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Asymptotic Regularity and Existence of Time-Dependent Attractors for Second-Order Undamped Evolution Equations with Memory

Xuan Wang [1], Didi Hu [2,*] and Chenghua Gao [1]

[1] College of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, China; wangxuan@nwnu.edu.cn (X.W.); gaochenghua@nwnu.edu.cn (C.G.)
[2] State Key Laboratory of Cryospheric Science, Northwest Institute of Eco-Environment and Resources, Chinese Academy of Sciences, Lanzhou 730070, China
* Correspondence: hudidi@lzb.ac.com

Abstract: Our purpose in this article is to study the asymptotic behavior of undamped evolution equations with fading memory on time-dependent spaces. By means of the theory of processes on time-dependent spaces, asymptotic a priori estimate and the technique of operator decomposition and the existence and asymptotic regularity of time-dependent attractors are, respectively, established in the critical case. At the same time, we also obtain the asymptotic regularity of the solution.

Keywords: asymptotic regularity; time-dependent attractor; undamped evolution equation; fading memory

MSC: 34Q35; 35B40; 35B41

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1. Introduction

In this paper, we study the following n-dimensional system of undamped abstract evolution equations with memory:

$$\begin{cases} \varepsilon(t)u_{tt}+k(0)A^{\theta}u+\int_0^{\infty}k'(s)A^{\theta}u(t-s)\mathrm{d}s+f(u)=g(x), & (x,t)\in\Omega\times[\tau,+\infty), \\ u(x,t)=0, & x\in\partial\Omega,\ t\in[\tau,+\infty), \\ u(x,t)=u_0(x,t),\ u_t(x,t)=u_1(x,t), & x\in\Omega,\ t\in(-\infty,\tau], \end{cases} \tag{1}$$

where Ω is a bounded domain of $\mathbb{R}^n$ $(n\geqslant 3)$ with a smooth boundary $\partial\Omega$, A is a Laplacian operator with the Dirichlet boundary condition, with domain $D(A)=H^2(\Omega)\cap H_0^1(\Omega)$, and

$$\theta\in(\frac{2n}{n+2},\frac{n}{2}),\ n\geqslant 3, \tag{2}$$

Through the linear time convolution of function $A^{\theta}u(\cdot)$ and memory kernel $k(\cdot)$, the fading memory term replaces the damping term and plays the role of energy dissipation in system (1). It follows that the solution semigroup (or the solution process) of undamped evolution equations with fading memory can generate a dissipative dynamical system.

Especially in recent years, one of the key problems in the study of abstract evolution equations with fading memory has been the asymptotic behavior of the solutions when time tends to infinity. Therefore, it has attracted the attention and research interest of many scholars (see, e.g., [1–4] and the references therein). Influenced by this, we also carried out a study of this issue. The problem in (1) we studied arises from isothermal viscoelasticity theory and describes the energy dissipation of an isotropic viscoelastic material (see, e.g., [5–10]). Therefore, it has a strong background in mathematical physics, and it can be naturally transformed into many concrete mathematical models such as the semilinear wave equation, Sine–Gordon equation, relative quantum mechanical equation, semilinear hyperbolic equation and the floating beam equation (see, e.g., [11–15]).

However, as far as we know, the undamped abstract evolution equations with fading memory is less considered. This is mainly because it is more difficult to verify the compactness of the solution semigroup (or solution process) and estimate the asymptotic regularity of solutions than in the damping case. Moreover, it is worth emphasizing that the energy dissipation of the system is only dependent on the fading memory term.

Regarding the abstract evolution equations with fading memory, known results are all in the case of $\varepsilon(t) \equiv 1$ and the asymptotic behavior of the solution can be studied by applying the usual dynamical system theory. Nevertheless, when $\varepsilon(\cdot)$ is a positive decreasing function, the standard theory fails to discuss the dissipative property involved in evolution equations. Therefore, the time-dependent terms are at a functional level; this can be found in (4).

To this end, let $\varepsilon(t)$ be a positive decreasing function which vanishes at infinity and satisfies:

$$\sup_{t\in\mathbb{R}}[|\varepsilon(t)| + |\varepsilon'(t)|] \leqslant L, \tag{3}$$

where $L > 0$. In this case, the natural energy functional associated with the system is defined in the standard way:

$$\mathfrak{E}(t) = \int_\Omega |A^{\frac{\theta}{2}}u|^2 \mathrm{d}x + \varepsilon(t)\int_\Omega |u_t|^2 \mathrm{d}x + \int_0^\infty \mu(s)\int_\Omega |A^{\frac{\theta}{2}}\eta^t|^2 \mathrm{d}x\mathrm{d}s, \tag{4}$$

which shows a structural dependence on time. Moreover, it is not hard to see that the vanishing property of $\varepsilon(\cdot)$ transforms the dissipative property and holds back the existence of absorbing sets in the usual sense, that is, the bounded sets of the phase space absorb all the trajectories after a certain period of time. In such a case, Conti et al. [16,17] put forward the notions and established theories of time-dependent attractors (the modified pullback attractors theory). The main idea is to obtain the existence of absorbing set and attractors by restricting the attraction domain of the compact pullback attracting family in the phase space.

By using the ideas in [16,17], some breakthrough progress was made in the research of the existence of time-dependent attractors and the regularity of solutions for the evolution equation problems. The semilinear wave equations have been treated in many papers, see, for example, [16,18–21]. Conti et al. [16] proved the existence and regularity of a time-dependent attractor, and they [18] obtained the asymptotic structure of a time-dependent attractor. Meng et al. [21,22] discussed and investigated the longtime dynamical behavior for the semilinear wave equation with nonlinear damping and the extensible Berger equation via a contractive function method, respectively. In addition, Meng et al. [20] gave some necessary and sufficient conditions to guarantee the existence of a time-dependent attractor. Liu et al. [23,24] considered the longtime dynamical behavior and achieved the existence of time-dependent attractor for the plate equation on a bounded domain or unbounded domain via an operator decomposition or a contractive function method, respectively. Furthermore, the time-dependent asymptotic behavior of the nonclassical reaction–diffusion equation was studied in [25,26].

Motivated by the ideas in [9,16,17], we were interested in analyzing the dynamical behavior of the undamped abstract evolution equations with fading memory, under the assumption that the nonlinear term satisfies critical growth. It is worth mentioning that the asymptotic regularity of the solutions and time-dependent attractors for the problem in (1) are discussed and investigated firstly in our paper.

The main goal of the present paper was to study the asymptotic behavior of the solutions of system (1). For the existence of time-dependent attractors, the compactness verification of the family of processes is a key ingredient. However, the critical nonlinearity, the memory space that lacks compactness and A^θ that is a fractional operator all contribute to the essential difficulties of the compactness verification. Furthermore, it seems hard to directly apply the methods of [9,16,17] to verify the asymptotic compactness in the time-dependent function space. Therefore, it is very important to study how to handle these natural difficulties brought by the critical nonlinearity, noncompact memory term and fractional operator in the undamped model when verifying the asymptotic compactness. At the same time, this is also a main problem in the research of the asymptotic behavior of nonlinear dynamical systems. By applying the process theory of time-dependent space, asymptotic a priori estimate and the technique of operator decomposition, we conquer the above difficulties, verify the compactness of the process and obtain our main results (see Theorem 5, Lemma 9 and Theorem 6).

The structure of the paper is as follows: in preliminary Section 2, we give a definition of some function sets, present the assumptions and recall some known abstract results; in Section 3, we state and prove our main results on the existence and regularity of a time-dependent attractor and the asymptotic regularity of solutions for system (1).

2. Preliminaries

In this section, we introduce some notations and abstract results about a time-dependent dynamical system.

Let $H = L^2(\Omega)$ and let $A = -\Delta$. A can be viewed as a self-adjoint and unbounded operator in H with domain $D(A)$.

We presume that $\{\lambda_j\}_{j\in\mathbb{N}}$ and $\{\omega_j\}_{j\in\mathbb{N}}$ are eigenvalues and eigenvectors of A, then $\{\omega_j\}_{j\in\mathbb{N}}$ can form a group of orthogonal basis of H, and

$$\begin{cases} A\omega_j = \lambda_j \omega_j, \\ 0 < \lambda_1 < \lambda_2 \leqslant \ldots \leqslant \lambda_j,\ \lambda_j \to \infty, \text{ as } j \to \infty. \end{cases}$$

Define the powers A^θ of A with domain $D(A^\theta) \subset H$ as follows:

$$D(A^\theta) = \{u \in H, \sum_{j=1}^{\infty} \lambda_j^{2\theta}(u,\omega_j)^2 < \infty\}, \tag{5}$$

and

$$\langle u, v\rangle_{2\theta} = \langle A^\theta \cdot, A^\theta \cdot \rangle, \quad \|u\|_{2\theta}^2 = \|A^\theta \cdot\|^2, \tag{6}$$

here, $\langle \cdot,\ \cdot \rangle_{2\theta}$ and $\|\cdot\|_{2\theta}$ are the inner product and norm in $D(A^\theta)$. Obviously, A^θ is also unbounded and self-adjoint.

Set $V_\theta = D(A^{\frac{\theta}{2}})$, for $\theta \in (\frac{2n}{n+2}, \frac{n}{2})$. Then, $V_0 = H = L^2(\Omega)$, $D(A^{\frac{\theta}{2}}) = V_\theta$, $D(A^{-\frac{\theta}{2}}) = V_{-\theta}$. In the paper, we assume that the forcing term $g(x)$ only belongs to $V_{-\theta}$. The spaces H and V_θ are endowed with the following inner products and norms, respectively:

$$\langle u, v\rangle = \int_\Omega u(x)v(x)\mathrm{d}x, \quad \|u\|^2 = \int_\Omega |u(x)|^2 \mathrm{d}x, \quad \forall\, u,\, v \in H; \tag{7}$$

$$\langle u, v\rangle_\theta = \int_\Omega A^{\frac{\theta}{2}}u(x)A^{\frac{\theta}{2}}v(x)\mathrm{d}x, \quad \|u\|_\theta^2 = \int_\Omega |A^{\frac{\theta}{2}}u(x)|^2\mathrm{d}x, \quad \forall\, u,\, v \in V_\theta. \tag{8}$$

Therefore, we know that the compact embedding is

$$V_{\theta_1} \hookrightarrow V_{\theta_2}, \text{ as } \theta_1 > \theta_2, \tag{9}$$

the continuous embedding is

$$V_\theta \hookrightarrow L^{\frac{2n}{n-2\theta}}, \tag{10}$$

and the following Poincaré inequality holds:

$$\lambda_1^\theta \int_\Omega |v|^2 \mathrm{d}x \leqslant \int_\Omega |A^{\frac{\theta}{2}} v|^2 \mathrm{d}x, \quad \forall\, v \in V_\theta. \tag{11}$$

In addition, concerning the memory kernel function in system (1), we presume that $k(\infty) = 1$, $k'(s) < 0$, $\forall\, s \in \mathbb{R}^+$.

