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On De la Peña Type Inequalities for Point Processes

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Abstract: There has been a renewed interest in exponential concentration inequalities for stochastic processes in probability and statistics over the last three decades. De la Peña established a nice exponential inequality for a discrete time locally square integrable martingale. In this paper, we obtain de la Peña's inequalities for a stochastic integral of multivariate point processes. The proof is primarily based on Doléans–Dade exponential formula and the optional stopping theorem. As an application, we obtain an exponential inequality for block counting process in $\Lambda-$coalescent.

Keywords: de la Peña's inequalities; purely discontinuous local martingales; stochastic integral of multivariate point processes; Doléans–Dade exponential

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1. Introduction

Let $S=(S_n)_{n\geq 0}$ be a locally square integrable martingale on $(\Omega,\mathcal{F},(\mathcal{F}_n)_{n\geq 1},\mathbb{P})$. The predictable quadratic variation of $S=(S_n)_{n\geq 0}$ is given by

$$<S,S>_n=\sum_{i=1}^{n}\mathbb{E}[((S_i-S_{i-1})^2|\mathcal{F}_{i-1}].$$

Many authors studied the upper bound of

$$\mathbb{P}(S_n\geq x,<S,S>_n\leq y).$$

The celebrated Freedman inequality is as follows.

Theorem 1 (Freedman [1]). *Let $S=(S_n)_{n\geq 0}$ be a locally square integrable martingale on $(\Omega,\mathcal{F},(\mathcal{F}_n)_{n\geq 1},\mathbb{P})$.If $|S_k-S_{k-1}|\leq c$ for each $1\leq k\leq n$, then*

$$\mathbb{P}\big(S_n\geq x,<S,S>_n\leq y\big)\leq \exp\{-\frac{x^2}{2(y+cx)}\}.$$

This result can be regarded as an extension of Hoeffding [2]. Fan, Grama and Liu [3,4], and Rio [5] obtained a series of remarkable extensions of Freedman inequality [1]. See also Bercu et al. [6] for a recent review in this field.

De la Peña [7] establishes a nice exponential inequality for discrete time locally square integrable martingales.

Theorem 2 (De la Peña [7]). *Let $S = (S_n)_{n\geq 0}$ be a locally square integrable and conditionally symmetric martingale on $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geq 1}, \mathbb{P})$. Then,*

$$\mathbb{P}\big(S_n \geq x, \sum_{i=1}^{n}(S_i - S_{i-1})^2 \leq y\big) \leq \exp\{-\frac{x^2}{2y}\}.$$

This result is quite different from the classical Freedman's inequality. The challenge for obtaining Theorem 1 is to find an approach based on the use of the exponential Markov's inequality. De la Peña constructed a supermartingale to get Theorem 1. Furthermore, Bercu and Touati [8] established the following result for self-normalized martingales, which are similar to Theorem 1.

Theorem 3 (Bercu and Touati [8]). *Let $S = (S_n)_{n\geq 0}$ be a locally square integrable martingale on $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geq 1}, \mathbb{P})$. Then, for all $x, y > 0$, $a \geq 0$ and $b > 0$,*

$$\mathbb{P}\big(\frac{|S_n|}{a + b < S, S >_n} \geq x, < S, S >_n \geq \sum_{i=1}^{n}(S_i - S_{i-1})^2 + y\big) \leq 2\exp\{-x^2(ab + \frac{b^2 y}{2})\}.$$

It is natural to ask what will happen when we study the continuous-time processes for the above cases? Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a stochastic basis. $M = (M_t)_{t\geq 0}$ is a continuous locally square integrable martingale. The predictable quadratic variation of M, $< M, M >$, is a continuous increasing process, such that $(M_t^2 - < M, M >_t)_{t\geq 0}$ is a local martingale. However, we cannot define an analogy for M like $\sum_{i=1}^{n}(S_i - S_{i-1})^2$ in Theorems 1 and 3. Since $M = (M_t)_{t\geq 0}$ has jumps, we can replace $\sum_{i=1}^{n}(S_i - S_{i-1})^2$ by $\sum_{s\leq t} |\triangle M_s|^2$. It is an interesting problem to consider De la Peña type inequalities for continuous-time local square integrable martingale with jumps. Some authors obtained the concentration inequalities for continuous-time stochastic processes. Bernstein's inequality for local martingales with jumps was given by van der Geer [9]. Khoshnevisan [10] found some concentration inequalities for continuous martingales. Dzhaparidze and van Zanten [11] extended Khoshnevisan's results to martingales with jumps.

This paper focuses on the De la Peña type inequalities for stochastic integrals of multivariate point processes. Stochastic integrals of multivariate point processes are an essential example of purely discontinuous local martingales. Some useful facts and results essential for this paper's proofs will be collected in Section 2. Section 3 will present our main results and give their proofs, while Section 4 will derive an exponential inequality for block counting process in $\Lambda-$coalescent as applications. Usually, $c, C, K, \cdots$ denote positive constants, which very often may be different at each occurrence.

