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Article

Algebraic Hyperstructure of Multi-Fuzzy Soft Sets Related to Polygroups

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Abstract: The combination of two elements in a group structure is an element, while, in a hypergroup, the combination of two elements is a non-empty set. The use of hypergroups appears mainly in certain subclasses. For instance, polygroups, which are a special subcategory of hypergroups, are used in many branches of mathematics and basic sciences. On the other hand, in a multi-fuzzy set, an element of a universal set may occur more than once with possibly the same or different membership values. A soft set over a universal set is a mapping from parameters to the family of subsets of the universal set. If we substitute the set of all fuzzy subsets of the universal set instead of crisp subsets, then we obtain fuzzy soft sets. Similarly, multi-fuzzy soft sets can be obtained. In this paper, we combine the multi-fuzzy soft set and polygroup structure, from which we obtain a new soft structure called the multi-fuzzy soft polygroup. We analyze the relation between multi-fuzzy soft sets and polygroups. Some algebraic properties of fuzzy soft polygroups and soft polygroups are extended to multi-fuzzy soft polygroups. Some new operations on a multi-fuzzy soft set are defined. In addition to this, we investigate normal multi-fuzzy soft polygroups and present some of their algebraic properties.

Keywords: multi-fuzzy soft set; multi-fuzzy soft polygroup; normal multi-fuzzy soft polygroup

MSC: 20N20; 20N25; 08A72

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1. Introduction

The concept of a hyperstructure was first introduced by Marty [1], at the 8th Congress of Scandinavian Mathematicians in 1934, when he defined hypergroups and started to analyze their properties. Indeed, the notion of hypergroups is a generalization of groups. Let H be a non-empty set and $\circ$ be a function (hyperoperation) from $H \times H$ to the family of non-empty subsets of H. Then, $(H, \circ)$ is a hypergroup, if $\circ$ is associative and $a \circ H = H \circ a = H$, for all $a \in H$. The hypergroup is a very general structure. Some researchers considered hypergroups with additional axioms. One of the axioms is the transposition axiom. This axiom is considered by Prenowitz [2–4], and then Jantosciak introduced the notion of transposition hypergroups [5]. A transposition hypergroup that has a scalar identity is called a quasicanonical hypergroup [6,7] or polygroup [8–11]. One can consider the quasicanonical hypergroups as a generalization of canonical hypergroups, introduced in [12]. Examples of polygroups, such as double set algebras, Prenowitz algebras, conjugacy class polygroups and character polygroups, can be found in [11]. This book contains the principal definitions, illustrated with examples and basic results of the theory. The category of polygroups is a category between the category of groups and transposition hypergroups; see Figure 1. More precisely, each group is a polygroup, and each polygroup is a transposition hypergroup. Recently, in [13], an excellent review of the several types of hypergroups was presented. Interesting results can be also found in [14]. The theory of algebraic hyperstructures has become a well-established branch in algebraic theory and it has extensive applications in many branches of mathematics and applied sciences; see [15–19].

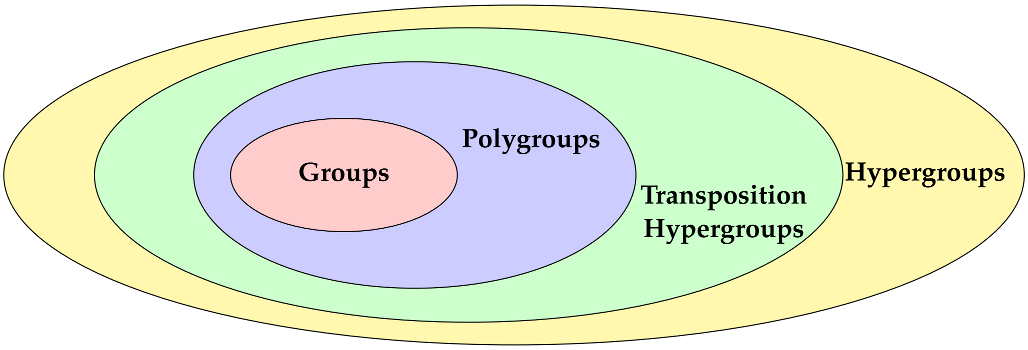

Figure 1. Each group is a polygroup, each polygroup is a transposition hypergroup, and each transposition hypergroup is a hypergroup.

The theory of fuzzy sets proposed by Zadeh [20] has achieved great success in many fields. Many researchers have applied the theory of fuzzy sets to hyperstructures. Firstly, Zahedi [21] discussed the subject of polygroups and fuzzy subpolygroups, and then Davvaz [22] presented the fuzzy subhypergroup concept, which is a generalization of Rosenfeld's fuzzy subgroup [23]. There are many articles dealing with the link between fuzzy sets and hyperstructures; see [24–26].

Soft set theory, introduced by Molodtsov [27], has been considered as an effective mathematical tool for modeling uncertainties. After Molodsov's work, different applications of soft sets were investigated in [28,29]. The idea of a fuzzy soft set, which is more general than fuzzy sets and soft sets, was first introduced by Maji et al. [30], and the algebraic properties of this concept were examined. Both of these theories have been applied to algebraic structures and algebraic hyperstructures—for instance, see [31,32].

Sebastian et al. in [33] proposed the concept of the multi-fuzzy set, which is a more general fuzzy set using ordinary fuzzy sets as building blocks; its membership function is an ordered sequence of ordinary fuzzy membership functions. Later, Yang et al. [34] introduced the concept of the multi-fuzzy soft set, which is a combination of the multi-fuzzy set and soft set, and studied its basic operations. They also introduced the application of this concept in decision making. In recent years, multi-fuzzy sets have become a subject of great interest to researchers and have been widely applied to algebraic structures. Some researchers—for instance, Onasanya and Hoskova-Mayerova [35]—studied the concept of multi-fuzzy groups, while Hoskova-Mayerova et al. [36] studied fuzzy multi-hypergroups and also fuzzy multi-polygroups in [37]. Akın [38] studied the concept of multi-fuzzy soft groups as a generalization of fuzzy soft groups, and Kazancı et al. [39] introduced a novel soft hyperstructure called the multi-fuzzy soft hyperstructure and investigated the notion of multi-fuzzy soft hypermodules and some of their structural properties on a hypermodule.

In a multi-fuzzy set, an element of a universal set U may occur more than once with possibly the same or different membership values. For example, if $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, then the set $A = \{< x_1, (0.3, 0.8) >, < x_2, (0.5, 0.7) >, < x_3, (0.1, 0.3) >, < x_4, (0.5, 0.4) >, < x_5, (0.8, 0.6) >, < x_6, (0.4, 0.7) >\}$ is a multi-fuzzy set. A soft set over a universe U is a mapping F from parameters to $\mathcal{P}(U)$. For example, let $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ be a set of apartments under consideration, and $A = \{e_1, e_2, e_3, e_4\}$ be a set of parameters such that e_1 = beautiful, e_2 = expensive, e_3 = a good view, and e_4 = near to the city center. If $F(e_1) = \{x_1, x_3\}$, $F(e_2) = \{x_1, x_2, x_5\}$, $F(e_3) = \{x_4, x_6\}$ and $F(e_4) = \{x_2, x_3, x_6\}$, then (F, A) is a soft set. If we substitute the set of all fuzzy subsets of U instead of crisp subsets of U, then we obtain fuzzy soft sets. Similarly, we can define multi-fuzzy soft sets.

In this paper, we combine three separated concepts: polygroups (or quasicanonical hypergroups), soft sets and multi-fuzzy sets (as a generalization of fuzzy sets). Previously, the authors have worked only on the one of these subjects or at most two of them. Indeed, we combine the multi-fuzzy soft set and polygroup structure, from which we obtain a new soft structure called the multi-fuzzy soft polygroup. The relation between the generalization of polygroups is indicated in Figure 2. To facilitate our discussion, we first review some basic concepts of the soft set, fuzzy soft set, multi-fuzzy set and polygroup in Section 2. In Section 3, we apply these to the notion of multi-fuzzy soft sets and polygroups and introduce multi-fuzzy soft polygroups. Then, we study some of their structural characterizations in Sections 4 and 5. Finally, we give the concept of a normal multi-fuzzy soft polygroup and discuss some of their structural characteristics. Finally, some conclusions are pointed out in Section 6.

