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Hyers–Ulam Stability of a System of Hyperbolic Partial Differential Equations

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Abstract: In this paper, we study Hyers–Ulam and generalized Hyers–Ulam–Rassias stability of a system of hyperbolic partial differential equations using Gronwall's lemma and Perov's theorem.

Keywords: system of hyperbolic partial differential equations; Hyers–Ulam stability; Gronwall's lemma; Perov's theorem

MSC: 35B35; 35B20

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1. Introduction

In many practical applications, there are problems modeled by differential equations, partial differential equations, differential inequalities, and systems of differential and partial differential equations. Ulam's stability of differential and partial differential equations has been studied by many mathematicians since 1940, when Ulam posed this problem [1]. In 1941, Hyers [2] established the first result regarding this type of stability. This was followed by the work of Obloza and Ger [3,4] on the stability of differential equations. The field then continued to grow rapidly. We mention the works [5–19]. A summary of certain works can be consulted in [20,21]. Ulam stability of systems of differential equations began with the paper by Prastaro and Rassias [22]. Systems have also been studied, for example, in [23,24].

We specify that Hyers–Ulam stability for coupled fixed points of contractive-type operators have been studied, for example, in [25,26] and Hyers–Ulam stability for coupled systems of fractional differential equations in [27–29].

In this paper, we study the Ulam stability of a system of second-order hyperbolic partial differential equations for functions of two variables. The system we consider is nonlinear and has the form

$$\begin{cases} \frac{\partial^2 u_1}{\partial x \partial y} = f_1(x, y, u_1(x, y), u_2(x, y)) \\ \frac{\partial^2 u_2}{\partial x \partial y} = f_2(x, y, u_1(x, y), u_2(x, y)) \end{cases}, \tag{1}$$

satisfying the conditions

$$\begin{cases} u_1(x, 0) = \varphi_1(x) \\ u_1(0, y) = \psi_1(y) \\ u_2(x, 0) = \varphi_2(x) \\ u_2(0, y) = \psi_2(y) \\ u_1(0, 0) = u_2(0, 0) = \varphi_1(0) = \psi_1(0) = \varphi_2(0) = \psi_2(0) = \alpha_0, \end{cases} \tag{2}$$

where $u_1, u_2 \in C^2(D, \mathbb{R})$, $f_1, f_2 \in C(D \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\varphi_1, \varphi_2 \in C[0, a], \psi_1, \psi_2 \in C[0, b]$, $a \in [0, +\infty), b \in [0, +\infty), D = [0, a] \times [0, b], \alpha_0 \in \mathbb{R}$.

The outline of the paper is as follows: in Section 2, we present the stability notions and prove several auxiliary results that are useful in the following sections (Remarks 2–5). The first main result (Theorem 3) is given in the next section and concerns the Hyers–Ulam stability of system (1) and (2), using Perov's theorem and Gronwall's lemma. The second main result (Theorem 4), regarding the generalized Hyers–Ulam–Rassias stability of system (1) and (2), using Gronwall's lemma, is proved in Section 4. Example 1 is also given in Section 3 to illustrate the results of Theorem 3.

2. Preliminary Notions and Results

Let $n \in \mathbb{N}, n \geq 2, x, y \in \mathbb{R}^n, x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n)$. By $x \leq y$, we understand $x_i \leq y_i$, for all $i = 1, 2, \ldots, n$. We will make an identification between column and row vectors in $\mathbb{R}^n$.

In the following, we recall the definition of generalized metric space.

Definition 1 ([30]). *Let X be a nonempty set. A mapping $d : X \times X \longrightarrow \mathbb{R}^n$ is called a generalized metric on X (or vector-valued metric) if the following properties are satisfied:*

(i) $d(x, y) \geq 0$ *for all* $x, y \in X$; $d(x, y) = 0 \Leftrightarrow x = y$;
(ii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$;
(iii) $d(x, y) \leq d(x, z) + d(z, y)$ *for all* $x, y, z \in X$.

A set endowed with a generalized metric d is called a generalized metric space. In a generalized metric space, the notions of convergent sequence, Cauchy sequence, completeness, open subset, and closed subset are similar to those for the usual metric space.

We denote by $\mathcal{M}_{nn}(\mathbb{R})$ the set of all $n \times n$ matrices and by $\mathcal{M}_{nn}(\mathbb{R}_+)$ the set of all $n \times n$ matrices with nonegative elements. Let be O the zero $n \times n$ matrix and I the identity $n \times n$ matrix.