2. Preliminaries

Let$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a stochastic basis. A stochastic process $M = (M_t)_{t\geq 0}$ is called a purely discontinuous local martingale if $M_0 = 0$ and M is orthogonal to all continuous local martingales. The reader is referred to the classic book [12] due to Jacod and Shiryayev for more information. We shall restrict ourselves to the integer-valued random measure μ on $\mathbb{R}_+ \times \mathbb{R}$ induced by a $\mathbb{R}_+ \times \mathbb{R}$-valued multivariate point process. In particular, let $(T_k, Z_k), k \geq 1$, be a multivariate point process, and define

$$\mu(dt, dx) = \sum_{k\geq 1} \mathbf{1}_{\{T_k<\infty\}}\varepsilon_{(T_k,Z_k)}(dt, dx), \tag{1}$$

where $\varepsilon_{(T_k,Z_k)}$ is the delta measure at point (T_k, Z_k). Then $\mu(\omega; [0, t] \times \mathbb{R}) < \infty$ for all $(\omega, t) \in \Omega \times \mathbb{R}$. Let $\tilde{\Omega} = \Omega \times \mathbb{R}_+ \times \mathbb{R}$, $\tilde{\mathcal{P}} = \mathcal{P} \otimes \mathcal{B}$, where $\mathcal{B}$ is a Borel σ-field on $\mathbb{R}$ and $\mathcal{P}$ a σ-field generated by all left continuous adapted processes on $\Omega \times \mathbb{R}_+$. The predictable function is a $\tilde{\mathcal{P}}$-measurable function on $\tilde{\Omega}$. Let ν be the unique predictable compensator of μ (up to a $\mathbb{P}$-null set). Namely, ν is a predictable random measure such that for any predictable function W, $W * \mu - W * \nu$ is a local martingale, where the $W * \mu$ is defined by

$$W * \mu_t = \begin{cases} \int_0^t \int_{\mathbb{R}} W(s,x)\mu(ds,dx), & \text{if } \int_0^t \int_{\mathbb{R}} |W(s,x)|\mu(ds,dx) < \infty, \\ +\infty, & \text{otherwise.} \end{cases}$$

Note the ν admits the disintegration

$$\nu(dt,dx) = dA_t K(\omega,t;dx), \tag{2}$$

where $K(\cdot,\cdot)$ is a transition kernel from $(\Omega \times \mathbb{R}_+, \mathcal{P})$ into $(\mathbb{R}, \mathcal{B})$, and $A = (A_t)_{t \geq 0}$ is an increasing càdlág predictable process. For μ in (1), which is defined through multivariate point process, ν admits

$$\nu(dt,dx) = \sum_{n \geq 1} \frac{1}{G_n([t,\infty] \times \mathbb{R})} \mathbf{1}_{\{t \leq T_{n+1}\}} G_n(dt,dx),$$

where $G_n(\omega,ds,dx)$ is a regular version of the conditional distribution of (T_{n+1}, Z_{n+1}) with respect to $\sigma\{T_1, Z_1, \cdots, T_n, Z_n\}$. In particular, if $F_n(dt) = G_n(dt \times \mathbb{R})$, the point process $N = \sum_{n \geq 1} \mathbf{1}_{[T_n,\infty)}$ has the compensator $A_t = \nu([0,t] \times \mathbb{R})$, which satisfies

$$A_t = \sum_{n \geq 1} \int_0^{T_{n+1} \wedge t} \frac{1}{F_n([s,\infty])} F_n(ds).$$

Now, we define the stochastic integrals of multivariate point processes. For a stopping time T, $[T] = \{(\omega,t) : T(\omega) = t\}$ is the graph of T. For μ in (1), define $D = \bigcup_{n=1}^{\infty} [T_n]$. With any measurable function W on $\tilde{\Omega}$, we define $a_t = \nu(\{t\} \times \mathbb{R})$, and

$$\hat{W} * \nu_t = \begin{cases} \int_{\mathbb{R}} W(t,x)\nu(\{t\} \times dx), & \text{if } \int_{\mathbb{R}} |W(t,x)|\nu(\{t\} \times dx) < \infty, \\ +\infty, & \text{otherwise.} \end{cases}$$

We denote by $G_{loc}(\mu)$ the set of all $\tilde{\mathcal{P}}-$measurable real-valued functions W such that $[\sum_{s \leq t} (\widetilde{W}_s)^2]^{1/2}$ is local integrable variation process, where $\widetilde{W}_t = W 1_D(\omega,t) - \hat{W}_t$.

Definition 1. *If $W \in G_{loc}(\mu)$, the stochastic integral of W with respect to $\mu - \nu$ is defined as a purely discontinuous local martingales, the jump process of which is indistinguishable from $\widetilde{W}$.*

We denote the stochastic integral of W with respect to $\mu - \nu$ by $W * (\mu - \nu)$. For a given predictable function W, $W * (\mu - \nu)$ is a purely discontinuous local martingale, which is defined through jump process. It is easy to prove that $W * (\mu - \nu) = W * \mu - W * \nu$. Denote $M = W * (\mu - \nu)$.

Itô's formula for a purely discontinuous local martingale is essential for our proofs. Now, we present Itô's formula for M.