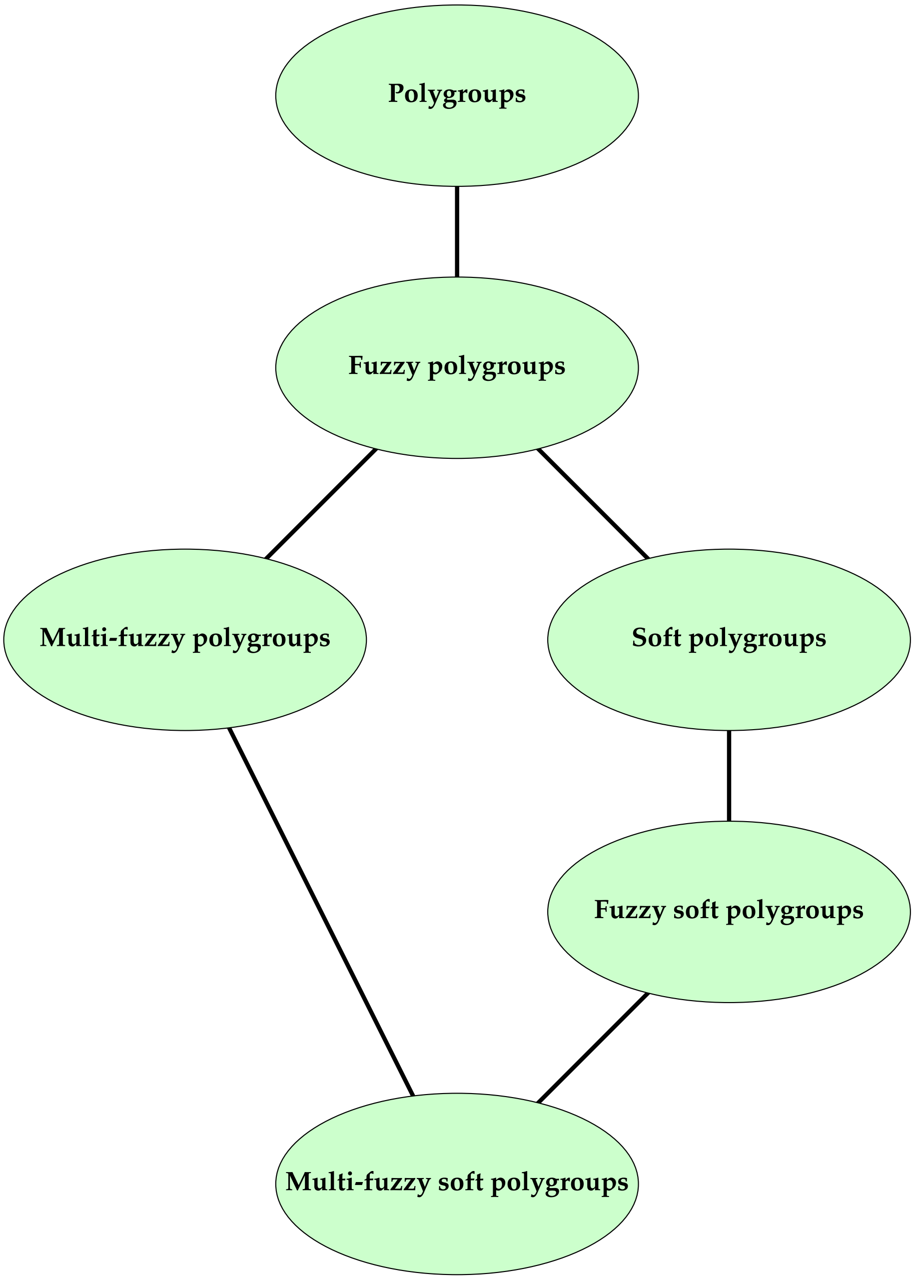

Figure 2. The relation between generalizations of polygroups.

2. Preliminaries

In this section, we provide some definitions and results of soft set theory that will help in understanding the content of the article [27,28,31,32,40]. Let $\mathcal{P}(U)$ denote the power set of U, where U is an initial universe set, E is a set of parameters and $A \subseteq E$.

Definition 1 ([27]). *Let $A \subseteq E$ and $F : A \to \mathcal{P}(U)$ be a set-valued function. Then, the pair (F, A) is called a soft set over U. For all $x \in A$ $F(x) = \{y \in U \mid (x, y) \in R\}$ and R stand for an arbitrary binary relation between an element of A and an element of U—that is, $R \subseteq A \times U$. In fact, a soft set over U is a parameterized family of subsets of the universe U.*

Definition 2 ([30,31]). *Let $A \subseteq E$ and $f : A \to FS(U)$ be a mapping. Then, the pair (f, A) is called a fuzzy soft set over U, where $FS(U)$ is the collection of all fuzzy subsets of U. That is, for each $a \in A$, $f(a)$ is a fuzzy set on U.*

Definition 3 ([33]). *A multi-fuzzy set (MF-set) $\widetilde{A}$ in U is a set of ordered sequences*

$$\widetilde{A} = \{< u, (\mu_i(u)) >: u \in U, \mu_i \in FS(U), i = 1, 2, ..., k\} \text{ and } k \text{ is a positive integer.}$$

The function $\mu_{\widetilde{A}} = (\mu_i(u))$ is said to be the multi membership function of $\widetilde{A}$ denoted by $MF_{\widetilde{A}}$, and k is called dimension of $\widetilde{A}$. The set of all MF-sets of dimension k in U is denoted by $M^kFS(U)$.

It is obvious that the one-dimensional MF-set is Zadeh's fuzzy set, and Atanassov's intuotionistic fuzzy set is a two-dimensional MF-set with $\mu_1(u) + \mu_2(u) \leq 1$.

Definition 4 ([33]). *Let $\widetilde{A} \in M^kFS(U)$. If $\widetilde{A} = \{u/(0, 0, ..., 0) : u \in U\}$, then $\widetilde{A}$ is said to be the null MF-set, defined by $\widetilde{\Phi_k}$. If $\widetilde{A} = \{u/(1, 1, ..., 1) : u \in U\}$, then $\widetilde{A}$ is said to be the absolute MF-set, denoted by $\widetilde{1}_k$.*

Definition 5 ([33]). *Let*

$$\widetilde{A} = \{< u, (\mu_i(u)) >: i = 1, 2, ..., k\} \text{ and } \widetilde{B} = \{< u, (\nu_i(u)) >: i = 1, 2, ..., k\} \in M^kFS(U).$$

Then

- *(i)* $\widetilde{A} \sqsubseteq \widetilde{B}$ *if and only if* $MF_{\widetilde{A}} \leq MF_{\widetilde{B}}$, *i.e* $\mu_i(u) \leq \nu_i(u), \forall u \in U$ *and* $1 \leq i \leq k$.
- *(ii)* $\widetilde{A} = \widetilde{B}$ *if and only if* $MF_{\widetilde{A}} = MF_{\widetilde{B}}$, *i.e* $\mu_i(u) = \nu_i(u), \forall u \in U$ *and* $1 \leq i \leq k$.
- *(iii)* $\widetilde{A} \sqcup \widetilde{B} = \{< u, (\mu_i(u) \vee \nu_i(u)) >: i = 1, 2, ..., k\}$. *That is* $MF_{\widetilde{A} \sqcup \widetilde{B}} = MF_{\widetilde{A}} \vee MF_{\widetilde{B}}$.
- *(iv)* $\widetilde{A} \sqcap \widetilde{B} = \{< u, (\mu_i(u) \wedge \nu_i(u)) >: i = 1, 2, ..., k\}$. *That is* $MF_{\widetilde{A} \sqcap \widetilde{B}} = MF_{\widetilde{A}} \wedge MF_{\widetilde{B}}$.

Definition 6 ([34]). *Let $\widetilde{f} : A \to M^kFS(U)$. Then, we call a pair $(\widetilde{f}, A)$ a multi-fuzzy soft set (MFS-set) of dimension k over U. That is, for every $a \in A$, $\widetilde{f}(a) = MF_{\widetilde{f}(a)} \in M^kFS(U)$. Here, $\widetilde{f}(a)$ may be considered a set of a-approximate elements of the multi-fuzzy soft set $(\widetilde{f}, A)$ for $a \in A$.*

Let $A \subseteq E$. Denote the set of all MFS-sets of dimension k over U by $M^kF^S_S(U, E)$

Definition 7 ([34]). *Let $A, B \subseteq E$ and $(\widetilde{f}, A), (\widetilde{g}, B) \in M^kF^S_S(U, E)$. Then, $(\widetilde{f}, A) \sqsubseteq (\widetilde{g}, B)$ if and only if $A \subseteq B$ and $MF_{\widetilde{f}(a)} \sqsubseteq MF_{\widetilde{g}(a)}$ for all $a \in A$.*

Definition 8 ([34]). *Let $(\widetilde{f}, A) \in M^kF^S_S(U, E)$. Then, $(\widetilde{f}, A)$ is said to be a null MFS-set, denoted by $\widetilde{\Phi^k_A}$, if $MF_{\widetilde{f}(a)} = \widetilde{\Phi_k}$ for all $a \in A$.*

$(\widetilde{f}, A)$ is said to be an absolute MFS-set defined by $\widetilde{U^k_A}$ if $MF_{\widetilde{f}(a)} = \widetilde{1}_k$ for each $a \in A$.