Definition 2. *A matrix $A \in \mathcal{M}_{nn}(\mathbb{R})$ is called convergent to zero if $A^m \to O$, as $m \longrightarrow +\infty$.*

Let us remember the following properties of matrices convergent to zero:

Theorem 1 ([30]). *Let $A \in \mathcal{M}_{nn}(\mathbb{R}_+)$. The following assertions are equivalents:*

(i) *A is convergent to zero;*
(ii) *the eigenvalues of A are in the open unit disc, i.e., $|\lambda| < 1$, for every $\lambda \in C$ with* $\det(A - \lambda I) = 0$;
(iii) *the matrix $(I - A)$ is nonsingular and*

$$(I - A)^{-1} = I + A + \cdots + A^n + \cdots ;$$

(iv) *the matrix $(I - A)$ is nonsingular and $(I - A)^{-1}$ has nonegative elements.*

Remark 1. *Every matrix $A = \begin{pmatrix} a & a \\ b & b \end{pmatrix} \in \mathcal{M}_{22}(\mathbb{R}_+)$ with $a + b < 1$ converges to O.*

The following Perov's theorem is used in the paper:

Theorem 2 (Perov [31]). *Let (X, d) be a complete generalized metric space, $f : X \longrightarrow X$ and $A \in \mathcal{M}_{nn}(\mathbb{R}_+)$ a matrix convergent to zero such that*

$$d(f(x), f(y)) \leq Ad(x, y) \text{ for all } x, y \in X.$$

Then:

(i) *f has a unique fixed point $x^* \in X$;*

(ii) *the sequence of succesive approximation* $(x_m)_{m\in N}$, $x_m = f^m(x_0)$, *is convergent to* x^*, *for all* $x_0 \in X$;

(iii) $d(x_m, x^*) \leq A^m(I-A)^{-1}d(x_0, x_1)$, *for all* $x_0 \in X$ *and* $m \geq 1$.

Let $\varepsilon_1 > 0, \varepsilon_2 > 0$ and $\alpha, \beta \in C(D, R_+)$. We consider the following systems of inequalities:

$$\begin{cases} \left|\frac{\partial^2 u_1}{\partial x \partial y} - f_1(x,y,u_1(x,y),u_2(x,y))\right| \leq \varepsilon_1 \\ \\ \left|\frac{\partial^2 u_2}{\partial x \partial y} - f_2(x,y,u_1(x,y),u_2(x,y))\right| \leq \varepsilon_2 \end{cases} \tag{3}$$

and

$$\begin{cases} \left|\frac{\partial^2 u_1}{\partial x \partial y} - f_1(x,y,u_1(x,y),u_2(x,y))\right| \leq \alpha(x,y) \\ \\ \left|\frac{\partial^2 u_2}{\partial x \partial y} - f_2(x,y,u_1(x,y),u_2(x,y))\right| \leq \beta(x,y) \end{cases}. \tag{4}$$

Definition 3. *System* (1) *is called Hyers–Ulam-stable if there exists a real number* $c > 0$ *so that for any solution* (u_1, u_2) *of system* (3), *satisfying* (2), *there is a solution* (u_1^0, u_2^0) *of system* (1), *satisfying* (2), *such that*

$$\left|u_1(x,y) - u_1^0(x,y)\right| + \left|u_2(x,y) - u_2^0(x,y)\right| \leq c \cdot (\varepsilon_1 + \varepsilon_2), \text{ for all} (x,y) \in D.$$

Definition 4. *System* (1) *is called generalized Hyers–Ulam–Rassias-stable if there exists a real number* $c > 0$ *so that for any solution* (u_1, u_2) *of system* (4), *satisfying* (2), *there is a solution* (u_1^0, u_2^0) *of system* (1), *satisfying* (2), *such that*

$$\left|u_1(x,y) - u_1^0(x,y)\right| + \left|u_2(x,y) - u_2^0(x,y)\right| \leq c \cdot (\alpha(x,y) + \beta(x,y)), \text{ for all} (x,y) \in D.$$

In Definitions 3 and 4, (u_1, u_2) *is called an approximate solution and* (u_1^0, u_2^0) *is called an exact solution of* (1).

