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Noise-to-State Stability in Probability for Random Complex Dynamical Systems on Networks

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Abstract: This paper studies noise-to-state stability in probability (NSSP) for random complex dynamical systems on networks (RCDSN). On the basis of Kirchhoff's matrix theorem in graph theory, an appropriate Lyapunov function which combines with every subsystem for RCDSN is established. Moreover, some sufficient criteria closely related to the topological structure of RCDSN are given to guarantee RCDSN to meet NSSP by means of the Lyapunov method and stochastic analysis techniques. Finally, to show the usefulness and feasibility of theoretical findings, we apply them to random coupled oscillators on networks (RCON), and some numerical tests are given.

Keywords: noise-to-state stability in probability; random complex dynamical systems on network; lyapunov method; Kirchhoff's matrix theorem

MSC: 05C81; 60H10

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1. Introduction

The past few decades have witnessed the evolution and development of research about complex dynamical systems on networks (CDSN). In fact, CDSN is widely applied in various fields such as social networks [1], neural networks [2–4], multiagent networks [5,6], and communication networks [7,8]. Since CDSN is often located in a complex communication environment [9], it is inevitable to receive noise interference, which is expressed by the white noise process in general when describing actual phenomena with CDSN. However, due to the bandwidth of the white noise being unlimited and not differentiable almost everywhere, many scholars turned to using a stationary random process to simulate external noise disturbance [10,11]. Thus, complex dynamical systems under the general noise disturbance (RCDSN) are usually employed to describe actual systems, and it is shown as following

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = f_i(x_i(t),t) + \sum_{j=1}^{h} a_{ij}T_{ij}(x_i(t),x_j(t),t) + g_i(x_i(t),t)\xi_i(t),\ i \in \mathbb{H}, \tag{1}$$

where $x_i \in \mathcal{R}^{n_i}$ is the system state and $\xi_i(t) \in \mathcal{R}^{m_i}$ is a second-order moment stochastic process satisfying $\mathcal{F}_t$-adapted. Functions $f_i : \mathcal{R}^{n_i} \times \mathcal{R}^+ \to \mathcal{R}^{n_i}$ and $g_i : \mathcal{R}^{n_i} \times \mathcal{R}^+ \to \mathcal{R}^{n_i \times m_i}$ are manifested as the drift coefficient and diffusion coefficient of the i-th subsystem, severally. Real numbers $a_{ij} \geq 0$ are expressed as the coupling strength, and functions $T_{ij} : \mathcal{R}^{n_i} \times \mathcal{R}^{n_j} \times \mathcal{R}^+ \to \mathcal{R}^{n_i}$ represent the coupling form from the j-th subsystem to the i-th one.

The stability of the system is the primary consideration in the analysis of system performance. For systems under white noise disturbance, namely Ito stochastic systems, global asymptotic stability in mean, input state stability, stochastic input state stability and finite-time stability have been extensively studied by scholars, and the relevant results have been published in [12–15]. Compared with the Ito stochastic system, there are few stability results about RCDSN (1). Inspired by the definition of input state stability, Wu introduced noise-to-state stability in probability (NSSP) and gave some adequacy criterions for the NSSP of random nonlinear systems [16]; following this study, some significant conclusions were obtained (more details can be seen in [10,11]). However, in the existing conclusions, only the stability of individual systems is studied, and few scholars consider the coupling structure between them, which is one of the motivations for us to study NSSP for RCDSN (1).

It is well known that the Lyapunov method is a powerful tool when we study the stability of the system. Unfortunately, due to the high dimensionality, nonlinearity and complexity of RCDSN (1), which are caused by the large number of subsystems, nonlinearity of subsystems and complex forms of connections between subsystems, studying NSSP for RCDSN (1) is a difficult problem, and one of the most challenging tasks is how to construct a suitable Lyapnov function for RCDSN (1). Recently, by combining graph theory, Li et al. provided a technique to study global stability for a coupling system [17], and many important results were obtained after this study [6,9]. However, to the best of our knowledge, there exist few results so far about the NSSP for RCDSN (1), which motivates us to try to investigate NSSP of RCDSN (1) in this paper by using this skill.

Compared with the existing literature, especially in [12,13], our contributions in this paper are as follows:

1. A suitable Lyapunov function for RCDSN (1) is established by using Kirchhoff's matrix tree theorem in graph theory. Combining with the Lyapunov method and stochastic analysis techniques, NSSP for RCDSN (1) is studied.
2. The main result is applied to random coupled oscillators on networks (RCON), and its usefulness and effectiveness can be fully demonstrated in some numerical tests.

The rest of this paper is organized as follows. Notations are given in Section 2. In Section 3, some preliminaries and model descriptions are provided. The main results are given in Section 4. An application to random coupled oscillators on networks is given in Sections 5 and 6, the numerical simulations are presented to explain the validity of our work. Finally, the conclusion is drawn in Section 7.

