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Existence and Multiplicity of Solutions for a Class of Particular Boundary Value Poisson Equations

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Abstract: In this paper, a special class of boundary value problems, $-\Delta u = \lambda u^q + u^r$, in Ω , $u > 0$, in Ω , $\mathbf{n} \cdot \nabla u + g(u)u = 0$, on $\partial\Omega$, where $0 < q < 1 < r < \frac{N+2}{N-2}$ and $g : [0, \infty) \rightarrow (0, \infty)$ is a nondecreasing C^1 function. Here, $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$ and $\lambda > 0$ is a parameter. The existence of the solution is verified via sub- and super-solutions method. In addition, the influences of parameters on the minimum solution are also discussed. The second positive solution is obtained by using the variational method.

Keywords: sub and super solutions method; comparison principle; variational method; mountain pass theorem

MSC: 35J66; 35K57



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1. Introduction

This paper deals with the nonlinear boundary value problems:

$$\begin{cases} -\Delta u = \lambda u^q + u^r & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \mathbf{n} \cdot \nabla u + g(u)u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1)$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$. Δ and λ are the Laplace operator and the real parameter, respectively. This problem arises in thermal explosion theory. In recent years, this kind of problem has no longer been limited to mathematical research. It involves many fields, such as physics, biology, environmental systems and economic systems (see [1–4] and the references therein). The nonlinear boundary condition is inspired by the following Dirichlet boundary problem. For example, Rey in [5] proved the existence of the solution of

$$\begin{cases} -\Delta v = \varepsilon f(x, v) + |v|^{\frac{4}{N-2}}v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \quad (2)$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain. In addition, $f(x, v)$ is a term of lesser order than $v^{\frac{N+2}{N-2}}$. When ε tends to zero, the asymptotic behavior of the solution of (2) is obtained. In [6], Tarantello showed the non-uniqueness of solutions for

$$\begin{cases} -\Delta v = f + |v|^{p-2}v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

and $p = \frac{2N}{N-2}$ ($N \geq 3$), $f \neq 0$. Denote by $(H_0^1(\Omega))^{-1}$ the dual space of $H_0^1(\Omega)$; then, $f \in (H_0^1(\Omega))^{-1}$ will be

$$\int_{\Omega} f v \leq c_N (\|\nabla v\|_2)^{\frac{N+2}{2}}, \text{ for } \forall v \in H_0^1(\Omega), \|v\|_p = 1,$$

where $c_N = \frac{4}{N-2} \left(\frac{N-2}{N+2}\right)^{\frac{N+2}{4}}$. When $N = 3$, Huang [7] proved that the problem

$$\begin{cases} -\Delta_p v = \lambda |v|^{s-2} v + |v|^{t-2} v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{3}$$

has a positive solution, where $\lambda^* < \lambda < \lambda_1$ and $1 < s < p < t$ ($t \leq \frac{NP}{N-p}$), when $N > 3$ and $2 \leq s < p < t \leq \frac{Np}{N-p}$. For the case of $0 < \lambda < \lambda_1$, Huang also proved the existence of the solution of (3).

In addition, Ambrosetti et al. [8] discussed the existence of the below question.

$$\begin{cases} -\Delta v = \lambda v^q + v^p & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

with $0 < q < 1 < p \leq \frac{N+2}{N-2}$.

Some other studies of the existence of Dirichlet boundary value problems can be found in [4,9–14] and the references therein. For the Poisson equations with nonlinear boundary conditions, we recall the following works presented in the literature (see [15–18] and the references therein). In [15], Garcia-Azorero and others discussed the concave–convex problem with the nonlinear boundary conditions.

$$\begin{cases} -\Delta v + v = |v|^{p-2} v & \text{in } \Omega, \\ \mathbf{n} \cdot \nabla v = \lambda |v|^{q-2} v & \text{on } \partial\Omega, \end{cases}$$

where $q \in (1, \frac{2(N-1)}{N-2})$, $p \in (1, \frac{2N}{N-2}]$ and $\lambda \in (0, +\infty)$. If $1 < q < 2 < p$ and λ is small, there exist two positive solutions, and for large λ there is no positive solution.

In thermal explosion theory, Ko and Prashanth [17] proved that the two-dimensional elliptic equations

$$\begin{cases} -\Delta v = \lambda e^{v^\alpha} & \text{in } \Omega, \\ \mathbf{n} \cdot \nabla v + g(v)v = 0 & \text{on } \partial\Omega, \end{cases}$$

have a positive solution which is not unique, for $\alpha \in (0, 2]$. In [18], Yu and Yan showed that there is a positive solution of the problem

$$\begin{cases} -\Delta v + \frac{K(x)}{v^\alpha} = \lambda v^p & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ \mathbf{n} \cdot \nabla v + g(v)v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\alpha, p \in (0, 1)$. Among them, the authors discuss three cases of $K(x)$ (positive function, negative function and sign changing function).

Gordon et al. [16] proved the uniqueness and variety of positive solutions for the problem below.

$$\begin{cases} -\Delta v = \lambda f(v) & \text{in } \Omega, \\ \mathbf{n} \cdot \nabla v + c(v)v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f : [0, \infty) \rightarrow (0, \infty) \in C^1, \lambda > 0$ is a parameter and $c \in C^{1,\gamma}$ is a non subtractive function defined on $[0, \infty)$, which satisfies $c(0) > 0$. In addition, $\lim_{s \rightarrow \infty} \frac{f(s)}{s} = 0$.

Differently from the above papers, consider the problem (1) in which $f(x, u) = \lambda u^q + u^r$, with $1 < q < 1 < r < \frac{N+2}{N-2}$ and $g(u)$ satisfies the following assumptions.

Hypothesis 1 (H1). $g : [0, \infty) \rightarrow (0, \infty)$ is a increasing C^1 function and satisfying $g(0) = g_0 > 0$.

Hypothesis 2 (H2). There exists C_0 such that for any $y > 0$, we have

$$\int_0^y g(s)ds \geq \frac{1}{r+1}g(y)y^2 + C_0y^2.$$

Hypothesis 3 (H3). g satisfying

$$\lim_{s \rightarrow +\infty} \frac{g(s)}{s^{r-1}} = 0.$$

Remark 1. (H3) indicates that the highest power of g is less than $r - 1$. The function g satisfying this assumption exists. For example, $g(s) = s^k + 1$ with $k < r - 1$, so there exists $0 < C_0 < \frac{r-1}{2(r+1)}$ such that

$$\begin{aligned} \int_0^y g(s)ds &= \frac{y^2}{2} + \frac{y^{k+2}}{k+2} \geq \frac{y^2}{2} + \frac{y^{k+2}}{r+1} \\ &= C_0y^2 + \frac{y^2}{2} - C_0y^2 + \frac{y^{k+2}}{r+1} > C_0y^2 + \left(\frac{1}{2} - \frac{r-1}{2(r+1)}\right)y^2 + \frac{y^{k+2}}{r+1} \\ &= \frac{1}{r+1}y^2 + \frac{y^{k+2}}{r+1} + C_0y^2 \\ &= \frac{1}{r+1}g(y)y^2 + C_0y^2 \end{aligned}$$

is true.

It is well known that the sub- and super-solutions method is an important tool for solving the existence of initial and boundary value problems (see [19–23]). In this paper, using the sub- and super-solutions method, we present some new results on the existence of positive solutions for problem (1).

The definition of energy functional corresponding to the problem (1) is introduced.

$$I_\lambda(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \int_\Omega F(u) dx + \int_{\partial\Omega} G(u) d\sigma, \tag{4}$$

with $0 < q < 1 < r < \frac{N+2}{N-2}$. $F(u) := \int_0^u \lambda s^q + s^r ds$ and $G(u) := \int_0^{u^+} g(s)ds$, where the symbol $d\sigma$ denotes the surface measure of $\partial\Omega$.

Definition 1. $u \in H^1(\Omega)$ is a weak solution of (1) if it satisfies

$$\int_{\Omega} \nabla u \cdot \nabla \phi dx = \int_{\Omega} (\lambda u^q \phi + u^r \phi) dx - \int_{\partial\Omega} g(u) u \phi d\sigma,$$

for any $\phi \in H^1(\Omega)$.

More precisely, $u \in H^1(\Omega)$ is a weak solution of (1) if and only if $u \in H^1(\Omega)$ is a critical point of I_{λ} and u is a positive solution.

Finally, the following results are obtained.

Theorem 1. Let $q \in (0, 1)$, $r \in (1, \frac{N+2}{N-2})$ and g satisfy (H1), (H2) and (H3)

(i) There exists $\Lambda \in \mathbb{R}$ and $\Lambda > 0$, such that (1) has at least one positive solution for $\lambda = \Lambda$. There is no positive solution in (1), for $\lambda \in (\Lambda, +\infty)$. There are at least two positive solutions in (1), for $0 < \lambda < \Lambda$.

(ii) For $0 < \lambda < \Lambda$, (1) has a minimal positive solution u_{λ} , and the map $\lambda \mapsto u_{\lambda}(\cdot)$ is increasing. Moreover, for $\lambda \in (0, \Lambda)$, I_{λ} has a local minimum near zero.

This paper is divided into the following sections. In the second section, we list and show several lemmas that can be widely applied. The Lemmas proposed in the third and fourth part are proved under the condition of Theorem 1 and prepare us for the proof of Theorem 1. The fifth part focuses on proving our results.

