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Article

Analysis of a Stochastic Inventory Model on Random Environment with Two Classes of Suppliers and Impulse Customers

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Abstract: This paper explores the random environment with two classes of suppliers and impulse customers. The system's greatest inventory size is S, and it has an infinitely large orbit. In this case, there are two categories of suppliers: temporary suppliers and regular suppliers. Whenever the inventory approaches r, we place on order Q_1 $(=S - r)$ unit items to a temporary supplier. Similarly, when the inventory level drops to s $(<Q_1 < r)$, we place an order for Q_2 $(=S - s > s + 1)$ units of items to our regular supplier. Two types of suppliers' lead times are considered to be exponentially distributed. Here, the customers who arrive from different states of the random environment (RE) are followed by the Markovian arrival process. If there is no inventory in the system when the customer arrives, they are automatically assigned to an orbit. The model was examined in steady state by using the matrix-analytic approach. Finally, the numerical examples for our structural model are discussed.

Keywords: random environment; two classes of suppliers; impulse customers; Markovian arrival process

MSC: 60K25; 90B05

1. Introduction

Inventory management faces many replenishment difficulties. If the supplier's activities are affected during replenishment, customers may be unable to get vital supplies. As a result, we use two suppliers, referred to in this paper as "regular suppliers" and "temporary suppliers". For instance, let's consider textile shops. The system always gives the order to the regular supplier, but the temporary supplier supplies trending dresses, so the system decides to make a trial for a temporary supplier at the same time to ensure the system doesn't lose the regular supplier. Good connections with servers and customers are among these objectives, and customers should avoid being sent back due to inventory shortage and helping to minimise the total cost. Following that, all arriving customers have the option of purchasing or not purchasing their items after the detailed inquiries in N different states of random environment (RE) in this paper.

Soujanyaa and Vijaya Laxmi [1] investigated perishable inventory systems with dual supply chains and negative customers. They considered the lead time and perishable rate to be exponentially distributed, the arrival process followed by Poisson, and found a limiting time distribution. Finally, they demonstrate numerical examples for cost rate function and system performance parameters. Yang and Tseng [2] considered a three-echelon inventory model with a backorder and permissible delay in payments under a controllable lead time. The queueing-inventory system literature has more ordering policies such as [3–6].

The system's behavior is determined by the current state of the Markovian random process with a finite state space. The technique of this operation in a fixed state is called a "random environment". The system makes an effort to avoid crowds while also being mindful of not losing customers. The system also has several modalities for receiving the demand. This effort helps to satisfy all the customers without any customer accumulation. Kim and Dudin [7] considered a multi-server and impatient customer in the queueing system with a random environment. The service time is a phase-type distribution. They derived ergodicity conditions for this system with an infinite buffer. Some numerical results are presented. Kim et al. [8] investigated a multi-server queueing system and two types of customers with infinite buffers. The system is operating in a random environment, and the arrival stream is marked as a Markovian arrival process. Non-preemptive priority is given to type 1 customers over type 2 customers. The service time for two types of customers has a phase-type distribution with different parameters. The correlation in the arrival process is depicted numerically in the section.

In this article, we consider the arrival process as MAP under the fixed state of a random environment. Whenever the states of a random environment are changed, the parameters of the arrival flow and retrying flow are changed as well. Here, we give some reviews about the random environment (see [9–14]). In the 1970s, Neuts introduced the Versatile Markovian Point Process (VMPP) as the Markovian arrival process. The VMPP had extensive notations when it was first developed; however, it was considerably simplified by Lucantoni [15], and this process has been known as the Markovian arrival process ever since. The Markovian arrival process was greatly vetted in the literature [16–21]. See Krishnamurthy et al. [22] for details on the batch Markovian arrival process in the queueing-inventory system.

In this paper, all the arriving customers go into orbit at inventory level zero, and the arriving customers from orbit may or may not buy the product. For example, customers wait for the trending dresses at a nearby textile shop. At the textile shop, after coming across the trending dresses, the customer went to the shop, but the dress colour or size did not fit the customer. At that point, the customer didn't buy the product and left the system. The responses to the last poll inspired our investigation, and there is no research into random environments with two supply chains to the best of our knowledge.

1.1. Research Gap

The authors above work with RE in the queuing system. This article examines RE in the inventory. The authors above handle the two-echelon chain, and both orders are obtained. In this article, we place an order with a temporary supplier, in case we have not received the order from the temporary supplier when the inventory level has reached some reordering point, we cancel that order and place the order with our regular supplier.