Suppose also that $\mu(s) = -k'(s)$ and that it satisfies:

$$\mu(s) \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+),\ \mu(s) \geqslant 0,\ \mu'(s) \leqslant 0, \quad \forall\, s \in \mathbb{R}^+; \tag{12}$$

$$\int_0^\infty \mu(s)\mathrm{d}s = k_0; \tag{13}$$

$$\mu'(s) + \delta\mu(s) \leqslant 0, \quad \forall\, s \in \mathbb{R}^+, \tag{14}$$

where k_0, δ are two positive constants. Furthermore, consequently, the kernels $k(s)$ and $\mu(s)$ decay to zero with an exponential rate.

Hereafter, we introduce a new unknown function $\eta^t(x,s)$ and let it be equal to $u(x,t) - u(x,t-s)$, $(x,s) \in \Omega \times \mathbb{R}$, $t \in [\tau, +\infty)$. In virtue of the presumption $k(\infty) = 1$, then the problem in (1) can be written in the form:

$$\begin{cases} \varepsilon(t)u_{tt} + A^\theta u + \int_0^\infty \mu(s) A^\theta \eta^t(s)\mathrm{d}s + f(u) = g(x),\ t \in [\tau, +\infty), \\ \eta_t^t = -\eta_s^t + u_t,\ t \in [\tau, +\infty), \end{cases} \tag{15}$$

with the initial-boundary conditions are:

$$\begin{cases} u(x,t) = 0,\ \eta^t(x,s) = 0, & x \in \partial\Omega, t \in [\tau, +\infty), \\ u(x,t) = u_0(x,t),\ u_t(x,t) = u_1(x,t), & x \in \Omega,\ t \in (-\infty, \tau], \\ \eta^\tau(x,s) = u_0(x,\tau) - u_0(x,\tau - s), & (x,s) \in \Omega \times \mathbb{R}^+, \end{cases} \tag{16}$$

where the unknown function $u(\cdot)$ satisfies the condition as follows: there exist two positive constants $\mathcal{R}$ and $\varrho = \min\{\frac{\delta}{2}, \frac{\lambda_1}{2}\}$, such that

$$\int_0^\infty e^{-\varrho s} \|\nabla u(-s)\|^2 \mathrm{d}s \leqslant \mathcal{R}, \tag{17}$$

were $\|\cdot\|$ denotes L^2-norm, and λ_1 is the first eigenvalue of the operator $-\Delta$ with a Dirichlet boundary condition.

We assume that $g \in V_{-\theta}$ and the nonlinear function $f(v) \in C^1(\mathbb{R})$ with $f(0) = 0$ satisfies the following conditions.

Growth condition:

$$|f'(s)| \leqslant C(1 + |s|^p), \forall s \in \mathbb{R}, \begin{cases} 0 \leqslant p \leqslant \frac{4}{n-2}, & n \geqslant 3, \\ p \geqslant 0 \text{ and is arbitrary }, & n = 1, 2. \end{cases} \tag{18}$$

The assumption in (18) will be used to verify the compactness about the solution process.

Dissipation condition:

$$\liminf_{|s| \to \infty} \frac{f(s)}{s} > -\lambda_1^\theta, \tag{19}$$

and in view of (19), we obtain

$$2\langle F(s), 1\rangle \geqslant -(1-\nu)\|s\|_\theta^2 - C^*,$$

and we also presume that

$$2\langle f(s), s\rangle \geqslant 2\langle F(s), 1\rangle - (1-\nu)\|s\|_\theta^2 - C^*, \tag{20}$$

where $0<\nu<1,\ C^*>0$, $F(u)=\int_0^u f(r)\mathrm{d}r$ and $\|\cdot\|_\theta$ is the norm of V_θ.

Considering the assumption about memory kernel $\mu(\cdot)$, let $L^2_\mu(\mathbb{R}^+;V_\theta)$ be the family of Hilbert spaces of the V_θ-valued functions on $\mathbb{R}^+$. The scalar product and norm are defined by the formula:

$$\langle\varphi,\psi\rangle_{\mu,\theta} = \int_0^\infty \mu(s)\int_\Omega A^{\frac{\theta}{2}}\varphi A^{\frac{\theta}{2}}\psi \mathrm{d}x\mathrm{d}s, \quad \|\varphi\|^2_{\mu,\theta} = \int_0^\infty \mu(s)\int_\Omega |A^{\frac{\theta}{2}}\varphi|^2\mathrm{d}x\mathrm{d}s. \tag{21}$$

Then, we introduce the family of Hilbert spaces $\mathcal{H}_t^{\theta+\sigma}$

$$\mathcal{H}_t^{\theta+\sigma} = V_{\theta+\sigma}\times V_\sigma \times L^2_\mu(\mathbb{R}^+;V_{\theta+\sigma}),$$

and endowed norm

$$\|z\|^2_{\mathcal{H}_t^{\theta+\sigma}} = \|(u,u_t,\eta^t)\|^2_{\mathcal{H}_t^{\theta+\sigma}} = \|u\|^2_{\theta+\sigma} + \varepsilon(t)\|u_t\|^2_\sigma + \|\eta^t\|^2_{\mu,\theta+\sigma}. \tag{22}$$

Clearly, when $\sigma\equiv 0$, the family of Hilbert spaces $\mathcal{H}_t^\theta$ is defined by:

$$\mathcal{H}_t^\theta = V_\theta \times H \times L^2_\mu(\mathbb{R}^+;V_\theta), \tag{23}$$

endowed with the norm:

$$\|z\|^2_{\mathcal{H}_t^\theta} = \|(u,u_t,\eta^t)\|^2_{\mathcal{H}_t^\theta} = \|u\|^2_\theta + \varepsilon(t)\|u_t\|^2 + \|\eta^t\|^2_{\mu,\theta}. \tag{24}$$

By use of assumptions (12)–(14), we can gain the preliminary result as follows ([27]).

Lemma 1. *If assumptions (12)–(14) about the memory kernel function $\mu(s)$ hold, then for any $\eta^t\in C^1([\tau,t];L^2_\mu(\mathbb{R}^+;V_r))$, $0<r\leqslant 2\theta$, $\forall t\geqslant\tau$, $\theta\in(\frac{2n}{n+2},\frac{n}{2})$, there exists a positive constant δ, such that $\langle\eta^t,\eta^t_s\rangle_{\mu,r}\geqslant\dfrac{\delta}{2}\|\eta^t\|^2_{\mu,r}$.*

We also need the following abstract results to prove the existence of time-dependent attractors.

Lemma 2 ([28])**.** *Let $(\mathcal{M},\mathrm{d})$ be a metric space and also let $U(t,\tau)$ be a Lipschitz continuous dynamical process in $\mathcal{M}$, i.e.,*

$$\mathrm{d}(U(t,\tau)m_1, U(t,\tau)m_2)\leqslant C\mathrm{e}^{K(t-\tau)}\mathrm{d}(m_1,m_2),$$

for appropriate constants C and K which are independent of m_i, t and τ. Assume further that there exist three subsets $M_1,M_2,M_3\subset\mathcal{M}$ such that

$$\mathrm{d}(U(t,\tau)M_1, U(t,\tau)M_2)\leqslant L_1\mathrm{e}^{-\nu_1(t-\tau)},$$

$$\mathrm{d}(U(t,\tau)M_2, U(t,\tau)M_3)\leqslant L_2\mathrm{e}^{-\nu_2(t-\tau)},$$

for some $\nu_1,\nu_2>0$ and $L_1,L_2>0$. Then, it follows that

$$\mathrm{d}(U(t,\tau)M_1, U(t,\tau)M_3)\leqslant L\mathrm{e}^{-\nu(t-\tau)},$$

where $\nu=\frac{\nu_1\nu_2}{K+\nu_1+\nu_2}$ and $L=CL_1+L_2$.

Lemma 3 ([13,29,30]). *Let $\mu(s) \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$ be a nonnegative function that satisfies the following: if there exists $s_0 \in \mathbb{R}^+$ such that $\mu(s_0) = 0$, then $\mu(s) \equiv 0$, for all $s \geqslant s_0$. Moreover, let B_0, B_1 and B_2 be Banach spaces satisfying*

$$B_0 \hookrightarrow B_1 \hookrightarrow B_2,$$

where B_0 and B_1 are reflexive, and the embedding $B_0 \hookrightarrow B_1$ is compact. Assume that $\mathcal{C} \subset L^2_\mu(\mathbb{R}^+; B_1)$ and it satisfies

(i) $\mathcal{C} \subset L^2_\mu(\mathbb{R}^+; B_0) \cap H^1_\mu(\mathbb{R}^+; B_2)$;
(ii) $\sup_{\eta \in \mathcal{C}} \|\eta(s)\|^2_{B_1} \leqslant h(s),\ \forall s \in \mathbb{R}^+,\ h(s) \in L^1_\mu(\mathbb{R}^+)$.

Then, $\mathcal{C}$ is relatively compact in $L^2_\mu(\mathbb{R}^+; B_1)$.

Subsequently, we review some basic concepts and abstract results about a process on a time-dependent system ([16,18,25]), which are used to study the long-time behavior of solutions.

Definition 1. *Let X_t be a family of normed spaces. A two-parameter family of operators $\{U(t,\tau) : X_\tau \to X_t, \tau \leqslant t, \tau \in \mathbb{R}\}$ is said to be a process, if for any $\tau \in \mathbb{R}$,*

(i) $U(\tau,\tau) = \mathrm{Id}$ *is the identity operator on X_τ;*
(ii) $U(t,s)U(s,\tau) = U(t,\tau),\ \forall \tau \leqslant s \leqslant t$.

Assume that X_t is a family of normed spaces. For every $t \in \mathbb{R}$, we introduce the R-ball of X_t:

$$\mathbb{B}_t(R) = \{z \in X_t \,|\, \|z\|_{X_t} \leqslant R\}.$$

The Hausdorff semidistance of sets A, B of X_t is denoted by:

$$\delta_t(A,B) = \sup_{x \in A} \mathrm{dist}_{X_t}(x,B) = \sup_{x \in A} \inf_{y \in B} \|x - y\|_{X_t}.$$

Definition 2. *A family $\mathfrak{C} = \{C_t\}_{t \in \mathbb{R}}$ of bounded sets $C_t \subset X_t$ is called uniformly bounded, if there exists a constant $R > 0$ such that $C_t \subset \mathbb{B}_t(R),\ \forall t \in \mathbb{R}$.*

Definition 3. *A uniformly bounded family $\mathfrak{B}_t = \{\mathbb{B}_t(R_0)\}_{t \in \mathbb{R}}$ is called a time-dependent absorbing set for the process $U(t,\tau)$, if for every $R > 0$, there exist a $t_0 = t_0(R) \leqslant t$ and $R_0 > 0$ such that*

$$\tau \leqslant t - t_0 \Rightarrow U(t,\tau)\mathbb{B}_\tau(R) \subset \mathbb{B}_t(R_0).$$

The process $U(t,\tau)$ is said to be dissipative as it possesses a time-dependent absorbing set.