2. Notations

In this paper, $\mathcal{R}^{n_i}$, $\mathcal{R}^{m_i\times n_i}$ denote the n_i-dimensional real vector space and the set of $m_i \times n_i$-dimensional real matrix, severally, and $\mathcal{R}^+ = [0, +\infty)$. The Euclidean norm is defined as $|x| = (\sum_{i=1}^{m} x_i^2)^{\frac{1}{2}}$ for vector $x = (x_1, \cdots, x_m)^{\mathrm{T}} \in \mathcal{R}^m$. A complete probability space is expressed as $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, where $\{\mathcal{F}_t\}_{t\geq 0}$ is a filtration satisfying the usual conditions, $\mathbb{P}$ is a probability measure, and $\mathbb{E}$ represents the expectation of $\mathbb{P}$. Let $\xi(t) \in \mathcal{R}^n$ be a stochastic process vector defined on the complete probability space. Define $N = \sum_{k=1}^{h} n_k$, $M = \sum_{k=1}^{h} m_k$, where n_k and m_k are positive natural numbers, $\mathbb{H} = \{1, 2, \cdots, h\}$, $\mathcal{K}_\infty = \{\alpha(\cdot) : \alpha(\cdot) : \mathcal{R}^+ \to \mathcal{R}^+,\ \alpha(\cdot)$ is strictly increasing, unbounded and $\alpha(0) = 0\}$ and $\mathcal{KL} = \{\beta(\cdot,\cdot) : \beta(\cdot,\cdot) : \mathcal{R}^+ \times \mathcal{R}^+ \to \mathcal{R}^+,\ \beta(\cdot, y) \in \mathcal{K}_\infty$ for each fixed $y \geq 0$, $\beta(x, \cdot)$ is strictly decreasing to 0 as $y \to +\infty$ for each fixed $x \geq 0\}$. The family of all non-negative functions $V(x, t)$ on $\mathcal{R}^n \times \mathcal{R}^+$ that are continuous once differentiable in x and t is represented by $C^{1,1}(\mathcal{R}^n \times \mathcal{R}^+; \mathcal{R}^+)$. Other symbols will be introduced where they first appear.

3. Model Description and Preliminaries

As we all know, a weighted digraph is an effective tool to describe RCDSN (1). By using a weighted directed graph $(\mathcal{G}, A)$ with h $(h \geq 2)$ vertices, where $\mathcal{G}$ is a digraph containing a vertex set as well as a directed arc set and $A = (a_{ij})_{h\times h}$ denotes the coupling configuration of RCDSN (1), we can describe RCDSN (1) clearly and naturally. In weighted digraph $(\mathcal{G}, A)$, each vertex represents a subsystem of RCDSN (1), and the i-th subsystem is represented as follows

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = f_i(x_i(t), t) + g_i(x_i(t), t)\xi_i(t).$$

Furthermore, the directed arc between two vertices represents the influence between two subsystems, and the influence of the j-th subsystem on the i-th one is written by $a_{ij}T_{ij}(x_i, x_j, t)$. More specially, if $a_{ij} = 0$, there is no effect between the two subsystems. In addition, for the convenience of the reader, we have abbreviated RCDSN (1) as follows

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = F(x(t), t) + G(x(t), t)\xi(t). \tag{2}$$

where $x = (x_1, \cdots, x_h)^{\mathrm{T}} \in \mathcal{R}^N$, the stochastic process $\xi = (\xi_1, \cdots, \xi_h)^{\mathrm{T}} \in \mathcal{R}^M$ is $\mathcal{F}_t$-adapted, satisfying $\sup_{0 \le s \le t} \mathbb{E}|\xi(s)|^2 < K_0$ with K_0 being a positive constant.

$$F(x, t) = \left(f_1(x_1, t) + \sum_{j=1}^{h} a_{1j}T_{1j}(x_1, x_j, t), \cdots, f_h(x_h, t) + \sum_{j=1}^{h} a_{hj}T_{hj}(x_h, x_j, t) \right)^{\mathrm{T}} \in \mathcal{R}^N$$

$$G(x, t) = \begin{pmatrix} g_1(x_1, t) & 0 & \cdots & 0 \\ 0 & g_2(x_2, t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & g_h(x_h, t) \end{pmatrix} \in \mathcal{R}^{N \times M}$$

are piecewise continuous in t and locally Lipschitz in x; moreover, $F(0, 0) = 0$ and $G(0, 0) = 0$.

At the end of this section, we give a definition and two useful lemmas.

Definition 1 (See the work of Deng et al. [18]). *If for any $\varepsilon \in (0, 1)$ and the initial value $x_0 \in \mathcal{R}^N$, there exists $\beta(\cdot, \cdot) \in \mathcal{KL}$ and $\Gamma(\cdot) \in \mathcal{K}_\infty$ such that*

$$\mathbb{P}\left\{ |x(t)| \le \beta(|x_0|, t) + \Gamma\left(\sup_{0 \le s \le t} \mathbb{E}|\xi(s)|^2 \right) \right\} \ge 1 - \varepsilon,$$

then, RCDSN (1) is said to be noise-to-state stability in probablity (NSSP).