Remark 2. *A pair of functions* (u_1, u_2), $u_1, u_2 \in C^2(D, \mathbb{R})$, *is a solution of system* (3) *if and only if there exist a pair of functions* (g_1, g_2), $g_1, g_2 \in C^2(D, \mathbb{R})$ *such that*

1. $|g_1(x,y)| \leq \varepsilon_1$, $|g_2(x,y)| \leq \varepsilon_2$, *for all* $(x,y) \in D$;

   and

2. $$\begin{cases} \frac{\partial^2 u_1}{\partial x \partial y} = f_1(x,y,u_1(x,y),u_2(x,y)) + g_1(x,y) \\ \frac{\partial^2 u_2}{\partial x \partial y} = f_2(x,y,u_1(x,y),u_2(x,y)) + g_2(x,y) \end{cases}, \text{ for all} (x,y) \in D.$$

Remark 3. *A pair of functions* (u_1, u_2), $u_1, u_2 \in C^2(D, \mathbb{R})$, *is a solution of system* (4) *if and only if there exist a pair of functions* (g_1, g_2), $g_1, g_2 \in C^2(D, \mathbb{R})$ *such that*

1. $|g_1(x,y)| \leq \alpha(x,y)$, $|g_2(x,y)| \leq \beta(x,y)$, *for all* $(x,y) \in D$;

   and

2. $$\begin{cases} \frac{\partial^2 u_1}{\partial x \partial y} = f_1(x,y,u_1(x,y),u_2(x,y)) + g_1(x,y) \\ \frac{\partial^2 u_2}{\partial x \partial y} = f_2(x,y,u_1(x,y),u_2(x,y)) + g_2(x,y) \end{cases}, \text{ for all } (x,y) \in D.$$

Remark 4. *If* (u_1, u_2), $u_1, u_2 \in C^2(D, \mathbb{R})$, *is a solution of system* (3), *then*

$$\left|u_1(x,y) - u_1(x,0) - u_1(0,y) + u_1(0,0) - \int_0^x \int_0^y f_1(x,y,u_1(x,y),u_2(x,y))dsdt\right| \leq \varepsilon_1 xy, \text{ for all} (x,y) \in D,$$

$$\left|u_2(x,y) - u_2(x,0) - u_2(0,y) + u_2(0,0) - \int_0^x \int_0^y f_2(x,y,u_1(x,y),u_2(x,y))dsdt\right| \le \varepsilon_2 xy, \text{ for all}(x,y) \in D.$$

Indeed, from Remark 2 we have:

$$u_1(x,y) = u_1(x,0) + u_1(0,y) - u_1(0,0) + \int_0^x \int_0^y f_1(s,t,u_1(s,t),u_2(s,t))dsdt + \int_0^x \int_0^y g_1(s,t)dsdt$$

$$u_2(x,y) = u_2(x,0) + u_2(0,y) - u_2(0,0) + \int_0^x \int_0^y f_2(s,t,u_1(s,t),u_2(s,t))dsdt + \int_0^x \int_0^y g_2(s,t)dsdt,$$

for all$(x,y) \in D$. *Hence*

$$\left|u_1(x,y) - u_1(x,0) - u_1(0,y) + u_1(0,0) - \int_0^x \int_0^y f_1(x,y,u_1(x,y),u_2(x,y))dsdt\right|$$
$$\le \int_0^x \int_0^y |g_1(s,t)|dsdt \le \varepsilon_1 xy, \text{ for all}(x,y) \in D,$$

$$\left|u_2(x,y) - u_2(x,0) - u_2(0,y) + u_2(0,0) - \int_0^x \int_0^y f_2(x,y,u_1(x,y),u_2(x,y))dsdt\right|$$
$$\le \int_0^x \int_0^y |g_2(s,t)|dsdt \le \varepsilon_2 xy, \text{ for all}(x,y) \in D.$$

Remark 5. *If* $(u_1,u_2),\ u_1,u_2 \in C^2(D,\mathbb{R})$, *is a solution of system* (4), *then*

$$\left|u_1(x,y) - u_1(x,0) - u_1(0,y) + u_1(0,0) - \int_0^x \int_0^y f_1(x,y,u_1(x,y),u_2(x,y))dsdt\right|$$
$$\le \int_0^x \int_0^y \alpha(s,t)dsdt, \text{ for all}(x,y) \in D,$$

$$\left|u_2(x,y) - u_2(x,0) - u_2(0,y) + u_2(0,0) - \int_0^x \int_0^y f_2(x,y,u_1(x,y),u_2(x,y))dsdt\right|$$
$$\le \int_0^x \int_0^y \beta(s,t)dsdt, \text{ for all}(x,y) \in D.$$