1.2. The Perspective of This Work

The model representation of the stochastic inventory model with two classes of suppliers and impulse customers on RE is portrayed in Section 2. In Section 3 the model analysis is portayed. The joint probability distribution of the number of customers in the orbit of infinite size, and the inventory level is evaluated in Section 4. We measure some important system peculiarities and construct the cost function in Section 5. Some numerical examples are provided in Section 6 and, finally, we describe a conclusion in Section 7.

2. Mathematical Framework of the Model

In this section, we discuss the random environment in the stochastic inventory system with two classes of suppliers named "temporary suppliers" and "regular suppliers". In this system, an infinite-sized orbit is added, and here there is no waiting space attached to the system. The system has a maximum S units of items. Whenever the inventory level approaches r, we place an order with the temporary supplier for Q_1 items, where $Q_1 = S - r$. Similarly, whenever the inventory level reaches $s(s < Q_1 < r)$, we will order $Q_2 = S - s(> s+1)$ items to the regular supplier and immediately cancel the order for Q_1 items from the temporary supplier. Both lead times of temporary and regular suppliers are exponentially distributed with β_1 and β_2 respectively.

The customers arrive according to a Markovian arrival process (MAP) from N different states of RE. Customers who require a single item have the option to choose any arrival mode (state) of the RE. The behaviour of the model depends on the state of the RE. The RE is used on the stochastic process $J_2(t)$, which is an irreducible continuous time Markov chain with the state space $\{1, 2, \ldots, N\}$ and the infinitesimal generator H. The stationary row vector η_1 of the RE is to be obtained by using $\eta_1 H = \mathbf{0}$, $\eta_1 \mathbf{e} = 1$. The underlying Markov chain $J_4(t)$ of the MAP has a generator $D^{(j_2)}$, which is a square matrix of dimension m with $D^{(j_2)} = D_0^{(j_2)} + D_1^{(j_2)}$ in a fixed state $j_2(= 1, 2, \ldots, N)$. In this case, $D_0^{(j_2)}$ denotes no arrival matrix of size m and $D_1^{(j_2)}$ denotes an arrival matrix of size m under fixed state j_2. The arrival process of customer representation is $(D_0^{(j_2)}, D_1^{(j_2)})$ under fixed state j_2. The average arrival rate $\lambda^{(j_2)}$ of a customer is defined by $\lambda^{(j_2)} = \eta_2^{(j_2)} D_1^{(j_2)} \mathbf{e}$, where stationary row vector $\eta_2^{(j_2)}$ of size $1 \times m$ is to be obtained by using $\eta_2^{(j_2)} D^{(j_2)} = \mathbf{0}$ and $\eta_2^{(j_2)} \mathbf{e} = 1$ under fixed state j_2.

Not all incoming customers buy the product, those customers are referred to as impulse customers. Assume that customers buy the product with probability $p^{(j_2)}$ and the complementary probability $q^{(j_2)}$ under fixed state j_2. The service process of the system is assumed to be instantaneous when the inventory level is positive. When the arriving customers find the stock stage is empty, they should enter into orbit. Even though the mode of approach to entering the system is different for each customer, once the customer enters the orbit, they will be considered retrial customers. Retrial customers have the option of choosing another arriving mode of RE during the retry without considering the previous choice of different states of RE. The retrial customers will get into service only if the stock stage is positive. The time between two successive retrials is exponentially distributed with a transition rate $\theta^{(j_2)}$.

3. Model Analysis

In this sector, we construct the transition rate matrix on the stochastic inventory system. The Markov process of the form $\{(J_1(t), J_2(t), J_3(t), J_4(t)), t \geq 0\}$ with state space

$$E = \{(j_1, j_2, j_3, j_4) : j_1 \geq 0;\ 1 \leq j_2 \leq N;\ 0 \leq j_3 \leq S;\ 1 \leq j_4 \leq m\},$$

where

$J_1(t)$: The number of customers in the orbit of infinite size waiting place at time t.
$J_2(t)$: The state of random environment at time t.
$J_3(t)$: The number of items in the inventory at time t.
$J_4(t)$: Phase of the arrival process at time t.

