveness of the original models are greatly improved in this way, and the two new models have strong desirability and applicability in many fields, such as data mining, machine learning, and group decision making.

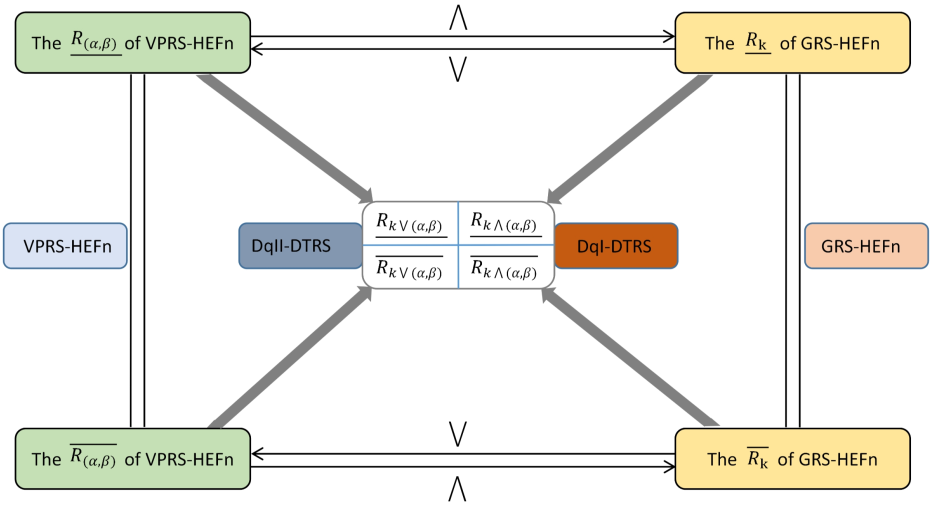

Figure 2. Formation of two double-quantitative models.

4.1. Model 1: A Double-Quantitative Method for Decision Models Based on Conjunction Operation in HFEn

Based on the above analysis, we need to take the conjunction operation into account to combine the previous two single-quantitative models, i.e., the VPRS-HFEn model and the GRS-HFEn model. Then, a new model named Model 1 will be formed and denoted by $(U, \overline{R_I}(H), \underline{R_I}(H))$.

Definition 4. *Let $IT = (U, CS \cup DS, V, F)$ be a hesitant fuzzy decision information system where U is a non-empty finite universe. For any $H \in H(U)$, $x \in U$, $0 \leqslant \beta < \alpha \leqslant 1$, furthermore, k is a non-negative integer, which satisfies the condition $0 \leqslant k \leqslant \Im\psi_S$. In Model 1, the upper and lower approximations with respect to H based on the equivalence relation R are given as follows:*

$$\overline{R_I}(H) = \overline{R_{k\wedge(\alpha,\beta)}}(H) = \{x \in U : P(H|[x]_R) > \beta \bigwedge \sum_{j=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x) > k\},$$

$$\underline{R_I}(H) = \underline{R_{k\wedge(\alpha,\beta)}}(H) = \{x \in U : P(H|[x]_R) \geqslant \alpha \bigwedge \sum_{j=1}^{\Im\psi_S} (\psi_S^{\sigma(j)}(x))^C \leqslant k\},$$

where $\psi_S(x) = \{S(H(x)) : x \in [x]_R\}$. $\psi_S^{\sigma(j)}(x)$ *indicates the* j_{th} *minimum value in* $\psi_S(x)$ *and* $\Im\psi_S$ *represents the number of elements in* $\psi_S(x)$. $\overline{R_I}$ *and* $\underline{R_I}$ *refer to the upper and lower approximation operators, respectively. Furthermore, the positive region, negative region, upper boundary region, lower boundary region, and boundary region with respect to H are obtained by the upper approximation* $\overline{R_I}(H)$ *and lower approximation* $\underline{R_I}(H)$*:*

$POS_I(H) = \overline{R_I}(H) \cap \underline{R_I}(H)$;
$NEG_I(H) = (\overline{R_I}(H) \cup \underline{R_I}(H))^C$;
$UBN_I(H) = \overline{R_I}(H) - \underline{R_I}(H)$;
$LBN_I(H) = \underline{R_I}(H) - \overline{R_I}(H)$;
$BND_I(H) = UBN_I(H) \cup LBN_I(H)$.

Remark 2. *It is obvious that* $\overline{R_I}(H) = \overline{R_k}(H) \cap \overline{R_{(\alpha,\beta)}}(H)$ *and* $\underline{R_I}(H) = \underline{R_k}(H) \cap \underline{R_{(\alpha,\beta)}}(H)$.

Theorem 2. *By definitions of* $P(H|[x]_R)$ *and the complementary set of the HFS above, the previous upper and lower approximations with respect to set H under equivalence relation R in Model 1 can be re-arranged as follows:*

$$\overline{R_I}(H) = \{x \in U : \sum_{j=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x) > \{k \vee \beta\Im\psi_S\}\},$$

$$\underline{R_I}(H) = \{x \in U : \sum_{j=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x) \geqslant \{(\Im\psi_S - k) \vee \alpha\Im\psi_S\}\},$$

where $\psi_S(x) = \{S(H(x)) : x \in [x]_R\}$. $\psi_S^{\sigma(j)}(x)$ *indicates the* j_{th} *minimum value in* $\psi_S(x)$ *and* $\Im\psi_S$ *represents the number of elements in* $\psi_S(x)$.

Proof. Here is only a proof of the lower approximation. Since $P(H|[x]_R) = \frac{\sum_{j=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x)}{\Im\psi_S}$, $\sum_{i=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x) > \alpha\Im\psi_S$ can be easily obtained. Then, considering the definition of the complementary set of the HFS, the equation $\sum_{j=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x) = \Im\psi_S - \sum_{j=1}^{\Im\psi_S} (\psi_S^{\sigma(j)}(x))^C$ holds. Therefore, it is clear that the above inequalities are true. □

Corollary 4. *If H is degenerated into a original fuzzy set* $\widetilde{X} \subseteq \mathscr{F}(U)$*, then*

$$\overline{R_I}(\widetilde{X}) = \overline{R_{k\wedge(\alpha,\beta)}}(\widetilde{X}) = \{x \in U : P(\widetilde{X}|[x]_R) > \beta \bigwedge \sum_{y\in[x]_R} \widetilde{X}(y) > k\},$$

$$\underline{R_I}(\widetilde{X}) = \underline{R_{k\wedge(\alpha,\beta)}}(\widetilde{X}) = \{x \in U : P(\widetilde{X}|[x]_R) \geqslant \alpha \bigwedge \sum_{y\in[x]_R} (1 - \widetilde{X}(y)) \leqslant k\},$$

where the $\mathscr{F}(U)$ *means all fuzzy sets of* U*. Obviously, it is consistent with the research conclusion of the* $\bigwedge -Dq - DTRFS$ *model defined in [35], so Model 1 in this paper is a generalization of the* $\bigwedge -Dq - DTRFS$ *model. From Corollary 3.1 in [35], it can be seen that when* $\alpha = 1, \beta = 0$*, and* $k = 0$*, if a fuzzy set* $\widetilde{X}$ *degenerates into a classical set X, then the* $\bigwedge -Dq - DTRFS$ *model will degenerate into a Pawlak rough set. Therefore, Model 1 in this article is an extended model of a Pawlak rough set.*

By Theorem 1 and the idea of minimum risk decision in the Bayesian decision procedure, it can be concluded that the decision rules in Model 1 can be organized as follows:

(1) When $k \geqslant \alpha\Im\psi_S$, it is easy to validate that $k \geqslant \alpha\Im\psi_S > \beta\Im\psi_S \geqslant \Im\psi_S - k$ and $UBN_I(H)$ cannot be determined in this situation, so we can obtain decision rules as below:

(POS_I) If $\sum_{j=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x) > k$, then decide $x \in POS_I(H)$;

(NEG_I) If $\sum_{j=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x) < \alpha\Im\psi_S$, then decide $x \in NEG_I(H)$;

(LBN_I) If $\alpha\Im\psi_S \leqslant \sum_{j=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x) \leqslant k$, then decide $x \in LBN_I(H)$.