Indeed, we have:

$$\left|u_1(x,y) - u_1(x,0) - u_1(0,y) + u_1(0,0) - \int_0^x \int_0^y f_1(x,y,u_1(x,y),u_2(x,y))dsdt\right|$$
$$\le \int_0^x \int_0^y |g_1(s,t)|dsdt \le \int_0^x \int_0^y \alpha(s,t)dsdt, \text{ for all}(x,y) \in D,$$

$$\left|u_2(x,y) - u_2(x,0) - u_2(0,y) + u_2(0,0) - \int_0^x \int_0^y f_2(x,y,u_1(x,y),u_2(x,y))dsdt\right|$$
$$\le \int_0^x \int_0^y |g_2(s,t)|dsdt \le \int_0^x \int_0^y \beta(s,t)dsdt, \text{ for all}(x,y) \in D.$$

3. Hyers–Ulam Stability

We present below a result regarding Hyers–Ulam stability of system (1), with the conditions (2), using Gronwall's lemma .

Let $a \in [0,+\infty), b \in [0,+\infty), D = [0,a] \times [0,b], \alpha_0 \in \mathbb{R}$.

Let $\varphi_1,\varphi_2 \in C[0,a], \psi_1,\psi_2 \in C[0,b], u_1,u_2,\overline{u_1},\overline{u_2} \in C^2(D), u = \binom{u_1}{u_2}, \overline{u} = \binom{\overline{u_1}}{\overline{u_2}}, f = \binom{f_1}{f_2}$, $\varphi = \binom{\varphi_1}{\varphi_2}, \psi = \binom{\psi_1}{\psi_2}, \alpha = \binom{\alpha_0}{\alpha_0}, \alpha_0 \in \mathbb{R}, f_1,f_2 \in C(D \times \mathbb{R} \times \mathbb{R},\mathbb{R})$. Consider a column vector equal to the corresponding line vector.

Theorem 3. *We suppose that*

(i) *There exists the matrix function* $l : D \longrightarrow \mathcal{M}_{22}(\mathbb{R}_+)$, $l(x,y) = \left(l_{ij}(x,y)\right)_{\substack{1\le i\le 2,\\ 1\le j\le 2}}$ $l \in C(D, \mathcal{M}_{22}(\mathbb{R}_+))$ *such that*

$$\begin{aligned} &|f_1(x,y,u_1(x,y),u_2(x,y)) - f_1(x,y,\overline{u_1}(x,y),\overline{u_2}(x,y))| \\ &\le l_{11}(x,y)|u_1(x,y) - \overline{u_1}(x,y)| + l_{12}(x,y)|u_2(x,y) - \overline{u_2}(x,y)|, \end{aligned}$$

$$\begin{aligned} &|f_2(x,y,u_1(x,y),u_2(x,y)) - f_2(x,y,\overline{u_1}(x,y),\overline{u_2}(x,y))| \\ &\le l_{21}(x,y)|u_1(x,y) - \overline{u_1}(x,y)| + l_{22}(x,y)|u_2(x,y) - \overline{u_2}(x,y)|, \end{aligned}$$

for all $x \in [0,a]$, $y \in [0,b]$ *and* $u_1, u_2, \overline{u_1}, \overline{u_2} \in C^2(D)$.

(ii) *We denote by* $A_{ij} = \max_D \int_0^x \int_0^y l_{ij}(s,t)dsdt$. *Let* $A = \left(A_{ij}\right)_{\substack{1\le i\le 2\\ 1\le j\le 2}} \in \mathcal{M}_{22}(\mathbb{R}_+)$. *We suppose that the matrix* A *converges to* $O \in \mathcal{M}_{22}(\mathbb{R})$.