Definition 9 ([34]). *Let $(\widetilde{f}, A), (\widetilde{g}, B) \in M^kF^S_S(U, E)$.*

- *(i)* *The $\widetilde{\wedge}$ -intersection $(\widetilde{f}, A)\widetilde{\wedge}(\widetilde{g}, B)$ is defined as $(\widetilde{h}, A \times B)$, where $\widetilde{h}(a, b) = \widetilde{f}(a) \sqcap \widetilde{g}(b)$, for all $(a, b) \in A \times B$.*
- *(ii)* *The $\widetilde{\vee}$ -union $(\widetilde{f}, A)\widetilde{\vee}(\widetilde{g}, B)$ is defined as $(\widetilde{h}, A \times B)$, where $\widetilde{h}(a, b) = \widetilde{f}(a) \sqcup \widetilde{g}(b)$, for all $(a, b) \in A \times B$.*

(iii) *The union* $(\widetilde{f}, A)\widetilde{\sqcup}(\widetilde{g}, B)$ *is defined as* $(\widetilde{h}, C)$, *where* $C = A \cup B$ *and for all* $c \in C$ $\widetilde{h}(c) = \widetilde{f}(c)$ *if* $c \in A - B$, $\widetilde{h}(c) = \widetilde{g}(c)$ *if* $c \in B - A$ *and* $\widetilde{h}(c) = \widetilde{f}(c) \sqcup \widetilde{g}(c)$ *if* $c \in A \cap B$.

Definition 10. *Let* $(\widetilde{f}, A), (\widetilde{g}, B) \in M^k F^S_{\mathcal{S}}(U, E)$.

(i) *The restricted intersection of* $(\widetilde{f}, A)$ *and* $(\widetilde{g}, B)$ *is the MFS-set* $(\widetilde{h}, C)$ *with* $A \cap B \neq \emptyset$ *where* $C = A \cap B$, *and for all* $c \in C$, $\widetilde{h}(c) = \widetilde{f}(c) \sqcap \widetilde{g}(c)$. *The situation is denoted by* $(\widetilde{f}, A) \sqcap_{\Re} (\widetilde{g}, B) = (\widetilde{h}, C)$.

(ii) *The extended intersection of* $(\widetilde{f}, A)$ *and* $(\widetilde{g}, B)$ *is the MFS-set* $(\widetilde{h}, C)$, *where* $C = A \cup B$ *and for all* $c \in C$, $\widetilde{h}(c) = \widetilde{f}(c)$ *if* $c \in A - B$, $\widetilde{h}(c) = \widetilde{g}(c)$ *if* $c \in B - A$ *and* $\widetilde{h}(c) = \widetilde{f}(c) \sqcap \widetilde{g}(c)$ *if* $c \in A \cap B$. *In this case, we write* $(\widetilde{f}, A) \sqcap_{\Im} (\widetilde{g}, B) = (\widetilde{h}, C)$.

Definition 11. *Let* H *be a non-empty set and let* $\mathcal{P}^*(H)$ *be the set of all non-empty subsets of* H. *A hyperoperation on* H *is a map* $\circ : H \times H \to \mathcal{P}^*(H)$ *and the pair* $(H, \circ)$ *is called a hypergroupoid.*

Definition 12 ([11,21]). *A multi-valued system* $P =< P, \circ, e, ^{-1} >$ *is called a polygroup where* $e \in P$, $^{-1} : P \longrightarrow P$, $\circ : P \times P \longrightarrow \mathcal{P}^*(P)$ *if the following axioms hold for all* x, y, z *in* P.

(i) $x \circ (y \circ z) = (x \circ y) \circ z$,

(ii) $x \circ e = e \circ x = x$,

(iii) $x \in y \circ z$ *implies* $y \in x \circ z^{-1}$ *and* $z \in y^{-1} \circ x$.

The following elementary properties follow from the axioms:

$$e \in x \circ x^{-1} \cap x^{-1} \circ x, \quad e^{-1} = e, \quad (x^{-1})^{-1} = x, \ \textit{and} \ (x \circ y)^{-1} = y^{-1} \circ x^{-1},$$

where $A^{-1} = \{a^{-1} \mid a \in A\}$.

Let P be a polygroup and K a non-empty subset of P; then, K is called a *subpolygroup* of P if $e \in K$ and $< K, \circ, e, ^{-1} >$ is a polygroup.

A subhypergroup N of a hypergroup is normal if $aN = Na$ [5]. According to [7], a quasicanonical subhypergroup N of a quasicanonical hypergroup H is called *normal* if and only if it is a member of an appreciated quotient system of H by some congruence relation.

Example 1. *Suppose that* H *is a subgroup of a group* G. *Define a system* $G//H =< \{HgH \mid g \in G\}, *, H, ^{-I} >$, *where* $(HgH)^{-I} = Hg^{-1}H$ *and*

$$(Hg_1H) * (Hg_2H) = \{Hg_1hg_2H \,|h \in H\}.$$

The algebra of double cosets $G//H$ *is a polygroup introduced in (Dresher and Ore* [41]*).*

Example 2. *Consider* $P = \{0, 1, 2, a, b\}$ *and define* $\circ$ *on* P *by the following table:.*

$\circ$	0	1	2	a	b
0	0	1	2	a	b
1	1	$\{0,2\}$	$\{1,2\}$	a	b
2	2	$\{1,2\}$	$\{0,1\}$	a	b
a	a	a	a	$\{0,1,2,b\}$	$\{a,b\}$
b	b	b	b	$\{a,b\}$	$\{0,1,2,a\}$

Then, P *is a canonical hypergroup. Suppose that* S_3 *is the symmetric group on a set with three elements. We consider*

$$P \times S_3 = \{(p, x) \mid p \in P \text{ and } x \in S_3\},$$

with the usual hyperoperation

$$(p_1, x_1) \odot (p_2, x_2) \mid p \in p_1 \circ p_2 \text{ and } x = x_1 \cdot x_2\},$$

for all (p_1, x_1), $(p_2, x_2) \in P \times S_3$. *Then,* $P \times S_3$ *is a non-commutative polygroup or quasicanonical hypergroup.*

3. Multi-Fuzzy Soft Polygroups

The concept of the MF-set was introduced by Sebastian et al. in [33]. By combining the MF-set and soft set, Yang et al. introduced the concept of the MFS-set [34]. Both of these theories have been applied to algebraic structures. At this point, we give a new type of polygroup named the multi-fuzzy soft polygroup (MFS-polygroup). Since the concepts of uncertainty and fuzziness can be better expressed with MFS-sets, their applications in hyperalgebraic structures are extremely important. Thus, in this section, we provide a new connection between the polygroup structure and MFS-set.

Definition 13. *Let P be a polygroup and* $(\widetilde{f}, A) \in M^k F^S_S(P, E)$. *Then,* $(\widetilde{f}, A)$ *is said to be an MFS-polygroup of dimension k over P if and only if, for all* $a \in A$ *and* $x, y \in P$,

(i) $\min\{MF_{\widetilde{f}(a)}(x), MF_{\widetilde{f}(a)}(y)\} \leq \inf_{z \in x \circ y}\{MF_{\widetilde{f}(a)}(z)\}$,

(ii) $MF_{\widetilde{f}(a)}(x) \leq MF_{\widetilde{f}(a)}(x^{-1})$.

That is, for each $a \in A$, $MF_{\widetilde{f}(a)}$ *is a multi-fuzzy subpolygroup.*

The first condition requires that the polygroup is closed under multi-fuzzy soft hyperoperation $\circ$ and the second condition is a generalization of the inverse element under $\circ$.

To better understand this new algebraic structure, consider the following examples.

Example 3. *Let* $P = \{e, a, b, c\}$ *be a polygroup with the Cayley table:*

$\circ$	e	a	b	c
e	e	a	b	c
a	a	a	{e,a,b,c}	c
b	b	{e,a,b}	b	{b,c}
c	c	{a,c}	c	{e,a,b,c}

Let $A = \{e_1, e_2, e_3\}$ *be the set of parameters.*
Consider the MF-set $\widetilde{f} : A \to M^3FS(P)$ *defined as follows.* $\widetilde{f} : A \to M^3FS(P)$ *as follows.*

$$\begin{aligned} MF_{\widetilde{f}(e_1)} &= \{e/(0.9, 0.8, 0.7), a/(0.6, 0.5, 0.6), b/(0.4, 0.1, 0.2), c/(0.4, 0.1, 0.2)\}, \\ MF_{\widetilde{f}(e_2)} &= \{e/(0.8, 0.5, 0.6), a/(0.7, 0.4, 0.5), b/(0.6, 0.3, 0.1), c/(0.6, 0.3, 0.1)\}, \\ MF_{\widetilde{f}(e_3)} &= \{e/(0.8, 0.8, 0.7), a/(0.5, 0.6, 0.3), b/(0.3, 0.6, 0.2), c/(0.2, 0.5, 0.1)\}. \end{aligned}$$

Then, $(\widetilde{f}, A)$ *is not an MFS-polygroup of dimension* 3 *over P since*

$$\inf_{c \in c \circ b}\{MF_{\widetilde{f}(e_3)}(c)\} \ngeq \min\{MF_{\widetilde{f}(e_3)}(c), MF_{\widetilde{f}(e_3)}(b)\}.$$