The process's infinitesimal generator P is generated by

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & \cdots \end{array} \\ \begin{pmatrix} A_{00} & A_{01} & \mathbf{0} & \mathbf{0} & \cdots \\ A_{10} & A_{11} & A_{01} & \mathbf{0} & \cdots \\ \mathbf{0} & A_{10} & A_{11} & A_{01} & \cdots \\ \mathbf{0} & \mathbf{0} & A_{10} & A_{11} & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix} \end{array}$$

$$A_{01} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ N \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & \cdots & N \end{array} \\ \begin{pmatrix} A^{(1)} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & A^{(2)} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A^{(3)} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & A^{(N)} \end{pmatrix} \end{array}.$$

Here, $j_2 = 1, 2, \ldots, N$:

$$A^{(j_2)} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ S \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & \cdots & S \end{array} \\ \begin{pmatrix} D_1^{(j_2)} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \end{array},$$

$$A_{10} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ N \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & \cdots & N \end{array} \\ \begin{pmatrix} B^{(1)} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & B^{(2)} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & B^{(3)} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & B^{(N)} \end{pmatrix} \end{array}.$$

Here, $j_2 = 1, 2, \ldots, N$:

$$B^{(j_2)} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ S \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & \cdots & S-1 & S \end{array} \\ \begin{pmatrix} q^{(j_2)}\theta^{(j_2)} I_m & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ p^{(j_2)}\theta^{(j_2)} I_m & q^{(j_2)}\theta^{(j_2)} I_m & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & p^{(j_2)}\theta^{(j_2)} I_m & q^{(j_2)}\theta^{(j_2)} I_m & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & p^{(j_2)}\theta^{(j_2)} I_m & \ddots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & p^{(j_2)}\theta^{(j_2)} I_m & q^{(j_2)}\theta^{(j_2)} I_m \end{pmatrix} \end{array},$$

$$A_{00} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ N \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & \cdots & N \end{array} \\ \begin{pmatrix} C_1^{(1)} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & C_1^{(2)} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_1^{(3)} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & C_1^{(N)} \end{pmatrix} \end{array} + H \otimes I_{k_2},$$

$$A_{11} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ N \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & \dots & N \end{array} \\ \begin{pmatrix} C_2^{(1)} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & C_2^{(2)} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_2^{(3)} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & C_2^{(N)} \end{pmatrix} \end{array} + H \otimes I_{k_2}.$$

For $k = 1$ and 2 and $j_2 = 1, 2, \dots, N$:

$$C_k^{(j_2)} = \begin{array}{c|ccccccccccccccccc} & 0 & 1 & \dots & s & s+1 & \dots & Q_1 & Q_1+1 & \dots & r & r+1 & \dots & Q_2 & Q_2+1 & \dots & S-1 & S \\ \hline 0 & d_0^k & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \beta_2 I_m & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ 1 & b_1 & d_1^k & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \beta_2 I_m & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ s & \mathbf{0} & \mathbf{0} & \dots & d_1^k & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \beta_2 I_m \\ s+1 & \mathbf{0} & \mathbf{0} & \dots & b_1 & d_2^k & \dots & \beta_1 I_m & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ s+2 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \beta_1 I_m & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ Q_1 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & d_2^k & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ Q_1+1 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & b_1 & d_2^k & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ r & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & d_2^k & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \beta_1 I_m \\ r+1 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & b_1 & d_3^k & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ Q_2 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & d_3^k & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ Q_2+1 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & b_1 & d_3^k & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ S & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & b_1 & d_3^k \end{array},$$

where $b_1 = p^{(j_2)} D_1^{(j_2)}$ and $d_i^k = D_0^{(j_2)} + \bar{\delta}_{i0} q^{(j_2)} D_1^{(j_2)} - (\delta_{i0}\beta_2 + \delta_{i1}\beta_2 + \delta_{i2}\beta_1 + \delta_{k2}\bar{\delta}_{i0}\theta^{(j_2)} + \delta_{k2}\delta_{i0} q^{(j_2)}\theta^{(j_2)}) I_m$, $i = 0, 1, 2$ and 3.

It may be noted that the A_{01}, A_{10}, A_{00}, and A_{11} are all square matrices of dimension k_1 and the matrices $A^{(j_2)},\ B^{(j_2)},\ C_1^{(j_2)}$ and, $C_2^{(j_2)}$ are dimension k_2, where $j_2 = 1, 2, \dots, N$.