2. Preliminaries

In this section, we rephrase problem (1) in the general form

$$\begin{cases} -\Delta u = f(x, u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \mathbf{n} \cdot \nabla u + g(u)u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

The corresponding definitions of sub-solution and super-solution are given as follows:

Definition 2. A function $\chi_2 : \bar{\Omega} \rightarrow \mathbb{R}$ is called a super-solution of (5) if $\chi_2 \in C^2(\Omega) \cap C(\bar{\Omega})$ and

$$\begin{cases} \Delta \chi_2 + f(x, \chi_2) \leq 0 & \text{in } \Omega, \\ \mathbf{n} \cdot \nabla \chi_2 + g(\chi_2)\chi_2 \geq 0 & \text{on } \partial\Omega. \end{cases}$$

Definition 3. A function $\chi_1 : \bar{\Omega} \rightarrow \mathbb{R}$ is called a sub-solution of (5) if $\chi_1 \in C^2(\Omega) \cap C(\bar{\Omega})$ and

$$\begin{cases} \Delta \chi_1 + f(x, \chi_1) \geq 0 & \text{in } \Omega, \\ \mathbf{n} \cdot \nabla \chi_1 + g(\chi_1)\chi_1 \leq 0 & \text{on } \partial\Omega. \end{cases}$$

Lemma 1 (see [24]). Let $f \in \bar{\Omega} \times \mathbb{R}$ and $\frac{\partial f}{\partial u}$ be continuous and g satisfy (H1). If μ and χ are the sub- and super-solutions of problem (5), respectively, such that $\mu \leq \chi$, then problem (5) has at least one solution u satisfying

$$\mu \leq u \leq \chi, \quad \text{on } \bar{\Omega}.$$

From the above lemma, it can be seen that if you want to obtain the solution by the sub- and super-solution method, you must prove that the sub-solution is less than or equal to the super-solution.

In order to compare the sub- and the super-solution more conveniently, the following comparison lemma is proposed.

Lemma 2 (see [17]). Let $w_1, w_2 \in C^{2,\beta}(\Omega) \cap C^{1,\beta}(\bar{\Omega})$ satisfy $-\Delta w_1 \leq -\Delta w_2$, in Ω , $\mathbf{n} \cdot \nabla w_1 + g(w_1)w_1 \leq \mathbf{n} \cdot \nabla w_2 + g(w_2)w_2$, on $\partial\Omega$. Then $w_1 < w_2$ in $\bar{\Omega}$.

Lemma 3 (see [18]). Let $f : \bar{\Omega} \times (0, \infty) \rightarrow \mathbb{R}$ be a continuous function such that $\frac{f(x, s)}{s}$ is strictly decreasing in $(0, \infty)$. Let $\omega_1, \omega_2 \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfy:

- (a) $\Delta\omega_2 + f(x, \omega_2) \geq 0 \geq \Delta\omega_1 + f(x, \omega_1)$, in Ω ;
- (b) $\mathbf{n} \cdot \nabla\omega_1 + g(\omega_1)\omega_1 \geq c \geq \mathbf{n} \cdot \nabla\omega_2 + g(\omega_2)\omega_2$, on $\partial\Omega$ with c a nonnegative constant and $\omega_1, \omega_2 > 0$ in Ω ;
- (c) $\Delta\omega_2 \in L^1(\Omega)$. Then $\omega_1 \geq \omega_2$ in $\bar{\Omega}$.

Let $H^1(\Omega) = \{u : u \in L^2(\partial\Omega), \nabla u \in L^2(\Omega)\}$. We have the following norm:

$$\|u\|_{H^1(\Omega)}^2 := \int_{\Omega} |\nabla u|^2 dx + b \int_{\partial\Omega} |u|^2 d\sigma,$$

where b is a positive constant. For convenience, take $b = 1$ in the following proof process.

Remark 2. Thanks to the trace imbedding and the imbedding of Cherrier (see [25–27]), it follows that $\|\cdot\|_{H^1}$ is indeed an equivalent norm in $H^1(\Omega)$. In other words, there are M_1 and $M_2 > 0$ such that

$$M_1 \|v\|_{H_0^1(\Omega)} \leq \|v\|_{H^1(\Omega)} \leq M_2 \|v\|_{H_0^1(\Omega)}, \forall v \in H^1(\Omega)$$

where $\|v\|_{H_0^1(\Omega)}^2 = \int_{\Omega} (|\nabla v|^2 + |v|^2) dx$ and $H_0^1(\Omega) = \{v : v \in L^2(\Omega), \nabla v \in L^2(\Omega)\}$.

In the proof, we will apply the next result.

Lemma 4 (see [28]). (Rellich–Kondrachov Compactness Theorem) Assume Ω is a bounded open subset of \mathbb{R}^N and $\partial\Omega$ is C^1 . Suppose $1 \leq p < N$. Then,

$$W^{1,p}(\Omega) \subset\subset L^q(\Omega),$$

for each $1 \leq q < p^* = \frac{Np}{N-p}$.

When studying the nonlinear problems on the boundary, we should also pay attention to the following embedding conditions on the boundary.

Lemma 5 (see [29]). Let Ω be a smooth bounded domain in \mathbb{R}^N , $N \geq 2$. For any $p > 1$, with $p \leq \frac{N}{N-2}$ if $N \geq 3$, we have the validity of the Sobolev trace embedding of $H^1(\Omega)$ into $L^p(\partial\Omega)$; namely, there exists a positive constant S such that

$$S \|u\|_{L^{p+1}(\partial\Omega)} \leq \|u\|_{H^1(\Omega)},$$

for all $u \in H^1(\Omega)$.

In order to construct a sub-solution, the following boundary value problem will be used.

$$\begin{cases} -\Delta v = \lambda v^q & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ \mathbf{n} \cdot \nabla v + g(v)v = \rho & \text{on } \partial\Omega \end{cases} \tag{6}$$

where $q \in (0, 1)$ and $\rho \geq 0$.

Lemma 6 (see [18]). *Let $q \in (0, 1)$, $\lambda > 0$. The problem (6) has only solution $\vartheta \in C(\overline{\Omega}) \cap C^{2,\beta}(\Omega)$ and $\vartheta > 0$ on $\partial\Omega$.*

The second solution of (1) is proved by variational method. The following lemma will be used.

Lemma 7 (see [30,31]). *Let F be a functional on a Banach space X , $F \in C^1(X, \mathbb{R})$. Let us assume that there exists $r, R > 0$ such that*

- (i) $F(u) > r$ and $\forall u \in X$ with $\|u\| = R$;
- (ii) $F(0) = 0$ and $F(w_0) < r$ for some $w_0 \in X$ with $\|w_0\| > R$.

Let us define $\Gamma = \gamma \in C([0, 1], X) : \gamma(0) = 0, \gamma(1) = w_0$ and

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} F(\gamma(t)).$$

Then, there exists a sequence $\{u_j\} \in X$ such that $F(u_j) \rightarrow c$ and $F'(u_j) \rightarrow 0$ in X^* (dual of X).

3. Constraints of λ When Solutions Exist

We define

$$\Lambda = \sup\{\lambda > 0 : (1) \text{ has a positive solution}\}. \tag{7}$$

Lemma 8. $0 < \Lambda < \infty$.

Proof. Let e be a solution of

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega, \\ \mathbf{n} \cdot \nabla u + g_0 u = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $0 < q < 1 < r$, we can seek out λ_0 such that for all $0 < \lambda \leq \lambda_0$ there exists $M = M(\lambda) > 0$ satisfying

$$M \geq \lambda M^q e^q + M^r e^r.$$

Then, the function $Me > 0$ verifies

$$-\Delta(Me) = M \geq \lambda(Me)^q + (Me)^r, \quad \text{in } \Omega$$

and

$$\mathbf{n} \cdot \nabla Me + g(Me)Me \geq M(\mathbf{n} \cdot \nabla e + g_0 e) = 0, \quad \text{on } \partial\Omega.$$

It guarantees that Me is a super-solution of (1).

In addition, in order to apply Lemma 1, the existence of sub-solutions needs to be confirmed. For $\varepsilon > 0$ small enough, the above discussion can deduce

$$\lambda(\varepsilon\vartheta)^q = (\varepsilon\vartheta)^q = -\Delta(\varepsilon\vartheta) \leq \lambda\varepsilon^q\vartheta^q + \varepsilon^r\vartheta^r.$$

On $\partial\Omega$, since g is a nondecreasing C^1 function,

$$\mathbf{n} \cdot \nabla\varepsilon\vartheta + g(\varepsilon\vartheta)\varepsilon\vartheta \leq \varepsilon(\mathbf{n} \cdot \nabla\vartheta) + g(\vartheta)\varepsilon\vartheta = \varepsilon(\mathbf{n} \cdot \nabla\vartheta + g(\vartheta)\vartheta) = 0.$$

Therefore, $\varepsilon\vartheta$ is a sub-solution of problem (1).

Let ε be sufficiently small to satisfy $\varepsilon\vartheta < Me$. Therefore, by Lemma 1, problem (1) admits a positive solution u such that

$$\varepsilon\vartheta \leq u \leq Me,$$

whenever $\lambda \leq \lambda_0$ and thus $\Lambda \geq \lambda_0$.

Next, prove that Λ is finite; namely, there is a positive constant $\bar{\lambda}$ such that $\Lambda < \bar{\lambda}$.