Definition 4. *The smallest family $\mathfrak{A} = \{A_t\}_{t \in \mathbb{R}}$ is called a time-dependent attractor for the process $U(t,\tau)$, if $\mathfrak{A}$ satisfies the following properties:*

(i) *Each A_t is compact in X_t ;*
(ii) *$\mathfrak{A}$ is pullback attracting, that is, it is uniformly bounded, and the limit*

$$\lim_{\tau \to -\infty} \delta_t(U(t,\tau)C_\tau, A_t) = 0$$

holds for every uniformly bounded family $\mathfrak{C} = \{C_t\}_{t \in \mathbb{R}}$ and every $t \in \mathbb{R}$.

Theorem 1 ([16]). *If $U(t,\tau)$ is asymptotically compact, that is, the set*

$$\mathbb{K} = \{\mathfrak{K} = \{K_t\}_{t \in \mathbb{R}} | \text{ Each } K_t \text{ is compact in } X_t,\ \mathfrak{K} \text{ is pullback attracting}\}$$

is not empty, then the time-dependent attractor $\mathfrak{A}$ *exists and coincides with* $\mathfrak{A} = \{A_t\}_{t\in\mathbb{R}}$. *In particular, it is unique.*

Definition 5. *A function* $t \to Z(t)$ *and* $Z(t) \in X_t$ *is a complete bounded trajectory* (CBT) *of the process* $U(t,\tau)$, *if and only if*

(i) $\sup_{t\in\mathbb{R}} \|Z(t)\|_{X_t} < \infty$;
(ii) $Z(t) = U(t,\tau)Z(\tau), \forall\, \tau \leqslant t,\ \tau \in \mathbb{R}$.

Definition 6. *A time-dependent attractor* $\mathfrak{A} = \{A_t\}_{t\in\mathbb{R}}$ *is invariant, if for all* $\tau \leqslant t$,

$$U(t,\tau)A_\tau = A_t.$$

Theorem 2. *If the time-dependent attractor* $\mathfrak{A} = \{A_t\}_{t\in\mathbb{R}}$ *of the process* $U(t,\tau)$ *is invariant, then it coincides with the set of all* CBTs *of the process* $U(t,\tau)$, *that is,*

$$\mathfrak{A} = \{Z | t \to Z(t) \in X_t \text{ and } Z(t) \text{ is a CBT of the process } U(t,\tau)\}.$$

3. Time-Dependent Global Attractor in $\mathcal{H}_t^\theta$

3.1. Well-Posedness

We start with the general existence and uniqueness of the solutions of the problem in (15) and (16). Based on the standard Faedo–Galerkin approximation method, see, e.g., [14], the following results can be easily obtained and the time-dependent function $\varepsilon(t)$ does not bring about any essential difficulties.

Theorem 3. *Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *with smooth boundary* $\partial\Omega$. *If* (12)–(14) *and* (3) *hold,* $g \in V_{-\theta}$ *and* f *satisfies* (18)–(20), *then for any initial data* $z(\tau) = (u(\tau), u_t(\tau), \eta^\tau(s)) \in \mathcal{H}_\tau^\theta$, *there exists a unique solution* $z(t) = (u(t), u_t(t), \eta^t(s)) \in L^\infty([\tau,t];\mathcal{H}_t^\theta) \cap C([\tau,t];\mathcal{H}_t^\theta)$ *of the problem in* (15) *and* (16), *in the sense that*

$$\begin{aligned} &u \in C([\tau,t];V_\theta), \quad u_t \in L^2([\tau,t];V_\theta), \quad \eta^t \in C([\tau,t];L^2_\mu(\mathbb{R}^+;V_\theta)),\\ &\eta_t^t + \eta_s^t \in L^\infty([\tau,t];L^2_\mu(\mathbb{R}^+;H)) \cap L^2([\tau,t];L^2_\mu(\mathbb{R}^+;V_\theta)) \end{aligned}$$

and

$$\begin{cases} \langle \varepsilon(t)u_{tt}, v\rangle + \langle u, v\rangle_\theta + \langle \eta^t(s), v\rangle_{\mu,\theta} + \langle f(u), v\rangle = \langle g, v\rangle,\\ \langle \eta_t^t(s) + \eta_s^t(s), \varphi(s)\rangle_{\mu,\theta} = \langle u_t, \varphi(s)\rangle_{\mu,\theta}, \end{cases}$$

for all $t \geqslant \tau$ *and any* $v \in V_\theta$, $\varphi \in L^2_\mu(\mathbb{R}^+;V_\theta)$.

Moreover, for any $t \geqslant \tau$, the mapping $z(\tau) \mapsto z(t)$ is continuous from $\mathcal{H}_\tau^\theta$ to $\mathcal{H}_t^\theta$.
By Theorem 3, we can define a process $U(t,\tau)$ as follows:

$$z(t) = U(t,\tau)z(\tau) : \mathcal{H}_\tau^\theta \to \mathcal{H}_t^\theta,$$

which is continuous from $\mathcal{H}_\tau^\theta$ to $\mathcal{H}_t^\theta$.

In the next subsection, we prove that $U(t,\tau)$ satisfies the continuous dependence property on the initial data.

Lemma 4. *Let* $z_i(t)$, $i = 1,2$, *be the corresponding solutions of the problem in* (15) *and* (16) *with* $z_i(\tau) \in \mathcal{H}_\tau^\theta$ *satisfying* $\|z_i(\tau)\|_{\mathcal{H}_\tau^\theta} \leqslant R$, $i = 1,2$. *Suppose that* (12)–(14) *and* (3) *hold. If* $g \in V_{-\theta}$ *and* f *satisfies* (18)–(20), *then there exists a positive constant* K, *such that the following estimate holds:*

$$\begin{aligned}\|z_1(t)-z_2(t)\|^2_{\mathcal{H}^\theta_t} &= \|U(t,\tau)z_1(\tau)-U(t,\tau)z_2(\tau)\|^2_{\mathcal{H}^\theta_t}\\ &\leqslant Ce^{K(t-\tau)}\|z_1(\tau)-z_2(\tau)\|^2_{\mathcal{H}^\theta_\tau}, \quad \forall\, \tau\leqslant t.\end{aligned}$$

3.2. Time-Dependent Absorbing Set in $\mathcal{H}^\theta_t$

In the subsequent estimates, we presume that $0<\rho<1$. Furthermore, we prove the following dissipative estimate.

Theorem 4. *Under the assumption of Lemma 4, for any initial data $z(\tau)\in\mathbb{B}_\tau(R)\subset\mathcal{H}^\theta_\tau$, then there exists $R_0>0$, such that the process $U(t,\tau)$ corresponding to the problem in (15) and (16) possesses a time-dependent absorbing set, namely, the family $\mathfrak{B}_t=\{\mathbb{B}_t(R_0)\}_{t\in\mathbb{R}}$.*

Proof. Multiplying (15) by $2(u_t+\rho u)$ and integrating over Ω, we obtain

$$\begin{aligned}&\langle\varepsilon(t)u_{tt},2(u_t+\rho u)\rangle+\langle A^\theta u,2(u_t+\rho u)\rangle+\langle\int_0^\infty\mu(s)A^\theta\eta^t(s)\mathrm{d}s,2(u_t+\rho u)\rangle\\ &\quad+\langle f(u),2(u_t+\rho u)\rangle=\langle g,2(u_t+\rho u)\rangle.\end{aligned}\tag{25}$$

Thanks to *Lemma* 1, we have

$$\begin{aligned}\langle\int_0^\infty\mu(s)A^\theta\eta^t(s)\mathrm{d}s,2u_t\rangle&=\int_\Omega\int_0^\infty 2(\eta^t_t+\eta^t_s)\mu(s)A^\theta\eta^t(s)\mathrm{d}s\mathrm{d}x\\ &\geqslant\frac{\mathrm{d}}{\mathrm{d}t}\|\eta^t\|^2_{\mu,\theta}+\delta\|\eta^t\|^2_{\mu,\theta},\end{aligned}\tag{26}$$

combining with the Hölder inequality, Cauchy inequality and (13), we obtain that

$$\begin{aligned}\langle\int_0^\infty\mu(s)A^\theta\eta^t(s)\mathrm{d}s,2\rho u\rangle&=2\rho\int_\Omega u\int_0^\infty\mu(s)A^\theta\eta^t(s)\mathrm{d}s\mathrm{d}x\\ &\geqslant-\frac{\rho\nu}{2}\int_\Omega|A^{\frac{\theta}{2}}u|^2\mathrm{d}x-\frac{2\rho}{\nu}\int_\Omega(\int_0^\infty\mu(s)|A^{\frac{\theta}{2}}\eta^t(s)|\mathrm{d}s)^2\mathrm{d}x\\ &\geqslant-\frac{\rho\nu}{2}\|u\|^2_\theta-\frac{2\rho k_0}{\nu}\|\eta^t\|^2_{\mu,\theta}.\end{aligned}\tag{27}$$

Therefore, we obtain from (20) that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|^2_\theta+\varepsilon(t)\|u_t\|^2+\|\eta^t\|^2_{\mu,\theta}+2\rho\varepsilon(t)\langle u_t,u\rangle+2\langle F(u),1\rangle-2\langle g,u\rangle+C)\\ &\quad+\rho(\|u\|^2_\theta+\varepsilon(t)\|u_t\|^2+\|\eta^t\|^2_{\mu,\theta}+2\rho\varepsilon(t)\langle u_t,u\rangle+2\langle F(u),1\rangle-2\langle g,u\rangle+C)\\ &\quad+\frac{\rho\nu}{2}\|u\|^2_\theta-(\varepsilon'(t)+3\rho\varepsilon(t))\|u_t\|^2+(\delta-\frac{2k_0\rho}{\nu}-\rho)\|\eta^t\|^2_{\mu,\theta}\\ &\quad-2\rho(\varepsilon'(t)+\rho\varepsilon(t))\langle u_t,u\rangle\leqslant\rho(C+C^*).\end{aligned}\tag{28}$$

The functional is defined by the formula:

$$\mathcal{M}(t)=\|u\|^2_\theta+\varepsilon(t)\|u_t\|^2+\|\eta^t\|^2_{\mu,\theta}+2\rho\varepsilon(t)\langle u_t,u\rangle+2\langle F(u),1\rangle-2\langle g,u\rangle+C,\tag{29}$$

where $C=\frac{2}{\nu\rho}\|g\|^2_{V_{-\theta}}+C^*$.