Then:

1. *System* (1) *has a unique solution satisfying* (2).
2. *System* (1), *with the conditions* (2), *is Hyers–Ulam-stable.*

Proof. 1. We consider the matrix form of the system:

$$\frac{\partial^2 u}{\partial x \partial y} = f(x,y,u(x,y)),$$

with the boundary conditions

$$\begin{cases} u(x,0) = \varphi(x), \\ u(0,y) = \psi(y), \\ u(0,0) = \varphi(0) = \psi(0) = \alpha. \end{cases}$$

The problem is equivalent to the following system of integral equations:

$$u(x,y) = u(x,0) + u(0,y) - u(0,0) + \int_0^x \int_0^y f(s,t,u(s,t))dsdt.$$

For the proof, we use the generalized norm in $\mathbb{R}^2$, $\|a\| = \begin{pmatrix} |a_1| \\ |a_2| \end{pmatrix}$ for $a = (a_1, a_2) \in \mathbb{R}^2$ (see [30]). We consider the operator $T : C^2(D, R^2) \longrightarrow C(D, \mathbb{R}^2)$ defined by

$$(Tu)(x,y) = u(x,0) + u(0,y) - u(0,0) + \int_0^x \int_0^y f(s,t,u(s,t))dsdt,$$

for all $u \in C^2(D, \mathbb{R}^2)$ and $(x,y) \in D$.

We prove that the operator T is a contraction . We have:

$$\begin{aligned} &\|(Tu)(x,y) - (T\overline{u})(x,y)\| \\ \le\ &\int_0^x \int_0^y \|f(s,t,u(s,t)) - f(s,t,\overline{u}(s,t))\|dsdt \\ \le\ &\int_0^x \int_0^y l(s,t)\|u - \overline{u}\|dsdt \le A\|u - \overline{u}\|, \end{aligned}$$

hence

$$\|Tu - T\overline{u}\| \le A\|u - \overline{u}\|.$$

Since the matrix A converges to the null matrix, from Perov's Theorem 2, it follows that the operator T has a unique fixed point (u_1^0, u_2^0), which is the solution of the integral system and therefore of the problem (1) and (2).

2. Let (u_1, u_2) be a solution of system (3), satisfying (2) and (u_1^0, u_2^0) the unique solution of system (1), satisfying (2). Let $l_1(x,y) = \max\{l_{11}(x,y), l_{12}(x,y)\}$, $l_2(x,y) = \max\{l_{21}(x,y), l_{22}(x,y)\}$, $(x,y) \in D$.

We have

$$\begin{aligned}\left|u_1(x,y) - u_1^0(x,y)\right| &\leq \left|u_1(x,0) - \varphi_1(x) - \psi_1(y) - \alpha_0 + \int_0^x \int_0^y f_1(s,t,u_1(s,t),u_2(s,t))dsdt\right| \\ &+ \int_0^x \int_0^y \left|f_1(s,t,u_1(s,t),u_2(s,t)) - f_1\left(s,t,u_1^0(s,t),u_2^0(s,t)\right)\right| dsdt \\ &\leq \varepsilon_1 xy + \int_0^x \int_0^y l_1(s,t)\left(\left|u_1(s,t) - u_1^0(s,t)\right| + \left|u_2(s,t) - u_2^0(s,t)\right|\right)dsdt.\end{aligned}$$

Analog

$$\begin{aligned}\left|u_2(x,y) - u_2^0(x,y)\right| &\leq \left|u_2(x,0) - \varphi_2(x) - \psi_2(y) - \alpha_0 + \int_0^x \int_0^y f_2(s,t,u_1(s,t),u_2(s,t))dsdt\right| \\ &+ \int_0^x \int_0^y \left|f_2(s,t,u_1(s,t),u_2(s,t)) - f_2\left(s,t,u_1^0(s,t),u_2^0(s,t)\right)\right| dsdt \\ &\leq \varepsilon_2 xy + \int_0^x \int_0^y l_2(s,t)\left(\left|u_1(s,t) - u_1^0(s,t)\right| + \left|u_2(s,t) - u_2^0(s,t)\right|\right)dsdt.\end{aligned}$$

Adding these relations, we obtain

$$\begin{aligned}&\left|u_1(x,y) - u_1^0(x,y)\right| + \left|u_2(x,y) - u_2^0(x,y)\right| \\ &\leq (\varepsilon_1 + \varepsilon_2)xy + \int_0^x \int_0^y (l_1(s,t) + l_2(s,t))\left(\left|u_1(s,t) - u_1^0(s,t)\right| + \left|u_2(s,t) - u_2^0(s,t)\right|\right)dsdt.\end{aligned}$$