Lemma 1 (Kirchhoff's matrix tree theorem [19]). *Assume that $h \ge 2$. Let l_i denote the cofactor of the i-th diagonal element of the Laplacian matrix of the weighted digraph $(\mathcal{G}, A)$. Then*

$$l_i = \sum_{\mathcal{T} \in \mathbb{T}_i} W(\mathcal{T}),\ i \in \mathbb{H},$$

where $\mathbb{T}_i$ is the set of all spanning trees $\mathcal{T}$ of the weighted digraph $(\mathcal{G}, A)$ that are rooted at vertex i, and $W(\mathcal{T})$ is the weight of $\mathcal{T}$. Particularly, if the weighted digraph $(\mathcal{G}, A)$ is strongly connected, then $l_i > 0$.

Lemma 2 (See the work of Wu [16]). *For RCDSN (2), if there exists positive numbers C_0, D_0 and function $V(x, t) \in C^{1.1}(\mathcal{R}^N \times \mathcal{R}^+; \mathcal{R}^+)$ such that*

$$\lim_{k \to \infty} \inf_{|x| > k} V(x, t) = \infty,$$

$$\mathbb{E}[V(x(t \wedge \eta_k), t \wedge \eta_k)] \le C_0 e^{D_0 t},$$

where $\eta_k = inf\{t \ge 0 : |x(t)| \ge k\}$ $(k = 1, 2, \cdots)$, then RCDSN (4) has a unique global solution.

4. Main Results

In recent years, the study of NSSP for stochastic nonlinear systems has become a hot topic, and many results have been published in [10,11]. In this section, we will give sufficient criteria for RCDSN (1) to satisfy NSSP, as shown in the following theorem.

Theorem 1. *Suppose every subsystem of RCDSN (1) has functions $V_i(x_i, t) \in C^{1,1}(R^{n_i} \times R^+; R^+)$ $(i \in \mathbb{H})$ which meets the following conditions C1, C2 and C3.*

C1. There exist functions $\Gamma_i^1(\cdot)$, $\Gamma_i^2(\cdot) \in \mathcal{K}_\infty$ which satify

$$\Gamma_i^1(|x_i|) \leq V_i(x_i, t) \leq \Gamma_i^2(|x_i|),$$

where $\Gamma_i^1(\cdot)$ is a convex function.

C2. There exist positive numbers q, λ_i and functions $Q_{ij}(x_i, x_j, t)$ such that

$$\begin{aligned} &\frac{\partial V_i(x_i,t)}{\partial t} + \frac{\partial V_i(x_i,t)}{\partial x_i}\left[f_i(x_i,t) + \sum_{j=1}^{h} a_{ij} T_{ij}(x_i, x_j, t)\right] + q\left|\frac{\partial V_i(x_i,t)}{\partial x_i} g_i(x_i,t)\right|^2 \\ \leq \quad & -\lambda_i V_i(x_i,t) + \sum_{j=1}^{h} a_{ij} Q_{ij}(x_i, x_j, t), \end{aligned}$$

where $\frac{\partial V_i(x_i,t)}{\partial x_i} = \left(\frac{\partial V_i(x_i,t)}{\partial x_i^{(1)}}, \cdots, \frac{\partial V_i(x_i,t)}{\partial x_i^{(n_i)}}\right)$, and coupling configuration matrix $A = (a_{ij})_{h\times h}$ is irreducible, which means the weighted digraph $(\mathcal{G}, A)$ is strongly connected.

C3. Along each directed cycle $\mathcal{C}_Q$ of weighted digraph $(\mathcal{G}, A)$, for all $x_i \in \mathcal{R}^{n_i}$ and $x_j \in \mathcal{R}^{n_j}$, there is

$$\sum_{(j,i)\in E(\mathcal{C}_Q)} Q_{ij}(x_i, x_j, t) \leq 0.$$

Then, RCDSN (1) is NSSP.

Proof. We construct a Lyapunov function as follows

$$V(x,t) = e^{\tau t} \sum_{i=1}^{h} l_i V_i(x_i, t),$$

where positive number $\tau < \min_{1\leq i\leq h}\{\lambda_i\}$ and l_i represents the cofactor of the i-th diagonal element of the Laplacian matrix

$$L = \begin{pmatrix} \sum_{k\neq 1} a_{1k} & -a_{12} & \cdots & -a_{1h} \\ -a_{21} & \sum_{k\neq 2} a_{2k} & \cdots & -a_{2h} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{h1} & -a_{h2} & \cdots & \sum_{k\neq h} a_{hk} \end{pmatrix}$$

of the weighted digraph $(\mathcal{G}, A)$. According to Lemma 1, we can conclude that $l_i > 0$.