Then, we deduce from (3) and (11) that

$$2\rho\varepsilon(t)\langle u_t,u\rangle\leqslant 2\rho\varepsilon(t)|\langle u_t,u\rangle|\leqslant\frac{\rho\nu}{2}\|u\|^2_\theta+\frac{2\rho L}{\nu\lambda_1^\theta}\varepsilon(t)\|u_t\|^2,$$

and

$$\pm 2\langle g,u\rangle \leqslant 2|\langle g,u\rangle| \leqslant \frac{\rho\nu}{2}\|u\|_\theta^2 + \frac{2}{\nu\rho}\|g\|_{V_{-\theta}}^2.$$

Using (19), for some $0<\nu<1$, we have

$$2\langle F(u),1\rangle \geqslant -(1-\nu)\|u\|_\theta^2 - C^*.$$

Thus, choosing ρ small enough, we obtain that $\mathcal{M}(t)\geqslant 0$.

Namely,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}(t)+\rho\mathcal{M}(t)+\frac{\rho\nu}{2}\|u\|_\theta^2-(\varepsilon'(t)+3\rho\varepsilon(t))\|u_t\|^2+(\delta-\frac{2k_0\rho}{\nu}-\rho)\|\eta^t\|_{\mu,\theta}^2 \\ -2\rho(\varepsilon'(t)+\rho\varepsilon(t))\langle u_t,u\rangle \leqslant \rho(C+C^*). \tag{30}$$

By (3) and (11), we find that

$$-2\rho(\varepsilon'(t)+\rho\varepsilon(t))\langle u_t,u\rangle \geqslant -\frac{\rho\nu}{2}\|u\|_\theta^2 - \frac{2\rho}{\nu\lambda_1^\theta}L^2\|u_t\|^2. \tag{31}$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}(t)+\rho\mathcal{M}(t)\leqslant \rho(C+C^*),$$

that is,

$$\mathcal{M}(t)\leqslant \mathrm{e}^{-\rho(t-\tau)}\mathcal{M}(\tau)+C+C^*. \tag{32}$$

For $\epsilon>0$, from (18), $\theta\in(\frac{2n}{n+2},\frac{n}{2})$ and the interpolation inequality, we have

$$\begin{aligned}\langle F(u),1\rangle &= \int_\Omega F(u)\mathrm{d}x \\ &\leqslant C\int_\Omega(|u|^2+|u|^{\frac{2n}{n-2}})\mathrm{d}x \\ &\leqslant C(\|u\|^2+\|\nabla u\|^{\frac{2n}{n-2}}) \\ &\leqslant C\|u\|^2+C(C_\epsilon\|u\|+\epsilon\|u\|_\theta)^{\frac{2n}{n-2}} \\ &\leqslant C\|u\|_\theta^{\frac{2n}{n-2}}.\end{aligned}$$

Thus, for a small enough ρ there exist positive constants C, C_1 and C_2, such that

$$C\|z(t)\|_{\mathcal{H}_t^\theta}^2 - C_1 \leqslant \mathcal{M}(t)\leqslant C\|z(t)\|_{\mathcal{H}_t^\theta}^{\frac{2n}{n-2}}+C_2. \tag{33}$$

Combining with (32), there exists a constant $N_1>0$ such that

$$\|U(t,\tau)z(\tau)\|_{\mathcal{H}_t^\theta}^2 \leqslant Q_1(\|z(\tau)\|_{\mathcal{H}_\tau^\theta})\mathrm{e}^{-\rho(t-\tau)}+N_1,$$

where $Q_1(\cdot)$ is an increasing positive function. Because $z(\tau)\in\mathbb{B}_\tau(R)$, the following inequality is valid

$$\|U(t,\tau)z(\tau)\|_{\mathcal{H}_t^\theta}^2 \leqslant Q_1(R)\mathrm{e}^{-\rho(t-\tau)}+N_1\leqslant 2N_1=R_0^2,$$

provided that $\tau\leqslant t-t_0$, where $t_0=\max\{0,\frac{1}{\rho}\ln\frac{2Q_1(R)}{N_1}\}$.

This completes the proof. □

Proof. Proof of Lemma 4:

Assume that the initial data $z_i(\tau)$, $i = 1, 2$, satisfy $\|z_i(\tau)\|_{\mathcal{H}_\tau^\theta} \leqslant R$. It follows from Theorem 4 that

$$\|z_i(t)\|_{\mathcal{H}_t^\theta} = \|U(t,\tau)z_i(\tau)\|_{\mathcal{H}_t^\theta} \leqslant R_0. \tag{34}$$

We substitute $\bar{z}(t) = (\bar{u}(t), \bar{u}_t(t), \bar{\eta}^t(s)) = z_1(t) - z_2(t)$ into (15). Then,

$$\varepsilon(t)\bar{u}_{tt} + A^\theta \bar{u} + \int_0^\infty \mu(s) A^\theta \bar{\eta}^t(s) \mathrm{d}s + f(u_1) - f(u_2) = 0 \tag{35}$$

and $\bar{z}(\tau) = z_1(\tau) - z_2(\tau)$.

Taking the scalar product of (35) with $2\bar{u}_t(t)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{z}\|^2_{\mathcal{H}_t^\theta} - \varepsilon'(t)\|\bar{u}_t\|^2 + \delta\|\bar{\eta}^t\|^2_{\mu,\theta} \leqslant -2\langle f(u_1) - f(u_2), \bar{u}_t\rangle. \tag{36}$$

From (18), (10) and (34), we have

$$\begin{aligned}
&-2\langle f(u_1) - f(u_2), \bar{u}_t\rangle \\
&\leqslant 2\int_\Omega |f'(\xi)||\bar{u}||\bar{u}_t| dx \\
&\leqslant 2C\Big(\int_\Omega (1 + |u_2|^{\frac{4}{n-2}} + |u_1|^{\frac{4}{n-2}})^{\frac{n}{\theta}} dx\Big)^{\frac{\theta}{n}} \Big(\int_\Omega |\bar{u}|^{\frac{2n}{n-2\theta}} dx\Big)^{\frac{n-2\theta}{2n}} \Big(\int_\Omega |\bar{u}_t|^2 dx\Big)^{\frac{1}{2}} \\
&\leqslant 2C(1 + \|u_2\|_\theta^{\frac{4}{n-2}} + \|u_1\|_\theta^{\frac{4}{n-2}})\|\bar{u}\|_\theta \|\bar{u}_t\| \\
&\leqslant C(\|\bar{u}\|_\theta^2 + \|\bar{u}_t\|^2),
\end{aligned} \tag{37}$$

where $\frac{2n}{n-2\theta} > \frac{4n}{(n-2)\theta}$. Substituting (37) into (36), we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{z}(t)\|^2_{\mathcal{H}_t^\theta} &\leqslant C(\|\bar{u}\|_\theta^2 + \|\bar{u}_t\|^2) \\
&= \frac{C}{\varepsilon(t)}(\varepsilon(t)\|\bar{u}\|_\theta^2 + \varepsilon(t)\|\bar{u}_t\|^2) \\
&\leqslant \frac{C}{\varepsilon(t)}(L\|\bar{u}\|_\theta^2 + \varepsilon(t)\|\bar{u}_t\|^2) \\
&\leqslant \frac{C(L+1)}{\varepsilon(t)}\|\bar{z}(t)\|^2_{\mathcal{H}_t^\theta}.
\end{aligned}$$

Applying the Gronwall lemma, we finally have

$$\begin{aligned}
\|z_1(t) - z_2(t)\|^2_{\mathcal{H}_t^\theta} &\leqslant \|\bar{z}(\tau)\|^2_{\mathcal{H}_\tau^\theta} \cdot \mathrm{e}^{C(L+1)\int_\tau^t \frac{1}{\varepsilon(s)}\mathrm{d}s} \\
&\leqslant C\mathrm{e}^{K(t-\tau)}\|z_1(\tau) - z_2(\tau)\|^2_{\mathcal{H}_\tau^\theta}.
\end{aligned} \tag{38}$$

The proof is completed. □

3.3. The Existence of a Time-Dependent Attractor in $\mathcal{H}_t^\theta$

Devoted to the difficulties arising from the critical exponent and noncompact memory space, in this subsection, we use the method of asymptotic a priori estimates and the technique of operator decomposition to verify the necessary compactness.

Since $H \hookrightarrow V_{-\theta}$ is dense, for every $g \in V_{-\theta}$ and any $\varrho > 0$, there exists $g^\varrho \in H$ which depends on g and ϱ, such that

$$\|g - g^\varrho\|_{V_{-\theta}} \leqslant \varrho. \tag{39}$$

Assuming (18)–(20) hold, we write $f = f_0 + f_1$, where f_0, $f_1 \in C^2(\mathbb{R})$ fulfill

$$f_0(0) = f_1(0) = 0, \tag{40}$$

$$2\langle f_0(s), s\rangle \geqslant 2\langle F_0(s), 1\rangle - \frac{1-\nu}{2}\|s\|_\theta^2 - \varrho, \tag{41}$$

from which we obtain

$$2\langle f_1(s), s\rangle \geqslant 2\langle F_1(s), 1\rangle - \frac{1-\nu}{2}\|s\|_\theta^2 - C^* + \varrho, \tag{42}$$

$$2\langle F_0(s), 1\rangle \geqslant -\frac{1-\nu}{2}\|s\|_\theta^2 - \varrho, \tag{43}$$

$$2\langle F_1(s), 1\rangle \geqslant -\frac{1-\nu}{2}\|s\|_\theta^2 - C^* + \varrho, \tag{44}$$

where $0 < 1-\nu < \lambda_1^\theta$, $C^* > 0$, $F_i(u) = \int_0^u f_i(r)\mathrm{d}r, i = 1,2,$ and $\|\cdot\|_\theta$ is the norm of V_θ. Furthermore, in the space $\mathcal{H}_t^\theta$, we assume that

$$|f_0'(s)| \leqslant C(1+|s|^p), \forall s \in \mathbb{R},\ 0 \leqslant p \leqslant \frac{4}{n-2},\ n \geqslant 3, \tag{45}$$

$$|f_1'(s)| \leqslant C(1+|s|^p), \forall s \in \mathbb{R},\ 0 \leqslant p < \frac{4}{n-2},\ n \geqslant 3. \tag{46}$$