Lemma 1 (Itô's formula, Jacod and Shiryaev [12])**.** *Let μ be a multivariate point process, ν be the predictable compensator of μ, W be a given predictable function on $\tilde{\Omega}$, and $W \in G_{loc}(\mu)$. Let f be a differentiable function, for $M = W * (\mu - \nu)$ and $t > 0$,*

$$f(M_t) = f(M_0) + \int_0^t f'(M_{s-})dM_s + \sum_{s \leq t} [f(M_s) - f(M_{s-}) - f'(M_{s-})\Delta M_s].$$

Under some conditions, Wang, Lin and Su [13] obtained

$$\mathbb{P}\Big(M_t \geq x, < M, M >_t \leq v^2 \text{ for some } t > 0\Big) \leq \exp\{-\frac{x^2}{2(v^2 + cx)}\} \tag{3}$$

where $< M, M >$ is the predictable quadratic variation process of $M = W * (\mu - \nu)$,

$$< M, M >_t = (W - \hat{W})^2 * \nu_t + \sum_{1 \leq s \leq t} (1 - a_s)\hat{W}_s^2.$$

When M is a purely discontinuous local martingale, $\sum_{s \leq \cdot} |\Delta M_s|^2 - < M, M >$ is a local martingale. There will be an interesting problem when the predictable quadratic variation $< M, M >$ in (3) is replaced by the quadratic variation $\sum_{s \leq \cdot} |\Delta M_s|^2$. In this paper, we will estimate the upper bound of two types of tail probabilities:

$$\mathbb{P}\Big(M_t \geq x, \sum_{s \leq t} |\Delta M_s|^2 \leq v^2 \text{ for some } t > 0\Big) \tag{4}$$

and

$$\mathbb{P}\Big(M_t \geq (\alpha + \beta \sum_{s \leq t} |\triangle M_s|^2)x,\ \sum_{s \leq t} |\triangle M_s|^2 \geq < M, M >_t + v^2 \text{ for some } t > 0\Big). \tag{5}$$

It is important to note that the continuity of A implies the quasi-left continuity of M. However, the quasi-left continuity of M can be destroyed easily by changing the filtration in the underlying space. For example, let N be a homogeneous Poisson process with respect to $\mathbb{F}$. Let $(T_n)_{n \geq 0}$ be the sequence of the jump-times of N. The process N is not quasi-left continuous in the filtration $\mathbb{G}$ obtained by enlarging $\mathbb{F}$ initially with the $\sigma-$field $\mathcal{R} = \sigma(T_1)$. ($\sigma_n = (1 - \frac{1}{2n})T_1$ is a sequence of $\mathbb{G}$ -stopping times announcing T_1). The main purpose of this paper consists in estimating (4) and (5) when M is not quasi-left continuous.

3. The Main Results and Their Proofs

Now, we present our first main result.

Theorem 4. *Let μ be a multivariate point process, ν be the predictable compensator of μ, $a_t = \nu(\{t\} \times \mathbb{R})$, W be a given predictable function on $\tilde{\Omega}$, and $W \in G_{loc}(\mu)$. $M = W * (\mu - \nu)$. Assume $\Delta M \geq -1$. Then, for $x > 0$, $v > 0$,*

$$\mathbb{P}\Big(M_t \geq x, \sum_{s \leq t} |\triangle M_s|^2 \leq v^2 \text{ for some } \ t > 0\Big) \leq \Big(\frac{v^2 + x}{v^2}\Big)^{v^2} e^{-x}.$$

Proof of Theorem 4. For simplicity of notation, put

$$\begin{aligned} S(\lambda)_t &= \int_0^t \int_{\mathbb{R}} (e^{[\lambda(W-\hat{W}) - (\lambda + \log(1-\lambda))(W-\hat{W})^2]} - 1 - \lambda(W - \hat{W}))\nu(ds, dx) \\ &\quad + \sum_{s \leq t} (1 - a_s)(e^{[-\lambda \hat{W}_s + (\lambda + \log(1-\lambda))(\hat{W}_s)^2]} - 1 + \lambda \hat{W}_s), \end{aligned}$$

where $\lambda \in [0, 1)$.

Furthermore,

$$\begin{aligned}\Delta S(\lambda)_t &= \int_{\mathbb{R}}\Big(e^{[\lambda(W-\hat{W})-(\lambda+\log(1-\lambda))(W-\hat{W})^2]}-1-\lambda(W-\hat{W})\Big)\nu(\{t\},dx)\\ &\quad+(1-a_t)(e^{[-\lambda\hat{W}_t+(\lambda+\log(1-\lambda))(\hat{W}_t)^2]}-1+\lambda\hat{W}_t)\\ &= e^{[-\lambda\hat{W}_t-(\lambda+\log(1-\lambda))(\hat{W}_t)^2]}\Big(\int_{\mathbb{R}}e^{[\lambda W-(\lambda+\log(1-\lambda))(W^2-2W\hat{W})]}\nu(\{t\},dx)+1-a_t\Big)\\ &\quad+(1-a_t)(-1+\lambda\hat{W}_t)-\int_{\mathbb{R}}\big(1+\lambda(W-\hat{W})\big)\nu(\{t\},dx)\\ &= e^{[-\lambda\hat{W}_t-(\lambda+\log(1-\lambda))(\hat{W}_t)^2]}\Big(\int_{\mathbb{R}}e^{[\lambda W-(\lambda+\log(1-\lambda))(W^2-2W\hat{W})]}\nu(\{t\},dx)+1-a_t\Big)\\ &\quad+(1-a_t)\lambda\hat{W}_t-1-\lambda\int_{\mathbb{R}}(W-\hat{W})\nu(\{t\},dx).\end{aligned}$$

In addition, it is easy to see by noting $a_t\le 1$,

$$\int_{\mathbb{R}}e^{[\lambda W-(\lambda+\log(1-\lambda))(W^2-2W\hat{W})]}\nu(\{t\},dx)+1-a_t\ge 0,$$

and

$$(1-a_t)\lambda\hat{W}_t=\lambda\int_{\mathbb{R}}(W-\hat{W})\nu(\{t\},dx).$$

In combination, we have for all $t>0$

$$\Delta S(\lambda)_t>-1,$$

where $\lambda\in[0,1)$. For any semimartingale $S(\lambda)_t$, the Doléans–Dade exponential is

$$\mathcal{E}(S(\lambda))_t=e^{S(\lambda)_t-S(\lambda)_0-\frac{1}{2}<S(\lambda)^c,S(\lambda)^c>_t}\prod_{s\le t}(1+\Delta S(\lambda)_t)e^{-\Delta S(\lambda)_t}.$$