The following eigenvalue problem,

$$\begin{cases} -\Delta v = \lambda v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{8}$$

and λ_1 and φ_1 are the corresponding minimum eigenvalue and eigenfunction respectively. If u is a positive solution of (1) corresponding to parameter λ , then

$$\int_{\Omega} -\Delta u \cdot \varphi_1 dx = \int_{\Omega} \lambda u^q \varphi_1 dx + \int_{\Omega} u^r \varphi_1 dx,$$

where φ_1 is solution of (8).

Through the computations and $\varphi_1 = 0$ on $\partial\Omega$, obtain

$$\begin{aligned} & \int_{\Omega} \lambda u^q \varphi_1 dx + \int_{\Omega} u^r \varphi_1 dx \\ &= \int_{\Omega} -\Delta u \cdot \varphi_1 dx \\ &= \int_{\partial\Omega} -\frac{\partial u}{\partial \mathbf{n}} \cdot \varphi_1 d\sigma + \int_{\Omega} \nabla u \nabla \varphi_1 dx \\ &= 0 + \int_{\partial\Omega} u \frac{\partial \varphi_1}{\partial \mathbf{n}} d\sigma - \int_{\Omega} u \Delta \varphi_1 dx \\ &= \int_{\partial\Omega} u \frac{\partial \varphi_1}{\partial \mathbf{n}} d\sigma + \int_{\Omega} u \lambda_1 \varphi_1 dx. \end{aligned}$$

By taking into account that $\frac{\partial \varphi_1}{\partial \mathbf{n}} < 0$, we have

$$\begin{aligned} \lambda_1 \int_{\Omega} u \varphi_1 dx &= \lambda \int_{\Omega} u^q \varphi_1 dx + \int_{\Omega} u^r \varphi_1 dx - \int_{\partial\Omega} u \frac{\partial \varphi_1}{\partial \mathbf{n}} d\sigma, \\ &\geq \lambda \int_{\Omega} u^q \varphi_1 dx + \int_{\Omega} u^r \varphi_1 dx. \end{aligned} \tag{9}$$

Let $\bar{\lambda}$ satisfy

$$\bar{\lambda}t^q + t^r > \lambda_1 t, \quad \forall t \in \mathbb{R}, t > 0.$$

Since $\varphi_1 > 0$ in Ω ,

$$\int_{\Omega} (\bar{\lambda}u^q + u^r) \varphi_1 dx > \int_{\Omega} \lambda_1 u \varphi_1 dx. \tag{10}$$

The previous relations (9) and (10) imply that

$$\int_{\Omega} (\bar{\lambda}u^q + u^r)\varphi_1 dx > \int_{\Omega} \lambda_1 u \varphi_1 dx \geq \lambda \int_{\Omega} u^q \varphi_1 dx + \int_{\Omega} u^r \varphi_1 dx$$

and $\lambda < \bar{\lambda}$. Hence, $\Lambda \leq \bar{\lambda}$ and $0 < \Lambda < \infty$. \square

Lemma 9. Under the assumptions of Theorem 1, the solution of problem (1) exists for all $0 < \lambda < \Lambda$.

Proof. Given $\lambda < \Lambda$, from the definition of upper bound, there exists $\lambda_0 > 0$ such that $\lambda < \lambda_0 < \Lambda$ when $\lambda = \lambda_0$.

Since
$$-\Delta u_{\lambda_0} = \lambda_0 u_{\lambda_0}^q + u_{\lambda_0}^r > \lambda u_{\lambda_0}^q + u_{\lambda_0}^r, \quad \text{in } \Omega$$

and

$$\mathbf{n} \cdot \nabla u_{\lambda_0} + g(u_{\lambda_0})u_{\lambda_0} = 0, \quad \text{on } \partial\Omega;$$

therefore, u_{λ_0} is a super-solution for (1) when the parameter is λ .

As $\varepsilon > 0$ is small enough to ensure $\varepsilon\vartheta < u_{\lambda_0}$, by exploiting Lemma 1, there is a positive solution u of (1) satisfying $\varepsilon\vartheta \leq u \leq u_{\lambda_0}$ for all $0 < \lambda < \Lambda$. \square

When using variational method to solve such problems, we can usually refer to weak solutions and to the energy functional (4) associated with problem (1).

It is easy to verify $I_{\lambda} \in C^1(H^1(\Omega), \mathbb{R})$ and

$$(I'_{\lambda}(u), v) = \int_{\Omega} \nabla u \cdot \nabla v dx - \int_{\Omega} (\lambda u^q + u^r) v dx + \int_{\partial\Omega} g(u) u v d\sigma,$$

for any $u, v \in H^1(\Omega)$. Here $\lambda \in (0, \Lambda)$, Λ is given by the definition (7).

Lemma 10. Suppose that the minimal positive solution of problem (1) exists. Then, $u_1 < u_2$ for $l_1 < l_2$ and $l_1, l_2 \in (0, \Lambda)$. Here, u_1 is the minimal positive solution of problem (1) for $\lambda = l_1$.

Proof. Indeed, if $l_1 < l_2$ then

$$-\Delta u_2 = l_2 u_2^q + u_2^r \geq l_1 u_2^q + u_2^r, \quad \text{in } \Omega$$

and u_2 is a sub-solution of (1) satisfying $\varepsilon\vartheta < u_2$ for a small enough $\varepsilon > 0$. By Lemma 1, problem (1) has a positive solution. We obtain $\varepsilon\vartheta \leq u_1 \leq u_2$ by Lemma 1.

Hence, we get $u_1 < u_2$ by $u_1 \not\equiv u_2$ and the strong maximum principle.

From

$$-\Delta u_2 = l_2 u_2^q + u_2^r \geq l_1 u_1^q + u_1^r = -\Delta u_1, \quad \text{in } \Omega$$

and

$$\mathbf{n} \cdot \nabla u_2 + g(u_2)u_2 = 0 = \mathbf{n} \cdot \nabla u_1 + g(u_1)u_1, \quad \text{on } \partial\Omega,$$

it can be deduced that $u_1 < u_2$ in $\bar{\Omega}$ by Lemma 2. \square

Lemma 11. For all $\lambda \in (0, \Lambda)$, problem (1) has a positive solution u . Thus, I_{λ} obtains a local minimum in the C^1 topology.

Proof. There exists λ_1 such that $\underline{\lambda} < \lambda_1 < \Lambda$ and the minimal positive solution $u_1 = u_{\lambda_1}$ defined in Lemma 10 and $u_1 > 0$ in $\bar{\Omega}$ by (7). Let $u_{\underline{\lambda}}$ be the unique positive solution of (6) with $\rho = 0$

$$\begin{cases} -\Delta u = \underline{\lambda} u^q & \text{in } \Omega, \\ \mathbf{n} \cdot \nabla u + g(u)u = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $\frac{s^q}{s} = s^{q-1}$ is strictly decreasing for each $0 < s < \infty$, we have $\Delta u_\lambda + \lambda u_\lambda^q = 0 \geq -u_1^r = \lambda_1 u_1^q + \Delta u_1$ in Ω and $\mathbf{n} \cdot \nabla u_\lambda + g(u_\lambda)u_\lambda = 0 = \mathbf{n} \cdot \nabla u_1 + g(u_1)u_1$ on $\partial\Omega$. Therefore, $u_\lambda \leq u_1$ in $\bar{\Omega}$ by the Hopf maximum principle and Lemma 3.

Let us define the following cut-off nonlinear function:

$$\hat{f}_\lambda(x, t) = \begin{cases} \lambda(u_\lambda(x))^q + (u_\lambda(x))^r & \text{if } t \leq u_\lambda(x), \\ \lambda t^q + t^r & \text{if } u_\lambda(x) < t < u_1(x), \\ \lambda(u_1(x))^q + (u_1(x))^r & \text{if } t \geq u_1(x) \end{cases}$$

and $\hat{F}_\lambda(x, u) = \int_0^u \hat{f}_\lambda(x, t) dt, x \in \Omega$. Then, $\hat{I}_\lambda : H^1(\Omega) \rightarrow \mathbb{R}$ is given by

$$\hat{I}_\lambda(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \int_\Omega \hat{F}_\lambda(x, u) dx + \int_{\partial\Omega} G(u) d\sigma.$$

This functional is coercive and bounded from below. Obviously, $\hat{I}_\lambda(u) = I_\lambda(u)$ when $u_\lambda(x) < u(x) < u_1(x)$. Let u_λ be a global minimizer of \hat{I}_λ on $H^1(\Omega)$. Then, u_λ is the solution of

$$\begin{cases} -\Delta u = \hat{f}_\lambda(x, u) & \text{in } \Omega, \\ \mathbf{n} \cdot \nabla u + g(u)u = 0 & \text{on } \partial\Omega. \end{cases}$$

Through the define of $\hat{f}_\lambda(x, t)$, we obtain $\hat{f}_\lambda(x, u_\lambda) \leq \hat{f}_\lambda(x, u_\lambda) \leq \hat{f}_\lambda(x, u_1)$ and $u_\lambda < u_\lambda < u_1$ in $\bar{\Omega}$ by Lemma 2. In addition, u_λ is a solution of (3).

Let $\Xi = \min\{\min_{x \in \bar{\Omega}} |u_\lambda(x) - u_\lambda(x)|, \min_{x \in \bar{\Omega}} |u_1(x) - u_\lambda(x)|\}$. On the set $\{u : \|u - u_\lambda\|_{C^1} < \frac{\Xi}{2}\}$, $\hat{I}_\lambda = I_\lambda$. Hence, u_λ is a local minimal for I_λ . \square

Remark 3. We observe that u_λ has negative energy $I_\lambda(u_\lambda) < 0$.