Let $\mathfrak{B}_t = \{\mathbb{B}_t(R_0)\}_{t\in\mathbb{R}}$ be a time-dependent absorbing set obtained in Theorem 4. For a fixed $\tau \in \mathbb{R}$ and any $\|z(\tau)\|_{\mathcal{H}_\tau^\theta} \leqslant R$, we decompose the solution $z(t) = (u(t), u_t(t), \eta^t)$ of the problem in (15) and (16) as follows:

$$z(t) = U(t,\tau)z(\tau) = V(t,\tau)z_1(\tau) + W(t,\tau)z_2(\tau) = z_1(t) + z_2(t),$$

where

$$z_1(t) = (v(t), v_t(t), \zeta^t(s)), \quad z_2(t) = (w(t), w_t(t), \xi^t(s))$$

satisfy

$$\begin{cases} \varepsilon(t)v_{tt} + A^\theta v + \int_0^\infty \mu(s)A^\theta\zeta^t(s)\mathrm{d}s + f_0(v) = g - g^\varrho, \\ \zeta_t^t = -\zeta_s^t + v_t, \\ v(x,t)|_{\partial\Omega} = 0,\ \ \zeta^t(x,t)|_{\partial\Omega} = 0, \\ v(x,\tau) = u_0(x,t),\ \ v_t(x,\tau) = u_1(x,t),\ x \in \Omega,\ t \leqslant \tau, \\ \zeta^\tau(x,s) = u_0(x,\tau) - u_0(x,\tau - s),\ (x,s) \in \Omega \times \mathbb{R}^+, \end{cases} \tag{47}$$

and

$$\begin{cases} \varepsilon(t)w_{tt} + A^\theta w + \int_0^\infty \mu(s)A^\theta\xi^t(s)\mathrm{d}s + f(u) - f_0(v) = g^\varrho, \\ \xi_t^t = -\xi_s^t + w_t, \\ w(x,t)|_{\partial\Omega} = 0,\ \ \xi^t(x,t)|_{\partial\Omega} = 0, \\ w(x,\tau) = 0,\ \ w_t(x,\tau) = 0,\ \ \xi^\tau(x,s) = 0. \end{cases} \tag{48}$$

By the Galerkin approximation method, the existence and uniqueness of the solution of (47) and (48) can be obtained.

Furthermore, akin to the proof of *Theorem* 4, for the solution $z_1(t)$ of (47), we get

Lemma 5. *Let $z_1(t)$ be the solution of problem (47) with initial data $z_1(\tau) = z(\tau)$ satisfying $\|z(\tau)\|^2_{\mathcal{H}^\theta_\tau} \leqslant R$. Suppose that $g \in V_{-\theta}$ and (12)–(14), (3) hold. If f_0 satisfies (40), (41), (43) and (45), for $\epsilon > 0$, then there exists a positive constant $\varrho = \varrho(\epsilon)$, such that the solution of problem (47) satisfies*

$$\|V(t,\tau)z(\tau)\|^2_{\mathcal{H}^\theta_t} \leqslant Q_2(\|z(\tau)\|_{\mathcal{H}^\theta_\tau})\mathrm{e}^{-\rho_1(t-\tau)} \leqslant 2\epsilon, \text{ as } \tau \leqslant t - t_1. \tag{49}$$

where $Q_2(\cdot)$ is an increasing positive function, $\rho_1 = \rho_1(\|\mathfrak{B}_t\|_{\mathcal{H}^\theta_t})$ is small enough and $t_1 = t_1(\epsilon, Q_2(R), \rho_1)$.

Proof. Taking the inner product of (47) with $2(v_t(t) + \rho_1 v(t))$, we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|v\|^2_\theta + \varepsilon(t)\|v_t\|^2 + \|\zeta^t\|^2_{\mu,\theta} + 2\rho_1\varepsilon(t)\langle v, v_t\rangle + 2\langle F_0(v), 1\rangle - 2\langle g - g^\varrho, v\rangle + C)\\
&\quad + \rho_1(\|v\|^2_\theta + \varepsilon(t)\|v_t\|^2 + \|\zeta^t\|^2_{\mu,\theta} + 2\rho_1\varepsilon(t)\langle v, v_t\rangle + 2\langle F_0(v), 1\rangle - 2\langle g - g^\varrho, v\rangle + C)\\
&\quad + \frac{\rho_1\nu}{2}\|v\|^2_\theta - (\varepsilon'(t) + 3\rho_1\varepsilon(t))\|v_t\|^2 + (\delta - 2\rho_1 k_0 - \rho_1)\|\zeta^t\|^2_{\mu,\theta}\\
&\quad - 2\rho_1(\varepsilon'(t) + \rho_1\varepsilon(t))\langle v, v_t\rangle\\
&\quad \leqslant \rho_1(\varrho + C),
\end{aligned} \tag{50}$$

where $F_0(s) = \int_0^s f_0(r)\mathrm{d}r$.

We define the functional as follows:

$$\mathcal{M}_1(t) = \|v\|^2_\theta + \varepsilon(t)\|v_t\|^2 + \|\zeta^t\|^2_{\mu,\theta} + 2\rho_1\varepsilon(t)\langle v, v_t\rangle + 2\langle F_0(v), 1\rangle - 2\langle g - g^\varrho, v\rangle + C, \tag{51}$$

where $C = 8\|g - g^\varrho\|^2_{V_{-\theta}} + \varrho = 8\varrho^2 + \varrho$.

In fact, by virtue of (41) and (45), we have $-\frac{1-\nu}{2}\|v\|^2_\theta - \varrho \leqslant 2\langle F_0(v), 1\rangle \leqslant C\|v\|_\theta^{\frac{2n}{n-2}}$. Similarly, we can deduce that

$$2\rho_1\varepsilon(t)\langle v, v_t\rangle \leqslant 2\rho_1\varepsilon(t)|\langle v, v_t\rangle| \leqslant \frac{1}{8}\|v\|^2_\theta + \frac{8\rho_1^2 L}{\lambda_1^\theta}\varepsilon(t)\|v_t\|^2,$$

$$2\langle g - g^\varrho, v\rangle \leqslant 2|\langle g - g^\varrho, v\rangle| \leqslant \frac{1}{8}\|v\|^2_\theta + 8\|g - g^\varrho\|^2_{V_{-\theta}}.$$

Then,

$$\frac{1}{4}\|V(t,\tau)z(\tau)\|^2_{\mathcal{H}^\theta_t} \leqslant \mathcal{M}_1(t) \leqslant C\|V(t,\tau)z(\tau)\|^{\frac{2n}{n-2}}_{\mathcal{H}^\theta_t} + 16\varrho^2 + \varrho. \tag{52}$$

From (3) and (11), we have

$$-2\rho_1(\varepsilon'(t) + \rho_1\varepsilon(t))\langle v, v_t\rangle \geqslant -\frac{\rho_1\nu}{4}\|v\|^2_\theta - \frac{4\rho_1 L^2}{\lambda_1^\theta\nu}\|v_t\|^2.$$

Choosing ρ_1 small enough, we have

$$-\varepsilon'(t) - 3\rho_1\varepsilon(t) - \frac{4\rho_1 L^2}{\lambda_1^\theta\nu} \geqslant 0, \quad \delta - 2\rho_1 k_0 - \rho_1 \geqslant 0.$$

Combining with the above estimates, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}_1(t)+\rho_1\mathcal{M}_1(t)\leqslant\rho_1(8\varrho^2+2\varrho). \tag{53}$$

Taking $\epsilon=32\varrho^2+2\varrho+C(64\varrho^2+4\varrho)$, we obtain from (52) and (53)

$$\|V(t,\tau)z_1(\tau)\|^2_{\mathcal{H}^\theta_t}\leqslant Q_2(\|z(\tau)\|_{\mathcal{H}^\theta_\tau})\mathrm{e}^{-\rho_1(t-\tau)}+\epsilon.$$

Due to $z(\tau)\in\mathbb{B}_\tau(R)$, we find the estimate

$$\|V(t,\tau)z(\tau)\|^2_{\mathcal{H}^\theta_t}\leqslant 2\epsilon,$$

provided that $\tau\leqslant t-t_1$, where $t_1=\max\{0,\frac{1}{\rho_1}\ln\frac{Q_2(R)}{\epsilon}\}$. This completes the proof of Lemma 6. □

All in all, the following uniformly bound estimate holds:

$$\sup_{\tau\leqslant t-t^*}\{\|U(t,\tau)z(\tau)\|_{\mathcal{H}^\theta_t}+\|V(t,\tau)z(\tau)\|_{\mathcal{H}^\theta_t}+\|W(t,\tau)z(\tau)\|_{\mathcal{H}^\theta_t}\}\leqslant 2R_0, \tag{54}$$

where $t^*=\max\{0,\frac{1}{\rho_1}\ln\frac{Q_2(R)}{2\epsilon},\frac{1}{\rho}\ln\frac{2Q_1(R)}{R_0^2}\}$.

Lemma 6. *Let $z_2(t)$ be the solution of (48) with initial data $z_2(\tau)$ satisfying $\|z_2(\tau)\|_{\mathcal{H}^\theta_\tau}=0$. If the assumptions (12)–(14), (18), (3) and (40)–(46) hold, then there exists $N_2=N_2(\mathfrak{B}_t)>0$, such that*

$$\sup_{\tau\leqslant t-t^*}\|z_2(t)\|_{\mathcal{H}^{\theta+\sigma}_t}=\sup_{\tau\leqslant t-t^*}\|W(t,\tau)z_2(\tau)\|_{\mathcal{H}^{\theta+\sigma}_t}\leqslant N_2, \tag{55}$$

where $\sigma=\min\{\frac{\theta}{4},\frac{(n+2)\theta-2n}{n-2},\frac{n}{2}-\theta\}$ and $t^*=t^*(R_0,\rho,\lambda_1,\|g^\varrho\|)$.

Proof. Note that $f=f_0+f_1$ implies that

$$f(u)-f_0(v)=f(u)-f(v)+f(v)-f_0(v)=f(u)-f(v)+f_1(v),$$

therefore, taking the inner product of (48) with $2A^\sigma(w_t(t)+\rho w(t))$, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\|w\|^2_{\theta+\sigma}+\varepsilon(t)\|w_t\|^2_\sigma+\|\xi^t\|^2_{\mu,\theta+\sigma}+2\rho\varepsilon(t)\langle w_t,A^\sigma w\rangle+2\langle f(u)-f_0(v),A^\sigma w\rangle-2\langle g^\varrho,A^\sigma w\rangle)\\&\quad+\frac{3\rho}{2}\|w\|^2_{\theta+\sigma}-(\varepsilon'(t)+2\rho\varepsilon(t))\|w_t\|^2_\sigma+(\delta-2\rho k_0)\|\xi^t\|^2_{\mu,\theta+\sigma}-2\rho\varepsilon'(t)\langle w_t,A^\sigma w\rangle\\&\quad+2\rho\langle f(u)-f_0(v),A^\sigma w\rangle-2\rho\langle g^\varrho,A^\sigma w\rangle\\&\leqslant 2\langle f'(u)u_t-f'(v)v_t,A^\sigma w\rangle+2\langle f_1'(v)v_t,A^\sigma w\rangle.\end{aligned} \tag{56}$$

Next, we will deal with each term on the right-hand of (56).