We shall first show that the process $\left(e^{[\lambda M_t-(\lambda+\log(1-\lambda))\sum_{s\le t}(\Delta M_s)^2)]}/\mathcal{E}(S(\lambda))_t\right)_{t\ge 0}$ is a local martingale. Denote $X_t=\lambda M_t-(\lambda+\log(1-\lambda))\sum_{s\le t}(\Delta M_s)^2$, $Y_t=\sum_{s\le t}(\Delta M_s)^2$.

The Itô formula yields

$$\begin{aligned}e^{X_t} &= 1+e^{X_{t-}}\cdot X+\sum_{s\le t}(e^{X_s}-e^{X_{s-}}-e^{X_{s-}}\Delta X_s)\\ &= 1+\lambda e^{X_{t-}}\cdot M-(\lambda+\log(1-\lambda))e^{X_{t-}}\cdot Y\\ &\quad+\sum_{s\le t}(e^{X_s}-e^{X_{s-}}-e^{X_{s-}}\Delta X_s)\\ &= 1+\lambda e^{X_{t-}}\cdot M+\sum_{s\le t}(e^{X_s}-e^{X_{s-}}-\lambda e^{X_{s-}}\Delta M_s).\end{aligned}$$

For X, the jump part of X is

$$\begin{aligned}\Delta X &= [\lambda(W-\hat{W})-(\lambda+\log(1-\lambda))(W-\hat{W})^2]1_D\\ &\quad-\lambda\hat{W}1_{D^c}+(\lambda+\log(1-\lambda))\hat{W}^2 1_{D^c}\end{aligned}$$

where D is the thin set, which is exhausted by $\{T_n\}_{n\ge 1}$. Thus,

$$\sum_{s\le t}(e^{\Delta X_s}-1-\lambda\Delta M_s)-S(\lambda)=:Z_t \tag{6}$$

is a local martingale. Denote $\Xi(\lambda)_t=\sum_{s\le t}(e^{\Delta X_s}-1-\lambda\Delta M_s)$, we have

$$\begin{aligned} & \sum_{s\le t}(e^{X_s}-e^{X_{s-}}-\lambda e^{X_{s-}}\Delta M_s)-e^{X_-}\cdot S(\lambda) \\ = \ & e^{X_-}\cdot \Xi(\lambda)-e^{X_-}\cdot S(\lambda)=e^{X_{t-}}\cdot Z. \end{aligned}$$

Thus,

$$\begin{aligned} & e^X-e^{X_-}\cdot S(\lambda) \\ = \ & 1+\lambda e^{X_{t-}}\cdot M+\sum_{s\le t}(e^{X_s}-e^{X_{s-}}-\lambda e^{X_{s-}}\Delta M_s)-e^{X_-}\cdot S(\lambda) \\ = \ & 1+\lambda e^{X_{t-}}\cdot M+e^{X_{t-}}\cdot Z=: N_1. \end{aligned}$$

N_1 is a local martingale. Following the similar arguments in Wang Lin and Su [13], we have $\left(e^{X_t}/\mathcal{E}(S(\lambda))_t\right)_{t\ge 0}$ is a local martingale. In fact, set $H=e^X$, $G=\mathcal{E}(S(\lambda))$, $A=S(\lambda)$ and $f(h,g)=\frac{h}{g}$. The Itô formula yields

$$\begin{aligned} f(H,G) \ = \ & 1+\frac{1}{G_-}\cdot H-\frac{H_-}{G_-^2}\cdot G \\ & +\sum_{s\le\cdot}\Big(\Delta f(H,G)_s-\frac{\Delta H_s}{G_{s-}}+\frac{f(H,G)_{s-}}{G_{s-}}\Delta G_s\Big). \end{aligned}$$

Since $\mathcal{E}(S(\lambda))=1+\mathcal{E}(S(\lambda))_-\cdot S(\lambda)$, we have

$$\begin{aligned} & \frac{1}{G_-}\cdot H-\frac{H_-}{G_-^2}\cdot G \\ = \ & \frac{1}{G_-}\cdot H-\frac{H_-}{G_-}\cdot S(\lambda)=\frac{1}{G_-}\cdot(e^X-e^{X_-}\cdot S(\lambda)) \\ = \ & \frac{1}{G_-}\cdot N_1. \end{aligned}$$

Noting that $\Delta G=G_-\Delta A$, $\Delta N_1=\Delta H-H_-\Delta A$, we have

$$\Delta f(H,G)_s-\frac{\Delta H_s}{G_{s-}}+\frac{f(H,G)_{s-}}{G_{s-}}\Delta G_s=-\frac{\Delta N_{1s}\Delta A_s}{G_{s-}(1+\Delta A_s)},$$

where A is a predictable process, and N is a local martingale. By the property of the Stieltjes integral, we have

$$\sum_{s\le\cdot}\Delta f(H,G)_s-\frac{\Delta H_s}{G_{s-}}+\frac{f(H,G)_{s-}}{G_{s-}}\Delta G_s=-\frac{\Delta A}{G_-(1+\Delta A)}\cdot N_1. \tag{7}$$

Thus,

$$\left(e^X/\mathcal{E}(S(\lambda))\right)=1+\frac{1}{G_-}\cdot N_1-\frac{\Delta A}{G_-(1+\Delta A)}\cdot N_1$$

is a local martingale.