In fact, $I_\lambda(u_\lambda) \leq I(u_1), \hat{I}_\lambda(u_1) = I(u_1)$ and

$$(\hat{I}'_\lambda(u_1), u_1) = \int_\Omega |\nabla u_1|^2 dx + \int_{\partial\Omega} g(u_1)u_1^2 d\sigma - \int_\Omega \lambda u_1^{q+1} + u_1^{r+1} dx = 0.$$

Hence we have

$$\begin{aligned} I_\lambda(u_\lambda) &\leq I_\lambda(u_1) = \hat{I}_\lambda(u_1) \\ &= \frac{1}{2} \int_\Omega |\nabla u_1|^2 dx + \int_{\partial\Omega} \int_0^{u_1} g(s) s ds d\sigma - \int_\Omega \int_0^{u_1} \lambda u_1^q + u_1^r ds dx \\ &\leq \frac{1}{2} \int_\Omega |\nabla u_1|^2 dx + \int_{\partial\Omega} g(|u_1|) \frac{u_1^2}{2} d\sigma - \int_\Omega \lambda u_1^{q+1} + u_1^{r+1} dx \\ &= \frac{1}{2} \int_\Omega \lambda u_1^{q+1} + u_1^{r+1} dx - \int_\Omega \lambda u_1^{q+1} + u_1^{r+1} dx \\ &= -\frac{1}{2} \int_\Omega \lambda u_1^{q+1} + u_1^{r+1} dx < 0. \end{aligned}$$

4. The Second Solution

The proof of the existence of the second solution is very long. For the convenience of readers, it will be proved separately. Next, let us prove an important result about bounded PS sequences.

Lemma 12. Let $\{u_n\} \subset H^1(\Omega)$ be a bounded (PS) sequence for \bar{I}_λ which is defined by Equation ((12)). Then, $u_n \rightarrow u$ in $H^1(\Omega)$.

Proof. Indeed, the (PS) sequence $\{u_n\}$ of \bar{I}_λ satisfies

$$|\bar{I}_\lambda(u_n)| \leq C \text{ and } \bar{I}'_\lambda(u_n) \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Since $\{u_n\} \subset H^1(\Omega)$ is bounded, we get that $\{u_n\}$ weakly converges to u ; i.e., $u_n \rightharpoonup u$, and

$$(\bar{I}'_\lambda(u), u_n - u) \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Obviously, it is

$$(\bar{I}'_\lambda(u_n), u_n - u) \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Combined with the previous two equations, there are

$$(\bar{I}'_\lambda(u_n) - \bar{I}'_\lambda(u), u_n - u) \rightarrow 0, \text{ as } n \rightarrow \infty.$$

$$\|u_n - u\|_{L^2(\Omega)} \rightarrow 0 \text{ and } \|u_n - u\|_{L^{r+1}(\Omega)} \rightarrow 0 \text{ since } 2 < r + 1 < \frac{2N}{N-2}.$$

For u_0 as the fixed parameter defined by Lemma 11, a fixed parameter λ and $|\lambda u^q| \leq \lambda(|u| + 1)$,

$$\begin{aligned} & \left| \int_{\Omega} (\lambda|u_n + u_0|^q - \lambda|u + u_0|^q)(u_n - u) dx \right| \\ & \leq \lambda \int_{\Omega} |u_n - u| (|u_n + u_0|^q + |u + u_0|^q) dx \\ & \leq \lambda \int_{\Omega} |u_n - u| (2 + |u_n + u_0| + |u + u_0|) dx \\ & \leq 2\lambda \cdot \text{meas}(\Omega) \|u_n - u\|_{L^2(\Omega)} + \lambda \left(\int_{\Omega} |u_n + u_0|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |u_n - u|^2 dx \right)^{\frac{1}{2}} \\ & \quad + \lambda \left(\int_{\Omega} |u + u_0|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |u_n - u|^2 dx \right)^{\frac{1}{2}} \\ & = 2\lambda \cdot \text{meas}(\Omega) \|u_n - u\|_{L^2(\Omega)} + \lambda (\|u_n + u_0\|_{L^2(\Omega)} + \|u + u_0\|_{L^2(\Omega)}) \|u_n - u\|_{L^2(\Omega)} \\ & \rightarrow 0, \text{ as } n \rightarrow \infty \end{aligned}$$

by Hölder inequality. Thus, we get

$$\int_{\Omega} (\lambda|u_n + u_0|^q - \lambda|u + u_0|^q)(u_n - u) dx = o(1), \text{ as } n \rightarrow \infty.$$

Similarly, it can be inferred that

$$\begin{aligned} & \left| \int_{\Omega} |u_n + u_0|^r - |u + u_0|^r (u_n - u) dx \right| \\ & \leq (\|u_n + u_0\|_{L^{r+1}(\Omega)}^r + \|u + u_0\|_{L^{r+1}(\Omega)}^r) \|u_n - u\|_{L^{r+1}(\Omega)} \\ & \rightarrow 0, \text{ as } n \rightarrow \infty. \end{aligned}$$

By exploiting the above relations, the Hölder inequality and assumption (H_1) , we obtain

$$[g(s_1) - g(s_2)](s_1 - s_2) \geq 0$$

and $\bar{g}(x, s) = 0$ for $s < 0$.

$$\begin{aligned}
 o(1) &= (I'_\lambda(u_n) - I'_\lambda(u), u_n - u) \\
 &= \int_\Omega |\nabla(u_n - u)|^2 dx + \int_{\partial\Omega} (\bar{g}(x, u_n) - \bar{g}(x, u))(u_n - u) d\sigma \\
 &\quad - \int_\Omega [\lambda|u_n + u_0|^q - \lambda u_0^q - (|u + u_0|^q - \lambda u_0^q)](u_n - u) dx \\
 &\quad - \int_\Omega [|u_n + u_0|^r - u_0^r - (|u + u_0|^r - u_0^r)](u_n - u) dx \\
 &= \int_\Omega |\nabla(u_n - u)|^2 dx + \int_{\partial\Omega} (g(u_n + u_0)(u_n + u_0) - g(u + u_0)(u + u_0))(u_n - u) d\sigma \\
 &\quad - \int_\Omega (\lambda|u_n + u_0|^q - |u + u_0|^q)(u_n - u) dx - \int_\Omega (|u_n + u_0|^r - |u + u_0|^r)(u_n - u) dx \\
 &\geq \int_\Omega |\nabla(u_n - u)|^2 dx + \int_{\partial\Omega} (g(u_n + u_0)(u_n + u_0) - g(u_n + u_0)(u + u_0))(u_n - u) d\sigma \\
 &\quad + \int_{\partial\Omega} (g(u_n + u_0)(u + u_0) - g(u + u_0)(u + u_0))(u_n - u) d\sigma \\
 &\geq \int_\Omega |\nabla(u_n - u)|^2 dx + \int_{\partial\Omega} g_0(u_n - u)^2 d\sigma \\
 &\quad + \int_{\partial\Omega} (g(u_n + u_0) - g(u + u_0))[(u_n + u_0) - (u + u_0)](u + u_0) d\sigma \\
 &\geq \min\{1, g_0\} \|u_n - u\|_{H^1(\Omega)}^2, \text{ as } n \rightarrow \infty.
 \end{aligned}$$

Thus, u_n strongly converges to u in $H^1(\Omega)$. The proof is concluded. \square

5. Proof of Theorem

Proof of Theorem 1(i). The first part (i) is divided into two steps: we first prove the existence of the solution, and then prove whether the solution is unique.

Through Lemmas 8–10, the solution of problem (1) exists, for any $\lambda \in (0, \Lambda)$.

Firstly, the following argument shows the second solution of (1) exists. Let us look for a second positive solution of the form $u = u_0 + v$, where $u_0 = u_\lambda$ is the positive solution found in Lemma 11. The function v satisfies

$$\begin{cases} -\Delta v = \bar{f}(x, v), & \text{in } \Omega, \\ \mathbf{n} \cdot \nabla v + \bar{g}(x, v) = 0, & \text{on } \partial\Omega \end{cases} \tag{11}$$

with

$$\begin{aligned}
 \bar{f}(x, s(x)) &= \begin{cases} \lambda(u_0 + s(x))^q + (u_0 + s(x))^r - \lambda u_0^q - u_0^r, & s(x) \geq 0, \\ 0, & s(x) < 0, \end{cases} \\
 \bar{g}(x, s(x)) &= \begin{cases} g(s(x) + u_0)(s(x) + u_0) - g(u_0)u_0, & s(x) \geq 0, \\ 0, & s(x) < 0 \end{cases}
 \end{aligned}$$

and

$$\bar{I}_\lambda(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx + \int_{\partial\Omega} \bar{G}(x, u) d\sigma - \int_\Omega \bar{F}(x, u) dx, \tag{12}$$

where $\bar{F}(x, u) = \int_0^u \bar{f}(x, s) ds$ and $\bar{G}(x, u) = \int_0^u \bar{g}(x, s) ds$. For convenience, we write $s(x)$ as s . The second component on $\bar{f}(x, u)$ and $\bar{g}(x, u)$ represents a function of $u = u(x)$.