4. Joint Probability Distribution under Steady State

We give the condition for stability through a theorem on the stochastic inventory system. Let us consider $G = A_{10} + A_{11} + A_{01}$. It is easy to visualize that G is given by

$$G = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ N \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & \dots & N \end{array} \\ \begin{pmatrix} F^{(1)} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & F^{(2)} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & F^{(3)} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & F^{(N)} \end{pmatrix} \end{array} + H \otimes I_{k_2}.$$

Here $j_2 = 1, 2, \dots, N$:

$$
F^{(j_2)} = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ s \\ s+1 \\ s+2 \\ \vdots \\ Q_1 \\ Q_1+1 \\ \vdots \\ r \\ r+1 \\ \vdots \\ Q_2 \\ Q_2+1 \\ \vdots \\ S \end{array}
\begin{array}{c}
\begin{array}{ccccccccccccccccc} 0 & 1 & \cdots & s & s+1 & \cdots & Q_1 & Q_1+1 & \cdots & r & r+1 & \cdots & Q_2 & Q_2+1 & \cdots & S-1 & S \end{array} \\
\left(\begin{array}{ccccccccccccccccc}
f_0 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \beta_2 I_m & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\
b_2 & f_1 & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \beta_2 I_m & \cdots & \mathbf{0} & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & f_1 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \beta_2 I_m \\
\mathbf{0} & \mathbf{0} & \cdots & b_2 & f_2 & \cdots & \beta_1 I_m & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \beta_1 I_m & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & f_2 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & b_2 & f_2 & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & f_2 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \beta_1 I_m \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & b_2 & f_3 & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & f_3 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & b_2 & f_3 & \cdots & \mathbf{0} & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & b_2 & f_3
\end{array} \right)
\end{array},
$$

where $b_2 = b_1 + p^{(j_2)}\theta^{(j_2)} I_m$ and $f_i = d_i^2 + \delta_{i0} D_1^{(j_2)} + q^{(j_2)}\theta^{(j_2)} I_m$, $i = 0, 1, 2$ and 3.

Here, $\mathfrak{Y}$ denotes the steady-state probability vector of G.

where

$$\mathfrak{Y} = (\mathfrak{Y}_1, \mathfrak{Y}_2, \ldots, \mathfrak{Y}_N),$$

$$\mathfrak{Y}_{j_2} = (\mathfrak{Y}_{(j_2,0)}, \mathfrak{Y}_{(j_2,1)}, \ldots, \mathfrak{Y}_{(j_2,S)}),\ 1 \leq j_2 \leq N.$$

The below system of equations are derived from $\mathfrak{Y}G = \mathbf{0}$:

$$\sum_{j_2=1}^{N} \mathfrak{Y}_{j_2}(\delta_{j_2 1} F^{(1)} + H_{j_2 1}) = 0$$

$$\sum_{j_2=1}^{N} \mathfrak{Y}_{j_2}(\delta_{j_2 2} F^{(2)} + H_{j_2 2}) = 0$$

$$\vdots$$

$$\sum_{j_2=1}^{N} \mathfrak{Y}_{j_2}(\delta_{j_2 N} F^{(N)} + H_{j_2 N}) = 0.$$

The steady-state probability vector $\mathfrak{Y}_i$ is obtained from the above system of equations and the normalizing condition ($\mathfrak{Y}\mathbf{e} = 1$).

4.1. Stability Condition

We give the condition for stability through theorem on the stochastic inventory system.

Theorem 1. *The stochastic inventory system under study is stable if and only if*

$$\sum_{j_2=1}^{N} \mathfrak{Y}_{j_2} A^{(j_2)} \mathbf{e} < \sum_{j_2=1}^{N} \mathfrak{Y}_{j_2} B^{(j_2)} \mathbf{e}. \quad (1)$$

Proof. From the standard results of Neuts [23] on the positive recurrence of P we have

$$\mathfrak{Y} A_{01} \mathbf{e} < \mathfrak{Y} A_{10} \mathbf{e}, \quad (2)$$

and by applying the structure of the matrices A_{10} and A_{01} and $\mathfrak{Y}$ the declared results follow. □

It can be seen from the structure of the rate matrix P and from the Theorem 1, that the Markov process $\{(J_1(t), J_2(t), J_3(t), J_4(t)), t \geq 1\}$ with the state space E is regular.