(2) When $k < \alpha\Im\psi_S$, we cannot accurately judge the four regions and $LBN_I(H)$ cannot be determined in this situation, and the following are discussed in two cases.

① When $k \leqslant \beta\Im\psi_S$, it is easy to validate that the decision rules can be expressed as:

(POS_I) If $\sum_{j=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x) \geqslant \Im\psi_S - k$, then decide $x \in POS_I(H)$;

(NEG_I) If $\sum_{j=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x) \leqslant \beta\Im\psi_S$, then decide $x \in NEG_I(H)$;

(LBN_I) If $\beta\Im\psi_S < \sum_{j=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x) < \Im\psi_S - k$, then decide $x \in UBN_I(H)$.

② When $k > \beta\Im\psi_S$, it is easy to validate that the decision rules are expressed as:

(POS_I) If $\sum_{j=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x) \geqslant \alpha\Im\psi_S$, then decide $x \in POS_I(H)$;

(NEG_I) If $\sum_{j=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x) \leqslant k$, then decide $x \in NEG_I(H)$;

(LBN_I) If $k < \sum_{j=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x) < \alpha\Im\psi_S$, then decide $x \in UBN_I(H)$,

where $\psi_S(x) = \{S(H(x)) : x \in [x]_R\}$, $\psi_S^{\sigma(j)}(x)$ indicates the j_{th} minimum value in $\psi_S(x)$ and $\Im\psi_S$ represents the number of elements in $\psi_S(x)$.

The reasoning process of the above classification is stated as follows. According to $\alpha + \beta = 1$ and $\alpha > \beta$, we can obtain that $\alpha\Im\psi_S > \beta\Im\psi_S$, and validate that $k \geqslant \alpha\Im\psi_S \Leftrightarrow \Im\psi_S - k \leqslant \beta\Im\psi_S$. Therefore, it is true for $k \geqslant \alpha\Im\psi_S$ that $k \geqslant \alpha\Im\psi_S > \beta\Im\psi_S \geqslant \Im\psi_S - k$; meanwhile, we can validate that $k < \alpha\Im\psi_S \Leftrightarrow \Im\psi_S - k > \beta\Im\psi_S$, and $k \leqslant \beta\Im\psi_S \Leftrightarrow \alpha\Im\psi_S \leqslant \Im\psi_S - k$. Then, it is true for $k < \alpha\Im\psi_S$ and $k \leqslant \beta\Im\psi_S$, thus $k \leqslant \beta\Im\psi_S < \alpha\Im\psi_S \leqslant \Im\psi_S - k$.

4.2. Model 2: A Double-Quantitative Method for Decision Models Based on Disjunction Operation in HFEn

In this subsection, we intend to take the disjunction operation into account to combine the previous two single-quantitative models. Then, a new model called Model 2 will be formed and denoted by $(U, \overline{R_{II}}(H), \underline{R_{II}}(H))$.

Definition 5. *Let $IT = (U, CS \cup DS, V, F)$ be a hesitant fuzzy decision information system where U is a non-empty finite universe. For any $H \in H(U)$, $x \in U$, $0 \leqslant \beta < \alpha \leqslant 1$ and, furthermore, k is a non-negative integer, which satisfies the condition $0 \leqslant k \leqslant \Im\psi_S$. In Model 2, the upper and lower approximations of set H based on the equivalence relation R are given as follows:*

$$\overline{R_{II}}(H) = \overline{R_{k\vee(\alpha,\beta)}}(H) = \{x \in U : P(H|[x]_R) > \beta \bigvee \sum_{j=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x) > k\},$$

$$\underline{R_{II}}(H) = \underline{R_{k\vee(\alpha,\beta)}}(H) = \{x \in U : P(H|[x]_R) \geqslant \alpha \bigvee \sum_{j=1}^{\Im\psi_S} (\psi_S^{\sigma(j)}(x))^c \leqslant k\},$$

where $\psi_S(x) = \{S(H(x)) : x \in [x]_R\}$. $\psi_S^{\sigma(j)}(x)$ indicates the j_{th} minimum value in ψ_S and $\Im\psi_S$ represents the number of elements in $\psi_S(x)$. $\overline{R_{II}}$ and $\underline{R_{II}}$ refer to the corresponding upper and lower approximation operators, respectively. Subsequently, the positive region, negative region, upper boundary region, lower boundary region, and boundary region with respect to H are obtained by the upper approximation $\overline{R_{II}}(H)$ and lower approximation $\underline{R_{II}}(H)$:

$POS_{II}(H) = \overline{R_{II}}(H) \cap \underline{R_{II}}(H)$;
$NEG_{II}(H) = (\overline{R_{II}}(H) \cup \underline{R_{II}}(H))^C$;
$UBN_{II}(H) = \overline{R_{II}}(H) - \underline{R_{II}}(H)$;
$LBN_{II}(H) = \underline{R_{II}}(H) - \overline{R_{II}}(H)$;
$BND_{II}(H) = UBN_{II}(H) \cup LBN_{II}(H)$.

Remark 3. *It obvious that $\overline{R_{II}}(H) = \overline{R_k}(H) \cup \overline{R_{(\alpha,\beta)}}(H)$ and $\underline{R_{II}}(H) = \underline{R_k}(H) \cup \underline{R_{(\alpha,\beta)}}(H)$.*

Theorem 3. *By the definitions of $P(H|[x]_R)$ and the complementary set of the HFS above, the previous upper and lower approximations with respect to set H under equivalence relation R in Model 2 can be re-arranged as follows:*

$$\overline{R_{II}}(H) = \{x \in U : \sum_{j=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x) > \{k \wedge \beta\Im\psi_S\}\},$$

$$\underline{R_{II}}(H) = \{x \in U : \sum_{j=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x) \geqslant \{\alpha\Im\psi_S \wedge (\Im\psi_S - k)\}\}.$$

Proof. The proof is similar to Theorem 2. □

Corollary 5. *If H is degenerated into an original fuzzy set $\widetilde{X} \subseteq \mathscr{F}(U)$, then*

$$\overline{R_{II}}(\widetilde{X}) = \overline{R_{k\vee(\alpha,\beta)}}(\widetilde{X}) = \{x \in U : P(\widetilde{X}|[x]_R) > \beta \bigvee \sum_{y\in[x]_R} \widetilde{X}(y) > k\},$$

$$\underline{R_{II}}(\widetilde{X}) = \underline{R_{k\vee(\alpha,\beta)}}(\widetilde{X}) = \{x \in U : P(\widetilde{X}|[x]_R) \geqslant \alpha \bigvee \sum_{y\in[x]_R} (1 - \widetilde{X}(y)) \leqslant k\},$$

where the $\mathscr{F}(U)$ represent all fuzzy sets of U. Obviously, it is the same as the research conclusion of the $\bigvee -Dq - DTRFS$ model which is defined in [35], so Model 2 in this paper is a generalization of the $\bigvee -Dq - DTRFS$ model. From Corollary 3.2 in [35], it can be obtained that when $\alpha = 1, \beta = 0$ and $k = 0$, if a fuzzy set $\widetilde{X}$ degenerates into a classical set X, then the $\bigvee -Dq - DTRFS$ model can degenerate into a Pawlak rough set. Accordingly, Model 2 in this paper is a generalized model of a Pawlak rough set.