Note that $\bar{I}_\lambda(0) = 0$ and $v = 0$ is a local minimum of \bar{I}_λ in $H^1(\Omega)$. Let v^+ be the positive part of v . As

$$I_\lambda(u_0) = \frac{1}{2} \int_\Omega |\nabla u_0|^2 dx + \int_{\partial\Omega} G(u_0) d\sigma - \int_\Omega F(u_0) dx$$

and

$$\begin{aligned} \bar{I}_\lambda(v) &= \frac{1}{2} \int_\Omega |\nabla v^+|^2 dx + \frac{1}{2} \int_\Omega |\nabla v^-|^2 dx + \int_{\partial\Omega} \bar{G}(v^+) d\sigma - \int_\Omega \bar{F}(v^+) dx \\ &= \frac{1}{2} \int_\Omega |\nabla v^+|^2 dx - \int_\Omega F(u_0 + v^+) dx + \int_\Omega \lambda u_0^q v^+ + \lambda u_0^r v^+ dx \\ &\quad + \frac{1}{2} \int_\Omega |\nabla v^-|^2 dx + \int_{\partial\Omega} \bar{G}(v^+) d\sigma + \int_\Omega F(u_0) dx. \end{aligned}$$

By computation, we have

$$\begin{aligned} I_\lambda(u_0 + v^+) &= \frac{1}{2} \int_\Omega |\nabla v^+|^2 dx + \frac{1}{2} \int_\Omega |\nabla u_0|^2 dx + \int_\Omega |\nabla u_0| |\nabla v^+| dx \\ &\quad + \int_{\partial\Omega} G(u_0 + v^+) d\sigma - \int_\Omega F(u_0 + v^+) dx \\ &= \frac{1}{2} \int_\Omega |\nabla v^+|^2 dx + \frac{1}{2} \int_\Omega |\nabla u_0|^2 dx + \int_\Omega |\nabla u_0| |\nabla v^+| dx \\ &\quad + \int_{\partial\Omega} G(u_0 + v^+) d\sigma - \int_\Omega F(u_0 + v^+) dx \\ &= \frac{1}{2} \int_\Omega |\nabla v^+|^2 dx - \int_\Omega F(u_0 + v^+) dx + \int_\Omega (\lambda u_0^q + \lambda u_0^r) v^+ dx \\ &\quad + \frac{1}{2} \int_\Omega |\nabla u_0|^2 dx + \int_{\partial\Omega} G(u_0 + v^+) d\sigma - \int_{\partial\Omega} g(u_0) u_0 v^+ d\sigma. \end{aligned}$$

Therefore,

$$\begin{aligned} \bar{I}_\lambda(v) &= \frac{1}{2} \int_\Omega |\nabla v^-|^2 dx + \int_{\partial\Omega} \bar{G}(v^+) d\sigma + \int_\Omega F(u_0) dx + I(u_0 + v^+) \\ &\quad + \int_{\partial\Omega} g(u_0) u_0 v^+ d\sigma - \frac{1}{2} \int_\Omega |\nabla u_0|^2 dx - \int_{\partial\Omega} \int_0^{u_0+v^+} g(s) s ds d\sigma \\ &= \frac{1}{2} \int_\Omega |\nabla v^-|^2 dx + \int_{\partial\Omega} \bar{G}(v^+) d\sigma + \int_{\partial\Omega} g(u_0) u_0 v^+ d\sigma + I(u_0 + v^+) \\ &\quad + \int_\Omega F(u_0) dx - \frac{1}{2} \int_\Omega |\nabla u_0|^2 dx - \int_{\partial\Omega} \int_0^{u_0} g(s) s ds d\sigma - \int_{\partial\Omega} \int_{u_0}^{u_0+v^+} g(s) s ds d\sigma \\ &= \frac{1}{2} \int_\Omega |\nabla v^-|^2 dx + \int_{\partial\Omega} \bar{G}(v^+) d\sigma + I(u_0 + v^+) - I(u_0) \\ &\quad + \int_{\partial\Omega} g(u_0) u_0 v^+ d\sigma - \int_{\partial\Omega} \int_{u_0}^{u_0+v^+} g(s) s ds d\sigma \\ &= \frac{1}{2} \int_\Omega |\nabla v^-|^2 dx + I(u_0 + v^+) - I(u_0) + \int_{\partial\Omega} g(u_0) u_0 v^+ d\sigma \\ &\quad + \int_{\partial\Omega} \int_0^{v^+} g(s + u_0)(s + u_0) - g(u_0) u_0 ds d\sigma - \int_{\partial\Omega} \int_0^{v^+} g(t + u_0)(t + u_0) dt d\sigma \\ &= \frac{1}{2} \|\nabla v^-\|_{L^2(\Omega)}^2 + I(u_0 + v^+) - I(u_0) \end{aligned}$$

with $t = s - u_0$. From the Lemma 11, $\bar{I}_\lambda(v) \geq \frac{1}{2} \|\nabla v^-\|_{L^1(\Omega)}^2 \geq 0 = \bar{I}_\lambda(0)$ is obtained and $v = 0$ is the local minimum of \bar{I}_λ in $H^1(\Omega)$.

Choose $L > 0$ so that

$$0 = \bar{I}_\lambda(0) \leq \bar{I}_\lambda(u), \text{ for all } \|u\|_{H^1(\Omega)} \leq L.$$

Since $q < 1 < r$ and the highest power of g is less than the $r - 1$ of (H3),

$$\begin{aligned}
 \bar{I}_\lambda(tu) &= \frac{t^2}{2} \int_\Omega |\nabla u|^2 dx + \int_{\partial\Omega} \bar{G}(u) d\sigma - \int_\Omega \bar{F}(u) dx \\
 &\leq \frac{t^2}{2} \int_\Omega |\nabla u|^2 dx + \int_{\partial\Omega} \int_0^{tu} g(s + u_0)(s + u_0) ds d\sigma \\
 &\quad - \int_\Omega \int_0^{tu} \lambda(s + u_0)^q + (s + u_0)^r - \lambda u_0^q - u_0^r ds dx \rightarrow -\infty
 \end{aligned}
 \tag{13}$$

as $t \rightarrow +\infty$. Fix $\mu \in H^1(\Omega) - \{0\}$ such that $\bar{I}_\lambda(\mu) < 0$. Necessarily, $\|\mu\|_{H^1(\Omega)} > L$. Set

$$\Gamma = \{\gamma : [0, 1] \rightarrow H^1(\Omega) : \gamma \text{ is continuous, } \gamma(0) = 0, \gamma(1) = \mu\}$$

and define the mountain-pass level

$$\beta = \inf_{\gamma \in \Gamma} \sup_{t \in [0, 1]} \bar{I}_\lambda(\gamma(t)).$$

Clearly, $\beta \geq 0$ since $\bar{I}_\lambda(0) = 0$. We recall the definition of the PS sequence around the closed set F .

Definition 4. We define the closed set $F = \{u_n \in H^1(\Omega) : \|u\|_{H^1(\Omega)} = \frac{L}{2}\}$ if $\beta = 0$ and $F = H^1(\Omega)$ if $\beta > 0$.

Definition 5. $(PS)_{F,\beta}$ means a sequence $\{u_n\} \in H^1(\Omega)$ such that

$$\lim_{n \rightarrow \infty} \text{dist}(u_n, F) = 0, \lim_{n \rightarrow \infty} \bar{I}_\lambda(u_n) = \beta \text{ and } \lim_{n \rightarrow \infty} \|\bar{I}'_\lambda(u_n)\|_{(H^1(\Omega))^*} = 0,$$

where $(H^1(\Omega))^*$ is the dual space of $H^1(\Omega)$.

We have the following two cases:

(i) $\inf\{\bar{I}_\lambda(u) : u \in H^1(\Omega) \text{ and } \|u\|_{H^1(\Omega)} = l\}$ for all $l < L$.

In this case, Ghoussoub and Preiss proved the existence of such a $(PS)_{F,\beta}$ sequence (see [32]). Next, we just need to prove that there is also a $(PS)_{F,\beta}$ sequence in the following case.

(ii) There exists $0 < l_1 < L$ such that

$$\inf\{\bar{I}_\lambda(u) : u \in H^1(\Omega) \text{ and } \|u\|_{H^1(\Omega)} = l_1\} > 0.$$

Note that $\beta = 0$ implies that (i) holds and (ii) implies $\beta > 0$. In the case (ii), we can find $r > 0$ such that $\bar{I}_\lambda(u) > r, \forall u \in H^1(\Omega)$ with $\|u\| = l_1, \bar{I}_\lambda(0) = 0$ and $\bar{I}_\lambda(\mu) < 0 < r$ for some $\mu \in H^1(\Omega)$ with $\|\mu\| > L > l_1$. By Lemma 7, $(PS)_{F,\beta}$ hold and there exists a sequence $\{v_n\} \in X$ such that $\bar{I}_\lambda(v_n) \rightarrow \beta$ and $\bar{I}'_\lambda(v_n) = \varepsilon_n \rightarrow 0$.

Hence, we have $|(\bar{I}'_\lambda(v_n), v_n)| \leq \varepsilon_n \|v_n\|_{H^1(\Omega)}$ and $(\bar{I}'_\lambda(v_n), v_n) \geq -\varepsilon_n \|v_n\|_{H^1(\Omega)}$. In addition, $(\bar{I}'_\lambda(v_n), u_0) \rightarrow 0$ for fixed $u_0 \in H^1(\Omega)$.