4.2. Steady-State Probability Vector

In this section, we calculate steady-state probability. Let Θ be the steady-state probability vector of the matrix P and it is obtained from $\Theta P = \mathbf{0}$, $\Theta \mathbf{e} = \mathbf{1}$, where

$$\Theta = (\Theta_0, \Theta_1, \Theta_2, \ldots),$$

where

$$\begin{aligned} \Theta_{j_1} &= (\Theta_{(j_1,1)}, \Theta_{(j_1,2)}, \Theta_{(j_1,3)}, \cdots, \Theta_{(j_1,N)}), j_1 = 0,1,2,\cdots, \\ \Theta_{(j_1,j_2)} &= (\Theta_{(j_1,j_2,0)}, \Theta_{(j_1,j_2,1)}, \Theta_{(j_1,j_2,2)}, \cdots, \Theta_{(j_1,j_2,S)}), j_1 = 0,1,2,\cdots, j_2 = 1,2,\cdots,N, \\ \Theta_{(j_1,j_2,j_3)} &= (\Theta_{(j_1,j_2,j_3,1)}, \Theta_{(j_1,j_2,j_3,2)}, \cdots, \Theta_{(j_1,j_2,j_3,m)}), j_1 = 0,1,2,\cdots, j_2 = 1,2,\cdots,N, \\ & \quad j_3 = 0,1,2,\cdots,S. \end{aligned}$$

The relation $\Theta P = \mathbf{0}$ leads to the following system of equations:

$$\Theta_0 A_{00} + \Theta_1 A_{10} = \mathbf{0} \tag{3}$$

$$\Theta_{j_1-1} A_{01} + \Theta_{j_1} A_{11} + \Theta_{j_1+1} A_{10} = \mathbf{0}, \ (j_1 \geq 1). \tag{4}$$

The Markov process $\{(J_1(t), J_2(t), J_3(t), J_4(t)), t \geq 0\}$ on the state space E and the limiting distribution

$$\Theta_{(j_1,j_2,j_3,j_4)} = \lim_{t\to\infty} Pr[J_1(t) = j_1, J_2(t) = j_2, J_3(t) = j_3, J_4(t) = j_4 | J_1(0), J_2(0), J_3(0), J_4(0)],$$

where $\Theta_{(j_1,j_2,j_3,j_4)}$ is the steady-state probability for the state (j_1, j_2, j_3, j_4), exists and is independent of the initial state.

Theorem 2. *When the stability condition $\{(J_1(t), J_2(t), J_3(t), J_4(t)), t \geq 0\}$ holds well, the steady-state probability vector Θ is given by*

$$\Theta_i = \Theta_0 \mathcal{R}^i, i \geq 1, \tag{5}$$

where the rate matrix $\mathcal{R}$, it is defined by

$$\mathcal{R} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ N \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & \ldots & N \end{array} \\ \begin{pmatrix} \mathcal{R}_{11} & \mathcal{R}_{12} & \ldots & \mathcal{R}_{1N} \\ \mathcal{R}_{21} & \mathcal{R}_{22} & \ldots & \mathcal{R}_{2N} \\ \mathcal{R}_{31} & \mathcal{R}_{32} & \ldots & \mathcal{R}_{3N} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{R}_{N1} & \mathcal{R}_{N2} & \ldots & \mathcal{R}_{NN} \end{pmatrix} \end{array}$$

and satisfies the matrix quadratic equation

$$\mathcal{R}^2 A_{10} + \mathcal{R} A_{11} + A_{01} = \mathbf{0}. \tag{6}$$

Proof. The proof follows from the well-known result on matrix geometric property given by Neuts [23]. □

Theorem 3. *The stationary probability vector Θ_0 is the unique solution of the system*

$$\Theta_0 (A_{00} + \mathcal{R} A_{10}) = \mathbf{0}, \tag{7}$$

and subject to the normalising condition

$$\Theta_0(I-\mathcal{R})^{-1}\mathbf{e}=1. \tag{8}$$

Proof. The proof is based on Neuts' [23] well-known matrix geometric property. And see [24] more details about the property. □

Once $\mathcal{R}$ is found, the boundary probability vector Θ_0 is computed by using the Equations (7) and (8), and the probability vectors $\Theta_i, i \geq 1$ can be obtained from Equation (5).

5. A Few Significant System Peculiarities

In the segment, we examine a few significant peculiarities measures.