Similar to the conclusion in Section 4.1, we can conclude that the decision rules in Model 2 are as follows:

(1) When $k \geqslant \alpha\Im\psi_S$, it is easy to validate that $k \geqslant \alpha\Im\psi_S > \beta\Im\psi_S \geqslant \Im\psi_S - k$ and $UBN_{II}(H)$ cannot be determined in this situation, so we can obtain decision rules as below:

(POS_{II}) If $\sum_{j=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x) > \beta\Im\psi_S$, then decide $x \in POS_{II}(H)$;

(NEG_{II}) If $\sum_{j=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x) < \Im\psi_S - k$, then decide $x \in NEG_{II}(H)$;

(LBN_{II}) If $\Im\psi_S - k \leqslant \sum_{j=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x) \leqslant \beta\Im\psi_S$, then decide $x \in LBN_{II}(H)$.

(2) When $k < \alpha\Im\psi_S$, we cannot accurately judge the four regions and $UBN_{II}(H)$ cannot be determined in this situation, and the following are discussed in two cases.

① When $k \leqslant \beta\Im\psi_S$, it is easy to validate that the decision rules can be expressed as:

(POS_{II}) If $\sum_{j=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x) \geqslant \alpha\Im\psi_S$, then decide $x \in POS_{II}(H)$;

(NEG_{II}) If $\sum_{j=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x) \leqslant k$, then decide $x \in NEG_{II}(H)$;

(LBN_{II}) If $k < \sum_{j=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x) < \alpha\Im\psi_S$, then decide $x \in UBN_{II}(H)$.

② When $k > \beta\Im\psi_S$, it is easy to validate that the decision rules are expressed as:

(POS_{II}) If $\sum_{j=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x) \geqslant \Im\psi_S - k$, then decide $x \in POS_{II}(H)$;

(NEG_{II}) If $\sum_{j=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x) \leqslant \beta\Im\psi_S$, then decide $x \in NEG_{II}(H)$;

(LBN_{II}) If $\beta\Im\psi_S < \sum_{j=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x) < \Im\psi_S - k$, then decide $x \in UBN_{II}(H)$.

4.3. The Comparison of the Models 1 and 2

In allusion to the definitions of the two double-quantitative models, we can find out the internal logic connections between the two models. Parameters α, β, and k in the two new models of this paper represent probability and the grade, respectively. It is obvious that $\alpha, \beta \in [0,1]$. In order to obtain a further understanding of the two new models, we will make a detailed analysis of the internal relationship between the two models under the different values of parameters α and β. The discussion falls into three categories: $\alpha + \beta = 1$, $\alpha + \beta < 1$, and $\alpha + \beta > 1$.

Case 1: Consider the relationship between α and β to meet the condition $\alpha + \beta = 1$.

Theorem 4. *If a loss function satisfies $h_{\lambda_{PP}} \preccurlyeq h_{\lambda_{NP}} \prec h_{\lambda_{BP}}$, $h_{\lambda_{BN}} \preccurlyeq h_{\lambda_{NN}} \prec h_{\lambda_{PN}}$, and $(S(h_{\lambda_{PN}}) - S(h_{\lambda_{BN}}))(S(h_{\lambda_{NP}}) - S(h_{\lambda_{BP}})) > (S(h_{\lambda_{BN}}) - S(h_{\lambda_{NN}}))(S(h_{\lambda_{BP}}) - S(h_{\lambda_{PP}}))$ for any $x \in U$, $H \in H(U)$ and $k \in N$, then it is true that $\alpha > 0.5 > \beta$, $\alpha = 1 - \beta$, and $\beta = 1 - \alpha$ and we can obtain that the following conclusions.*

(1) $\overline{R_I}(H^C) = (\underline{R_{II}}(H))^C$;
(2) $\underline{R_I}(H^C) = (\overline{R_{II}}(H))^C$.

Proof. The proof for these equations can be supported by Definitions and Theorems of Sections 3 and 4 as follows:

(1) For the left side of the equation, we know that $\overline{R_I}(H^C) = \overline{R_{(\alpha,\beta)}}(H^C) \cap \overline{R_k}(H^C)$ based on the Remark in Definition 4. By the definitions of the upper approximations of graded and variable precision rough sets, we deduce that $\overline{R_{(\alpha,\beta)}}(H^C) \cap \overline{R_k}(H^C) = \{x \in U : P(H^C|[x]_R) > \beta\} \cap \{x \in U : \Im\psi_S - \sum \psi_S^{\sigma(j)}(x) > k\}$. According to the definition of $P(H|[x]_R)$, we are able to confirm that $\overline{R_{(\alpha,\beta)}}(H^C) \cap \overline{R_k}(H^C) = \{x \in U : \Im\psi_S - \sum \psi_S^{\sigma(j)}(x) > \beta\Im\psi_S\} \cap \{x \in U : \Im\psi_S - \sum \psi_S^{\sigma(j)}(x) > k\} = \{x \in U : \sum \psi_S^{\sigma(j)}(x) < (1-\beta)\Im\psi_S\} \cap \{x \in U : \sum \psi_S^{\sigma(j)}(x) < \Im\psi_S - k\}$. After re-arranging the above formula, it can be transformed into $\{x \in U : \sum \psi_S^{\sigma(j)}(x) < \{(1-\beta)\Im\psi_S \wedge (\Im\psi_S - k)\}\} = \{x \in U : \sum \psi_S^{\sigma(j)}(x) < \{\alpha\Im\psi_S \wedge (\Im\psi_S - k)\}\}$.

For the right side, by Definition 5, Definition 3, and definitions of $P(H|[x]_R)$ and $\underline{R_{(\alpha,\beta)}}(H)$, $(\underline{R_{II}}(H))^C = (\underline{R_{(\alpha,\beta)}}(H) \cup \underline{R_k}(H))^C = (\underline{R_{(\alpha,\beta)}}(H))^C \cap (\underline{R_k}(H))^C = \{x \in U : P(H|[x]_R) < \alpha\} \cap \{x \in U : \Im\psi_S - \sum \psi_S^{\sigma(j)}(x) > k\} = \{x \in U : \sum \psi_S^{\sigma(j)}(x) < \alpha\Im\psi_S\} \cap \{x \in U : \sum \psi_S^{\sigma(j)}(x) < \Im\psi_S - k\} = \{x \in U : \sum \psi_S^{\sigma(j)}(x) < \{\alpha\Im\psi_S \wedge (\Im\psi_S - k)\}\}$.

With the above analysis, it is shown that Equation (1) is verified.