Note that $u_0 \in H^1(\Omega)$ is the local minimum positive solution of I_λ . Thus,

$$\langle I'_\lambda(u_0), \varphi \rangle = 0, \text{ for any } \varphi \in H^1(\Omega)$$

and

$$\int_\Omega \nabla u_0 \cdot \nabla \varphi dx + \int_{\partial\Omega} g(u_0) u_0 \varphi d\sigma = \int_\Omega (\lambda u_0^q + u_0^r) \varphi dx \leq T_1 \|\varphi\|_{L^2},
 \tag{14}$$

where $T_1 = (\int_\Omega (\lambda |u_0|^q + |u_0|^r)^2 dx)^{\frac{1}{2}} > 0$.

Since $1 < q + 1 < 2$ and (14)

$$\begin{aligned}
 \beta + o_n(1) &= \bar{I}_\lambda(v_n) - \frac{1}{r+1}(\bar{I}'_\lambda(v_n), u_0) + \frac{1}{r+1}(\bar{I}'_\lambda(v_n), v_n + u_0) - \frac{1}{r+1}(\bar{I}'_\lambda(v_n), v_n + u_0) \\
 &= \frac{1}{2} \int_\Omega |\nabla v_n|^2 dx + \int_{\partial\Omega} \int_0^{v_n} g(s + u_0)(s + u_0) - g(u_0)u_0 ds d\sigma + \frac{1}{r+1}(\bar{I}'_\lambda(v_n), v_n) \\
 &\quad - \int_\Omega \frac{\lambda|v_n + u_0|^{q+1} - \lambda u_0^{q+1}}{q+1} - \lambda u_0^q |v_n| dx - \int_\Omega \frac{|v_n + u_0|^{r+1} - u_0^{r+1}}{r+1} - u_0^r |v_n| dx \\
 &\quad - \frac{1}{r+1} \int_\Omega |\nabla v_n| |\nabla(v_n + u_0)| dx - \frac{1}{r+1} \int_{\partial\Omega} (g(v_n + u_0)(v_n + u_0) - g(u_0)u_0)(v_n + u_0) d\sigma \\
 &\quad + \frac{\lambda}{r+1} \int_\Omega (v_n + u_0)^{q+1} - u_0^q (v_n + u_0) dx + \frac{1}{r+1} \int_\Omega (v_n + u_0)^{r+1} - u_0^r (v_n + u_0) dx \\
 &\geq \frac{1}{2} \int_\Omega |\nabla v_n|^2 dx - \frac{1}{r+1} \int_\Omega |\nabla v_n|^2 dx - \frac{1}{r+1} \int_\Omega |\nabla v_n| |\nabla u_0| dx \\
 &\quad + \int_{\partial\Omega} \int_{u_0}^{v_n+u_0} g(t) t dt d\sigma - \int_{\partial\Omega} g(u_0)u_0 v_n d\sigma - \frac{1}{r+1} \varepsilon_n \|v_n\|_{H^1(\Omega)} \\
 &\quad - \frac{1}{r+1} \int_{\partial\Omega} g(v_n + u_0)(v_n + u_0)^2 d\sigma + \frac{1}{r+1} \int_{\partial\Omega} g(u_0)u_0 (v_n + u_0) d\sigma \\
 &\quad - \frac{\lambda}{q+1} \int_\Omega |v_n + u_0|^{q+1} dx + \frac{\lambda}{q+1} \int_\Omega u_0^{q+1} dx + \lambda \int_\Omega u_0^q |v_n| dx \\
 &\quad - \frac{1}{r+1} \int_\Omega \lambda |v_n + u_0|^{r+1} dx + \frac{1}{r+1} \int_\Omega u_0^{r+1} dx + \int_\Omega u_0^r |u_n| dx \\
 &\quad + \frac{\lambda}{r+1} \int_\Omega |v_n + u_0|^{q+1} dx - \frac{\lambda}{r+1} \int_\Omega u_0^{q+1} dx - \frac{\lambda}{r+1} \int_\Omega u_0^q |u_n| dx \\
 &\quad + \frac{1}{r+1} \int_\Omega |v_n + u_0|^{r+1} dx - \frac{1}{r+1} \int_\Omega u_0^{r+1} dx - \frac{1}{r+1} \int_\Omega u_0^r |u_n| dx \\
 &\geq \frac{r-1}{2(r+1)} \int_\Omega |\nabla v_n|^2 dx + C_0 \int_{\partial\Omega} |v_n + u_0|^2 d\sigma - \int_{\partial\Omega} \int_0^{u_0} g(t) t dt d\sigma - \frac{1}{r+1} \int_\Omega |\nabla v_n| |\nabla u_0| dx \\
 &\quad - \int_{\partial\Omega} g(u_0)u_0 v_n d\sigma + \frac{r}{r+1} \int_\Omega (\lambda u_0^q + u_0^r) |u_n| dx - \frac{\varepsilon_n}{r+1} \|v_n\|_{H^1(\Omega)} \\
 &\quad - \lambda \frac{r-q}{(q+1)(r+1)} \int_\Omega |v_n + u_0|^{q+1} dx + \lambda \frac{r-q}{(q+1)(r+1)} \int_\Omega |u_0|^{q+1} dx \\
 &\geq \frac{r-1}{2(r+1)} \int_\Omega |\nabla v_n|^2 dx + C_0 \int_{\partial\Omega} |v_n|^2 d\sigma - \int_{\partial\Omega} g(u_0) \frac{u_0^2}{2} d\sigma - \frac{1}{r+1} \int_\Omega |\nabla v_n| |\nabla u_0| dx \\
 &\quad + \int_\Omega |\nabla v_n| |\nabla u_0| dx - \int_\Omega (\lambda u_0^q + u_0^r) |u_n| dx + \frac{r}{r+1} \int_\Omega (\lambda u_0^q + u_0^r) |u_n| dx \\
 &\quad - \lambda \frac{r-q}{(q+1)(r+1)} \int_\Omega \max\{2|v_n|, 2|u_0|\}^{q+1} dx - \frac{\varepsilon_n}{r+1} \|v_n\|_{H^1(\Omega)} \\
 &\geq \min\left\{\frac{r-1}{2(r+1)}, C_0\right\} \|v_n\|_{H^1(\Omega)}^2 - \int_{\partial\Omega} g(u_0) \frac{u_0^2}{2} d\sigma - \frac{1}{r+1} \int_\Omega (\lambda u_0^q + u_0^r) |u_n| dx \\
 &\quad - \lambda \frac{2^{q+1}(r-q)}{(q+1)(r+1)} \max\{\|v_n\|_{L^{q+1}(\Omega)}^{q+1}, \|u_0\|_{L^{q+1}(\Omega)}^{q+1}\} - \frac{\varepsilon_n}{r+1} \|v_n\|_{H^1(\Omega)} \\
 &\geq \min\left\{\frac{r-1}{2(r+1)}, C_0\right\} \|v_n\|_{H^1(\Omega)}^2 - C_2 - \frac{1}{r+1} T_1 \|v_n\|_{L^2(\Omega)} - \frac{\varepsilon_n}{r+1} \|v_n\|_{H^1(\Omega)} \\
 &\quad - \lambda \frac{2^{q+1}(r-q)}{(q+1)(r+1)} \max\{\|v_n\|_{L^{q+1}(\Omega)}^{q+1}, \|u_0\|_{L^{q+1}(\Omega)}^{q+1}\} \\
 &\geq C_1 \|v_n\|_{H^1(\Omega)}^2 - C_2 - C_3 \|v_n\|_{H^1(\Omega)} - C_4 \max\{\|v_n\|_{H^1(\Omega)}^{q+1}, \|u_0\|_{H^1(\Omega)}^{q+1}\},
 \end{aligned}
 \tag{15}$$

with $C_1 = \min\{\frac{r-1}{2(r+1)}, C_0\}$, $C_2 = \int_{\partial\Omega} g(u_0) \frac{u_0^2}{2} d\sigma$, $C_3 = \frac{1}{r+1} T_1 S_2 + \frac{\varepsilon_n}{r+1}$ and $C_4 = \lambda \frac{2^{q+1}(r-q)}{(q+1)(r+1)} S^{q+1}$, where S_i satisfies $\|u\|_{L^i(\Omega)} \leq S_i \|u\|_{H^1(\Omega)}$ and C_2 is a constant value.

Therefore, we get the bounded (PS) sequence $\{v_n\}$ of I_λ . Accordingly with the properties of bounded sequences and Lemma 12, we have

$$u_n \rightarrow u, \text{ in } H^1(\Omega).$$

Next, let us verify the conditions of the Mountain Pass Theorem (see [33]). Obviously, $\bar{I}_\lambda(0) = 0$. In the previous proof, we found $r > 0$ such that $\bar{I}_\lambda(u) > r, \forall u \in H^1(\Omega)$ with $\|u\| = l_1$.

In Equation (13),

$$\bar{I}_\lambda(tu) \rightarrow -\infty, \text{ as } t \rightarrow +\infty, \text{ for any } u \in H^1(\Omega) - \{0\}.$$

Choose a sufficiently large $t = t^\infty$ such that $\bar{I}_\lambda(u^\infty)|_{\|u^\infty\|_{H^1(\Omega)} > L > l_1} < 0$. For $\beta = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} \bar{I}_\lambda(\gamma(t))$ and

$$\Gamma = \{\gamma : [0,1] \rightarrow H^1(\Omega) : \gamma \text{ is continuous, } \gamma(0) = 0, \gamma(1) = u\}$$

and the critical value $\beta > r$ by the mountain pass theorem (see [33]).