Let

$$B_1=\{M_t\ge x,\sum_{s\le t}|\triangle M_s|^2\le v^2 \text{ for some } t>0\}$$

and

$$\tau_1=\inf\{t>0: M_t\ge x,\sum_{s\le t}|\triangle M_s|^2\le v^2\}.$$

Note by (4.12) in [4], for $\lambda \in [0,1)$ and $x \geq -1$,

$$\exp\{\lambda x + x^2(\lambda + \log(1-\lambda))\} \leq 1 + \lambda x.$$

This implies

$$\int_0^t \int_{-1}^{\infty} \exp\{\lambda x + (\lambda + \log(1-\lambda))x^2\}\nu^M(ds, dx) \leq \int_0^t \int_{-1}^{\infty} (1+\lambda x)\nu^M(ds, dx), \quad (8)$$

because $\Delta M_t \geq -1$ for any $t > 0$, where ν^M is the predictable compensate jump measure of M. Inequality (8) implies $S(\lambda) \leq 0$. Since $e^x \geq x + 1$ and $e^{S(\lambda)_t} \geq \mathcal{E}(S(\lambda)_t)$,

$$\mathbb{E}[\frac{e^{\lambda X_T}}{e^{S(\lambda)_T}}] \leq \mathbb{E}[\frac{e^{\lambda X_T}}{\mathcal{E}(S(\lambda))_T}] = 1 \quad (9)$$

for any stopping time T. Thus, $U = (U_t)_{t \geq 0}$ is a supermartingale, where

$$U_t = \frac{\exp\{\lambda M_t + (\lambda + \log(1-\lambda))\sum_{s \leq t}(\triangle M_s)^2\}}{\exp\{S(\lambda)_t\}}.$$

Thus, on B_1

$$U_{\tau_1} \geq \exp\{\lambda x + (\lambda + \log(1-\lambda))v^2\}.$$

We have

$$\begin{aligned}\mathbb{P}(B_1) &\leq \inf_{\lambda \in [0,1)} \exp\{-\lambda x - (\lambda + \log(1-\lambda))v^2\} \\ &= \left(\frac{v^2 + x}{v^2}\right)^{v^2} e^{-x}. \quad (10)\end{aligned}$$

□

Put

$$\begin{aligned}L(\lambda)_t &= \int_0^t \int_{\mathbb{R}} (e^{[\lambda(W-\hat{W}) + f(\lambda)(W-\hat{W})^2]} - 1 - \lambda(W - \hat{W}))\nu(ds, dx) \\ &\quad + \sum_{s \leq t}(1 - a_s)(e^{[-\lambda \hat{W}_s + f(\lambda)(\hat{W}_s)^2]} - 1 + \lambda \hat{W}_s),\end{aligned}$$

where $f(\lambda) \geq 0$ for $\lambda \geq 0$. We have the following proposition from the proof of Theorem 4.

Proposition 1. *Let μ be a multivariate point process, ν be the predictable compensator of μ, $a_t = \nu(\{t\} \times \mathbb{R})$, W be a given predictable function on $\tilde{\Omega}$. $M = W * (\mu - \nu)$. Denote $\tilde{X}_t = \lambda M_t - f(\lambda)\sum_{s \leq t}(\Delta M_s)^2$, for $\lambda \geq 0$. Then, $e^{\tilde{X}}/\mathcal{E}(L(\lambda))$ is a local martingale.*

In Theorem 4, the condition $\triangle M \geq -1$ plays an important role. In the following theorem, we will present another result, which is the analogy of Theorem 1 in continuous time case.

Theorem 5. *Let μ be a multivariate point process, ν be the predictable compensator of μ, $a_t = \nu(\{t\} \times \mathbb{R})$, W be a given predictable function on $\tilde{\Omega}$, and $W \in G_{loc}(\mu)$. $M = W * (\mu - \nu)$, In addition, define*

$$\begin{aligned}\widetilde{S}(\lambda)_t &=: \int_0^t \int_{\mathbb{R}} (e^{[\lambda(W-\hat{W}) - \frac{\lambda^2}{2}(W-\hat{W})^2]} - 1 - \lambda(W - \hat{W}))\nu(ds, dx) \\ &\quad + \sum_{s \leq t}(1 - a_s)(e^{[-\lambda \hat{W}_s + \frac{\lambda^2}{2}(\hat{W}_s)^2]} - 1 + \lambda \hat{W}_s),\end{aligned}$$

and assume that for any $t > 0$ and $\lambda > 0$, $\widetilde{S}(\lambda)_t \leq 0$. Then, for $x > 0$, $v > 0$,

$$\mathbb{P}\Big(M_t \geq x, \sum_{s\leq t} |\triangle M_s|^2 \leq v^2 \text{ for some } \; t > 0\Big) \leq \exp\{-\frac{x^2}{2v^2}\}$$

Proof of Theorem 5. Define

$$V_t = \frac{\exp\{\lambda M_t - \frac{\lambda^2}{2}\sum_{s\leq t}|\triangle M_s|^2\}}{\mathcal{E}(\widetilde{S}(\lambda))_t}.$$

By Proposition 1, V is a local martingale. Note $\widetilde{S}(\lambda)_t \leq 0$ for any $t > 0$ and $\lambda > 0$. We have

$$\mathbb{E}[\frac{\exp\{\lambda M_T - \frac{\lambda^2}{2}\sum_{s\leq T}|\triangle M_s|^2\}}{e^{\tilde{S}(\lambda)_T}}] \leq \mathbb{E}[V_T] = 1 \tag{11}$$

for any stopping time T.