(2) Analogously, the proof is similar to (1). □

Theorem 5. *If a loss function satisfies $h_{\lambda_{PP}} \preccurlyeq h_{\lambda_{NP}} \prec h_{\lambda_{BP}}$, $h_{\lambda_{BN}} \preccurlyeq h_{\lambda_{NN}} \prec h_{\lambda_{PN}}$ and $(S(h_{\lambda_{PN}}) - S(h_{\lambda_{BN}}))(S(h_{\lambda_{NP}}) - S(h_{\lambda_{BP}})) > (S(h_{\lambda_{BN}}) - S(h_{\lambda_{NN}}))(S(h_{\lambda_{BP}}) - S(h_{\lambda_{PP}}))$ for any $x \in U$, $H \in H(U)$, and $k \in N$, then it is true that $\alpha > 0.5 > \beta$, $\alpha = 1 - \beta$, and $\beta = 1 - \alpha$ and, furthermore, we can obtain the following conclusions:*

(1) $POS_I(H^C) = NEG_{II}(H)$;
(2) $NEG_I(H^C) = POS_{II}(H)$;
(3) $UBN_I(H^C) = UBN_{II}(H)$;
(4) $LBN_I(H^C) = LBN_{II}(H)$.

Proof. We can prove it easily by Theorem 4. □

Corollary 6. *If loss functions satisfy $h_{\lambda_{PP}} \preccurlyeq h_{\lambda_{NP}} \prec h_{\lambda_{BP}}$, $h_{\lambda_{BN}} \preccurlyeq h_{\lambda_{NN}} \prec h_{\lambda_{PN}}$, and the condition $(S(h_{\lambda_{PN}}) - S(h_{\lambda_{BN}}))(S(h_{\lambda_{NP}}) - S(h_{\lambda_{BP}})) > (S(h_{\lambda_{BN}}) - S(h_{\lambda_{NN}}))(S(h_{\lambda_{BP}}) - S(h_{\lambda_{PP}}))$, then the semantic interpretation of the decision about H can be interconversion between acceptance and rejection of Models 1 and 2. The specific operations are as follows:*

(1) Accepting an object for H^C in Model 1 with $\sum_{j=1}^{\Im\psi_S} (\psi_S^{\sigma(j)}(x))^C \geqslant \alpha\Im\psi_S$ and $(\sum_{j=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x))^C > \{k \vee (\Im\psi_S - k)\}$ also means rejecting an object for H in Model 2 with $\sum_{j=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x) \geqslant \alpha\Im\psi_S$ and $\sum_{j=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x) < \{k \wedge (\Im\psi_S - k)\}$ and vice versa;

(2) *rejecting an object for H^C in Model 1 with $\sum_{j=1}^{\Im\psi_S}(\psi_S^{\sigma(j)}(x))^C \leqslant \{k \vee \beta\Im\psi_S\}$ and $\sum_{j=1}^{\Im\psi_S}(\psi_S^{\sigma(j)}(x))^C < \{\alpha\Im\psi_S \vee (\Im\psi_S - k)\}$ is equivalent to accepting an object for H in Model 2 with $\sum_{j=1}^{\Im\psi_S}\psi_S^{\sigma(j)}(x) < \alpha\Im\psi_S\}$ and $\sum_{j=1}^{\Im\psi_S}\psi_S^{\sigma(j)}(x) > \{k \wedge (\Im\psi_S - k)\}$.*

Proof. It is easy to obtain by Theorem 5. □

Remark 4. *For Case 1, loss functions must satisfy $h_{\lambda_{PP}} \preccurlyeq h_{\lambda_{NP}} \prec h_{\lambda_{BP}}$, $h_{\lambda_{BN}} \preccurlyeq h_{\lambda_{NN}} \prec h_{\lambda_{PN}}$, and the condition $(S(h_{\lambda_{PN}}) - S(h_{\lambda_{BN}}))(S(h_{\lambda_{BN}}) - S(h_{\lambda_{NN}})) = (S(h_{\lambda_{NP}}) - S(h_{\lambda_{BP}}))(S(h_{\lambda_{BP}}) - S(h_{\lambda_{PP}}))$. The classical rough set is a special case when $\alpha = 1$, $\beta = 0$, $k = 0$, and H degenerates into a fuzzy set $\widetilde{X}$.*

Case 2: Consider the relationship between α and β to meet the condition $\alpha + \beta < 1$.

Then, we can obtain that $\alpha > \beta$ and $\beta < 0.5$ hold for Case 2, and a loss function must satisfy the following condition:

$$(S(h_{\lambda_{PN}}) - S(h_{\lambda_{BN}}))(S(h_{\lambda_{BN}}) - S(h_{\lambda_{NN}})) < (S(h_{\lambda_{NP}}) - S(h_{\lambda_{BP}}))(S(h_{\lambda_{BP}}) - S(h_{\lambda_{PP}})).$$

It is worth noting that $\alpha \neq 1 - \beta$ in Case 2, so we no longer have the equivalence of accepting H^C in Model 1 and rejecting H in Model 2, nor the equivalence of rejecting H^C in Model 1 and accepting H in Model 2. Moreover, the unilateral mirror implications in Case 1 are also uncertain in Case 2. The specific reasons are as follows: Here, we can easily deduce accepting an object for H^C in Model 1 with $\sum_{j=1}^{\Im\psi_S}\psi_S^{\sigma(j)}(x) < (1-\alpha)\Im\psi_S$, $\sum_{j=1}^{\Im\psi_S}\psi_S^{\sigma(j)}(x) \leqslant k$, and $\sum_{j=1}^{\Im\psi_S}\psi_S^{\sigma(j)}(x) < \Im\psi_S - k$, and rejecting an object for H in Model 2 with $\sum_{j=1}^{\Im\psi_S}\psi_S^{\sigma(j)}(x) \leqslant \beta\Im\psi_S$, $\sum_{j=1}^{\Im\psi_S}\psi_S^{\sigma(j)}(x) \leqslant k$, and $\sum_{j=1}^{\Im\psi_S}\psi_S^{\sigma(j)}(x) < \Im\psi_S - k$. However, we cannot determine the relationship between $1 - \alpha$ and β, so the above statements are supported.

Case 3: Consider the relationship between α and β to meet the condition $\alpha + \beta > 1$.

Then, we can obtain that $\alpha > \beta$ and $\alpha > 0.5$ hold for Case 3, and a loss function must satisfy the following condition:

$$(S(h_{\lambda_{PN}}) - S(h_{\lambda_{BN}}))(S(h_{\lambda_{BN}}) - S(h_{\lambda_{NN}})) > (S(h_{\lambda_{NP}}) - S(h_{\lambda_{BP}}))(S(h_{\lambda_{BP}}) - S(h_{\lambda_{PP}})).$$

Similar to the above, the equivalence relationship between accepting H^C in Model 1 and rejecting H in Model 2 is the same as the equivalence relationship between rejecting H^C in Model 1 and accepting H in Model 2. Moreover, the unilateral mirror implications in Case 1 are also uncertain in Case 3. The specific reasons are the same as those described in Case 2.

5. A Practical Example and Analysis

With the rapid development of social economy, the annual increase in housing prices has become the main vexation of the masses. There is an ingrained thought for most families that buying a home has always been a very important thing. On the one hand, they need to consider whether they can afford the house prices; on the other hand, the views of each family member need to be taken into account. That is to say, everyone is a decision maker. There are a lot of influence factors when buying a house, such as interior design, daylighting, traffic conditions, etc. Everyone has different preferences for various factors when making a decision. For example, the investigation shows that ninety percent of families with children prefer the lighting conditions of the house; families with old people tend to choose a place where the traffic is convenient; and most newlyweds care more about the unique interior design of the house when choosing a house. In this way, it is difficult for family members to reach consensus on decisions. In response to this phenomenon, this paper adopts the idea of three-way decisions. In this section, we use the above example scenario as the basis for the problem description and solution, and further elaborate the details of these utility models with an illustrative example.