Through the above argument, there is a solution $u_1 \in H^1(\Omega)$ such that

$$\bar{I}'_\lambda(u_1) = 0 \text{ and } \bar{I}_\lambda(u_1) = \beta \geq r > 0$$

and u_1 is a critical point of functional \bar{I}_λ .

Assuming $u_1 = 0$, the contradiction is obtained from $\bar{I}_\lambda(u_1) = 0 \neq \beta = \bar{I}_\lambda(u_1)$. Since $-\Delta u_1 = \bar{f}(x, u_1) \geq 0$ in Ω and $\mathbf{n} \cdot \nabla u_1 + g(u_1)u_1 \geq 0$ on $\partial\Omega$, we get $u_1 > 0$ in $\bar{\Omega}$ by Lemma 2. Therefore, $u_0 + u_1$ is the second positive solution of the problem (1).

When $\lambda = \Lambda$, the problem has a positive solution.

Let $\{\lambda_n\}$ be a sequence satisfying $\lambda_n \rightarrow \Lambda$, where $\lambda_n \leq \Lambda$. Then there exists a sequence of solutions $\{u_{\lambda_n}\} \subset H^1(\Omega)$ to problem (1) for λ_n fulfilling

$$\sup I_{\lambda_n}(u_{\lambda_n}) < +\infty, \text{ and } I'_{\lambda_n}(u_{\lambda_n}) = 0.$$

From Lemma 11, $\{u_{\underline{\lambda}}\}_{0 \leq \underline{\lambda} \leq \Lambda}$ is uniformly bounded in $C^1(\bar{\Omega})$ and $I_\lambda(u_\lambda) \leq 0$. In fact, $-\Delta u_{\underline{\lambda}} = \underline{\lambda} u_{\underline{\lambda}}^q < \Lambda u_{\underline{\lambda}}^q = -\Delta u_\Lambda$ in Ω and $\mathbf{n} \cdot \nabla u_{\underline{\lambda}} + g(u_{\underline{\lambda}})u_{\underline{\lambda}} = 0 = \mathbf{n} \cdot \nabla u_\Lambda + g(u_\Lambda)u_\Lambda$ on $\partial\Omega$, where u_Λ is the unique positive solution of

$$\begin{cases} -\Delta u = \Lambda u^q & \text{in } \Omega, \\ \mathbf{n} \cdot \nabla u + g(u)u = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence, for all $0 \leq \underline{\lambda} \leq \Lambda$, $\{u_{\underline{\lambda}}\}$ is less than u_Λ . $u_{\lambda_n} \leq u_\Lambda$, for any $\lambda_n \in [0, \Lambda]$. Therefore, $\{u_{\lambda_n}\}$ is a bounded sequence in $H^1(\Omega)$, so $u_{\lambda_n} \rightharpoonup u_\Lambda$. For parameter $\lambda = \Lambda$, it is proved that the problem has at least one positive solution.

Thirdly, accordingly to (7), there are no positive solutions for $\lambda > \Lambda$.

This concludes the proof of the first part (i) of Theorem 1. \square

Proof of Theorem 1(ii). Assuming $\rho = 0$, (6) has an only positive solution ϑ by Lemma 6 with

$$-\Delta \vartheta = \lambda \vartheta^q \leq \lambda \vartheta^q + \vartheta^r, \quad \text{in } \Omega$$

and

$$\mathbf{n} \cdot \nabla \vartheta + g(\vartheta)\vartheta = 0, \quad \text{on } \partial\Omega.$$

Since problem (1) has a positive solution for any $\lambda \in (0, \Lambda)$,

$$\Delta u + \lambda u^q \leq 0 = \Delta \vartheta + \lambda \vartheta^q, \quad \text{in } \Omega,$$

$$\mathbf{n} \cdot \nabla \vartheta + g(\vartheta)\vartheta = 0 = \mathbf{n} \cdot \nabla u + g(u)u, \quad \text{on } \partial\Omega$$

and $\vartheta \leq u$ by Lemma 3.

Construct the following monotone iteration:

$$\begin{cases} -\Delta u_{k+1} = \lambda u_k^q + u_k^r, & \text{in } \Omega, \\ \mathbf{n} \cdot \nabla u_{k+1} + g(u_{k+1})u_{k+1} = 0, & \text{on } \partial\Omega, \end{cases}$$

with $u_0 = \vartheta$. Let u be an arbitrary solution; then, $u \geq \vartheta$ is a super-solution of (1).

For $k = 0$,

$$-\Delta u_1 = \lambda u_0^q + u_0^r \geq \lambda u_0^q = -\Delta u_0 = -\Delta \vartheta, \quad \text{in } \Omega$$

and

$$\mathbf{n} \cdot \nabla u_1 + g(u_1)u_1 = 0 = \mathbf{n} \cdot \nabla u_0 + g(u_0)u_0, \quad \text{on } \partial\Omega.$$

Hence, $\vartheta = u_0 \leq u_1$, by Lemma 2.

Since $(\lambda s^q + s^r)' = q\lambda s^{q-1} + rs^{r-1} > 0$, the function $\lambda s^q + s^r$ is strictly increasing. For $k = 1$,

$$-\Delta u_2 = \lambda u_1^q + u_1^r > \lambda u_0^q + u_0^r = -\Delta u_1, \quad \text{in } \Omega$$

and

$$\mathbf{n} \cdot \nabla u_2 + g(u_2)u_2 = 0 = \mathbf{n} \cdot \nabla u_1 + g(u_1)u_1, \quad \text{on } \partial\Omega.$$

This implies by Lemma 2 that $u_1 \leq u_2$.

For any $k \in \mathbb{N}$,

$$\vartheta = u_0 \leq u_1 \leq u_2 \leq \dots \leq u_k \leq \dots \leq u.$$

Thus, $u_k \leq u$ by iterating this process. In addition,

$$u_{\lambda_0} = \lim_{k \rightarrow +\infty} u_k \leq \lim_{k \rightarrow +\infty} u = u,$$

so u_{λ_0} is a minimal positive solution of (1).

Denote by u_{λ_0} the minimal positive solution of (1) for $\lambda_0 \in (0, \Lambda)$. Moreover, by Lemma 10 we get the strict inequality $u_{\lambda_0} < u_{\lambda_1}$ for $\lambda_0 < \lambda_1$.

Let λ as defined above,

$$\begin{aligned} I_\lambda(tv) &= \frac{1}{2} \int_\Omega |\nabla tv|^2 dx - \lambda \int_\Omega dx \int_0^{tv} |s|^q ds - \int_\Omega \int_0^{tv} |s|^r ds + \int_{\partial\Omega} d\sigma \int_0^{tv} g(s)s ds \\ &= \frac{t^2}{2} \int_\Omega |\nabla v|^2 dx - \frac{\lambda t^{q+1}}{q+1} \int_\Omega |v|^{q+1} dx - \frac{t^{r+1}}{r+1} \int_\Omega |v|^{r+1} dx + \int_{\partial\Omega} d\sigma \int_0^{tv} g(s)s ds, \end{aligned}$$

for any $v \in H^1$. In a sufficiently small neighborhood near the zero point,

$$\begin{aligned}
 & I_\lambda(tv) \\
 &= \frac{t^2}{2} \int_\Omega |\nabla v|^2 dx - \frac{\lambda t^{q+1}}{q+1} \|v\|_{L^{q+1}(\Omega)}^{q+1} - \frac{t^{r+1}}{r+1} \|v\|_{L^{r+1}(\Omega)}^{r+1} + \int_{\partial\Omega} d\sigma \int_0^{tv} (g_0 + o(1))s ds \\
 &= \frac{t^2}{2} \int_\Omega |\nabla v|^2 dx + \frac{g_0 t^2}{2} \int_{\partial\Omega} v^2 d\sigma + \frac{o(t^2)}{2} \int_{\partial\Omega} v^2 d\sigma - \frac{\lambda t^{q+1}}{q+1} \|v\|_{L^{q+1}(\Omega)}^{q+1} \\
 &\quad - \frac{t^{r+1}}{r+1} \|v\|_{L^{r+1}(\Omega)}^{r+1} \\
 &\leq \max\left\{\frac{1}{2}, \frac{g_0}{2}\right\} t^2 \left(\int_\Omega |\nabla v|^2 dx + \int_{\partial\Omega} v^2 d\sigma\right) + \frac{o(t^2)}{2} \int_{\partial\Omega} v^2 d\sigma - \frac{\lambda t^{q+1}}{q+1} \|v\|_{L^{q+1}(\Omega)}^{q+1} \\
 &\quad - \frac{t^{r+1}}{r+1} \|v\|_{L^{r+1}(\Omega)}^{r+1} \\
 &= \max\left\{\frac{1}{2}, \frac{g_0}{2}\right\} t^2 \|v\|_{H^1(\Omega)}^2 + \frac{o(t^2)}{2} \int_{\partial\Omega} v^2 d\sigma - \frac{\lambda t^{q+1}}{q+1} \|v\|_{L^{q+1}(\Omega)}^{q+1} - \frac{t^{r+1}}{r+1} \|v\|_{L^{r+1}(\Omega)}^{r+1} \\
 &< 0, \text{ as } t \rightarrow 0.
 \end{aligned}$$