Applying Gronwall's Lemma, we obtain

$$\begin{aligned}\left|u_1(x,y) - u_1^0(x,y)\right| + \left|u_2(x,y) - u_2^0(x,y)\right| &\leq (\varepsilon_1 + \varepsilon_2)xy \exp\left(\int_0^x \int_0^y (l_1(s,t) + l_2(s,t))dsdt\right) \\ &\leq (\varepsilon_1 + \varepsilon_2)ab \exp\left(\int_0^a \int_0^b (l_1(s,t) + l_2(s,t))dsdt\right) \\ &\leq c(\varepsilon_1 + \varepsilon_2),\end{aligned}$$

where $c = ab \exp\left(\int_0^a \int_0^b (l_1(s,t) + l_2(s,t))dsdt\right)$; that is, system (1) is Hyers–Ulam-stable. □

Example 1. *Consider* $D = [0,1] \times [0,1]$, $f_1, f_2 \in C(D \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\varphi_2 \in C[0,1]$, $\psi_1, \psi_2 \in C[0,1]$, $\alpha_0 \in \mathbb{R}$,

$$f_1(x,y,u_1,u_2) = \frac{1}{20}xy\cos(u_1 + u_2),$$

$$f_2(x,y,u_1,u_2) = \frac{1}{10}x^2y^2\sin(u_1 + u_2).$$

We remark that

$$|f_1(x,y,u_1,u_2) - f_1(x,y,\overline{u_1},\overline{u_2})| \leq \frac{1}{20}|u_1 - \overline{u}_1| + \frac{1}{20}|u_2 - \overline{u}_2|$$

and

$$|f_2(x,y,u_1,u_2) - f_2(x,y,\overline{u_1},\overline{u_2})| \leq \frac{1}{10}|u_1 - \overline{u}_1| + \frac{1}{10}|u_2 - \overline{u}_2|,$$

for all $u_1, u_2, \overline{u_1}, \overline{u_2} \in \mathbb{R}$. *Using Remark 1, we obtain the result that the matrix* $A = \begin{pmatrix} \frac{1}{20} & \frac{1}{20} \\ \frac{1}{10} & \frac{1}{10} \end{pmatrix}$ *converges to* O.

We consider the system

$$\begin{cases} \frac{\partial^2 u_1}{\partial x \partial y} = \frac{1}{20} xy \cos(u_1 + u_2) \\ \frac{\partial^2 u_2}{\partial x \partial y} = \frac{1}{10} x^2 y^2 \sin(u_1 + u_2) \end{cases}. \tag{5}$$

Let $\varepsilon_1 > 0, \varepsilon_2 > 0$. *We consider also the following system of inequalities:*

$$\begin{cases} \left| \frac{\partial^2 u_1}{\partial x \partial y} - \frac{1}{20} xy \cos(u_1 + u_2) \right| \leq \varepsilon_1 \\ \left| \frac{\partial^2 u_2}{\partial x \partial y} - \frac{1}{10} x^2 y^2 \sin(u_1 + u_2) \right| \leq \varepsilon_2 \end{cases}. \tag{6}$$

From Theorem 3, we have that system (5) with the conditions (2) is Hyers–Ulam-stable; that is, for each solution (u_1, u_2) *of system (6) satisfying (2) and for* (u_1^0, u_2^0)*, which is the unique solution of system (5) satisfying (2), we have*

$$\left| u_1(x,y) - u_1^0(x,y) \right| + \left| u_2(x,y) - u_2^0(x,y) \right| \leq (\varepsilon_1 + \varepsilon_2) \exp\left(\frac{3}{20} \right).$$

4. Generalized Hyers–Ulam–Rassias Stability

We present below a result regarding generalized Hyers–Ulam–Rassias stability of system (1), with the conditions (2), using Gronwall's lemma .

Let $a \in [0, +\infty), b \in [0, +\infty), D = [0,a] \times [0,b], \alpha_0 \in \mathbb{R}$.

Let $\varphi_1, \varphi_2 \in C[0,a], \psi_1, \psi_2 \in C[0,b], u_1, u_2 \in C^2(D), f_1, f_2 \in C(D \times \mathbb{R} \times \mathbb{R}, \mathbb{R}), \alpha, \beta \in C(D, R_+)$.