Example 4. *Consider the polygroup given in Example 3 and define the MF-set* $\widetilde{f} : A \to M^3FS(P)$ *as follows.*

$$\begin{aligned} MF_{\widetilde{f}(e_1)} &= \{e/(0.8, 0.6, 0.7), a/(0.4, 0.5, 0.6), b/(0.3, 0.4, 0.2), c/(0.3, 0.4, 0.2)\}, \\ MF_{\widetilde{f}(e_2)} &= \{e/(0.8, 0.5, 0.6), a/(0.6, 0.4, 0.5), b/(0.4, 0.3, 0.4), c/(0.4, 0.3, 0.4)\}, \\ MF_{\widetilde{f}(e_3)} &= \{e/(0.8, 0.8, 0.7), a/(0.5, 0.6, 0.3), b/(0.3, 0.6, 0.2), c/(0.3, 0.6, 0.2)\}. \end{aligned}$$

Then, for all $a \in A$, $MF_{\widetilde{f}(a)}$ *is an MF-subpolygroup of P. By Definition 13,* $(\widetilde{f}, A)$ *is an MFS-polygroup of dimension* 3 *over P.*

Example 5. *Consider the polygroup given in Example 3 and define the MF-set $\widetilde{f} : A \to M^3FS(P)$ as follows.*

$$\begin{aligned} MF_{\widetilde{f}(e_1)} &= \{e/(0.9, 0.8, 0.6), a/(0.8, 0.7, 0.6), b/(0.7, 0.6, 0.5), c/(0.7, 0.6, 0.5)\}, \\ MF_{\widetilde{f}(e_2)} &= \{e/(0.8, 0.8, 0.6), a/(0.7, 0.6, 0.5), b/(0.4, 0.5, 0.2), c/(0.4, 0.5, 0.2)\}, \\ MF_{\widetilde{f}(e_3)} &= \{e/(0.6, 0.7, 0.5), a/(0.5, 0.6, 0.4), b/(0.4, 0.3, 0.1), c/(0.3, 0.4, 0.1)\}. \end{aligned}$$

Then, it is clear to see that $MF_{\widetilde{f}(e_1)}$ and $MF_{\widetilde{f}(e_2)}$ are MF-subpolygroups of P. However, $MF_{\widetilde{f}(e_3)}$ is not an MF-subpolygroup of P since

$$\inf_{b \in c \circ c} \{MF_{\widetilde{f}(e_3)}(b)\} \ngeq \min\{MF_{\widetilde{f}(e_3)}(c), MF_{\widetilde{f}(e_3)}(c)\} = MF_{\widetilde{f}(e_3)}(c).$$

By Definition 13 $(\widetilde{f}, A)$ is not an MFS-polygroup of dimension 3 over P.

The following example shows that every soft set (F, A) over P can be seen as an MFS-set of dimension k over P.

Example 6. *Let $A \subset E$ and (F, A) be a soft set over P. For all $a \in A$, the MF-set $\widetilde{\chi}_{F(a)} : A \to M^kFS(P)$ defined by*

$$MF_{\widetilde{\chi}_{F(a)}}(b) = \begin{cases} \widetilde{1}_k & \text{if} \quad b \in F(a) \\ \widetilde{\Phi}_k, & \text{otherwise} \end{cases}$$

for all $b \in A$. Then, $(\widetilde{\chi}_{F(a)}, A) \in M^kF^S_S(P, E)$.

Proposition 1. *Let $(\widetilde{f}, A) \in M^kF^S_S(P, E)$. If $(\widetilde{f}, A)$ is an MFS-polygroups, then, for all $a \in A$ and $x, y \in P$,*

(i) $MF_{\widetilde{f}(a)}(x^{-1}) = MF_{\widetilde{f}(a)}(x)$,

(ii) $\inf_{e \in x \circ x^{-1}} \{MF_{\widetilde{f}(a)}(e)\} \geq MF_{\widetilde{f}(a)}(x)$.

Proof. (i) By Definition 13, $MF_{\widetilde{f}(a)}(x) \leq MF_{\widetilde{f}(a)}(x^{-1})$ for all $a \in A$ and $x \in P$. Moreover, $MF_{\widetilde{f}(a)}(x) = MF_{\widetilde{f}(a)}(x^{-1})^{-1} \leq MF_{\widetilde{f}(a)}(x^{-1})$. This completes the proof of (i).

(ii) Suppose that $x \in P$. Since $e \in x \circ x^{-1}$ and $(\widetilde{f}, A)$ is an MFS-polygroup, then, for all $a \in A$, we obtain

$$\begin{aligned} \inf_{e \in x \circ x^{-1}} \{MF_{\widetilde{f}(a)}(e)\} &\geq \min\{MF_{\widetilde{f}(a)}(x), MF_{\widetilde{f}(a)}(x^{-1})\} \\ &= MF_{\widetilde{f}(a)}(x). \end{aligned}$$

□

The relationship between soft polygroups and MFS-polygroups is given in the following theorem.

Theorem 1. *Let $F : A \to \mathcal{P}^*(P)$ be a soft set over P. Then, (F, A) is a soft polygroup over P if and only if $(\widetilde{\chi}_{F(a)}, A) \in M^kF^S_S(P, E)$ is an MFS-polygroup.*

Proof. The proof follows by Example 6. □

In Theorem 2, we show that the restricted intersection and the extended intersection of two MFS-polygroups are also an MFS-polygroup.

Theorem 2. *Let $(\widetilde{f},A),(\widetilde{g},B)\in M^kF_S^S(P,E)$ be two MFS-polygroups.*

(i) $(\widetilde{f},A)\sqcap_{\Re}(\widetilde{g},B)\in M^kF_S^S(P,E)$ *is an MFS-polygroup.*

(ii) $(\widetilde{f},A)\sqcap_{\Im}(\widetilde{g},B)\in M^kF_S^S(P,E)$ *is an MFS-polygroup.*

Proof. (i) By Definition 10 (i), let $(\widetilde{f},A)\sqcap_{\Re}(\widetilde{g},B)=(\widetilde{h},C)$, where $C=A\cap B$ and for all $c\in C$, $\widetilde{h}(c)=\widetilde{f}(c)\sqcap\widetilde{g}(c)$. Since $(\widetilde{f},A)$ and $(\widetilde{g},B)$ are MFS-polygroups, we have for arbitrary $c\in C$ and for all $x,y\in P$

$$\begin{aligned}
\inf_{z\in x\circ y}\{MF_{\widetilde{f}(c)}(z)\} &\geq \min\{MF_{\widetilde{f}(c)}(x),MF_{\widetilde{f}(c)}(y)\},\\
MF_{\widetilde{f}(c)}(x) &\leq MF_{\widetilde{f}(c)}(x^{-1}) and\\
\inf_{z\in x\circ y}\{MF_{\widetilde{g}(c)}(z)\} &\geq \min\{MF_{\widetilde{g}(c)}(x),MF_{\widetilde{g}(c)}(y)\},\\
MF_{\widetilde{g}(c)}(x) &\leq MF_{\widetilde{g}(c)}(x^{-1}).
\end{aligned}$$

For arbitrary $c\in C$ and for all $x,y\in P$,

$$\begin{aligned}
\inf_{z\in x\circ y}\{MF_{\widetilde{h}(c)}(z)\} &= \inf_{z\in x\circ y}\{MF_{\widetilde{f}(c)\sqcap\widetilde{g}(c)}(z)\}\\
&= \inf_{z\in x\circ y}\{MF_{\widetilde{f}(c)}(z)\wedge MF_{\widetilde{g}(c)}(z)\}\\
&= \inf_{z\in x\circ y}\{MF_{\widetilde{f}(c)}(z)\}\wedge\inf_{z\in x\circ y}\{MF_{\widetilde{g}(c)}(z)\}\\
&\geq \min\{MF_{\widetilde{f}(c)}(x),MF_{\widetilde{f}(c)}(y)\}\wedge\min\{MF_{\widetilde{g}(c)}(x),MF_{\widetilde{g}(c)}(y)\}\\
&= \min\{MF_{\widetilde{f}(c)}(x),MF_{\widetilde{g}(c)}(x)\}\wedge\min\{MF_{\widetilde{f}(c)}(y),MF_{\widetilde{g}(c)}(y)\}\\
&= \min\{MF_{\widetilde{f}(c)\sqcap\widetilde{g}(c)}(x),MF_{\widetilde{f}(c)\sqcap\widetilde{g}(c)}(y)\}\\
&= \min\{MF_{\widetilde{h}(c)}(x),MF_{\widetilde{h}(c)}(y)\}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
MF_{\widetilde{h}(c)}(x) &= MF_{\widetilde{f}(c)\sqcap\widetilde{g}(c)}(x)\\
&= \min\{MF_{\widetilde{f}(c)}(x),MF_{\widetilde{g}(c)}(x)\}\\
&\leq \min\{MF_{\widetilde{f}(c)}(x^{-1}),MF_{\widetilde{g}(c)}(x^{-1})\}\\
&= MF_{\widetilde{f}(c)\sqcap\widetilde{g}(c)}(x^{-1})\\
&= MF_{\widetilde{h}(c)}(x^{-1}).
\end{aligned}$$

Therefore, $(\widetilde{f},A)\sqcap_{\Re}(\widetilde{g},B)$ is an MFS-polygroup of dimension k over P.