Recall that

$$B_1 = \{M_t \geq x, \sum_{s\leq t} |\triangle M_s|^2 \leq v^2 \text{ for some } t > 0\}$$

and

$$\tau_1 = \inf\{t > 0 : M_t \geq x, \sum_{s\leq t} |\triangle M_s|^2 \leq v^2\}.$$

We have

$$\begin{aligned} \mathbb{P}(B_1) &\leq \inf_{\lambda\geq 0} \exp\{-\lambda x + \frac{\lambda^2}{2}v^2\} \\ &= \exp\{-\frac{x^2}{2v^2}\}. \end{aligned} \tag{12}$$

□

Remark 1. *For integrable random variable ξ and a positive number $a > 0$, define*

$$T_a(\xi) = \min(|\xi|, a) sign(\xi).$$

If $\mathbb{E}[\xi] = 0$, and for all $a > 0$, $\mathbb{E}[T_a(\xi)] \leq 0$. Then, ξ is called heavy on left. Bercu and Touati [14] extended Theorem 1 to general case. Let $S = (S_n)_{n\geq 0}$ be a locally square integrable on $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geq 1}, \mathbb{P})$. If

$$\mathbb{E}[T_a(S_n - S_{n-1})|\mathcal{F}_{n-1}] \leq 0 \tag{13}$$

for all $a > 0$ and $n > 0$, Bercu and Touati [14] obtained

$$\mathbb{P}(S_n \geq x, \sum_{i=1}^{n}(S_i - S_{i-1})^2 \leq y) \leq \exp\{-\frac{x^2}{2y}\}.$$

In fact, our condition, $\widetilde{S}(\lambda)_t \leq 0$, is analogy of (13) in continuous time case. Let $N = (N_t)_{t\geq 0}$ be a homogeneous Poisson point process with parameter κ, and let $(\eta_k)_{k\geq 1}$ be a sequence of i.i.d. r.v.'s with a common distribution function $F(x)$. Assume N is independent of $(\eta_k)_{k\geq 1}$. Define

$$Y_t = \sum_{k=1}^{N_t} \eta_k, \quad t \geq 0. \tag{14}$$

This is a so-called compound Poisson process. The jump measure of Y is given by

$$\mu^Y(dt, dx) = \sum_{k\geq 1} \mathbf{1}_{\{T_k<\infty\}}\varepsilon_{(T_k,\eta_k)}(dt, dx), \tag{15}$$

and the predictable compensator ν^Y *is*

$$\nu^Y(dt, dx) = \kappa dt F(dx). \tag{16}$$

Thus, $(Y_t - x * \nu_t^Y)_{t\geq 0}$ *is a purely discontinuous local martingale. For* $(Y_t - x * \nu_t^Y)_{t\geq 0}$,

$$\widetilde{S}(\lambda)_t = \kappa \int_0^t \int_{\mathbb{R}} (e^{[\lambda x - \frac{\lambda^2}{2}x^2]} - 1 - \lambda x) F(dx) ds.$$

If $\mathbb{E}[\eta_k] = 0$ *for any* $\kappa \geq 1$, $\widetilde{S}(\lambda)_t \leq 0$ *implies that*

$$\int_{\mathbb{R}} e^{[\lambda x - \frac{\lambda^2}{2}x^2]} F(dx) \leq 1. \tag{17}$$

Bercu and Touati [14] found that if η_k *is heavy on the left, then (17) holds. Thus, our condition is an analogy of (13) in continuous time case.*

In [7,15], there were obtained a series of exponential inequalities for events involving ratios in the context of continuous martingales, which in turn extended the results in [10]. Su and Wang [16] extended a similar problem for purely discontinuous local martingales in quasi-left continuous case. In this subsection, we obtained the similar inequality for stochastic integrals of a multivariate point process.

Theorem 6. *Let* μ *be a multivariate point process,* ν *be the predictable compensator of* μ, $a_t = \nu(\{t\} \times \mathbb{R})$, W *be a given predictable function on* $\tilde{\Omega}$, *and* $W \in G_{loc}(\mu)$. *Denote* $M = W * (\mu - \nu)$. *Then, for all* $x \geq 0, \beta > 0, v > 0$ $\alpha \in \mathbb{R}$,

$$\begin{aligned} &\mathbb{P}\Big(M_t \geq (\alpha + \beta \sum_{s\leq t} |\triangle M_s|^2)x, \ \sum_{s\leq t} |\triangle M_s|^2 \geq < M, M >_t + v^2 \text{ for some } \ t > 0\Big) \\ \leq \quad & \exp\{-\frac{x^2}{2}(\alpha\beta + \frac{\beta^2 v^2}{2})\}. \end{aligned}$$

Proof of Theorem 6. Recall that $V = (V_t)_{t\geq 0}$ is a local martingale, where

$$V_t = \frac{\exp\{\lambda M_t - \frac{\lambda^2}{2}\sum_{s\leq t} |\triangle M_s|^2\}}{\mathcal{E}(\widetilde{S}(\lambda))_t}.$$