5.1. Problem Statement

In combination with the background of the house purchase and the new double-quantitative decision-making models proposed in Section 4, we need to give the definitions of the relevant parameters of the new models. In this context, the set of states $\Omega = \{H, H^C\}$ shows whether the house price is within the acceptable range of the family or not. For a given set of actions given by $\mathscr{A} = \{a_P, a_B, a_N\}$, a_P, a_B, a_N indicate that the buyer's attitude towards the house is satisfied, hesitant, or disappointed, respectively, when classifying object x. Suppose $P(H|[x])$ is the probability that the price of house x is within the affordability of the buyer, the functions $h_{\lambda_{PP}}$, $h_{\lambda_{BP}}$, $h_{\lambda_{NP}}$ in Table 1 denote the extent of loss sustained by taking action a_P, a_B, a_N when the price falls within the budget. Similarly, the functions $h_{\lambda_{PN}}$, $h_{\lambda_{BN}}$, $h_{\lambda_{NN}}$ represent the degree of loss sustained by taking action a_P, a_B, a_N, respectively, when the price is not within the scope of the budget.

In order to interpret the theories of the two new models, a hesitant fuzzy information system about houses is introduced in this paper. Let $IT = (U, CS \cup DS, V, F)$ be an information table with hesitant fuzzy decision information, where $U = \{x_1, x_2, \cdots, x_{40}\}$ indicate a finite non-empty domain of 40 houses, $CS = \{Design, traffic\}$ and $DS = \{get\}$ are a conditional attribute set and decision attribute set, respectively. Tables 2 and 3 provide statistical data on the classification of these 40 houses, where $(i, j)(i = Design, j = traffic)$ represents the rank of conditional attributes. The rough set regions will be calculated in the case that $k = 1, 2, 3, 4$.

Table 2. The information about the houses.

Houses	Design	Traffic	Get	Houses	Design	Traffic	Get
x_1	1	0	{0.2}	x_{21}	1	0	{0.5, 0.6}
x_2	0	0	{0, 0.1}	x_{22}	0	1	{0.3, 0.7}
x_3	1	2	{0.9, 0.7, 0.8}	x_{23}	2	1	{0.8, 0.4, 0.9}
x_4	1	1	{0.9, 1}	x_{24}	2	0	{0.2, 0.4, 0}
x_5	1	0	{0.6}	x_{25}	1	2	{0.7, 0.9, 0.5}
x_6	2	2	{1}	x_{26}	2	1	{0.9, 0.6, 0.7, 0.8}
x_7	2	1	{0.7, 0.6}	x_{27}	0	2	{0.4, 0.7}
x_8	0	2	{0.6, 0, 0.3}	x_{28}	2	1	{0.3, 0.6}
x_9	2	2	{0.9, 1}	x_{29}	0	1	{0, 1, 0.5, 0.3}
x_{10}	0	2	{0.6, 0.8}	x_{30}	0	0	{0, 0.1, 0.2}
x_{11}	1	1	{0.4, 0.1}	x_{31}	1	1	{0.2, 0.4, 0.3}
x_{12}	0	1	{0.7, 0.9}	x_{32}	2	2	{0.5, 0.6, 1}
x_{13}	2	0	{0.4, 0.5}	x_{33}	2	0	{0.6, 0.1, 0.2}
x_{14}	0	0	{0.1, 0.3, 0.2}	x_{34}	1	0	{0.3, 0.4}
x_{15}	1	0	{0.3}	x_{35}	1	1	{0.4, 0.6}
x_{16}	2	1	{0.9, 1}	x_{36}	2	0	{0.2, 0.3}
x_{17}	1	1	{0.7, 0.4}	x_{37}	1	0	{0.4, 0.6}
x_{18}	0	1	{0.4, 0.5}	x_{38}	1	2	{0.6, 0.4, 0.8}
x_{19}	2	0	{0.1, 0.6}	x_{39}	0	1	{0.1, 0.3}
x_{20}	2	2	{0.8, 1}	x_{40}	0	0	{0, 0.3}

Table 3. Statistical results of house classes under R and the related calculation results.

(i,j)	$[x]_R$	$\psi_S^{\sigma(j)}(x)$	$\Im\psi_S$	$\sum_{i=1}^{\Im\psi_S}\psi_S^{\sigma(i)}(x)$	$P(H\|[x]_R)$	$\sum_{i=1}^{\Im\psi_S}(\psi_S^{\sigma(i)}(x))^C$
$(0,0)$	$x_2, x_{14}, x_{30}, x_{40}$	$\{0.05, 0.1, 0.15, 0.2\}$	4	0.500	0.125	3.500
$(0,1)$	$x_{12}, x_{18}, x_{22}, x_{29}, x_{39}$	$\{0.2, 0.3, 0.45, 0.5, 0.8\}$	5	2.250	0.450	2.750
$(0,2)$	x_8, x_{10}, x_{27}	$\{0.3, 0.55, 0.7\}$	3	1.550	0.517	1.450
$(1,0)$	$x_1, x_5, x_{15}, x_{21}, x_{34}, x_{37}$	$\{0.2, 0.3, 0.35, 0.5, 0.55, 0.6\}$	6	2.500	0.417	3.500
$(1,1)$	$x_4, x_{11}, x_{17}, x_{31}, x_{35}$	$\{0.25, 0.3, 0.5, 0.55, 0.95\}$	5	2.550	0.510	2.450
$(1,2)$	x_3, x_{25}, x_{38}	$\{0.6, 0.7, 0.8\}$	3	2.100	0.700	0.900
$(2,0)$	$x_{13}, x_{19}, x_{24}, x_{33}, x_{36}$	$\{0.2, 0.25, 0.3, 0.35, 0.5\}$	5	1.550	0.310	3.450
$(2,1)$	$x_7, x_{16}, x_{23}, x_{26}, x_{28}$	$\{0.45, 0.65, 0.7, 0.75, 0.95\}$	5	3.500	0.700	1.500
$(2,2)$	x_6, x_9, x_{20}, x_{32}	$\{0.7, 0.85, 0.95, 1\}$	4	3.500	0.875	0.500

5.2. Analysis of the Problems in GRS-HFEn Model and VPRS-HFEn Model

First of all, we need to sort out the raw data of the houses in Table 2 and analyze them. Then, the related calculation results under the relation R are shown in Table 3.

From the previous relevant concepts and some of the results calculated in Table 3, the values of k are 0, 1, 2, 3, 4, so we can acquire the upper and lower approximations of the GRS-HFEn model under different values of the grade k. Furthermore, the corresponding four regions can also be calculated based on their definitions. Refer to the results in Tables 4 and 5, respectively.

Table 4. The upper and lower approximations under the different values of k in GRS-HFEn model.

k	0	1	2	3	4
$\overline{R_k}(H)$	U	$U-[x_2]_R$	$U-([x_2]_R\cup[x_8]_R\cup[x_{13}]_R)$	$[x_6]_R\cup[x_7]_R$	ϕ
$\underline{R_k}(H)$	ϕ	$[x_3]_R\cup[x_6]_R$	$[x_3]_R\cup[x_6]_R\cup[x_7]_R\cup[x_8]_R$	$U-([x_1]_R\cup[x_2]_R\cup[x_{13}]_R)$	U

Table 5. Related results of the GRS-HFEn model.