(ii) According to Definition 10 (ii), we can write $(\widetilde{f},A)\sqcap_{\Im}(\widetilde{g},B)=(\widetilde{h},C)$, $C=A\cup B$. If $c\in A-B$, then $\widetilde{h}(c)=\widetilde{f}(c)$ is an MF-subpolygroup of P, since $(\widetilde{f},A)$ is an MFS-polygroup over P; if $c\in B-A$, then $\widetilde{h}(c)=\widetilde{g}(c)$ is an MF-subpolygroup of P, since $(\widetilde{g},B)$ is an MFS-polygroup over P; if $c\in A\cap B$, then $\widetilde{h}(c)=\widetilde{f}(c)\sqcap\widetilde{g}(c)$ is an MF-subpolygroup of P by (i). Therefore, $(\widetilde{f},A)\sqcap_{\Im}(\widetilde{g},B)$ is an MFS-polygroup of dimension k over P. □

The following corollary follows from Theorem 2.

Corollary 1. *Let $\{(\widetilde{f}_i,A_i)\mid i\in I\}\in M^kF_S^S(P,E)$ be a family of MFS-polygroups. If $\cap_{i\in I}A_i\neq\emptyset$. Then,*

(i) $(\sqcap_{\Re})_{i\in I}(\widetilde{f}_i,A_i)\in M^kF_S^S(P,E)$ *is an MFS-polygroup.*

(ii) $(\sqcap_{\Im})_{i\in I}(\widetilde{f}_i,A_i)\in M^kF_S^S(P,E)$ *is an MFS-polygroup.*

The union of two MFS-polygroups is not an MFS-polygroup. In Theorem 3, we provide a condition for the union to be an MFS-polygroup as well.

Theorem 3. *Let $(\widetilde{f}, A), (\widetilde{g}, B) \in M^k F_S^S(P, E)$ be two MFS-polygroups. If $A \cap B = \emptyset$, then $(\widetilde{f}, A)\widetilde{\sqcup}(\widetilde{g}, B) \in M^k F_S^S(P, E)$ is an MFS-polygroup.*

Proof. By Definition 9(iii), we can write $(\widetilde{f}, A)\widetilde{\sqcup}(\widetilde{g}, B) = (\widetilde{h}, C)$, where $C = A \cup B$. Since $A \cap B = \emptyset$, it follows that either $c \in A - B$ or $c \in B - A$ for all $c \in C$. If $c \in A - B$, then $\widetilde{h}(c) = \widetilde{f}(c)$ is an MF-subpolygroup of P and if $c \in B - A$, then $\widetilde{h}(c) = \widetilde{g}(c)$ is an MF-subpolygroup of P. Therefore, $(\widetilde{f}, A)\widetilde{\sqcup}(\widetilde{g}, B)$ is an MFS-polygroup of dimension k over P. □

Theorem 4. *Let $(\widetilde{f}, A), (\widetilde{g}, B) \in M^k F_S^S(P, E)$ be two MFS-polygroups. Then, $(\widetilde{f}, A)\widetilde{\wedge}(\widetilde{g}, B) \in M^k F_S^S(P, E)$ is an MFS-polygroup.*

Proof. By Definition 9(i), let $(\widetilde{f} A)\widetilde{\wedge}(\widetilde{g}, B) = (\widetilde{h}, A \times B)$. We know that for all $a \in A$, $\widetilde{f}(a)$ is an MF-subpolygroup of P and for all $b \in B$, $\widetilde{g}(b)$ is an MF-subpolygroup of P and so is $\widetilde{h}(a, b) = MF_{\widetilde{h}(a,b)} = MF_{\widetilde{f}(a) \sqcap \widetilde{g}(b)}$ for all $(a, b) \in A \times B$, because the intersection of two multi-fuzzy subpolygroups is also an MF-subpolygroup. Hence, $(\widetilde{f}, A)\widetilde{\wedge}(\widetilde{g}, B)$ is an MFS-polygroup of dimension k over P. □

By Theorems 3 and 4, we obtain the following corollary.

Corollary 2. *Let $\{(\widetilde{f}_i, A_i) \mid i \in I\} \in M^k F_S^S(P, E)$ be a family of MFS-polygroups.*

(i) If $A_i \cap A_j = \emptyset$ for all $i, j \in I$ and $i \neq j$, then $\widetilde{\sqcup}_{i \in I}(\widetilde{f}_i, A_i) \in M^k F_S^S(P, E)$ is an MFS-polygroup.
(ii) $\widetilde{\wedge}_{i \in I}(\widetilde{f}_i, A_i) \in M^k F_S^S(P, E)$ is an MFS-polygroup.

The following theorem gives a condition for the $\widetilde{\vee}$-union of two MFS-polygroups to be an MFS-polygroup.

Theorem 5. *Let $(\widetilde{f}, A), (\widetilde{g}, B) \in M^k F_S^S(P, E)$ be two MFS-polygroups. If $(\widetilde{f}, A) \sqsubset (\widetilde{g}, B)$ or $(\widetilde{g}, B) \sqsubset (\widetilde{f}, A)$, then $(\widetilde{f}, A)\widetilde{\vee}(\widetilde{g}, B) \in M^k F_S^S(P, E)$ is an MFS-polygroup.*

Proof. Suppose that $(\widetilde{f}, A)$ and $(\widetilde{g}, B)$ are MFS-polygroups of dimension k over P. By Definition 9 (ii), we can write $(\widetilde{f}, A)\widetilde{\vee}(\widetilde{g}, B) = (\widetilde{h}, C)$, where $C = A \times B$, and $\widetilde{h}(a, b) = \widetilde{f}(a) \sqcup \widetilde{g}(b)$ for all $(a, b) \in C$. Since $(\widetilde{f}, A)$ and $(\widetilde{g}, B)$ are MFS-polygroups of dimension k over P, we obtain that for all $a \in A$, $\widetilde{f}(a)$ is an MF-subpolygroup of P and for all $b \in B$, $\widetilde{g}(b)$ is an MF-subpolygroup of P. By assumption, $\widetilde{h}(a, b) = \widetilde{f}(a) \sqcup \widetilde{g}(b)$ is an MF-subpolygroup of P for all $(a, b) \in C$. Hence, $(\widetilde{f}, A)\widetilde{\vee}(\widetilde{g}, B)$ is an MFS-polygroup. □

Definition 14. *The sum of two MFS-sets $(\widetilde{f}, A)$ and $(\widetilde{g}, B)$ of dimension k over P, denoted by $(\widetilde{f}, A) \oplus (\widetilde{g}, B)$, is the MFS-set $(\widetilde{h}, C)$, where $C = A \cup B$ and for all $c \in C$,*

$$\widetilde{h}(c) = \begin{cases} \widetilde{f}(c) & \text{if} \quad c \in A \backslash B \\ \widetilde{g}(c) & \text{if} \quad c \in B \backslash A \\ \widetilde{f}(c) \oplus \widetilde{g}(c) & \text{if} \quad c \in A \cap B \end{cases}$$

For every $z \in P$,

$$(\widetilde{f}(c) \oplus \widetilde{g}(c))(z) = \bigvee \{MF_{\widetilde{f}(c)}(x) \wedge MF_{\widetilde{g}(c)}(y), x, y \in P, z \in x \circ y\}.$$

The next theorem gives a condition for the sum of two MFS-polygroups to be an MFS-polygroup.

Theorem 6. *Let $(\widetilde{f}, A), (\widetilde{g}, B) \in M^k F_S^S(P, E)$ be two MFS-polygroups. If $(\widetilde{f}, A) \oplus (\widetilde{g}, B) = (\widetilde{g}, B) \oplus (\widetilde{f}, A)$, then $(\widetilde{f}, A) \oplus (\widetilde{g}, B) \in M^k F_S^S(P, E)$ is an MFS-polygroup.*

Proof. The proof is straightforward. □

Definition 15. *Let $(\widetilde{f}, A) \in M^k F_S^S(P, E)$. The soft set*

$$(\widetilde{f}, A)_t = \{(MF_{\widetilde{f}(a)})_t \mid a \in A\} \quad where \quad (MF_{\widetilde{f}(a)})_t = \{x \in P \mid MFS_{\widetilde{f}(a)}(x) \geq t\},$$

for all $t = (t_1, t_2, ..., t_k)$, $t_i \in (0,1] 1 \leq i \leq k$, is called a t-level soft set of the MFS-set $(\widetilde{f}, A)$, where $(MF_{\widetilde{f}(a)})_t$ is a t-level subset of the MF-set $MF_{\widetilde{f}(a)}$.

The following theorem explores the relation between MFS-polygroups and t-level soft sets.