1. Mean inventory level, η_I is given by
$$\eta_I=\sum_{j_1=0}^{\infty}\sum_{j_2=1}^{N}\sum_{j_3=1}^{S} j_3\Theta_{(j_1,j_2,j_3)}\mathbf{e}.$$
2. Mean reorder for temporary supplier, η_{R1} is given by
$$\eta_{R1}=\sum_{j_1=0}^{\infty}\sum_{j_2=1}^{N} p^{(j_2)}D_1^{(j_2)}[\Theta_{(j_1,j_2,r+1)}]\mathbf{e}$$
$$+\sum_{j_1=1}^{\infty}\sum_{j_2=1}^{N} p^{(j_2)}\theta^{(j_2)}[\Theta_{(j_1,j_2,r+1)}]\mathbf{e}.$$
3. Mean reorder for regular supplier, η_{R2} is given by
$$\eta_{R2}=\sum_{j_1=0}^{\infty}\sum_{j_2=1}^{N} p^{(j_2)}D_1^{(j_2)}[\Theta_{(j_1,j_2,s+1)}]\mathbf{e}$$
$$+\sum_{j_1=1}^{\infty}\sum_{j_2=1}^{N} p^{(j_2)}\theta^{(j_2)}[\Theta_{(j_1,j_2,s+1)}]\mathbf{e}.$$
4. Mean number of customers in the orbit
$\eta_{OC}=\Theta_0\mathcal{R}(I-\mathcal{R})^{-2}\mathbf{e}.$
5. Mean loss rate of arrival of impulse customers
$$\eta_{LC}=\sum_{j_1=0}^{\infty}\sum_{j_2=1}^{N}\sum_{j_3=1}^{S} q^{(j_2)}D_1^{(j_2)}\Theta_{(j_1,j_2,j_3)}\mathbf{e}$$
$$+\sum_{j_1=1}^{\infty}\sum_{j_2=1}^{N}\sum_{j_3=1}^{S} q^{(j_2)}\theta^{(j_2)}\Theta_{(j_1,j_2,j_3)}\mathbf{e}.$$
6. Expected number of time the replenishment is to be done from temporary supplier
$$\eta_{TS}=\sum_{j_1=0}^{\infty}\sum_{j_2=1}^{N}\sum_{j_3=s+1}^{r} \beta_1\Theta_{(j_1,j_2,j_3)}\mathbf{e}.$$
7. Expected number of time the replenishment is to be done from regular supplier
$$\eta_{RS}=\sum_{j_1=0}^{\infty}\sum_{j_2=1}^{N}\sum_{j_3=0}^{s} \beta_2\Theta_{(j_1,j_2,j_3)}\mathbf{e}.$$
8. Overall retrial rate
$$\eta_{OR}=\sum_{j_1=1}^{\infty}\sum_{j_2=1}^{N} \theta^{(j_2)}\Theta_{(j_1,j_2)}\mathbf{e}.$$
9. Success retrial rate
$$\eta_{SR}=\sum_{j_1=1}^{\infty}\sum_{j_2=1}^{N}\sum_{j_3=1}^{S} \theta^{(j_2)}\Theta_{(j_1,j_2,j_3)}\mathbf{e}.$$
10. Fractional success retrial rate
$\eta_{FSR}=\frac{\eta_{SR}}{\eta_{OR}}.$

Construction of the Cost Feature

To construct the expected total cost function per unit time is provided by $C(S,s) = c_h\eta_I + c_w\eta_{OC} + c_{TS}\eta_{R1} + c_{rs}\eta_{R2} + c_{il}\eta_{LC}$,
where c_h: The inventory carrying cost per unit time.
c_w: Waiting cost of a customer in the orbit per unit time.
c_{ts}: Setup cost per order for temporary supplier.
c_{rs}: Setup cost per order for regular supplier.
c_{il}: Lost cost of a impulse customer per unit time.

6. Numerical Analysis

We give a few descriptive numerical examples that expose the convexity of the expected cost rate and consider that orders are received with 2 different states of RE; these are confined by the infinitesimal generator

$$H = \begin{bmatrix} -0.10 & 0.10 \\ 0.01 & -0.01 \end{bmatrix}.$$

The MAP for appearance of customers are

1. Hyper-exponential(HEX):

$$D_0^{(1)} = \begin{bmatrix} -1.90 & 0 \\ 0 & -0.19 \end{bmatrix}, D_1^{(1)} = \begin{bmatrix} 1.710 & 0.190 \\ 0.171 & 0.019 \end{bmatrix};$$

$$D_0^{(2)} = \begin{bmatrix} -2.85 & 0 \\ 0 & -0.285 \end{bmatrix}, D_1^{(2)} = \begin{bmatrix} 2.565 & 0.285 \\ 0.2565 & 0.0285 \end{bmatrix}.$$