$Grade(k)$	$POS_k(H)$	$NEG_k(H)$	$UBN_k(H)$	$LBN_k(H)$
0	ϕ	ϕ	U	ϕ
1	$[x_3]_R\cup[x_6]_R$	$[x_2]_R$	$U-([x_2]_R\cup[x_3]_R\cup[x_6]_R)$	ϕ
2	$[x_3]_R\cup[x_6]_R\cup[x_7]_R$	$[x_2]_R\cup[x_{13}]_R$	$[x_1]_R\cup[x_4]_R\cup[x_{12}]_R$	$[x_8]_R$
3	$[x_6]_R\cup[x_7]_R$	$[x_1]_R\cup[x_2]_R\cup[x_{13}]_R$	ϕ	$[x_3]_R\cup[x_4]_R\cup[x_8]_R\cup[x_{12}]_R$
4	ϕ	ϕ	ϕ	U

It should be noted that the following semantic descriptions are applicable to the following models. An object in the positive region shows that the house is satisfactory; on the contrary, the house is unsatisfactory if an object is in the negative region. If an object is in the upper boundary region or in the lower boundary region, it shows that it is not yet determined whether the house is satisfactory, while the former indicates that the uncertainty tends to be satisfactory and the latter is closer to unsatisfactory. For the sake of the analysis of the regions of the VPRS-HFEn model by the losses from the Bayesian decision procedure, as shown in Table 6, we consider the values of the loss function in three cases.

Table 6. The values of loss function for three cases.

	Case 1		Case 2		Case 3	
a_P	$\{0\}$	$\{0.6,1\}$	$\{0\}$	$\{0.6,0.8\}$	$\{0\}$	$\{1\}$
a_B	$\{0.2,0.4,0.6\}$	$\{0.1,0.3\}$	$\{0.5\}$	$\{0,0.2\}$	$\{0.1,0.2,0.3\}$	$\{0.4\}$
a_N	$\{0.6,0.8\}$	$\{0\}$	$\{0.8,0.9,0.7\}$	$\{0\}$	$\{0.8,1\}$	$\{0\}$

Through the calculation of α and β, the following conclusions can be obtained: for Case 1, $\alpha=0.60$, $\beta=0.40$; for Case 2, $\alpha=0.55$, $\beta=0.25$; for Case 3, $\alpha=0.75$, $\beta=0.36$. It is clear that $\alpha+\beta=1$ in Case 1, $\alpha+\beta<1$ in Case 2, and $\alpha+\beta>1$ in Case 3. Then, we discuss these three cases in the VPRS-HFEn model with the idea of Bayesian decision theory. In these three cases, the corresponding upper and lower approximations of H and the four regions can be calculated by their definitions. For the convenience of comparative analysis, the upper and lower approximations and the results of the regions are recorded in Tables 7 and 8, respectively.

Table 7. The upper and lower approximations under different cases in the VPRS-HFEn model.

	$\overline{R}_{(\alpha,\beta)}(H)$	$\underline{R}_{(\alpha,\beta)}(H)$
Case 1	$U-([x_2]_R\cup[x_{13}]_R)$	$[x_3]_R\cup[x_6]_R\cup[x_7]_R$
Case 2	$U-[x_2]_R$	$[x_3]_R\cup[x_6]_R\cup[x_7]_R$
Case 3	$U-([x_2]_R\cup[x_{13}]_R)$	$[x_6]_R$

Table 8. The regions under different cases in the VPRS-HFEn model.

	$POS_{(\alpha,\beta)}(H)$	$NEG_{(\alpha,\beta)}(H)$	$BND_{(\alpha,\beta)}(H)$
Case 1	$[x_3]_R \cup [x_6]_R \cup [x_7]_R$	$[x_2]_R \cup [x_{13}]_R$	$[x_{12}]_R \cup [x_8]_R \cup [x_1]_R \cup [x_4]_R$
Case 2	$[x_3]_R \cup [x_6]_R \cup [x_7]_R$	$[x_2]_R$	$[x_{12}]_R \cup [x_8]_R \cup [x_1]_R \cup [x_4]_R \cup [x_{13}]_R$
Case 3	$[x_6]_R$	$[x_2]_R \cup [x_{13}]_R$	$U - ([x_2]_R \cup [x_6]_R \cup [x_{13}]_R)$

The VPRS-HFEn model and the GRS-HFEn model are two quantification models that describe the inclusion relation between the equivalence class and a basic concept from the perspective of relative and absolute quantitative information. However, it turns out that they are deficient, and the specific reasons to support this statement are expressed as follows.

For the GRS-HFEn model, Section 3 shows that the upper and lower approximations in this model can be defined by $\sum_{i=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x)$ and $\sum_{j=1}^{\Im\psi_S} (\psi_S^{\sigma(j)}(x))^C$, respectively. Therefore, a narrative should be established in theory: We can determine the consistency of the classification of two objects from different equivalence classes by the values of $\sum_{i=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x)$ and $\sum_{j=1}^{\Im\psi_S} (\psi_S^{\sigma(j)}(x))^C$. As shown in Tables 3 and 5, there are two types of classification mechanisms, and the specific performance is as follows: For one side, $\sum_{i=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x_8) = \sum_{i=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x_{13}) = 1.550$ while the values of $\sum_{j=1}^{\Im\psi_S} (\psi_S^{\sigma(j)}(x_8))^C$ and $\sum_{i=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x_{13})$ are not equal. Finally, $[x_8]_R$ and $[x_{13}]_R$ are divided into the lower boundary region and the negative region, respectively. For $[x_1]_R$ and $[x_2]_R$, $\sum_{j=1}^{\Im\psi_S} (\psi_S^{\sigma(j)}(x_1))^C = \sum_{j=1}^{\Im\psi_S} (\psi_S^{\sigma(j)}(x_2))^C = 3.500$ when the values of $\sum_{i=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x_1)$ and $\sum_{i=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x_2)$ are different, and $[x_1]_R$ which satisfies $\sum_{i=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x_1) = 2.500$ is divided into the upper boundary region while $[x_2]_R$ that satisfies $\sum_{i=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x_2) = 0.500$ belongs to the negative region. For the other side, $\sum_{i=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x_6) = \sum_{i=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x_7) = 0.700$ while the values of $\sum_{j=1}^{\Im\psi_S} (\psi_S^{\sigma(j)}(x_6))^C$ and $\sum_{j=1}^{\Im\psi_S} (\psi_S^{\sigma(j)}(x_7))^C$ are not equal for $[x_6]_R$ and $[x_7]_R$. Unlike the previous circumstances, both of them belong to the positive region.

For the VPRS-HFEn model, Section 3 shows that the upper and lower approximations in this model can be defined either by the conditional probability $P(H|[x]_R)$ or by $\sum_{j=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x)$ and $\Im\psi_S(x)$. Therefore, the indiscernible relationships between objects in different equivalence classes should be related to $P(H|[x]_R)$ or both $\sum_{j=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x)$ and $\Im\psi_S(x)$. We can easily verify the truth of the following narrative: For the values of $P(H|[x]_R)$, it is clear that $P(H|[x_3]_R) = 0.700 = P(H|[x_7]_R)$, while both $[x_3]_R$ and $[x_7]_R$ are divided into different regions in three different cases. Both of them belong to positive region in Case 1 and Case 2, however, they belong to the boundary region in Case 3. For values of $\sum_{j=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x)$ and $\Im\psi_S(x)$, $\sum_{j=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x_8) = \sum_{j=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x_{13})$ when $\Im\psi_S(x_8) \neq \Im\psi_S(x_{13})$, as shown in Table 8, it is clear that the classification results of $[x_8]_R$ and $[x_{13}]_R$ are as follows: $[x_8]_R$ belongs to the boundary region and $[x_{13}]_R$ belongs to the negative region in Case 1 and Case 3, while both of them belong to the boundary region. The same is true between $[x_6]_R$ and $[x_7]_R$, both $[x_7]_R$ and $[x_6]_R$ belong to the positive region in Case 1 and Case 2, while $[x_7]_R$ belongs to the boundary region and $[x_6]_R$ belongs to the positive region in Case 3.