Given condition C1, we can derive that

$$V(x,t) \leq e^{\tau t} \sum_{i=1}^{h} l_i \Gamma_i^{(2)}(|x_i|) \leq e^{\tau t} \sum_{i=1}^{h} l_i \max_{1\leq i\leq h}\{\Gamma_i^{(2)}\}(|x|) \triangleq e^{\tau t}\Gamma_2(|x|), \tag{3}$$

where $\Gamma_2(\cdot) = \sum_{i=1}^{h} l_i \max_{1\le i\le h}\{\Gamma_i^{(2)}\}(\cdot) \in \mathcal{K}_\infty$. Based on the condition C1, it is easy to obtain

$$\begin{aligned} V(x,t) &\ge e^{\tau t}\sum_{i=1}^{h} l_i \Gamma_i^{(1)}(|x_i|) \ge e^{\tau t}\min_{1\le i\le h}\{l_i\}\widehat{\Gamma}\left(\frac{1}{h}\sum_{i=1}^{h}|x_i|\right) \\ &\ge e^{\tau t}\min_{1\le i\le h}\{l_i\}\widehat{\Gamma}\left(\frac{|x|}{h}\right) \triangleq e^{\tau t}\Gamma_1(|x|), \end{aligned} \tag{4}$$

where $\widehat{\Gamma}(\cdot) = \min_{1\le i\le h}\{\Gamma_i^{(1)}(\cdot)\}$ and $\Gamma_1(\cdot) = \min_{1\le i\le h}\{l_i\}\widehat{\Gamma}(\frac{\cdot}{h}) \in \mathcal{K}_\infty$. Combining with inequalities (3) and inequalities (4), we can obtain that

$$e^{\tau t}\Gamma_1(|x|) \le V(x,t) \le e^{\tau t}\Gamma_2(|x|). \tag{5}$$

Then, in terms of condition C2, C3, the combination identical equation in graph theory [17] and the fact that $W(\mathcal{Q}) \ge 0$, taking the derivative for $V(x,t)$ along the trajectory of RCDSN (1), we have

$$\begin{aligned} \frac{\mathrm{d}V(x,t)}{\mathrm{d}t} &\le e^{\tau t}\sum_{i=1}^{h} l_i\Bigg(\tau V_i(x_i,t) + \frac{\partial V_i(x_i,t)}{\partial t} + \frac{\partial V_i(x_i,t)}{\partial x_i}\Bigg[f_i(x_i,t) + \sum_{j=1}^{h} a_{ij}T_{ij}(x_i,x_j,t)\Bigg] \\ &\quad + q\left|\frac{\partial V_i(x_i,t)}{\partial x_i} g_i(x_i,t)\right|^2 + \frac{1}{4q}|\xi(t)|^2\Bigg) \\ &\le e^{\tau t}\sum_{i=1}^{h} l_i\left[(\tau-\lambda)V_i(x_i,t) + \frac{1}{4q}|\xi(t)|^2\right] + e^{\tau t}\sum_{i,j=1}^{h} l_i a_{ij} Q_{ij}(x_i,x_j,t) \\ &\le e^{\tau t}\sum_{i=1}^{h} l_i \frac{1}{4q}|\xi(t)|^2, \end{aligned} \tag{6}$$

where $\lambda = \min_{1\le i\le h}\{\lambda_i\}$.

Next, we construct a stopping time sequence $\{\eta_k\}_{k=1}^{\infty}$. Each item in this stopping time sequence is regarded as $\eta_k = \inf\{t \ge 0 : |x(t)| \ge k\}$ $(k = 1, 2, \cdots)$, and it meets that $\eta_k \to \infty$ whenever $k \to \infty$. Taking the integrals in $[0, t \wedge \eta_k)$ and expectation on both sides of inequality (6), it yields that

$$\mathbb{E}V(x(t\wedge\eta_k), t\wedge\eta_k) \le V(x_0,0) + \int_0^{t\wedge\eta_k}\mathbb{E}\sum_{i=1}^{h} l_i\frac{1}{4q}|\xi(s)|^2 e^{\tau s}ds \tag{7}$$

$$\le \left(V(x_0,0) + \frac{1}{4\tau q}\sum_{i=1}^{h} l_i \sup_{0\le s\le t}\mathbb{E}|\xi(s)|^2\right)e^{\tau t} \tag{8}$$

In terms of inequality (8) and Lemma 2, it is easy to obtain that RCDSN (1) has a unique global solution. Let $k \to \infty$ in inequality (7); combining with inequality (5), we can derive that

$$e^{\tau t}\mathbb{E}\Gamma_1(|x(t)|) \le \mathbb{E}V(x(t),t) \le V(x_0,0) + \frac{1}{4q}\sum_{i=1}^{h} l_i \sup_{0\le s\le t}\mathbb{E}|\xi(s)|^2\int_0^t e^{\tau s}ds. \tag{9}$$