Theorem 7. *Let $(\widetilde{f}, A) \in M^k F_S^S(P, E)$. Then, $(\widetilde{f}, A)$ is an MFS-polygroup if and only, if for all $a \in A$ and for arbitrary $t \in (0,1]$ with $(MF_{\widetilde{f}(a)})_t \neq \emptyset$, the t-level soft set $(\widetilde{f}, A)_t$ is a soft polygroup over P in Wanga's sense [40].*

Proof. Let $(\widetilde{f}, A) \in M^k F_S^S(P, E)$ be an MFS-polygroup. Then, for each $a \in A$, $MF_{\widetilde{f}(a)}$ is an MF-subpolygroup of P. Suppose that $t \in (0,1]$ with $(MF_{\widetilde{f}(a)})_t \neq \emptyset$ and $x, y \in (MF_{\widetilde{f}(a)})_t$. Then, $MF_{\widetilde{f}(a)}(x) \geq t$, $MF_{\widetilde{f}(a)}(y) \geq t$. Thus,

$$t \leq \min\{MF_{\widetilde{f}(a)}(x), MF_{\widetilde{f}(a)}(y)\} \leq \inf_{z \in x \circ y}\{MF_{\widetilde{f}(a)}(z)\}.$$

which implies $MF_{\widetilde{f}(a)}(z) \geq t$ for all $z \in x \circ y$. Therefore, $x \circ y \subseteq (MF_{\widetilde{f}(a)})_t$. Moreover, for $x \in (MF_{\widetilde{f}(a)})_t$, we have $MF_{\widetilde{f}(a)}(x^{-1}) \geq MF_{\widetilde{f}(a)}(x) \geq t$. It follows that $x^{-1} \in (MF_{\widetilde{f}(a)})_t$. we obtain that $(MF_{\widetilde{f}(a)})_t$ is a subpolygroup of P for all $a \in A$. Consequently, $(\widetilde{f}, A)_t$ is a soft polygroup over P. Conversely, let $(\widetilde{f}, A)_t$ be a soft polygroup over P for all $t \in (0,1]$. Let $t_0 = \min\{MF_{\widetilde{f}(a)}(x), MF_{\widetilde{f}(a)}(y)\}$ for some $x, y \in P$. Then, obviously, $x, y \in (MF_{\widetilde{f}(a)})_{t_0}$; consequently, $x \circ y \subseteq (MF_{\widetilde{f}(a)})_{t_0}$. Thus,

$$\min\{MF_{\widetilde{f}(a)}(x), MFS_{\widetilde{f}(a)}(y)\} = t_0 \leq \inf_{z \in x \circ y}\{MF_{\widetilde{f}(a)}(z)\}.$$

Now, $t_0 = MF_{\widetilde{f}(a)}(x)$ for some $x \in P$. Since, by the assumption, every non-empty t-level soft set $(\widetilde{f}, A)_t$ is a soft polygroup over P, $x^{-1} \in (MF_{\widetilde{f}(a)})_{t_0}$. Hence, $MF_{\widetilde{f}(a)}(x^{-1}) \geq t_0 = MF_{\widetilde{f}(a)}(x)$. As a result, we obtain that $MF_{\widetilde{f}(a)}$ is an MF-subpolygroup of P for all $a \in A$. Consequently, $(\widetilde{f}, A)$ is an MFS-polygroup of dimension k over P. □

4. The Behavior Image and Inverse Image of MFS-Polygroups

Definition 16. *A pair (φ, ψ) is called an MF-soft function from P_1 to P_2, where $\varphi : P_1 \to P_2$ and $\psi : E_1 \to E_2$ are functions.*

Definition 17. *Let $(\widetilde{f}, A) \in M^k F_S^S(P_1, E_1)$,$(\widetilde{g}, B) \in M^k F_S^S(P_2, E_2)$ and (φ, ψ) be an MF- soft function from P_1 to P_2.*

(i) *The image of $(\widetilde{f}, A)$ under the MF-soft function (φ, ψ), denoted by $(\varphi, \psi)(\widetilde{f}, A)$, is the MFS-set $(\varphi(\widetilde{f}), \psi(A))$ such that the MF-set $\varphi(\widetilde{f})(t)$ for any $t \in \psi(A)$ is characterized by the following MF-membership function:*

$$MF_{\varphi(\widetilde{f})(t)}(y) = \begin{cases} \bigvee\limits_{\varphi(x)=y} \bigvee\limits_{\psi(a)=t} MF_{\widetilde{f}(a)}(x) & \text{if} \quad \exists x \in \varphi^{-1}(y) \\ 0, & \text{otherwise} \end{cases}$$

for all $y \in P_2$.

(ii) *The pre-image of $(\widetilde{g}, B)$ under the MF-soft function (φ, ψ), denoted by $(\varphi, \psi)^{-1}(\widetilde{g}, B)$, is the MFS-set $(\varphi^{-1}(\widetilde{g}), \psi^{-1}(B))$ such that the MF-set $\varphi^{-1}(\widetilde{g})(a)$ is characterized by the following MF-membership function:*

$$MF_{\varphi^{-1}(\widetilde{g})(a)}(x) = MF_{\widetilde{g}(\psi(a))}(\varphi(x))$$

for all $a \in \psi^{-1}(B)$ and $x \in P_1$.

If φ and ψ are injective (surjective), then (φ, ψ) is said to be injective (surjective).

Definition 18. *Let P_1, P_2, be two polygroups and (φ, ψ) be an MF-soft function from P_1 to P_2. If φ is a strong homomorphism of polygroups, then the pair (φ, ψ) is called an MF-soft homomorphism. If φ is an isomorphism and ψ is a one-to-one mapping, then (φ, ψ) is said to be an MF-soft isomorphism.*

Theorem 8. *Let P_1, P_2 be two polygroups and (φ, ψ) be an MF-soft homomorphism from P_1 to P_2. If $(\widetilde{f}, A) \in M^k F_S^S(P_1, E_1)$ is an MFS-polygroup, then $(\varphi, \psi)(\widetilde{f}, A) \in M^k F_S^S(P_2, E_2)$ is an MFS-polygroup.*

Proof. Let $k \in \psi(A)$, $u, v \in P_2$. If $\varphi^{-1}(u) = \emptyset$ or $\varphi^{-1}(v) = \emptyset$, the proof is straightforward. Assume that there exists $x, y \in P_1$, such that $\varphi(x) = u$ and $\varphi(y) = v$. Since $(\widetilde{f}, A) \in M^k F_S^S(P_1, E_1)$ is an MFS-polygroup, it follows that for each $a \in A$

$$\min\{MF_{\widetilde{f}(a)}(x), MF_{\widetilde{f}(a)}(y)\} \leq MF_{\widetilde{f}(a)}(z)$$

for all $z \in x \circ y$. Let $z^* \in u \circ v = \varphi(x \circ y)$. We obtain $z^* = \varphi(z)$. Then, we have

$$\min\{\bigvee_{\varphi(x)=u} MF_{\widetilde{f}(a)}(x), \bigvee_{\varphi(y)=v} MF_{\widetilde{f}(a)}(y)\} \quad \leq \quad \bigvee_{\varphi(x)=u} \bigvee_{\varphi(y)=v} MF_{\widetilde{f}(a)}(z).$$

Hence,

$$\begin{aligned} \min\{MF_{\varphi(\widetilde{f})(t)}(u), MF_{\varphi(\widetilde{f})(t)}(v)\} \quad &\leq \quad \bigvee_{\psi(a)=t} \bigvee_{\varphi(x)=u} \bigvee_{\varphi(y)=v} MF_{\widetilde{f}(a)}(z) \\ &= \quad \bigvee_{\psi(a)=t} \bigvee_{\varphi(z)=z^*} MF_{\varphi(\widetilde{f})(t)}(z) \end{aligned}$$

for all $z^* \in u \circ v$. Then, we have

$$\inf_{z^* \in u \circ v}\{MF_{\varphi(\widetilde{f})(t)}(z^*)\} \geq \min\{MF_{\varphi(\widetilde{f})(t)}(u), MF_{\varphi(\widetilde{f})(t)}(v)\}$$

Moreover, for all $u \in P_2$ where $\varphi(x) = u$ and $x \in P_1$, we have

$$\begin{aligned} MF_{\varphi(\widetilde{f})(t)}(u^{-1}) &= \bigvee_{\varphi(x^{-1})=u^{-1}} \bigvee_{\psi(a)=k} MF_{\widetilde{f}(a)}(x^{-1}) \\ &\geq \bigvee_{\varphi(x)=u} \bigvee_{\psi(a)=t} MF_{\widetilde{f}(a)}(x) \\ &= MF_{\varphi(\widetilde{f})(t)}(u) \end{aligned}$$

Consequently, $(\varphi, \psi)(\widetilde{f}, A) \in M^k F^S_S(P_2, E_2)$ is an MFS-polygroup. □