In accordance with the above analysis, we can conclude that the GRS-HFEn model and the VPRS-HFEn model are inadequate in distinguishing the relationship between the equivalent classes.

5.3. Decision Analysis of Models 1–2

Similarly, we first calculate the upper and lower approximations of three cases in Models 1 and 2, respectively. We choose the value of k to be 2 for convenience.

Case 1: Considering the relationship between α and β to meet the condition $\alpha + \beta = 0.60 + 0.40 = 1$, the upper and lower approximations in Models 1 and 2 are represented as follows:

$\overline{R_I^1}(H) = U - ([x_2]_R \cup [x_8]_R \cup [x_{13}]_R)$; $\overline{R_{II}^1}(H) = U - ([x_2]_R \cup [x_{13}]_R)$;
$\underline{R_I^1}(H) = [x_3]_R \cup [x_7]_R \cup [x_6]_R$; $\underline{R_{II}^1}(H) = [x_8]_R \cup [x_3]_R \cup [x_7]_R \cup [x_6]_R$.

Accordingly, we can obtain the four regions of Models 1 and 2:

$POS_I^1(H) = [x_3]_R \cup [x_7]_R \cup [x_6]_R$; $POS_{II}^1(H) = [x_8]_R \cup [x_3]_R \cup [x_7]_R \cup [x_6]_R$;
$NEG_I^1(H) = [x_2]_R \cup [x_8]_R \cup [x_{13}]_R$; $NEG_{II}^1(H) = [x_2]_R \cup [x_{13}]_R$;
$UBN_I^1(H) = [x_{12}]_R \cup [x_1]_R \cup [x_4]_R$; $UBN_{II}^1(H) = [x_{12}]_R \cup [x_1]_R \cup [x_4]_R$;
$LBN_I^1(H) = \phi$. $LBN_{II}^1(H) = \phi$.

With regard to the values of the three thresholds, $\alpha = 0.60$, $\beta = 0.40$, and $k = 2$, Models 1 and 2 have specific quantitative semantics for the relative and absolute degrees in this case. For Model 1, $POS_I^1(H) = [x_3]_R \cup [x_7]_R \cup [x_6]_R$ denotes that the relative degree of the customers belonging to the buyers set exceeds 0.6 and the external grade with respect to the buyers set does not exceed 2. In Model 2, $POS_{II}^1(H) = [x_8]_R \cup [x_3]_R \cup [x_7]_R \cup [x_6]_R$ denotes that the relative degree of the customers belonging to the buyers set is at least 0.40 and the internal grade with respect to the buyers set exceeds 2. The same analytic procedure can be applied to the negative region, upper boundary region, and lower boundary region in both of the two new models with the thresholds $\alpha = 0.60$, $\beta = 0.40$, and $k = 2$.

Case 2: Considering the relationship between α and β to meet the condition $\alpha + \beta = 0.55 + 0.25 = 0.80 < 1$, the upper and lower approximations in Models 1 and 2 are represented as follows:

$\overline{R_I^2}(H) = U - ([x_2]_R \cup [x_8]_R \cup [x_{13}]_R)$; $\overline{R_{II}^2}(H) = U - [x_2]_R$;
$\underline{R_I^2}(H) = [x_3]_R \cup [x_7]_R \cup [x_6]_R$; $\underline{R_{II}^2}(H) = [x_8]_R \cup [x_3]_R \cup [x_7]_R \cup [x_6]_R$.

Accordingly, we can obtain the four regions of Models 1 and 2:

$POS_I^2(H) = [x_3]_R \cup [x_7]_R \cup [x_6]_R$; $POS_{II}^2(H) = [x_8]_R \cup [x_3]_R \cup [x_7]_R \cup [x_6]_R$;
$NEG_I^2(H) = [x_2]_R \cup [x_8]_R \cup [x_{13}]_R$; $NEG_{II}^2(H) = [x_2]_R$;
$UBN_I^2(H) = [x_{12}]_R \cup [x_1]_R \cup [x_4]_R$; $UBN_{II}^2(H) = [x_{12}]_R \cup [x_1]_R \cup [x_4]_R \cup [x_{13}]_R$;
$LBN_I^2(H) = \phi$. $LBN_{II}^2(H) = \phi$.

For Model 1, $POS_I^2(H) = [x_3]_R \cup [x_7]_R \cup [x_6]_R$ denotes that the relative degree of the customers belonging to the buyers set exceeds 0.55 and the external grade with respect to the buyers set does not exceed 2. In Model 2, $POS_{II}^2(H) = [x_8]_R \cup [x_3]_R \cup [x_7]_R \cup [x_6]_R$ denotes that the relative degree of the customers belonging to the buyers set is at least 0.25 and the internal grade with respect to the buyers set exceeds 2. The same analytic procedure can be applied to the negative region, upper boundary region, and lower boundary region in both of the two new models with the thresholds $\alpha = 0.55$, $\beta = 0.25$, and $k = 2$.

Case 3: Considering the relationship between α and β to meet the condition $\alpha + \beta = 0.75 + 0.36 = 1.11 > 1$, the upper and lower approximations in Models 1 and 2 are represented as follows:

$\overline{R_I^3}(H) = U - ([x_2]_R \cup [x_8]_R \cup [x_{13}]_R)$; $\overline{R_{II}^3}(H) = U - ([x_2]_R \cup [x_{13}]_R)$;
$\underline{R_I^3}(H) = [x_6]_R$; $\underline{R_{II}^3}(H) = [x_8]_R \cup [x_3]_R \cup [x_7]_R \cup [x_6]_R$.

Accordingly, we can obtain the four regions of the Models 1 and 2:

$POS_I^3(H) = [x_6]_R$; $POS_{II}^3(H) = [x_8]_R \cup [x_3]_R \cup [x_7]_R \cup [x_6]_R$;
$NEG_I^3(H) = [x_2]_R \cup [x_8]_R \cup [x_{13}]_R$; $NEG_{II}^3(H) = [x_2]_R \cup [x_{13}]_R$;
$UBN_I^3(H) = [x_{12}]_R \cup [x_1]_R \cup [x_4]_R \cup [x_3]_R \cup [x_7]_R$; $UBN_{II}^3(H) = [x_{12}]_R \cup [x_1]_R \cup [x_4]_R$;
$LBN_I^3(H) = \phi$. $LBN_{II}^3(H) = \phi$.

For Model 1, $POS_I^3(H) = [x_6]_R$ denotes that the relative degree of the customers belonging to the buyers set exceeds 0.75 and the external grade with respect to the buyers set does not exceed 2. In Model 2, $POS_{II}^3(H) = [x_8]_R \cup [x_3]_R \cup [x_7]_R \cup [x_6]_R$ denotes that the relative degree of the customers belonging to the buyers set is at least 0.36 and the internal grade with respect to the buyers set exceeds 2. The same analytic procedure can be applied to the negative region, upper boundary region, and lower boundary region in both of the two new models with the thresholds $\alpha = 0.75$, $\beta = 0.36$, and $k = 2$.