Theorem 4. *We suppose that*

(i) *The conditions (i), (ii) from Theorem 3 are satisfied.*

(ii) *There exists* $\alpha_1, \beta_1 \in R_+$ *such that*

$$\int_0^x \int_0^y \alpha(s,t) dsdt \leq \alpha_1 \cdot \alpha(x,y), \textit{ for all} (x,y) \in D,$$

$$\int_0^x \int_0^y \beta(s,t) dsdt \leq \beta_1 \cdot \beta(x,y), \textit{ for all} (x,y) \in D.$$

(iii) $\alpha, \beta \in C(D, R_+)$ *are increasing.*

Then, system (1), with the conditions (2), is generalized Hyers–Ulam–Rassias-stable.

Proof. Let (u_1, u_2) be a solution of system (4) satisfying (2) and (u_1^0, u_2^0) the unique solution of system (1) satisfying (2) (this solution exists; see Theorem 3). Let $l_1(x,y) = \max\{l_{11}(x,y), l_{12}(x,y)\}, l_2(x,y) = \max\{l_{21}(x,y), l_{22}(x,y)\}, (x,y) \in D$.

We have

$$\begin{aligned} &\left| u_1(x,y) - u_1^0(x,y) \right| \\ &\leq \int_0^x \int_0^y \alpha(s,t) dsdt + \int_0^x \int_0^y l_1(s,t) \left(\left| u_1(s,t) - u_1^0(s,t) \right| + \left| u_2(s,t) - u_2^0(s,t) \right| \right) dsdt. \\ &\leq \alpha_1 \cdot \alpha(x,y) + \int_0^x \int_0^y l_1(s,t) \left(\left| u_1(s,t) - u_1^0(s,t) \right| + \left| u_2(s,t) - u_2^0(s,t) \right| \right) dsdt. \end{aligned}$$

Analog

$$
\begin{aligned}
&\left|u_2(x,y)-u_2^0(x,y)\right| \\
&\le \int_0^x\int_0^y \beta(s,t)dsdt+\int_0^x\int_0^y l_2(s,t)\left(\left|u_1(s,t)-u_1^0(s,t)\right|+\left|u_2(s,t)-u_2^0(s,t)\right|\right)dsdt \\
&\le \beta_1\cdot\beta(x,y)+\int_0^x\int_0^y l_2(s,t)\left(\left|u_1(s,t)-u_1^0(s,t)\right|+\left|u_2(s,t)-u_2^0(s,t)\right|\right)dsdt.
\end{aligned}
$$

Adding these relations, we obtain

$$
\begin{aligned}
&\left|u_1(x,y)-u_1^0(x,y)\right|+\left|u_2(x,y)-u_2^0(x,y)\right|\le(\alpha_1\cdot\alpha(x,y)+\beta_1\cdot\beta(x,y)) \\
&+\int_0^x\int_0^y (l_1(s,t)+l_2(s,t))\left(\left|u_1(s,t)-u_1^0(s,t)\right|+\left|u_2(s,t)-u_2^0(s,t)\right|\right)dsdt.
\end{aligned}
$$

Applying Gronwall's Lemma, we obtain

$$
\begin{aligned}
&\left|u_1(x,y)-u_1^0(x,y)\right|+\left|u_2(x,y)-u_2^0(x,y)\right| \\
&\le(\alpha_1\cdot\alpha(x,y)+\beta_1\cdot\beta(x,y))\exp\left(\int_0^x\int_0^y (l_1(s,t)+l_2(s,t))dsdt\right) \\
&\le(\alpha_1\cdot\alpha(x,y)+\beta_1\cdot\beta(x,y))\exp\left(\int_0^a\int_0^b (l_1(s,t)+l_2(s,t))dsdt\right) \\
&\le c\cdot(\alpha(x,y)+\beta(x,y)),
\end{aligned}
$$

where $c=max\{\alpha_1,\beta_1\}\cdot\exp\left(\int_0^a\int_0^b(l_1(s,t)+l_2(s,t))dsdt\right)$; that is, system (1) is generalized Hyers–Ulam–Rassias-stable. □

5. Conclusions

In this paper, we have studied Hyers–Ulam and generalized Hyers–Ulam–Rassias stability of systems (1) and (2) , using Gronwall's lemma (in Theorems 3 and 4, respectively). We also gave Example 1, to see the application of Theorem 3.

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