Theorem 9. *Let P_1, P_2 be two polygroups and (φ, ψ) be an MF-soft homomorphism from P_1 to P_2. If $(\widetilde{g}, B) \in M^k F^S_S(P_2, E_2)$ is an MFS-polygroup, then $(\varphi^{-1}(\widetilde{g}), \psi^{-1}(B)) \in M^k F^S_S(P_1, E_1)$ is an MFS-polygroup.*

Proof. Let $a \in \psi^{-1}(B)$, $x, y \in P_1$. For all $z \in x \circ y$, we have

$$\begin{aligned} \inf_{z \in x \circ y}\{MF_{\varphi^{-1}(\widetilde{g})(a)}(z)\} &= \inf_{z \in x \circ y}\{MF_{\widetilde{g}(\psi(a))}(\varphi(z))\} \\ &\geq \min\{MF_{\widetilde{g}(\psi(a))}(\varphi(x)), MF_{\widetilde{g}(\psi(a))}(\varphi(y))\} \\ &= \min\{MF_{(\varphi^{-1}\widetilde{g})(a)}(x), MF_{(\varphi^{-1}\widetilde{g})(a)}(y)\} \end{aligned}$$

Similarly, we obtain $MF_{(\varphi^{-1}\widetilde{g})(a)}(x^{-1}) \geq MF_{(\varphi^{-1}\widetilde{g})(a)}(x)$. Therefore, we conclude that $(\varphi^{-1}(\widetilde{g}), \psi^{-1}(B)) \in M^k F^S_S(P_1, E_1)$ is an MFS-polygroup. □

5. Normal MFS-Polygroups

In this section, we define normal MFS-polygroups and study some of their basic properties. We proved that the images of normal MFS-polygroups are the normal MFS-polygroups under some conditions.

Definition 19. *Let $(\widetilde{f}, A) \in M^k F^S_S(P, E)$ be an MFS-polygroup. Then, $(\widetilde{f}, A)$ is said to be normal if and only if*

$$\inf_{z \in x \circ y}\{MF_{\widetilde{f}(a)}(z)\} = \inf_{z' \in y \circ x}\{MF_{\widetilde{f}(a)}(z')\},$$

for all $a \in A$ and $x, y \in P$.

It is obvious that if $(\widetilde{f}, A)$ is a normal MFS-polygroup, then

$$\inf_{z \in x \circ y}\{MF_{\widetilde{f}(a)}(z) = \inf_{z' \in x \circ y}\{MF_{\widetilde{f}(a)}(z')\},$$

for all $a \in A$ and $x, y \in P$.

Theorem 10. *Let $(\widetilde{f}, A) \in M^k F^S_S(P, E)$ be an MFS-polygroup. Then, the following conditions are equivalent:*

(i) $(\widetilde{f}, A)$ *is a normal MFS-polygroup,*
(ii) $\inf\limits_{z \in x \circ y \circ x^{-1}}\{MF_{\widetilde{f}(a)}(z)\} = MF_{\widetilde{f}(a)}(y)$, *for all $a \in A$ and $x, y \in P$,*
(iii) $\inf\limits_{z \in x \circ y \circ x^{-1}}\{MF_{\widetilde{f}(a)}(z)\} \geq MF_{\widetilde{f}(a)}(y)$, *for all $a \in A$ and $x, y \in P$,*
(iv) $\inf\limits_{z \in y^{-1} \circ x^{-1} \circ y \circ x}\{MF_{\widetilde{f}(a)}(z)\} \geq MF_{\widetilde{f}(a)}(y)$, *for all $a \in A$ and $x, y \in P$.*

Proof. $(i) \Rightarrow (ii)$: For any $a \in A$, suppose that $x, y \in P$ and $z \in x \circ y \circ x^{-1}$. Then, $z \in x \circ s$, where $s \in y \circ x^{-1}$. Since $s \in y \circ x^{-1}$, then $y \in s \circ (x^{-1})^{-1} = s \circ x$. Thus, by hypothesis, we obtain

$$\inf_{z \in x \circ s}\{MF_{\widetilde{f}(a)}(z)\} = \inf_{y \in s \circ x}\{MF_{\widetilde{f}(a)}(y)\} = MF_{\widetilde{f}(a)}(y).$$

That is, $\inf\limits_{z \in x \circ y \circ x^{-1}}\{MF_{\widetilde{f}(a)}(z)\} = MF_{\widetilde{f}(a)}(y)$.

$(ii) \Rightarrow (iii)$: The proof is trivial.

$(iii) \Rightarrow (iv)$: For any $a \in A$, suppose that $x, y \in P$ and $z \in y^{-1} \circ x^{-1} \circ y \circ x$. Then, $z \in y^{-1} \circ s$, where $s \in x^{-1} \circ y \circ x$. By (iii), we obtain $\inf\limits_{s \in x^{-1} \circ y \circ x}\{MF_{\widetilde{f}(a)}(s)\} \geq MF_{\widetilde{f}(a)}(y)$. Since $z \in y^{-1} \circ s$ and letting $(\widetilde{f}, A) \in M^k FS(P)$ be a MFS-polygroup, then we have

$$\inf_{z \in y^{-1} \circ s}\{MF_{\widetilde{f}(a)}(z)\} \geq \min\{MF_{\widetilde{f}(a)}(y^{-1}), MF_{\widetilde{f}(a)}(s)\} = MF_{\widetilde{f}(a)}(y).$$

That is, $\inf\limits_{z \in y^{-1} \circ x^{-1} \circ y \circ x}\{MF_{\widetilde{f}(a)}(z)\} \geq MF_{\widetilde{f}(a)}(y)$ for all $a \in A$ and $x, y \in P$.

$(iv) \Rightarrow (i)$: For any $a \in A$, suppose that $x, y \in P$ and $u \in x^{-1} \circ y \circ x$. Then, $u \in x^{-1} \circ y \circ x \subset y \circ y^{-1} \circ x^{-1} \circ y \circ x$. Thus, $u \in y \circ s$, where $s \in y^{-1} \circ x^{-1} \circ y \circ x$. By (iv), we obtain $\inf\limits_{s \in y^{-1} \circ x^{-1} \circ y \circ x}\{MF_{\widetilde{f}(a)}(s)\} \geq MF_{\widetilde{f}(a)}(y)$. On the other hand,

$$\inf_{u \in y \circ s}\{MF_{\widetilde{f}(a)}(u)\} \geq \min\{MF_{\widetilde{f}(a)}(y), MF_{\widetilde{f}(a)}(s)\} = MF_{\widetilde{f}(a)}(y).$$

Now, let $\omega \in x \circ y$ and $v \in y \circ x$. Then, $y \in v \circ x^{-1}$ and so $\omega \in x \circ y \subset x \circ v \circ x^{-1}$. By the above result, $MF_{\widetilde{f}(a)}(\omega) \geq MF_{\widetilde{f}(a)}(v)$. Similarly, we obtain $MF_{\widetilde{f}(a)}(v) \geq MF_{\widetilde{f}(a)}(\omega)$. Therefore,

$$\inf_{\omega \in x \circ y}\{MF_{\widetilde{f}(a)}(\omega)\} = \inf_{v \in y \circ x}\{MF_{\widetilde{f}(a)}(v)\},$$

for all $a \in A$ and $x, y \in P$. Hence, $(\widetilde{f}, A)$ is a normal MFS-polygroup. □

Lemma 1. *Let $(\widetilde{f}, A) \in M^k F^S_S(P, E)$ be an MFS-polygroup. If $MF_{\widetilde{f}(a)}(x) < MF_{\widetilde{f}(a)}(y)$ for all $a \in A$ and $x, y \in P$, then*

$$\inf_{z \in x \circ y}\{MF_{\widetilde{f}(a)}(z)\} = \inf_{z' \in y \circ x}\{MFS_{\widetilde{f}(a)}(z')\} = MF_{\widetilde{f}(a)}(x).$$

Proof. Let $x, y \in P$ and $z \in x \circ y$. Then,

$$MF_{\widetilde{f}(a)}(z) \geq \min\{MF_{\widetilde{f}(a)}(x), MF_{\widetilde{f}(a)}(y)\} = MF_{\widetilde{f}(a)}(x)$$

for all $a \in A$. Since $z \in x \circ y$, then $x \in z \circ y^{-1}$. Thus,

$$\begin{aligned}\inf_{x \in z \circ y^{-1}}\{MF_{\widetilde{f}(a)}(x)\} &\geq \min\{MF_{\widetilde{f}(a)}(z), MF_{\widetilde{f}(a)}(y^{-1})\} \\ &= \min\{MF_{\widetilde{f}(a)}(z), MF_{\widetilde{f}(a)}(y)\}.\end{aligned}$$

If $\min\{MF_{\widetilde{f}(a)}(z), MF_{\widetilde{f}(a)}(y)\} = MF_{\widetilde{f}(a)}(y)$, then $MF_{\widetilde{f}(a)}(x) \geq MF_{\widetilde{f}(a)}(y)$, a contradiction. Thus, $\min\{MF_{\widetilde{f}(a)}(z), MF_{\widetilde{f}(a)}(y)\} = MF_{\widetilde{f}(a)}(z)$. Hence, $MF_{\widetilde{f}(a)}(x) \geq MF_{\widetilde{f}(a)}(z)$. Consequently, $\inf\limits_{z \in x \circ y}\{MF_{\widetilde{f}(a)}(z)\} = MF_{\widetilde{f}(a)}(x)$ for all $a \in A$ and $x, y \in P$. Similarly, we obtain $\inf\limits_{z' \in y \circ x}\{MF_{\widetilde{f}(a)}(z')\} = MF_{\widetilde{f}(a)}(x)$ for all $a \in A$ and $x, y \in P$. □

Theorem 11. *Let $(\widetilde{f}, A) \in M^k F_S^S(P, E)$ be an MFS-polygroup. Then, $(\widetilde{f}, A)$ is normal if and only if*

$$MF_{\widetilde{f}(a)}(x) = MF_{\widetilde{f}(a)}(y) \Rightarrow \inf_{z \in x \circ y}\{MF_{\widetilde{f}(a)}(z)\} = \inf_{z' \in y \circ x}\{MF_{\widetilde{f}(a)}(z')\},$$

for all $a \in A$ and $x, y \in P$.