According to Jensen's inequality and inequality (9), we can obtain

$$\begin{aligned} \mathbb{E}|x(t)| &\le \Gamma_1^{-1}(e^{-\tau t}\Gamma_2(|x_0|)) + \Gamma_1^{-1}\left(\tfrac{1}{4\tau q}\sum_{i=1}^{h} l_i \sup_{0\le s\le t}\mathbb{E}|\xi(s)|^2\right) \\ &\triangleq \tilde{\beta}(|x_0|,t) + \tilde{\Gamma}\left(\sup_{0\le s\le t}\mathbb{E}|\xi(s)|^2\right). \end{aligned} \tag{10}$$

For any $\varepsilon \in (0,1)$, in accordance with Chebyshev's inequality and inequality (10), it yields that

$$\mathbb{P}\left\{ |x(t)| > \beta(|x_0|,t) + \Gamma\left(\sup_{0\le s\le t} \mathbb{E}|\xi(s)|^2\right)\right\} \le \frac{\varepsilon\mathbb{E}|x(t)|}{\tilde{\beta}(|x_0|,t) + \tilde{\Gamma}\left(\sup_{0\le s\le t} \mathbb{E}|\xi(s)|^2\right)} \le \varepsilon,$$

where $\beta(|x_0|,t) = \frac{1}{\varepsilon}\tilde{\beta}(|x_0|,t) \in \mathcal{KL}$ and $\Gamma(\cdot) = \frac{1}{\varepsilon}\tilde{\Gamma}(\cdot) \in \mathcal{K}_\infty$. In summary, binding with Definition 1, RCDSN (1) is NSSP, which means the proof completes. □

Remark 1. *Based on the condition that the $\mathcal{F}_t$-adapted stochastic process $\xi(t)$ in RCDSN (2) satisfies $\sup_{0\le s\le t}\mathbb{E}|\xi(s)|^2 < K_0$ with K_0 being a positive constant, Theorem 1 is obtained. In fact, if the stochastic process $\xi(t)$ satisfies $\sup_{0\le s\le t}\mathbb{E}|\xi(s)|^2 < c_0e^{d_0t}$ where c_0, d_0 are positive numbers, Theorem 1 still holds. In addition, from the proof of Theorem 1, it can be seen easily that RCDSN (1) is asymptotically stability .*

Remark 2. *Since RCDSN (1) is very complex, some theorems in graph theory such as Kirchhoff's matrix tree theorem are used to connect the dynamic behavior and topological structure of RCDSN (1) in Theorem 1. Furthermore, condition C3 holds for every directed cycle $\mathcal{C}_Q$ of the weighted digraph $(\mathcal{G}, A)$, which is difficult to verify in many cases. Fortunately, the difficulties mentioned above can be overcome if we find some suitable functions Q_{ij} $(i,j \in \mathbb{H})$. In fact, if there exist functions T_i and T_j for every Q_{ij} such that*

$$Q_{ij}(x_i,x_j,t) \le T_j(x_j,t) - T_i(x_i,t).$$

Then, the following inequality can be naturally obtained.

$$\sum_{(j,i)\in E(\mathcal{C}_Q)} Q_{ij}(x_i,x_j,t) \le \sum_{(j,i)\in E(\mathcal{C}_Q)} \left[T_j(x_j,t) - T_i(x_i,t)\right] = 0.$$

Clearly, condition P3 could be verified easily.

Remark 3. *Recently, the use of mathematical models to solve real-world problems has attracted the attention of many scholars, and the relevant results have been published in [20–26]. In [20], Rahaman et al. studied a numerical solution method for initial value problems with initial singularities. In [21–25], scholars explored a semiconductor circuit breaker fluid dynamics model, a Fractional Order SIR Model with 2019-nCoV, mathematical models on the digestive system and COVID-19 pandemic, badminton players' trajectory and Lane–Emden differential equation in fair value analysis of financial accounting by using a numerical calculation method. A stochastic numerical computing framework based on Gudermannian neural networks together with the global and local search genetic algorithm and active-set approach has been presented by Sabir et al. in [26]. In this paper, we studied NSSP for RCDSN (1) by constructing a new Lyapunov function and applying the Lyapunov method and stochastic analysis skills.*

Remark 4. *The past decades have witnessed the evolution and development of stochastic complex dynamical systems stability research. For example, in [12], Gao et al. studied global asymptotic stability in mean for stochastic complex networked control systems. Asymptotic stability in probability for discrete-time stochastic coupled systems on networks with multiple dispersal has been researched by Wang et al. in [6]. What these studies have in common is to explore the dynamic behavior of complex dynamical systems under white noise perturbation. Compared with them, the innovation of this paper is that it gives the sufficiency criterion for NSSP of complex dynamical systems on networks under the disturbance of the second-order moment process (RCDSN) in Theorem 1, which broadens the scope of application of RCDSN.*