2. Negative Correlation (NC):

$$D_0^{(1)} = \begin{bmatrix} -1.00222 & 1.00222 & 0 \\ 0 & -1.00222 & 0 \\ 0 & 0 & -225.75 \end{bmatrix}, D_1^{(1)} = \begin{bmatrix} 0 & 0 & 0 \\ 0.01002 & 0 & 0.99220 \\ 223.49250 & 0 & 2.25750 \end{bmatrix};$$
$$D_0^{(2)} = \begin{bmatrix} -2.35 & 2.35 & 0 \\ 0 & -2.35 & 0 \\ 0 & 0 & -3.5 \end{bmatrix}, D_1^{(2)} = \begin{bmatrix} 0 & 0 & 0 \\ 0.0235 & 0 & 2.3265 \\ 3.465 & 0 & 0.035 \end{bmatrix}.$$

3. Positive Correlation (PC):

$$D_0^{(1)} = \begin{bmatrix} -1.00222 & 1.00222 & 0 \\ 0 & -1.00222 & 0 \\ 0 & 0 & -225.75 \end{bmatrix}, D_1^{(1)} = \begin{bmatrix} 0 & 0 & 0 \\ 0.99220 & 0 & 0.01002 \\ 2.25750 & 0 & 223.49250 \end{bmatrix};$$
$$D_0^{(2)} = \begin{bmatrix} -2.35 & 2.35 & 0 \\ 0 & -2.35 & 0 \\ 0 & 0 & -3.5 \end{bmatrix}, D_1^{(2)} = \begin{bmatrix} 0 & 0 & 0 \\ 2.3265 & 0 & 0.0235 \\ 0.035 & 0 & 3.465 \end{bmatrix}.$$

The demand process has negative (positive) correlated arrival with coefficient of variance $c_{var}^{(1)} = 2\lambda^{(1)}\eta_2^{(1)}(-D_0^{(1)})^{-1}\mathbf{e} - 1 = \mathbf{1.986}(\mathbf{1.986})$ and coefficient of correlation $c_{cor}^{(1)} = (\lambda^{(1)}\eta_2^{(1)})(-D_0^{(1)})^{-1}D_1^{(1)}(-D_0^{(1)})^{-1})\mathbf{e} - 1)/c_{var}^{(1)} = -\mathbf{0.4889}(\mathbf{0.4889})$ with an arrival rate $\lambda^{(1)} = 1.0$.

The demand process has negative (positive) correlated arrival with coefficient of variance $c_{var}^{(2)} = 2\lambda^{(2)}\eta_2^{(2)}(-D_0^{(2)})^{-1}\mathbf{e} - 1 = \mathbf{0.9342}(\mathbf{0.9342})$ and coefficient of correlation $c_{cor}^{(2)} = (\lambda^{(2)}\eta_2^{(2)})(-D_0^{(2)})^{-1}D_1^{(2)}(-D_0^{(2)})^{-1})\mathbf{e} - 1)/c_{var}^{(2)} = -\mathbf{0.2595}(\mathbf{0.2595})$ with an arrival rate $\lambda^{(2)} = 1.7594$.

Some discussion about numerical examples of our perspective model and its obtainment with the parameters $p^{(1)} = 0.8$, $q^{(1)} = 1 - p^{(1)}$, $\beta_1 = 0.2$, $\lambda_1^{(1)} = 1$, $\beta_2 =$

0.9, $\theta^{(1)} = 5$, $p^{(2)} = 0.6$, $q^{(2)} = 1 - p^{(2)}$, $\theta^{(2)} = 4.8$, $\lambda_1^{(2)} = 1.5$, $R = 2$ and cost values are $c_h = 0.1$, $c_w = 10$, $c_{ts} = 28$, $c_{rs} = 15$, $c_{il} = 10$ are listed in the Results and Discussion section.