For any stopping time T,

$$\mathbb{E}[\frac{\exp\{\lambda M_T - \frac{\lambda^2}{2}\sum_{s\leq T} |\triangle M_s|^2\}}{\exp\{\tilde{S}(\lambda)_T\}}] \leq \mathbb{E}[V_T] = 1. \tag{18}$$

By Markov's inequality, we obtain that for all $\lambda > 0$,

$$
\begin{aligned}
&\mathbb{P}\Big(M_t \geq (\alpha+\beta\sum_{s\leq t}|\triangle M_s|^2)x \text{ and } \sum_{s\leq t}|\triangle M_s|^2 \geq <M,M>_t+v^2 \text{ for some } t>0\Big)\\
\leq\ & \mathbb{E}[\exp\{\frac{\lambda}{4}M_{\tau_2}-(\frac{\alpha\lambda x}{4}+\frac{\beta\lambda x}{4}\sum_{s\leq\tau_2}|\triangle M_s|^2)\}1_{B_2}]\\
=\ & \exp\{-\frac{\alpha\lambda x}{4}\}\mathbb{E}[\exp\{\frac{\lambda}{4}M_{\tau_2}-\frac{\lambda^2}{8}(\sum_{s\leq\tau_2}|\triangle M_s|^2+<M,M>_{\tau_2})\\
&+(\frac{\lambda^2}{8}-\frac{\beta\lambda x}{4})\sum_{s\leq\tau_2}|\triangle M_s|^2+\frac{\lambda^2}{8}<M,M>_{\tau_2})\}1_{B_2}]\\
\leq\ & \exp\{-\frac{\alpha\lambda x}{4}\}\sqrt{\mathbb{E}[\exp\{\frac{\lambda}{2}M_{\tau_2}-\frac{\lambda^2}{4}(\sum_{s\leq\tau_2}|\triangle M_s|^2+<M,M>_{\tau_2})\}1_{B_2}]}\\
&\times\sqrt{\mathbb{E}[\exp\{(\frac{\lambda^2}{4}-\frac{\beta\lambda x}{2})\sum_{s\leq\tau_2}|\triangle M_s|^2+\frac{\lambda^2}{4}<M,M>_{\tau_2}\}1_{B_2}]},
\end{aligned}
$$

where

$$B_2=\{M_t\geq(\alpha+\beta\sum_{s\leq t}|\triangle M_s|^2)x,\sum_{s\leq t}|\triangle M_s|^2\geq<M,M>_t+v^2 \text{ for some } t>0\},$$

$$\tau_2=\inf\{t>0: M_t\geq(\alpha+\beta\sum_{s\leq t}|\triangle M_s|^2)x,\sum_{s\leq t}|\triangle M_s|^2\geq<M,M>_t+v^2\}.$$

In fact,

$$
\begin{aligned}
&\mathbb{E}[\exp\{\frac{\lambda}{2}M_{\tau_2}-\frac{\lambda^2}{4}(\sum_{s\leq\tau_2}|\triangle M_s|^2+<M,M>_{\tau_2})1_{B_2}]\}\\
\leq\ & \sqrt{\mathbb{E}[\frac{\exp\{\lambda M_{\tau_2}-\frac{\lambda^2}{2}\sum_{s\leq\tau_2}|\triangle M_s|^2\}}{\exp\{\widetilde{S}(\lambda)_{\tau_2}\}}1_{B_2}]}\sqrt{\mathbb{E}[\exp\{\widetilde{S}(\lambda)_{\tau_2}-\frac{\lambda^2}{2}<M,M>_{\tau_2}]}\}.
\end{aligned}
$$

By (18)

$$\mathbb{E}[\frac{\exp\{\lambda M_{\tau_2}-\frac{\lambda^2}{2}\sum_{s\leq\tau_2}|\triangle M_s|^2\}}{\exp\{\widetilde{S}(\lambda)_{\tau_2}\}}1_{B_2}]\leq 1.$$

Furthermore,

$$\mathbb{E}[\exp\{\widetilde{S}(\lambda)_{\tau_2}-\frac{\lambda^2}{2}<M,M>_{\tau_2}\}]\leq 1$$

by

$$|\exp\{x-\frac{1}{2}x^2\}-1-x|\leq\frac{1}{2}x^2,\quad x\in\mathbb{R}.$$

Taking $\lambda=\beta x$, we get

$$\mathbb{P}\Big(B_2\Big)\leq\exp\{-\frac{x^2}{4}(\alpha\beta+\frac{\beta^2v^2}{2})\}\times\sqrt{\mathbb{P}\Big(B_2\Big)}.$$

Thus

$$
\begin{aligned}
&\mathbb{P}\Big(M_t\geq(\alpha+\beta\sum_{s\leq t}|\triangle M_s|^2)x,\ \sum_{s\leq t}|\triangle M_s|^2\geq<M,M>_t+v^2 \text{ for some } t>0\Big)\\
\leq\ & \exp\{-\frac{x^2}{2}(\alpha\beta+\frac{\beta^2v^2}{2})\}.
\end{aligned}
$$

□

From the proof of Theorem 6, we can obtain the following results.