5. An Application to Random Coupled Oscillators on Networks

Coupling oscillators on networks (CON) are widely studied as important applications in engineering systems and power systems, and many results have been published [27,28]. In [27], Yin et al. investigated the Van der Pol-Duffing oscillator. The synchronization of coupled harmonic oscillators has been researched by Song et al. in [28]. However, some tiny noises will affect the stability of CON. Thus, in this section, to verify the validity of our theoretical results, NSSP for random coupled oscillators on networks (RCON) is studied. First, we give CON as the following.

$$\ddot{x}_i(t) + \psi_i(x_i(t))\dot{x}_i(t) + x_i(t) = 0,$$

where the Damping function $\psi_i(\cdot)$ is bounded, which satisfies $m_i \le \psi_i(x_i) \le M_i$, where $m_i > 0$. Based on a transform of $y_i(t) = \dot{x}_i(t) + \eta x_i(t)$ and considering the effects of the coupling structure and general noise disturbance, RCON is given as follows

$$\begin{cases} \dfrac{\mathrm{d}x_i(t)}{\mathrm{d}t} = y_i(t) - \eta x_i(t) + \mu_i \sin x_i(t)\xi_i^{(1)}(t), \\ \dfrac{\mathrm{d}y_i(t)}{\mathrm{d}t} = (\eta - \psi_i(x_i(t)))y_i(t) + (\eta\psi_i(x_i(t)) - \eta^2 - 1)x_i(t) \\ \qquad + \sum\limits_{j=1}^{h} b_{ij}\big(y_j(t) - y_i(t)\big) + \gamma_i \cos y_i(t)\xi_i^{(2)}(t),\ i \in \mathbb{H}, \end{cases} \tag{11}$$

where $x_i, y_i \in \mathcal{R}^1$ is the system state and η is a positive number. Positive numbers μ_i, γ_i represent the intensity of noise perturbation, and $\xi_i^{(j)}(t) \in \mathcal{R}^1$ $(j = 1,2)$ are second-order moment processes satisfying $\mathcal{F}_t$-adapted. The coupling configuration matrix $B = (b_{ij})_{h\times h}$ $(b_{ij} \ge 0)$ is irreducible, where b_{ij} is the coupling strength. Functions $y_j(t) - y_i(t)$ represent the linear coupling form.

For the convenience of proof, we simplify the RCON (11) as follows

$$\frac{\mathrm{d}\theta_i(t)}{\mathrm{d}t} = f_i(\theta_i(t),t) + \sum_{j=1}^{h} b_{ij}P_{ij}(\theta_i(t),\theta_j(t),t) + g_i(\theta_i(t),t)\xi_i(t), \quad i \in \mathbb{H},$$

where $\theta_i = (x_i, y_i)^{\mathrm{T}}$, $f_i = \big(y_i - \eta x_i,\ (\eta - \psi_i(x_i))y_i + (\eta\psi_i(x_i) - \eta^2 - 1)x_i\big)^{\mathrm{T}}$, $P_{ij} = (0, y_j - y_i)^{\mathrm{T}}$, $g_i = (\mu_i \sin x_i(t), \gamma_i \cos y_i(t))^{\mathrm{T}}$ and $\xi_i(t) = \Big(\xi_i^{(1)}(t),\ \xi_i^{(2)}(t)\Big)^{\mathrm{T}}$

Furthermore, a number of adequacy criterions for NSSP of RCON (11) are given.

Theorem 2. *RCON (11) is NSSP if the following condition holds.*

P1. $\max\limits_{1\le i\le h}\Big\{\Big[|\eta M_i - \eta^2 - 1| + 1 - 2\eta + \eta^2 + 8\mu_i^2\Big],\ \Big[|\eta M_i - \eta^2 - 1| + 2\eta - 2m_i + 1 + \sum\limits_{j=1}^{h} b_{ij} + 8\gamma_i^2\Big]\Big\} < 0.$

Proof. Let Lyapunov function $V_i(\theta_i, t) = |\theta_i|^2$ for the i-th subsystem of RCON (11). There exists convex functions $\Gamma_i^{(1)}(|\theta_i|) = \frac{1}{i}|\theta_i|^2$ and $\Gamma_i^{(2)}(|\theta_i|) = i|\theta_i|^2 \in \mathcal{K}_\infty$, which means condition C1 in Theorem 1 is satisfied. In terms of condition P1 in Theorem 2, we can obtain