Proof. The proof of Theorem 11 follows from Lemma 1. □

Theorem 12. *Let $(\widetilde{f}, A) \in M^k F_S^S(P, E)$. Then, $(\widetilde{f}, A)$ is a normal MFS-polygroup if and only if each of its non-empty level subsets is a normal soft polygroup over P.*

Proof. Let $(\widetilde{f}, A) \in M^k F_S^S(P, E)$ be a normal MFS-polygroup. By Theorem 7, $(MF_{\widetilde{f}(a)})_t$ is a soft polygroup over P for all $a \in A$. Now, we will show that $(MF_{\widetilde{f}(a)})_t$ is normal. Suppose that $y \in (MF_{\widetilde{f}(a)})_t$ and $x \in P$. Then, we have

$$\inf_{z \in x \circ y \circ x^{-1}}\{MF_{\widetilde{f}(a)}(z)\} \geq MF_{\widetilde{f}(a)}(y) \geq t.$$

It follows that $MF_{\widetilde{f}(a)}(z) \geq t$ for all $z \in x \circ y \circ x^{-1}$. That is, $x \circ y \circ x^{-1} \subset (MF_{\widetilde{f}(a)})_t$. We obtain that $(MF_{\widetilde{f}(a)})_t$ is a normal subpolygroup of P for all $a \in A$. Consequently, $(\widetilde{f}, A)_t$ is a normal soft polygroup over P. Conversely, let $(\widetilde{f}, A)_t$ be a normal soft polygroup over P for all $t \in [0, 1]$. By Theorem 7, $(\widetilde{f}, A) \in M^k F_S^S(P, E)$ is an MFS-polygroup. That is, $MF_{\widetilde{f}(a)}$ is an MF-subpolygroup of P for all $a \in A$. We will show that $MF_{\widetilde{f}(a)}$ is normal. Assume that $x, y \in P$, $t_0 = MF_{\widetilde{f}(a)}(y)$. Then, $MF_{\widetilde{f}(a)}(y) \geq t_0$. Since $(\widetilde{f}, A)_{t_0}$ is normal, we have $x \circ y \circ x^{-1} \subset (MFS_{\widetilde{f}(a)})_{t_0}$. Thus, $z \in (MF_{\widetilde{f}(a)})_{t_0}$ for all $z \in x \circ y \circ x^{-1}$. Therefore,

$$\inf_{z \in x \circ y \circ x^{-1}}\{MF_{\widetilde{f}(a)}(z)\} \geq t_0 = MF_{\widetilde{f}(a)}(y).$$

We obtain that $MF_{\widetilde{f}(a)}$ is a normal MF-subpolygroup of P for all $a \in A$. Consequently, $(\widetilde{f}, A)$ is a normal MFS-polygroup. □

Theorem 13. *Let $(\widetilde{f}, A), (\widetilde{g}, B) \in M^k F_S^S(P, E)$ be two normal MFS-polygroups. Then,*

(i) $(\widetilde{f}, A) \sqcap_{\Re} (\widetilde{g}, B)$ is a normal MFS-polygroup.
(ii) $(\widetilde{f}, A) \sqcap_{\Im} (\widetilde{g}, B)$ is a normal MFS-polygroup.
(iii) If $A \cap B = \emptyset$, then $(\widetilde{f}, A) \sqcup (\widetilde{g}, B)$ is a normal MFS-polygroup.
(iv) $(\widetilde{f}, A) \widetilde{\wedge} (\widetilde{g}, B)$ is a normal MFS-polygroup.

Theorem 14. *Let P_1, P_2 be two polygroups and (φ, ψ) be a surjective multi-fuzzy soft homomorphism from P_1 to P_2. If $(\widetilde{f}, A) \in M^k F_S^S(P_1, E_1)$ is a normal MFS-polygroup, then $(\varphi, \psi)(\widetilde{f}, A) \in M^k F_S^S(P_2, E_2)$ is a normal MFS-polygroup.*

Proof. For each $t \in \psi(A)$ and $u, v \in P_2$, there exists $x, y \in P_1$, such that $\varphi(x) = u$ and $\varphi(y) = v$. Since $(\widetilde{f}, A) \in M^k F_S^S(P, E)$ is a normal MFS-polygroup, it follows that for each $a \in A$

$$MF_{\widetilde{f}(a)}(y) \leq MF_{\widetilde{f}(a)}(z)$$

for all $z \in x \circ y \circ x^{-1}$. Let $z^* \in u \circ v \circ u^{-1} = \varphi(x \circ y \circ x^{-1})$. We obtain $z^* = \varphi(z)$. Then, we have

$$\bigvee_{\varphi(y)=v} MF_{\widetilde{f}(a)}(y) \leq \bigvee_{\varphi(x)=u} \bigvee_{\varphi(y)=v} \bigvee_{\varphi(x^{-1})=u^{-1}} MF_{\widetilde{f}(a)}(z).$$

Hence,

$$\begin{aligned} MF_{\varphi(\widetilde{f})(t)}(v) &\leq \bigvee_{\psi(a)=t} \bigvee_{\varphi(x)=u} \bigvee_{\varphi(y)=v} \bigvee_{\varphi(x^{-1})=u^{-1}} MF_{\widetilde{f}(a)}(z) \\ &= \bigvee_{\psi(a)=t} \bigvee_{\varphi(z)=z^*} MF_{\varphi(\widetilde{f})(t)}(z) \end{aligned}$$

for all $z^* \in u \circ v \circ u^{-1}$. Then, we have

$$\inf_{z^* \in u \circ v \circ u^{-1}} \{MF_{\varphi(\widetilde{f})(t)}(z^*)\} \geq MF_{\varphi(\widetilde{f})(t)}(v)\}$$

Consequently, $(\varphi, \psi)(\widetilde{f}, A)$ is a normal MFS-polygroup. $\square$

Theorem 15. *Let P_1, P_2 be two polygroups and (φ, ψ) be an MF-soft homomorphism from P_1 to P_2. If $(\widetilde{g}, B) \in M^k F_S^S(P_2, E_2)$ is a normal MFS-polygroup, then $(\varphi^{-1}(\widetilde{g}), \psi^{-1}(B)) \in M^k F_S^S(P_1, E_1)$ is a normal MFS-polygroup.*

Proof. Let $a \in \psi^{-1}(B)$, $x, y \in P_1$. For all $z \in x \circ y \circ x^{-1}$, we have

$$\begin{aligned} \inf_{z \in x \circ y \circ x^{-1}} \{MF_{(\varphi^{-1}(\widetilde{g}))(a)}(z)\} &= \inf_{z \in x \circ y \circ x^{-1}} \{MF_{\widetilde{g}(\psi(a))}(\varphi(z))\} \\ &\geq MF_{\widetilde{g}(\psi(a))}(\varphi(y)) \\ &= MF_{(\varphi^{-1}(\widetilde{g}))(a)}(y). \end{aligned}$$

Therefore, $(\varphi^{-1}(\widetilde{g}), \psi^{-1}(B))$ is a normal MFS-polygroup. $\square$

6. Conclusions

In real life, many problems often involve uncertainties that are difficult to describe and solve with traditional mathematical tools. To investigate these uncertainties, many researchers have proposed mathematical theory to address the problem of uncertainty. Currently, mathematical theories dealing with the problem of uncertainty include fuzzy set theory, soft set theory, multi-fuzzy set theory, probability theory and so on. The purpose of this paper is to apply the MFS-set theory to algebraic hyperstructures, motivated by the study of the algebraic structures of MF-sets. We generalized the concept of fuzzy polygroups and studied the algebraic properties of MFS-sets in polygroup structures. Thus, this paper provides a new connection between polygroup structures and MFS-sets. We hope that our work enhances the understanding of MFS-polygroups for future researchers. To extend this work, one should study the MFS-sets related to various hyperrings, which can be researched further. A solution to a decision-making problem can be investigated using a different algorithm in the future as well.

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