First, by virtue of (18), (10), (49) and (54), we have

$$\begin{aligned}&2\langle f'(u)u_t-f'(v)v_t,A^\sigma w)\rangle\\&\leqslant C(\int_\Omega(1+|u|^{\frac{4}{n-2}})^{\frac{n}{\theta-\sigma}}\mathrm{d}x)^{\frac{\theta-\sigma}{n}}(\int_\Omega|u_t|^2\mathrm{d}x)^{\frac12}(\int_\Omega|A^\sigma w|^{\frac{2n}{n-2(\theta-\sigma)}}\mathrm{d}x)^{\frac{n-2(\theta-\sigma)}{2n}}\\&\quad+C(\int_\Omega(1+|v|^{\frac{4}{n-2}})^{\frac{n}{\theta-\sigma}}\mathrm{d}x)^{\frac{\theta-\sigma}{n}}(\int_\Omega|v_t|^2\mathrm{d}x)^{\frac12}(\int_\Omega|A^\sigma w|^{\frac{2n}{n-2(\theta-\sigma)}}\mathrm{d}x)^{\frac{n-2(\theta-\sigma)}{2n}}\\&\leqslant C(1+\|u\|_\theta^{\frac{4}{n-2}})\|u_t\|\|w\|_{\theta+\sigma}+C(1+\|v\|_\theta^{\frac{4}{n-2}})\|v_t\|\|w\|_{\theta+\sigma}\\&\leqslant\frac{\rho}{8}\|w\|^2_{\theta+\sigma}+C,\end{aligned} \tag{57}$$

where $\frac{4}{n-2} \cdot \frac{n}{\theta-\sigma} \leqslant \frac{2n}{n-2\theta}$.

Second, by (46), (10) and (49), we get

$$-2\langle f_1'(v)v_t, A^\sigma w)\rangle \leqslant \frac{\rho}{8}\|w\|_{\theta+\sigma}^2 + C. \tag{58}$$

Choosing a suitable constant $C > 0$, we define the functional:

$$\begin{aligned}\mathcal{M}_2(t) = {}& \|w\|_{\theta+\sigma}^2 + \varepsilon(t)\|w_t\|_\sigma^2 + \|\xi^t\|_{\mu,\theta+\sigma}^2 + 2\rho\varepsilon(t)\langle w_t, A^\sigma w\rangle \\ &+ 2\langle f(u) - f_0(v), A^\sigma w\rangle - 2\langle g^\varrho, A^\sigma w\rangle + C,\end{aligned} \tag{59}$$

as ρ is small enough, we know that

$$\frac{1}{2}\|W(t,\tau)z_2(\tau)\|_{\mathcal{H}_t^{\theta+\sigma}}^2 \leqslant \mathcal{M}_2(t) \leqslant 2\|W(t,\tau)z_2(\tau)\|_{\mathcal{H}_t^{\theta+\sigma}}^2 + C. \tag{60}$$

Indeed, we can easily deduce that

$$\begin{aligned}2\rho\varepsilon(t)|\langle w_t, A^\sigma w\rangle| &\leqslant \rho\|w\|_{\theta+\sigma}^2 + \frac{\rho L}{\lambda_1^\theta}\varepsilon(t)\|w_t\|_\sigma^2, \\ 2|\langle g^\varrho, A^\sigma w\rangle| &\leqslant \rho\|w\|_{\theta+\sigma}^2 + \frac{1}{\rho\lambda_1^{\theta-\sigma}}\|g^\varrho\|^2.\end{aligned}$$

Thanks to (10), we have

$$\begin{aligned}&2|\langle f(u) - f_0(v), A^\sigma w\rangle| \\ &\leqslant 2|\langle f(u) - f(v), A^\sigma w\rangle| + 2|\langle f_1(v), A^\sigma w\rangle| \\ &\leqslant C\Big(\int_\Omega (1+|u|^{\frac{4}{n-2}} + |v|^{\frac{4}{n-2}})^{\frac{n}{\theta-\sigma}}dx\Big)^{\frac{\theta-\sigma}{n}}\Big(\int_\Omega |w|^2 dx\Big)^{\frac{1}{2}}\Big(\int_\Omega |A^\sigma w|^{\frac{2n}{n-2(\theta-\sigma)}}dx\Big)^{\frac{n-2(\theta-\sigma)}{2n}} \\ &\quad + C\Big(\int_\Omega (1+|v|^\gamma)^{\frac{n}{\theta-\sigma}}dx\Big)^{\frac{\theta-\sigma}{n}}\Big(\int_\Omega |v|^2 dx\Big)^{\frac{1}{2}}\Big(\int_\Omega |A^\sigma w|^{\frac{2n}{n-2(\theta-\sigma)}}dx\Big)^{\frac{n-2(\theta-\sigma)}{2n}} \\ &\leqslant C(1+\|u\|_\theta^{\frac{4}{n-2}} + \|v\|_\theta^{\frac{4}{n-2}})\|w\|_\theta\|w\|_{\theta+\sigma} + C(1+\|v\|_\theta^\gamma)\|v\|_\theta\|w\|_{\theta+\sigma} \\ &\leqslant \frac{1}{4}\|w\|_{\theta+\sigma}^2 + C,\end{aligned} \tag{61}$$

where $\frac{2n}{n-2\theta} \geqslant \frac{4n}{(n-2)(\theta-\sigma)} > \frac{n\gamma}{\theta-\sigma}$.

It follows from the above estimates that

$$\begin{aligned}&\frac{d}{dt}\mathcal{M}_2(t) + \rho\mathcal{M}_2(t) + \frac{\rho}{2}\|w\|_{\theta+\sigma}^2 - (\varepsilon'(t) + 3\rho\varepsilon(t))\|w_t\|_\sigma^2 + (\delta - \rho - 2\rho k_0)\|\xi^t\|_{\mu,\theta+\sigma}^2 \\ &- 2\rho(\varepsilon'(t) + \rho\varepsilon(t))\langle w_t, A^\sigma w\rangle \leqslant \frac{\rho}{4}\|w\|_{\theta+\sigma}^2 + C.\end{aligned} \tag{62}$$

Obviously, we can gain

$$-2\rho(\varepsilon'(t) + \rho\varepsilon(t))\langle w_t, A^\sigma w\rangle \geqslant -\frac{\rho}{4}\|w\|_{\theta+\sigma}^2 - \frac{4\rho L^2}{\lambda_1^\theta}\|w_t\|_\sigma^2. \tag{63}$$

Substituting (63) into (62), we have

$$\frac{d}{dt}\mathcal{M}_2(t) + \rho\mathcal{M}_2(t) - (\varepsilon'(t) + 3\rho\varepsilon(t) + \frac{4\rho L^2}{\lambda_1^\theta})\|w_t\|_\sigma^2 + (\delta - 2\rho k_0 - \rho)\|\xi^t\|_{\mu,\theta+\sigma}^2 \leqslant C. \tag{64}$$

Taking ρ small enough, we know

$$\frac{d}{dt}\mathcal{M}_2(t) + \rho\mathcal{M}_2(t) \leqslant C. \tag{65}$$

From (65) and (60), we obtain (55).

We completed the proof. □

To verify the asymptotic compactness of the process $U(t,\tau)$ corresponding to the problem in (15) and (16), we also need the following preliminary results.

For any $\xi_0 \in L^2_\mu(\mathbb{R}^+; V_\theta)$, the Cauchy problem (see [13,29,30])

$$\begin{cases} \xi^t_t = -\xi^t_s + w_t, \quad t \geqslant \tau, \\ \xi^\tau = \xi_0 \end{cases} \tag{66}$$

has a unique solution $\xi^t \in C([\tau,\infty); L^2_\mu(\mathbb{R}^+; V_\theta))$. Then, for (66), we have the explicit expression

$$\xi^t(x,s) = \begin{cases} w(x,t) - w(x,t-s), & \tau \leqslant s < t, \\ w(x,t), & s \geqslant t. \end{cases} \tag{67}$$

Let $\mathfrak{B}_t$ be the time-dependent absorbing set for the process $U(t,\tau)$ corresponding to the problem in (15) and (16) in $\mathcal{H}^\theta_t$ obtained from Theorem 4. Then,

Lemma 7. *For every given $\tau < T$, we set*

$$\mathcal{K}_T := \Pi W(T,\tau)\mathfrak{B}_\tau.$$

Assume that the forcing term $g \in V_{-\theta}$. If the assumptions (12)–(14), (18)–(20), (3) and (40)–(46) hold, then there exists a positive constant $N_4 = N_4(\|\mathcal{B}_0\|_{\mathcal{H}^\theta_\tau})$, such that

(i) *$\mathcal{K}_T$ is bounded in $L^2_\mu(\mathbb{R}^+; V^{\theta+\sigma}) \cap H^1_\mu(\mathbb{R}^+; V^\sigma)$;*
(ii) *$\sup_{\xi\in\mathcal{K}_T} \|\xi(s)\|^2_{V^\theta} \leqslant N_4$,*

where $\sigma = \min\{\frac{\theta}{4}, \frac{(n+2)\theta-2n}{n-2}, \frac{n}{2}-\theta\}$, $W(T,\tau)$ is a solution operator of (48) and $\Pi : V_{\theta+\sigma} \times V_\sigma \times L^2_\mu(\mathbb{R}^+; V_{\theta+\sigma}) \to L^2_\mu(\mathbb{R}^+; V_{\theta+\sigma})$ is a projection operator.

Proof. From (67), we conclude that

$$\xi^T_s(x,s) = \begin{cases} w_s(x,T-s), & \tau \leqslant s < T, \\ 0, & s \geqslant T. \end{cases} \tag{68}$$

Applying Lemma 6, we know (i) holds.