$$\begin{aligned} &\frac{\partial V_i(\theta_i,t)}{\partial t} + \frac{\partial V_i(\theta_i,t)}{\partial \theta_i}\Big[f_i(\theta_i,t) + \sum_{j=1}^{h} b_{ij}P_{ij}(\theta_i,\theta_j,t)\Big] + \Big|\frac{\partial V_i(\theta_i,t)}{\partial \theta_i}g_i(\theta_i,t)\Big|^2 \\ =\ & (2x_i, 2y_i)\begin{pmatrix} y_i - \eta x_i \\ (\eta - \psi_i(x_i))y_i + (\eta\psi_i(x_i) - \eta^2 - 1)x_i + \sum\limits_{j=1}^{h} b_{ij}(y_j - y_i) \end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
&+\Big[(2x_i,2y_i)(\mu_i\sin x_i,\gamma_i\cos y_i)^{\mathrm{T}}\Big]^2\\
\le\ &\Big(|\eta M_i-\eta^2-1|+1-2\eta+\eta^2+8\mu_i^2\Big)x_i^2+\Big(1+|\eta M_i-\eta^2-1|\\
&+2\eta-2m_i+8\gamma_i^2+\sum_{j=1}^{h}b_{ij}\Big)y_i^2+\sum_{j=1}^{h}b_{ij}(y_j^2-y_i^2)\\
\le\ &-\lambda_i V_i(\theta_i,t)+\sum_{j=1}^{h}b_{ij}Q_{ij}(\theta_i,\theta_j,t),
\end{aligned}
$$

where positive number $\lambda_i=\min\Big\{\big||\eta M_i-\eta^2-1|+1-2\eta+\eta^2+8\mu_i^2\big|,\big||\eta M_i-\eta^2-1|+2\eta-2m_i+1+\sum_{j=1}^{h}b_{ij}+8\gamma_i^2\big|\Big\}$ and function $Q_{ij}(\theta_i,\theta_j,t)=y_j^2-y_i^2$, finally combining with Remark 2, conditions C2 and C3 in Theorem 1 hold, which means RCON (11) is NSSP. □

Remark 5. *Recently, as important models in mechanical systems, RCON (11) has been widely studied, and many results have been published in [29,30]. In [29], stochastic switching in delay-coupled oscillators was studied. In [30], Wu et al. investigated the synchronization of discrete-time state-coupled stochastic oscillators. What these results have in common is the study of the dynamic behavior of coupled oscillators under white noise perturbation. Compared with them, the adequacy criterion of NSSP for RCON (11) is given under general noise disturbance in Theorem 2, which is related to the bounded damping functions $\psi_i(x_i)$ and the coupling configuration matrix $B=(b_{ij})_{h\times h}$ of RCON (11).*

6. Numerical Test

To verify the validity of our theoretical results, a numerical example is given as follows (see Figures 1 and 2). First, we let the number of oscillators in RCON be four. Subsequently, we choose the wide stationary process $\xi_i(t)=(10i\cos(t+\phi),10i\sin(t+\phi))^T$ where the random variable ϕ is uniformly distributed in $(0,2\pi)$. Let positive number $\eta=1.5$ and intensity of noise perturbation

$$\mu_1=0.05,\ \mu_2=0.04,\ \mu_3=0.03,\ \mu_4=0.02,$$

$$\gamma_1=0.02,\ \gamma_2=0.03,\ \gamma_3=0.04,\ \gamma_4=0.05.$$

Furthermore, the damping functions $\psi_i(\cdot)$ are chosen as

$$\psi_1(x_1)=0.005\cos x_1+2.165,\ \psi_2(x_2)=0.005\sin x_2+2.165,$$

$$\psi_3(x_3)=-0.005\cos x_3+2.165,\ \psi_4(x_4)=-0.005\sin x_4+2.165,$$

and the coupling configuration matrix is

$$B=\begin{pmatrix}0.01&0&0&0.04\\0&0.02&0.03&0\\0&0.02&0.03&0\\0.01&0&0&0.04\end{pmatrix}.$$

which is irreducible obviously. Through calculation, the condition P1 in Theorem 2 is met. Thus, RCON (11) is NSSP with initial values as following

$$\theta_1=(0.3,-0.1)^{\mathrm{T}},\ \theta_2=(-0.5,0.5)^{\mathrm{T}},\ \theta_3=(0.7,-2)^{\mathrm{T}},\ \theta_4=(0.1,6)^{\mathrm{T}}.$$

From Figures 1 and 2, we can observe that all the subsystems of RCON (11) states $\theta_i(t)$ ($i \in \mathbb{H}$) are bounded when the random process $\xi(t)$ meets $\sup_{0\le s\le t}\mathbb{E}|\xi(s)|^2 < K_0$ with K_0 being a positive constant, condition P1 in Theorem 2 is satisfied and coupling configuration matrix $B = (b_{ij})_{h\times h}$ of RCON (11) is irreducible, which brings into correspondence with the conclusion above and further shows the practicability of the results presented in the paper.