From Table 9, it can be seen that the two new double-quantitative decision models (Models 1 and 2) in the decision-making problems are obviously superior to the previous GRS-HFEn model and the VPRS-HFEn model. The targeted analysis on several pairs of objects extracted in Table 10 will be performed below. Table 10 is the decision results of several pairs of objects extracted specifically from Table 9, and the following is the targeted analysis.

Table 9. The four regions of Models 1 and 2.

Case	Model	Positive	Negative	Upper, Boundary	Lower, Boundary
1	1	$[x_3]_R \cup [x_7]_R \cup [x_6]_R$	$[x_2]_R \cup [x_8]_R \cup [x_{13}]_R$	$[x_{12}]_R \cup [x_1]_R \cup [x_4]_R$	ϕ
	2	$[x_8]_R \cup [x_3]_R \cup [x_7]_R \cup [x_6]_R$	$[x_2]_R \cup [x_{13}]_R$	$[x_{12}]_R \cup [x_1]_R \cup [x_4]_R$	ϕ
2	1	$[x_3]_R \cup [x_7]_R \cup [x_6]_R$	$[x_2]_R \cup [x_8]_R \cup [x_{13}]_R$	$[x_{12}]_R \cup [x_1]_R \cup [x_4]_R$	ϕ
	2	$[x_8]_R \cup [x_3]_R \cup [x_7]_R \cup [x_6]_R$	$[x_2]_R$	$[x_{12}]_R \cup [x_1]_R \cup [x_4]_R \cup [x_{13}]_R$	ϕ
3	1	$[x_6]_R$	$[x_2]_R \cup [x_8]_R \cup [x_{13}]_R$	$[x_{12}]_R \cup [x_1]_R \cup [x_4]_R \cup [x_3]_R \cup [x_7]_R$	ϕ
	2	$[x_8]_R \cup [x_3]_R \cup [x_7]_R \cup [x_6]_R$	$[x_2]_R \cup [x_{13}]_R$	$[x_{12}]_R \cup [x_1]_R \cup [x_4]_R$	ϕ

Table 10. Analysis of the advantages of Models 1 and 2.

Model	Case	$[x_7]_R$ and $[x_6]_R$	$[x_1]_R$ and $[x_2]_R$	$[x_3]_R$ and $[x_7]_R$	$[x_8]_R$ and $[x_{13}]_R$
1	1	$POS_I(H), POS_I(H)$	$UBN_I(H), NEG_I(H)$	$POS_I(H), POS_I(H)$	$NEG_I(H), NEG_I(H)$
	2	$POS_I(H), POS_I(H)$	$UBN_I(H), NEG_I(H)$	$POS_I(H), POS_I(H)$	$NEG_I(H), NEG_I(H)$
	3	$POS_I(H), UBN_I(H)$	$UBN_I(H), NEG_I(H)$	$UBN_I(H), UBN_I(H)$	$NEG_I(H), NEG_I(H)$
2	1	$POS_{II}(H), POS_{II}(H)$	$UBN_{II}(H), NEG_{II}(H)$	$POS_{II}(H), POS_{II}(H)$	$POS_{II}(H), NEG_{II}(H)$
	2	$POS_{II}(H), POS_{II}(H)$	$UBN_{II}(H), NEG_{II}(H)$	$POS_{II}(H), POS_{II}(H)$	$POS_{II}(H), UBN_{II}(H)$
	3	$POS_{II}(H), POS_{II}(H)$	$UBN_{II}(H), NEG_{II}(H)$	$POS_{II}(H), POS_{II}(H)$	$POS_{II}(H), NEG_{II}(H)$

In terms of $[x_8]_R$ and $[x_{13}]_R$, Tables 3, 5 and 8 show the threshold values $\alpha = 0.60$, $\beta = 0.40$, and $k = 2$ in Case 1, and it is clear that $\sum_{j=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x_8) = \sum_{j=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x_{13})$. Therefore, $[x_8]_R$ and $[x_{13}]_R$ are indiscernible and equal in the GRS-HFEn model. However, we find that they belong to different regions. In Table 10, $[x_8]_R$ and $[x_{13}]_R$ belong to the negative region based on Model 1 in Cases 1, 2, and 3. This implies that $[x_8]_R$ and $[x_{13}]_R$ are indiscernible in certain conditions. For $[x_7]_R$ and $[x_6]_R$, $\alpha = 0.75$, $\beta = 0.36$, and $k = 2$ in Case 3, it is clear that $\sum_{j=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x_6) = \sum_{j=1}^{\Im\psi_S} \psi_S^{\sigma(j)}(x_7)$, but we find that $[x_7]_R$ belongs to the boundary region and $[x_6]_R$ belongs to the positive region in the VPRS-HFEn model. Table 10 shows that both $[x_7]_R$ and $[x_6]_R$ belong to the positive region in Cases 1, 2, and 3 for Model 2; $[x_7]_R$ belongs to the positive region and $[x_6]_R$ belongs to the upper boundary region in Case 3 for Model 1. Regarding the result that $[x_7]_R$ belongs to the boundary region and $[x_6]_R$ belongs to the positive region in the VPRS-HFEn model, the results in Model 1 are more accurate and persuasive. This shows that the two new double-quantitative decision-making models in this paper have more practical application value.

For $[x_1]_R$ and $[x_2]_R$, shown in Tables 3, 5 and 10, it is clear that they all have the same results in the GRS-HFEn model, Models 1 and 2. To be specific, $[x_1]_R$ belongs to the upper boundary region and $[x_2]_R$ belongs to the negative region. This shows that the two new double-quantitative decision-making models do not violate the decision criteria of the GRS-HFEn model.

With regard to$[x_3]_R$ and $[x_7]_R$, shown in Tables 3, 8 and 10, it is clear that they have the same value of $P(H|[x]_R)$. That is, $P(H|[x_3]_R) = P(H|[x_7]_R)$. We can find that $[x_3]_R$ and $[x_7]_R$ belong to the positive region in Cases 1 and 2, while they belong to the boundary region in Case 3 for the VPRS-HFEn model. It can be seen from Table 10 that both $[x_3]_R$ and $[x_7]_R$ belong to the positive region in Cases 1, 2, and 3 for Model 2. Meanwhile, $[x_3]_R$ and $[x_7]_R$ belong to the positive region in Cases 1 and 2, while they belong to the upper boundary region in Case 3 for Model 1. It is once again verified that the decision results in Models 1 and 2 are more accurate than the results in the GRS-HFEn model and the VPRS-HFEn model.

Based on the above analysis, these double-quantitative decision-theoretic models analyze and solve problems from the perspective of relative quantitative information and absolute quantitative information, and provide valuable value for practical application problems in the field of decision analysis.

6. Conclusions

In order to improve the quality of decision making and enhance the effectiveness of decision making in the hesitant fuzzy environment, this paper first extends the classical graded rough set and conditional probability measure in the context of hesitation fuzzy information. On this basis, the variable precision rough set is derived through the Bayesian decision procedure and the idea of minimum decision rules in the hesitant fuzzy environment. Then, two double-quantitative decision-making models are considered systematically by integrating the variable precision rough set model and the graded rough set model in the hesitant fuzzy environment. Furthermore, we discuss the properties of two models and make a comparison from the internal relationship between the two models. The example analysis shows that the double-quantitative decision models not only significantly avoid the singleness of information selection and reduce the influence of the subjective arbitrariness of decision makers, but also effectively depict the target information from the perspective of relative quantification and absolute quantification. Based on the above results, we will further study the incremental feature selection and the dynamic approximation updating of the proposed model in hesitant fuzzy dynamic data sets.

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