Using (68) once again, we can easily deduce that

$$\|\xi^T(x,s)\|_{V_\theta} = \begin{cases} \|w(x,T) - w(x,T-s)\|_{V_\theta} \leqslant \|w(x,T)\|_{V_\theta} + \|w(x,T-s)\|_{V_\theta}, & \tau < s < T, \\ \|w(x,T)\|_{V_\theta}, & s \geqslant T. \end{cases} \tag{69}$$

Clearly, it implies (ii) holds. The proof is complete. □

Therefore, applying Lemma 3, we conclude that $\mathcal{K}_T$ is relatively compact in $L^2_\mu(\mathbb{R}^+; V_\theta)$. Moreover, by the compact embedding $V_{\theta+\sigma} \times V_\sigma \hookrightarrow V_\theta \times L^2(\Omega)$, we obtain:

Lemma 8. *Let $\{W(T,\tau)\}_{\tau\leqslant T}$ be the process corresponding to the problem (48). If the assumptions of Lemma 7 hold, then for any $\tau < T$ and given $R > 0$, $W(T,\tau)\mathbb{B}_\tau(R)$ is relatively compact in $\mathcal{H}^\theta_T$.*

Theorem 5. *Let $U(t,\tau) : \mathcal{H}^\theta_\tau \to \mathcal{H}^\theta_t$ be the process generated by the problem in (15) and (16). Assume that $g \in V_{-\theta}$. If (12)–(14), (18)–(20), (3) and (40)–(46) hold, then the process $U(t,\tau)$ possesses an invariant time-dependent global attractor $\mathfrak{A} = \{A_t\}_{t\in\mathbb{R}}$ in $\mathcal{H}^\theta_t$.*

Proof. According to Lemmas 6 and 8, we consider the family $\mathfrak{K} = \{K_t^{\theta+\sigma}\}_{t\in\mathbb{R}}$, where

$$K_t^{\theta+\sigma} = \{z(t) \in \mathcal{H}_t^{\theta+\sigma} : \|z(t)\|_{\mathcal{H}_t^{\theta+\sigma}} \leqslant M\}.$$

By the compact embedding $\mathcal{H}_t^{\theta+\sigma} \hookrightarrow \mathcal{H}_t^{\theta}$ and Lemma 8, $K_t^{\theta+\sigma}$ is compact in $\mathcal{H}_t^{\theta}$. In addition, since the injection constant M is independent of t, the set $\mathfrak{K}$ is uniformly bounded.

It follows from Theorem 4, Lemmas 5 and 6 that $\mathfrak{K}$ is pullback attracting. In fact,

$$\delta_t(U(t,\tau)\mathbb{B}_\tau(R), K_t^{\theta+\sigma}) \leqslant Ce^{-\rho(t-\tau)}, \quad \forall \tau \leqslant t,$$

here, $\delta_t(\cdot,\cdot)$ denotes the Hausdorff semidistance of two subsets of $\mathcal{H}_t^{\theta}$. Hence, the process $U(t,\tau)$ is asymptotically compact, which implies the existence of the unique time-dependent global attractor $\mathfrak{A} = \{A_t\}_{t\in\mathbb{R}}$ of the process $U(t,\tau)$. Finally, the invariance of $\mathfrak{A}$ can be concluded by Lemma 4 (the continuity of the process $U(t,\tau)$ in $\mathcal{H}_t^{\theta}$).

We completed the proof. □

3.4. The Regularity of the Time-Dependent Attractor

Here, we prove that the time-dependent attractor $\mathfrak{A} = \{A_t\}_{t\in\mathbb{R}}$ is bounded in $\mathcal{H}_t^{2\theta}$, where the bound is independent of t.

For any given $\tau \in \mathbb{R}$ and $z(\tau) \in A_\tau$, we give a decomposition of the solution $U(t,\tau)z(\tau)$:

$$U(t,\tau)z(\tau) = z(t) = z_1(t) + z_2(t) = V_1(t,\tau)z_1(\tau) + W_1(t,\tau)z_2(\tau),$$

where

$$V_1(t,\tau)z_1(\tau) = (v(t), v_t(t), \zeta^t(s)), \quad W_1(t,\tau)z_2(\tau) = (w(t), w_t(t), \xi^t(s))$$

solve the equations, respectively,

$$\begin{cases} \varepsilon(t)v_{tt} + A^\theta v + \int_0^\infty \mu(s)A^\theta\zeta^t(s)\mathrm{d}s = 0, \\ \zeta_t^t = -\zeta_s^t + v_t, \\ v(x,t)|_{\partial\Omega} = 0, \ \ \zeta^t(x,t)|_{\partial\Omega} = 0, \\ v(x,\tau) = u_0(x,t), \ \ v_t(x,\tau) = u_1(x,t), \ x \in \Omega, \ t \leqslant \tau, \\ \zeta^\tau(x,s) = u_0(x,\tau) - u_0(x,\tau - s), \ (x,s) \in \Omega \times \mathbb{R}^+, \end{cases} \tag{70}$$

and

$$\begin{cases} \varepsilon(t)w_{tt} + A^\theta w + \int_0^\infty \mu(s)A^\theta\xi^t(s)\mathrm{d}s + f(u) = g(x), \\ \xi_t^t = -\xi_s^t + w_t, \\ w(x,t)|_{\partial\Omega=0}, \ \ \xi^t(x,t)|_{\partial\Omega} = 0, \\ w(x,\tau) = 0, \ \ w_t(x,\tau) = 0, \ \ \xi^\tau(x,s) = 0, \ x \in \Omega, s \in \mathbb{R}^+. \end{cases} \tag{71}$$

As a special case of Lemma 5, we can get

$$\|V_1(t,\tau)z(\tau)\|_{\mathcal{H}_t^\theta} \leqslant Ce^{-\rho_1(t-\tau)}, \quad \forall \tau \leqslant t - \bar{t}_1, \tag{72}$$

where $\bar{t}_1 = \max\{0, \frac{1}{\rho_1}\ln\frac{Q_2(R)}{2\epsilon}\}$.

Lemma 9. *Let $z_2(t)$ be the solution of (71) with initial data $z_2(\tau) \in A_\tau$ satisfying $\|z_2(\tau)\|_{\mathcal{H}_\tau^\theta} = 0$. If the assumptions (12)–(14), (18), (3) and (40)–(46) hold, then $\{A_t\}_{t\in\mathbb{R}}$ is bounded in $\mathcal{H}_t^{2\theta}$ and the bound is independent of t.*

Proof. For $\theta \in (\frac{2n}{n+2}, \frac{n}{2})$, $n \geqslant 3$, we set

$$0 < \sigma_0 = \sigma < \min\{\frac{\theta}{4}, \frac{(n+2)\theta - 2n}{n-2}, \frac{n}{2} - \theta\}, \text{ and } \sigma_0 < \sigma_1 = \min\{\frac{(n+2)(\theta+\sigma_0) - 2n}{n-2}, \theta\}.$$

Taking the inner product of (71) with $2(A^{\sigma_1} w_t + \rho A^{\sigma_1} w)$, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|w\|^2_{\theta+\sigma_1} + \varepsilon(t)\|w_t\|^2_{\sigma_1} + \|\xi^t\|^2_{\mu,\theta+\sigma_1} + 2\rho\varepsilon(t)\langle w_t, A^{\sigma_1} w\rangle + 2\langle f(u) - g, A^{\sigma_1} w\rangle) \\
&\quad + \frac{3}{2}\rho\|w\|^2_{\theta+\sigma_1} - (\varepsilon'(t) + 2\rho\varepsilon(t))\|w_t\|^2_{\sigma_1} + (\delta - 2\rho k_0)\|\xi^t\|^2_{\mu,\theta+\sigma_1} \\
&\quad - 2\rho\varepsilon'(t)\langle w_t, A^{\sigma_1} w\rangle + 2\rho\langle f(u) - g, A^{\sigma_1} w\rangle \\
&\quad \leqslant 2\langle f'(u)u_t, A^{\sigma_1} w\rangle.
\end{aligned} \tag{73}$$

Set

$$\begin{aligned}
\mathcal{M}_3(t) = &\|w\|^2_{\theta+\sigma_1} + \varepsilon(t)\|w_t\|^2_{\sigma_1} + \|\xi^t\|^2_{\mu,\theta+\sigma_1} + 2\rho\varepsilon(t)\langle w_t, A^{\sigma_1} w\rangle \\
&+ 2\langle f(u) - g, A^{\sigma_1} w\rangle + C.
\end{aligned} \tag{74}$$

Similar to (60), for ρ small enough, we have

$$\frac{1}{2}\|W_1(t,\tau)z(\tau)\|^2_{\mathcal{H}_t^{\theta+\sigma_1}} \leqslant \mathcal{M}_3(t) \leqslant 2\|W_1(t,\tau)z(\tau)\|^2_{\mathcal{H}_t^{\theta+\sigma_1}} + C, \tag{75}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}_3(t) + \rho\mathcal{M}_3(t) + \frac{\rho}{4}\|w\|^2_{\theta+\sigma_1} \leqslant 2\langle f'(u)u_t, A^{\sigma_1} w\rangle + \rho C. \tag{76}$$

Due to the invariance of $\mathfrak{A}$, we have

$$\|U(t,\tau)z(\tau)\|_{\mathcal{H}_t^{\theta+\sigma_0}} \leqslant C,$$

where C is a generic constant depending on the size of A_t in $\mathcal{H}_t^{\theta+\sigma_0}$.

Using the embedding (10), we can deduce that

$$\begin{aligned}
&2\langle f'(u)u_t, A^{\sigma_1} w\rangle \\
&\leqslant C\Big(\int_\Omega (1 + |u|^{\frac{4}{n-2}})^{\frac{n}{\theta+\sigma_0-\sigma_1}}\mathrm{d}x\Big)^{\frac{\theta+\sigma_0-\sigma_1}{n}} \cdot \Big(\int_\Omega |u_t|^{\frac{2n}{n-2\sigma_0}}\mathrm{d}x\Big)^{\frac{n-2\sigma_0}{2n}} \\
&\quad \cdot \Big(\int_\Omega |A^{\sigma_1} w|^{\frac{2n}{n-2(\theta-\sigma_1)}}\mathrm{d}x\Big)^{\frac{n-2(\theta-\sigma_1)}{2n}} \\
&\leqslant C(1 + \|u\|^{\frac{4}{n-2}}_{\theta+\sigma_0})\|u_t\|_{\sigma_0}\|w\|_{\theta+\sigma_1} \\
&\leqslant \frac{\rho}{4}\|w\|^2_{\theta+\sigma_1} + C,
\end{aligned} \tag{77}$$

where $\frac{4}{n-2} \cdot \frac{n}{\theta+\sigma_0-\sigma_1} \leqslant \frac{2n}{n-2(\theta+\sigma_0)}$.

Therefore, we conclude that

$$\frac{d}{dt}\mathcal{M}_3(t) + \rho\mathcal{M}_3(t) \leqslant C.$$

Applying the Gronwall lemma and combining with (75), we obtain that there exists $N^{*_1} = N^{*_1}(\mathfrak{A}) > 0$, such that

$$\sup_{\tau \leqslant t-t^*} \|z_2(t)\|_{\mathcal{H}_t^{\theta+\sigma_1}} = \sup_{\tau \leqslant t-t^*} \|W_1(t,\tau)z_2(\tau)\|_{\mathcal{H}_t^{\theta+\sigma_1}} \leqslant N^{*_1}, \tag{78}$$

where $t^* = \max\{0, \frac{1}{\rho_1}\ln\frac{Q_2(R)}{2\epsilon}, \frac{1}{\rho}\ln\frac{2Q_1(R)}{R_0^2}\}$. Namely, $\|W_1(t,\tau)z_2(\tau)\|_{\mathcal{H}_t^{\theta+\sigma_1}}$ is uniformly bounded.