Accordingly to the calculation, there exists $\|u_m\|_{H^1(\Omega)} < \frac{1}{n}$. Let $\{u_n\} \subset B_{\frac{1}{n}}(0)$ be a minimizing sequence for I_λ , where $B_{\frac{1}{n}}(0) =: \{u \in H^1(\Omega) : \|u\|_{H^1(\Omega)} \leq \frac{1}{n}\}$. Since the bounded sequence $\{u_n\} \subset H^1(\Omega)$ has convergent subsequence $\{u_{n_k}\}$, there exists $u_m \in H^1(\Omega)$ such that $u_{n_k} \rightharpoonup u_m$. Clearly,

$$\int_\Omega |\nabla u|^2 dx \leq \liminf_{k \rightarrow \infty} \int_\Omega |\nabla u_{n_k}|^2 dx$$

is established. In addition, since $q, r < \frac{2N}{N-2}$,

$$\lambda \int_\Omega u_{n_k}^q dx \rightarrow \lambda \int_\Omega u_m^q dx$$

and

$$\int_\Omega u_{n_k}^r dx \rightarrow \int_\Omega u_m^r dx,$$

by Lemma 5 and the convergence theorem. It also follows that $\int_{\partial\Omega} G(u_{n_k}) d\sigma \rightarrow \int_{\partial\Omega} G(u_m) d\sigma$, according to the compactness of the tracked embedding. Hence, we have $I_\lambda(u_\lambda) \leq \liminf_{k \rightarrow \infty} I_\lambda(u_{n_k}) = \inf_{B_{\frac{1}{n}}(0)} I_\lambda$. Since $u_m \in B_{\frac{1}{n}}(0)$, there must be $I_\lambda(u_m) = \inf_{B_{\frac{1}{n}}(0)} I_\lambda$. Based on the

above discussion, u_m is the local minimum of I_λ in the set $\{u \in H^1(\Omega) : \|u\|_{H^1(\Omega)} \leq B_{\frac{1}{n}}(0)\}$. Notice that because $I_\lambda(0) = 0 > I_\lambda(u_m)$, there is $u_m \neq 0$. The local minimum value can be obtained near zero of I_λ . \square

Therefore, Theorem 1 has been fully proved.

6. Concluding Remarks

In this paper, we did not only prove the existence of an application of the sub- and super-solutions method, but also proved the existence of the second solution via variational method. The results show that the uniqueness of the positive solution of the elliptic equation with a special boundary is related to the parameters of the internal nonlinear equation.

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References

1. Az-Zo'bi, E.A.; Al-Khaled, K.; Darweesh, A. Numeric-Analytic Solutions for Nonlinear Oscillators via the Modified Multi-Stage Decomposition Method. *Mathematics* **2019**, *7*, 550. [[CrossRef](#)]
2. Botmart, T.; Sabir, Z.; Raja, M.A.Z.; Ali, M.R.; Sadat, R.; Aly, A.A.; Saad, A. A hybrid swarming computing approach to solve the biological nonlinear Leptospirosis system. *Biomed. Signal Process. Control* **2022**, *77*, 103789. [[CrossRef](#)]
3. Kiani, A.K.; Khan, W.U.; Raja, M.A.Z.; He, Y.; Sabir, Z.; Shoaib, M. Intelligent Backpropagation Networks with Bayesian Regularization for Mathematical Models of Environmental Economic Systems. *Sustainability* **2021**, *13*, 9537. [[CrossRef](#)]
4. Sabir, Z. Neuron analysis through the swarming procedures for the singular two-point boundary value problems arising in the theory of thermal explosion. *Eur. Phys. J. Plus* **2022**, *137*, 638. [[CrossRef](#)]
5. Rey, O. Concentration of solutions to elliptic equations with critical nonlinearity. *Annales de l'Institut Henri Poincaré. Non-Linear Anal.* **1992**, *9*, 201–218.
6. Tarantello, G. On nonhomogeneous elliptic equations involving critical Sobolev exponent. *Ann. L Inst. Henri-Poincaré-Anal. Non Lineaire* **1992**, *9*, 281–304. [[CrossRef](#)]
7. Huang, Y.X. Positive solutions of certain elliptic equations involving critical Sobolev exponents. *Nonlinear Anal. Theory Methods Appl.* **1998**, *33*, 617–636. [[CrossRef](#)]
8. Ambrosetti, A.; Brezis, H.; Cerami, G. Combined Effects of Concave and Convex Nonlinearities in Some Elliptic Problems. *J. Funct. Anal.* **1994**, *122*, 519–543. [[CrossRef](#)]
9. Liu, Y.; Yu, H. Bifurcation of positive solutions for a class of boundary value problems of fractional differential inclusions. *Abstr. Appl. Anal.* **2013**, *2013*, 103–128. [[CrossRef](#)]
10. Liu, Y. Positive solutions using bifurcation techniques for boundary value problems of fractional differential equations. *Abstr. Appl. Anal.* **2013**, *2013*, 764–787. [[CrossRef](#)]
11. Liu, Y. Bifurcation techniques for a class of boundary value problems of fractional impulsive differential equations. *J. Nonlinear Sci. Appl.* **2015**, *8*, 340–353. [[CrossRef](#)]
12. Usmanov, K.; Turmetov, B.; Nazarova, K. On the Solvability of Some Boundary Value Problems for the Nonlocal Poisson Equation with Boundary Operators of Fractional Order. *Fractal Fract.* **2022**, *6*, 308. [[CrossRef](#)]
13. Yan, B. Positive solutions for the singular nonlocal boundary value problems involving nonlinear integral conditions. *Bound. Value Probl.* **2014**, *2014*, 38. [[CrossRef](#)]
14. Yan, B.; O'Regan, D.; Agarwal, R.P. Multiplicity and uniqueness results for the singular nonlocal boundary value problem involving nonlinear integral conditions. *Bound. Value Probl.* **2014**, *2014*, 148. [[CrossRef](#)]
15. Garcia-Azorero, J.; Peral, I.; Rossi, J.D. A convex-concave problem with a nonlinear boundary condition. *J. Differ. Equ.* **2004**, *198*, 91–128. [[CrossRef](#)]
16. Gordon, P.V.; Ko, E.; Shivaji, R. Multiplicity and uniqueness of positive solutions for elliptic equations with nonlinear boundary conditions arising in a theory of thermal explosion. *Nonlinear Anal. Real World Appl.* **2014**, *15*, 51–57. [[CrossRef](#)]
17. Ko, E.; Prashanth, S. Positive solutions for elliptic equations in two dimensionals arising in a theory of thermal explosion. *Taiwan. J. Math.* **2015**, *19*, 1759–1775.
18. Yu, S.; Yan, B. Positive Solutions for a Singular Elliptic Equation Arising in a Theory of Thermal Explosion. *Mathematics* **2021**, *9*, 2173. [[CrossRef](#)]
19. Cheng, H.; Yuan, R. Existence and asymptotic stability of traveling fronts for nonlocal monostable evolution equations. *Discret. Contin. Dyn. Syst.-Ser. B* **2017**, *22*, 3007–3022. [[CrossRef](#)]
20. Cheng, H.; Yuan, R. Existence and stability of traveling waves for Leslie-Gower predator-prey system with nonlocal diffusion. *Discret. Contin. Dyn. Syst.-Ser. A* **2017**, *37*, 5433–5454. [[CrossRef](#)]
21. Cheng, H.; Yuan, R. Traveling waves of some Holling-Tanner predator-prey system with nonlocal diffusion. *Appl. Math. Comput.* **2018**, *338*, 12–24. [[CrossRef](#)]
22. Cheng, H.; Yuan, R. The stability of the equilibria of the Allen-Cahn equation with fractional diffusion. *Appl. Anal.* **2019**, *38*, 600–610. [[CrossRef](#)]
23. Liu, Y.; O'Regan, D. Controllability of impulsive functional differential systems with nonlocal conditions. *Electron. J. Differ. Equ.* **2013**, *194*, 1–10.

24. Inkmann, F. Existence and Multiplicity Theorems for Semilinear Elliptic Equations with Nonlinear Boundary Conditions. *Indiana Univ. Math. J.* **1982**, *31*, 213–221. [[CrossRef](#)]
25. Cherrier, P. Meilleures constantes dans des inegalites relatives aux espaces de Sobolev. *Bull. Math. Sci.* **1984**, *108*, 225–262.
26. Cherrier, P. Problemes de Neumann nonlineaires sur les varietes Riemanniennes. *Ann. Funct. Anal.* **1984**, *57*, 154–207. [[CrossRef](#)]
27. Li, Y.Y.; Zhu, M. Sharp Sobolev inequalities involving boundary terms. *Geom. Funct. Anal.* **1998**, *8*, 59–87. [[CrossRef](#)]
28. Evans, L.C. *Partial Differential Equations*; Interscience Publishers: Geneva, Switzerland, 1964.
29. del Pino, M.; Flores, C. Asymptotic Behavior of best Constants and Extremals for trace embeddings in expanding domains. *Commun. Part. Differ. Equ.* **2001**, *26*, 2189–2210. [[CrossRef](#)]
30. García-Azorero, J.; Alonso, I.P. Multiplicity of solutions for elliptic problems with critical exponent or with a nonsymmetric term. *Trans. Am. Math. Soc.* **1991**, *323*, 877–895. [[CrossRef](#)]
31. Aubin, J.P.; Ekeland, I. *Applied Nonlinear Analysis*; Wiley: New York, NY, USA, 1984.
32. Ghoussoub, N.; Preiss, D. A general mountain pass principle for locating and classifying critical points. *Ann. IHP Anal. Non Lineaire* **1989**, *6*, 321–330.
33. Badiale, E.M. *Semilinear Elliptic Equations for Beginners*; Springer: London, UK, 2011.