Theorem 7. *Let μ be a multivariate point process, ν be the predictable compensator of μ, $a_t = \nu(\{t\} \times \mathbb{R})$, W be a given predictable function on $\tilde{\Omega}$, and $W \in G_{loc}(\mu)$. Denote $M = W * (\mu - \nu)$. In addition, define*

$$\begin{aligned}\widetilde{S}(\lambda)_t \quad =: \quad & \int_0^t \int_{\mathbb{R}} (e^{[\lambda(W-\hat{W})-\frac{\lambda^2}{2}(W-\hat{W})^2]} - 1 - \lambda(W-\hat{W}))\nu(ds,dx) \\ & + \sum_{s\le t}(1-a_s)(e^{[-\lambda\hat{W}_s+\frac{\lambda^2}{2}(\hat{W}_s)^2]} - 1 + \lambda\hat{W}_s),\end{aligned}$$

and assume that for any $t > 0$ and $\lambda > 0$, $\widetilde{S}(\lambda)_t \le 0$. Then for all $x \ge 0, \beta > 0, v > 0, \alpha \in \mathbb{R}$,

$$\begin{aligned}& \mathbb{P}\Big(M_t \ge (\alpha + \beta\sum_{s\le t}|\triangle M_s|^2)x,\ \sum_{s\le t}|\triangle M_s|^2 \ge v^2 \text{ for some } \ t > 0\Big) \\ \le \quad & \exp\{-\frac{x^2}{4}(\alpha\beta + \frac{\beta^2 v^2}{2})\}.\end{aligned}$$

4. Application

In this section, we will derive exponential inequalities for block counting process in $\Lambda-$coalescent. The $\Lambda-$coalescent was introduced independently by Pitman [17] and Sagitov [18]. In this paper, the notation and details of $\Lambda-$coalescent are from Limic and Talarczyk [19].

Let Λ be an probability measure on $[0,1]$, $\Pi = (\Pi_t)_{t\ge 0}$ is a Markov jump process. Π takes values in the set of partition of $\{1,2,\cdots\}$. For any $n \ge 1$, the restriction Π^n of Π to $\{1,2,\cdots,n\}$ is a continuous time Markov chain with the following transitions: when Π^n has b blocks, any given $k-$tuples of blocks coalesces at rate

$$\lambda_{b,k} = \int_0^1 r^{k-2}(1-r)^{b-k}\Lambda(dr)$$

where $2 \le b \le n$. Let N_t be the number of blocks of Π_t at t. In fact, $N = (N_t)_{t\ge 0}$ is a point process. Limic and Talarczyk [19] presented integral equation for N. Define

$$\pi(dt,dy,d\mathbf{x}) = \sum_{k\ge 1}\varepsilon_{\{T_k,Y_k,\mathbf{X}^\mathbf{k}\}}(dt,dy,d\mathbf{x})$$

where $\{\mathbf{X}^\mathbf{k}\}$ is an independent array of i.i.d. random variables $(X_j^k)_{j,k\in\mathbb{N}}$, where X_j^k have uniform distribution on $[0,1]$. The multivariate point processes π have the compensator $dt\frac{\Lambda(dy)}{y^2}d\mathbf{x}$.

Limic and Talarczyk [19] found that

$$N_t = N_r - \int_r^t \int_0^1 \int_{[0,1]^{\mathbb{N}}} f(N_{s-},y,\mathbf{x})\pi(ds,dy,d\mathbf{x})$$

for all $0 < r < t$, where

$$f(k,y,\mathbf{x}) = \sum_{j=1}^k 1_{\{x_i\le y\}} - 1 + 1_{\cap_{j=1}^k\{x_j>y\}}.$$

Define

$$\Psi(k) = \int_0^1 \int_{[0,1]^{\mathbb{N}}} f(k,y,\mathbf{x})]\frac{\Lambda(dy)}{y^2}d\mathbf{x},$$

$$t=\int_{v_t}^{\infty}\frac{1}{\Psi(q)}dq,$$

and

$$M_t=\int_0^t\int_0^1\int_{[0,1]^{\mathbb{N}}}\frac{f(N_{s-},y,\mathbf{x})}{v_s}(\pi(dt,dy,d\mathbf{x})-ds\frac{\Lambda(dy)}{y^2}d\mathbf{x}).$$

$M=(M_t)_{t\geq 0}$ plays important role in the study of $\Lambda-$coalescent. Limic and Talarczyk [19] obtained that M is a square integrable martingale. It is not difficult to see that $\triangle M\geq 0$,

$$\sum_{s\leq t}|\triangle M_s|^2=\int_0^t\int_0^1\int_{[0,1]^{\mathbb{N}}}\frac{f^2(N_{s-},y,\mathbf{x})}{v_s^2}\pi(dt,dy,d\mathbf{x})$$

and

$$<M,M>_t=\int_0^t\int_0^1\int_{[0,1]^{\mathbb{N}}}\frac{f^2(N_{s-},y,\mathbf{x})}{v_s^2}ds\frac{\Lambda(dy)}{y^2}d\mathbf{x}.$$

We have the following result.

Theorem 8. *Let M be defined as above, we have*

$$\mathbb{P}\Big(M_t\geq x,\sum_{s\leq t}|\triangle M_s|^2\leq v^2\ \textit{for some}\quad t>0\Big)\leq\Big(\frac{v^2+x}{v^2}\Big)^{v^2}e^{-x}$$

and

$$\mathbb{P}\Big(M_t\geq(\alpha+\beta\sum_{s\leq t}|\triangle M_s|^2)x,\ \sum_{s\leq t}|\triangle M_s|^2\geq<M,M>_t+v^2\ \textit{for some}\ \ t>0\Big)$$
$$\leq\quad\exp\{-\frac{x^2}{2}(\alpha\beta+\frac{\beta^2v^2}{2})\}.$$

where $x\geq 0,\beta>0,v>0,\alpha\in\mathbb{R}$.

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