We denote

$$K_t^{\theta+\sigma_1} = \{z(t) \in \mathcal{H}_t^{\theta+\sigma_1} : \|z(t)\|_{\mathcal{H}_t^{\theta+\sigma_1}} \leqslant N_4\}.$$

It follows from (72) and Lemma 9 that

$$\lim_{\tau\to-\infty} \delta_t(U(t,\tau)A_\tau, K_t^{\theta+\sigma_1}) = 0, \forall\, t \in \mathbb{R}.$$

The invariance of $\mathfrak{A}$ implies that

$$\delta_t(A_t, K_t^{\theta+\sigma_1}) = 0.$$

Therefore, $A_t \subset \overline{K_t^{\theta+\sigma_1}} = K_t^{\theta+\sigma_1}$. We can deduce that A_t is bounded in $\mathcal{H}_t^{\theta+\sigma_1}$ (with a bound independent of $t \in \mathbb{R}$).

For $\theta \in (\frac{2n}{n+2}, \frac{n}{2})$, $n \geqslant 3$, we set

$$\sigma_1 < \sigma_2 = \frac{(n+2)(\theta+\sigma_1) - 2n}{n-2}.$$

Repeating the above process, we obtain that A_t is bounded in $\mathcal{H}_t^{\theta+\sigma_2}$ (with a bound independent of $t \in \mathbb{R}$).

We set $\sigma_{\min} = \min\{\sigma_i\}$, $i = 1, 2, \cdots$. Repeating the above process at most $[\frac{\theta}{\sigma_{\min}} + 1]$ times, we can finally obtain that A_t is bounded in $\mathcal{H}_t^{2\theta}$ (with a bound independent of $t \in \mathbb{R}$). □

3.5. The Asymptotic Regularity of the Solution

By using bootstrap methods, the following results can be obtained.

Lemma 10. *Assume that the forcing term $g \in V_{-\theta}$. Let the assumptions (12)–(14), (18)–(20), (3) and (40)–(46) hold. For any bounded (in $\mathcal{H}_\tau^{\theta+\sigma}$) set $B_\tau^{\theta+\sigma}$, there is a positive constant $N_{\|B_t^{\theta+\sigma}\|_{\mathcal{H}_t^{\theta+\sigma}}}$ that depends on $\|B_t^{\theta+\sigma}\|_{\mathcal{H}_t^{\theta+\sigma}}$, such that for any $\tau \in \mathbb{R}$ and $t_2 \leqslant t^* \leqslant t$,*

$$\|U(t,\tau)z_\tau\|^2_{\mathcal{H}_t^{\theta+\sigma}} \leqslant N_{\|B_{\theta+\sigma}\|_{\mathcal{H}_t^{\theta+\sigma}}},\ \text{as } \tau \leqslant t - t_2 \text{ and } z_\tau \in B_\tau^{\theta+\sigma},$$

where $\sigma = \min\{\frac{\theta}{4}, \frac{(n+2)\theta-2n}{n-2}, \frac{n}{2} - \theta\}$.

Lemma 11. *Let $\sigma < \iota = \min\{\theta, \frac{(n+2)(\theta+\sigma)-2n}{n-2}\}$. Under the assumptions of Lemma 10, for any bounded (in $\mathcal{H}_\tau^{\theta+\iota}$) set $B_\tau^{\theta+\iota}$, there is a positive constant $N_{\|B_t^{\theta+\iota}\|_{\mathcal{H}_t^{\theta+\iota}}}$ that depends on $\|B_t^{\theta+\iota}\|_{\mathcal{H}_t^{\theta+\iota}}$, such that for any $\tau \in \mathbb{R}$ and $t_3 \leqslant t_2 \leqslant t$,*

$$\|U(t,\tau)z_\tau\|^2_{\mathcal{H}_t^{\theta+\iota}} \leqslant N_{\|B_t^{\theta+\iota}\|_{\mathcal{H}_t^{\theta+\iota}}},\ \text{as } \tau \leqslant t - t_3 \text{ and } z_\tau \in B_\tau^{\theta+\iota}.$$

Lemma 12. *Assume that $\sigma < \iota = \min\{\theta, \frac{(n+2)(\theta+\sigma)-2n}{n-2}\}$. Under the assumptions of Lemma 10, for any bounded (in $\mathcal{H}_\tau^{\theta+\iota}$) set $B_\tau^{\theta+\iota}$, there is a positive constant $J_{\|B_t^{\theta+\iota}\|_{\mathcal{H}_t^{\theta+\iota}}}$ that depends on $\|B_t^{\theta+\iota}\|_{\mathcal{H}_t^{\theta+\iota}}$, such that for the solution $z_2(t)$ of Equation (48), for any $\tau \in \mathbb{R}$ and $t_4 \leqslant t_3 \leqslant t$,*

$$\|W(t,\tau)z_\tau\|^2_{\mathcal{H}_t^{\theta+\kappa_0}} \leqslant J_{\|B_t^{\theta+\iota}\|_{\mathcal{H}_t^{\theta+\iota}}},\ \text{as } \tau \leqslant t - t_4 \text{ and } z_\tau \in B_\tau^{\theta+\iota},$$

where $\iota < \kappa_0 = \min\{\theta, \frac{(n+2)(\theta+\iota)-2n}{n-2}\}$.

Theorem 6 (Asymptotic regularity of solution)**.** *Let* Ω *be a bounded domain in* $\mathbb{R}^n (n \geqslant 3)$ *with smooth boundary* $\partial\Omega$*. Under the assumptions of Lemma 11, then there exist a bounded (in* $\mathcal{H}_t^{2\theta}$*) set* $\mathcal{B}_t \subset \mathcal{H}_t^{2\theta}$*, a positive constant* ν *and a monotonically function* $Q(\cdot)$*, such that for any bounded (in* $\mathcal{H}_\tau^{\theta}$*) set* $B_\tau \subset \mathcal{H}_\tau^{\theta}$*, any* $\tau \in \mathbb{R}$*, the following estimate holds:*

$$\delta_t(U(t,\tau)B_\tau, \mathcal{B}_t) \leqslant Q(\|B_\tau\|_{\mathcal{H}_\tau^\theta})\mathrm{e}^{-\nu(t-\tau)}, \tag{79}$$

where δ_t *is the Hausdorff semidistance in* $\mathcal{H}_t^\theta$ *and* ν *is independent of* B_τ*,* g *and* τ*.*

Proof. Let $\mathfrak{B}_t = \{\mathbb{B}_t(R_0)\}_{t\in\mathbb{R}}$ be the time-dependent absorbing set in $\mathcal{H}_t^\theta$ obtained from Theorem 4. From Lemmas 5 and 6, we can deduce that there exists a bounded (in $\mathcal{H}_t^{\theta+\sigma}$) subset $A_t^{\theta+\sigma} \subset \mathcal{H}_t^{\theta+\sigma}$, such that

$$\begin{aligned}\delta_t(U(t,\tau)\mathbb{B}_\tau(R_0), A_t^{\theta+\sigma}) &\leqslant \delta_t(V(t,\tau)\mathbb{B}_\tau(R_0), A_t^{\theta+\sigma}) \\ &\leqslant Q_2(R_0)\mathrm{e}^{-\rho_1(t-\tau)}.\end{aligned} \tag{80}$$

In regard to $A_\tau^{\theta+\sigma}$, from Lemmas 5 and 12, it is easy to know that there exists a bounded set $A_t^{\theta+\kappa_0}$ in $\mathcal{H}_t^{\theta+\kappa_0}$, such that

$$\begin{aligned}\delta_t(U(t,\tau)A_\tau^{\theta+\sigma}, A_t^{\theta+\kappa_0}) &\leqslant \delta_t(V(t,\tau)\mathcal{A}_t^{\theta+\sigma}, A_t^{\theta+\kappa_0}) \\ &\leqslant Q_2(\|A_\tau^{\theta+\sigma}\|_{\mathcal{H}_\tau^\theta})\mathrm{e}^{-\rho_1(t-\tau)},\end{aligned} \tag{81}$$

where ρ_1 is positive and only depends on $\|A_{\theta+\sigma}\|_{\mathcal{H}_\tau^\theta}$, and $\kappa_0 = \min\{\theta, \frac{(n+2)(\theta+\iota)-2n}{n-2}\}$.

From (38), (80), (81) and Lemma 2, we obtain

$$\delta_t(U(t,\tau)\mathbb{B}_\tau(R_0), A_t^{\theta+\kappa_0}) \leqslant CQ_2(R_0)\mathrm{e}^{-\rho_2(t-\tau)}, \tag{82}$$

where C and ρ_2 are both positive constants.

Fix $\kappa_0 = \min\{\theta, \frac{(n+2)(\theta+\iota)-2n}{n-2}\}$ and $\sigma = \min\{\frac{\theta}{4}, \frac{(n+2)\theta-2n}{n-2}, \frac{n}{2}-\theta\}$. By a finite number of steps (no more than $[\frac{\theta}{\kappa_0}+1]$ steps), we can deduce that there exists a bounded (in $\mathcal{H}_t^{2\theta}$) set $\mathcal{B}_t \subset \mathcal{H}_t^{2\theta}$, such that

$$\delta_t(U(t,\tau)\mathbb{B}_\tau(R_0), \mathcal{B}_t) \leqslant Q(R_0)\mathrm{e}^{-\nu(t-\tau)}, \tag{83}$$

where ν is dependent of R_0.

For any bounded (in $\mathcal{H}_\tau^\theta$) set B_τ, by Theorem 4, there exists a t_0 such that

$$U(t,\tau)B_\tau \subset \mathbb{B}_t(R_0), \text{ as } \tau \leqslant t - t_0. \tag{84}$$

Therefore,

$$\delta_t(U(t,\tau)B_\tau, \mathbb{B}_t(R_0)) \leqslant N_3\mathrm{e}^{\nu t_0}\mathrm{e}^{-\nu(t-\tau)}, \tag{85}$$

where $N_3 = \sup\{\|U(t,\tau)B_\tau\|_{\mathcal{H}_t^\theta}, \tau \leqslant t - t_0\} < \infty$.

By Lemma 2 once more, we can deduce that (79). The proof is complete. □

4. Conclusions

For the undamped second-order abstract evolution equation with fading memory, when the nonlinear term satisfies the critical exponential growth, the existence and asymptotic regularity of the time-dependent attractor, as well as the asymptotic regularity of the solutions, can be obtained by using the process theory, asymptotic prior technique and decomposition technique. This result improves and generalizes some known results (see [9,23,30]).

We will continue to study the asymptotic behavior of the solutions of the equation in the strong topological space $V_{2\theta} \times V_\theta \times L^2_\mu(\mathbb{R}^+; V_{2\theta})$, and it is expected that corresponding research results will be obtained.

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