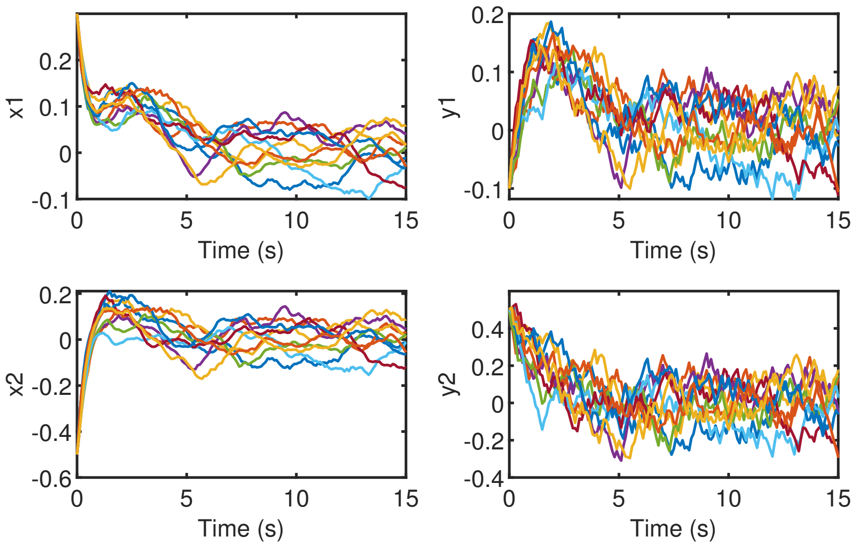

Figure 1. The sample path for $\theta_1(t)$ and $\theta_2(t)$ of RCON (11) with chosen initial values.

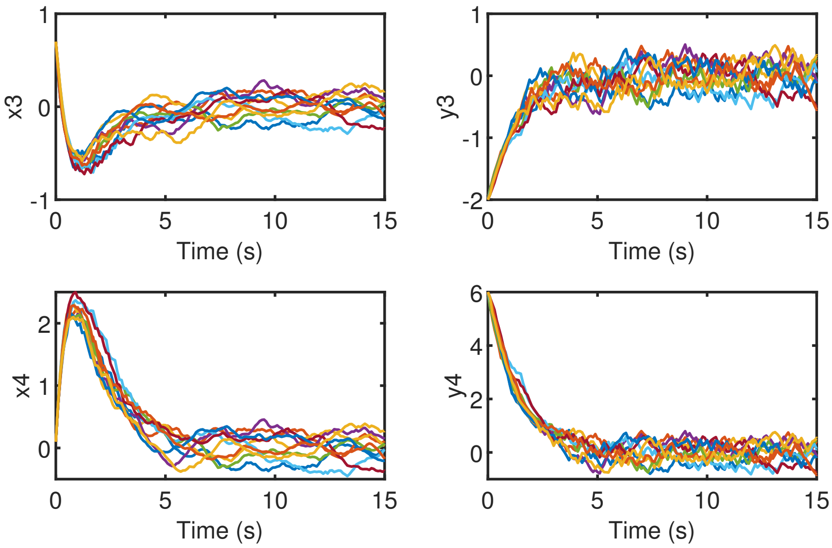

Figure 2. The sample path for $\theta_3(t)$ and $\theta_4(t)$ of RCON (11) with chosen initial values.

Remark 6. *As we all know, the analytical solution of stochastic differential equations under white noise perturbation is often difficult to find. Furthermore, solving random differential equations under perturbations of second-order moment processes is a more difficult task. In order to verify the validity of our theoretical results, constructing a suitable numerical method for numerical simulation is an important tool. However, in order to ensure the calculation accuracy of the numerical simulation of the system, it is often necessary to reduce the calculation step size, which greatly reduces the calculation efficiency. Therefore, it is necessary to choose the appropriate numerical method to balance the contradiction between calculation accuracy and efficiency. In Section 6, the numerical solutions of RCON (11) are given by using the Euler method with the step size of 0.1 (see Figures 1 and 2), which correctly reflects the dynamic behavior of RCON (11) and is highly operable in practice.*

7. Conclusions

In this paper, we have investigated NSSP for RCDSN (1), and an available Lyapunov function has been constructed on the basis of Kirchhoff's matrix theorem in graph theory. Combining some stochastic analysis skills and the Lyapunov method, some sufficient conditions guaranteeing RCDSN (1) to meet NSSP have been provided. In addition, the usefulness and feasibility of the theoretical findings have been demonstrated by applying them to RCON (11). Finally, by using the Euler method with a step size of 0.1, some sample paths of RCON (11) are given (see Figures 1 and 2). It can be seen from the figures that RCON (11) is NSSP when the random process $\xi(t)$ meets $\sup_{0\le s\le t}\mathbb{E}|\xi(s)|^2 < K_0$ with K_0 with K_0 being a positive constant and condition P1 in Theorem 2 is satisfied, which shows the validity of our theoretical results. In Theorem 1, the NSSP of RCDSN (1) is studied when weighted digraph $(\mathcal{G}, A)$ is strongly connected. However, in the real world, the topological structure of RCDSN (1) may be arbitrary. Therefore, it is valuable to study RCDSN (1) when the weighted digraph $(\mathcal{G}, A)$ is not strongly connected in the future.

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