Results and Discussion

- We discuss the behavior of the cost function of two variables, $C(S,s)$, under hyper-exponential distribution. The values are divulged in **bold** in each column to indicate the minimum cost rate, whereas the least cost rate is specified in each row by underlining the values. As a result, a value (bold and underlined) represents the local minimum of the function $C(S,s)$. At S^* = **28** and s^* = **5**, the optimal cost value $C^*(S,s)$ = **5.7025**. The function $C(S,s)$ is convex, as shown in Table 1 and Figure 1. Figure 2 depicts a contour plot of the total cost function, which also demonstrates that the function $C(S,s)$ is convex.
- Figures 3 and 4 show that the mean reorder for temporary supplier (η_{R1}) and regular supplier (η_{R2}) compare with s and r respectively. Figure 3 demonstrates that η_{R1} decreases and η_{R2} increases, whenever s increases. Figure 4 shows that η_{R1} is increased and regular suppliers η_{R2} is decreased whenever r is increased.
- Figures 5 and 6 show that the mean number of times the replenishment is to be done from a temporary supplier (η_{TS}) and regular supplier (η_{RS}). Here, Figure 5 shows that η_{TS} decreases and η_{RS} increases whenever s increases. Figure 6 shows that when r increases, η_{TS} increases and η_{RS} decreases.
- Figure 7 compares the lead time rate of temporary (β_1) and regular (β_2) suppliers with their total expected cost value. Here, the total expected cost value decreases whenever β_1 and β_2 rates are increased.
- Table 2 shows that when increasing the lead time rate of the temporary supplier (β_1) increases the optimal cost value $C^*(S,s)$, whereas when increasing the lead time rate of the regular supplier (β_2) decreases the $C^*(S,s)$. The $C^*(S,s)$ for the lead time rates of two suppliers decreases as the maximum inventory level (S) increases.
- When the setup cost for temporary supplier (c_{ts}) and regular supplier (c_{rs}) increases, the value of $C^*(S,s)$ increases in Table 3. When S increases, the value of $C^*(S,s)$ decreases. When the values of c_h, c_w, c_{il} increase, so does the value of $C^*(S,s)$ as shown in Table 4.
- As shown in Table 5, when s and s values rise, the mean inventory level (η_I) rises, but the mean number of customers in the orbit (η_{OC}) and the mean loss rate of arrival of impulse customers (η_{LC}) decreases.

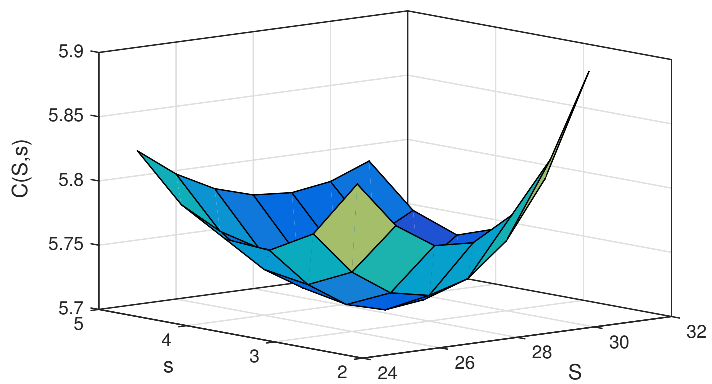

Figure 1. Total expected cost rate as a function of s and S.

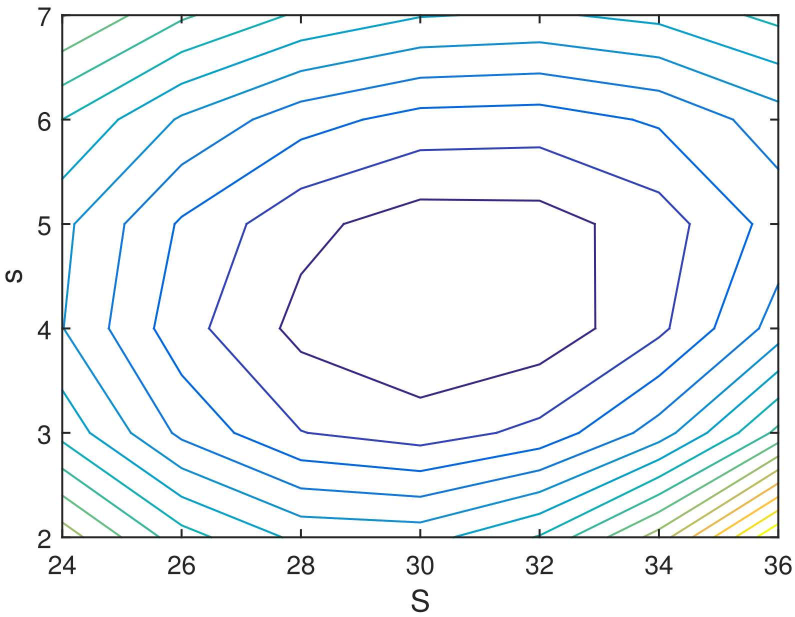

Figure 2. Contour plot of total expected cost rate.

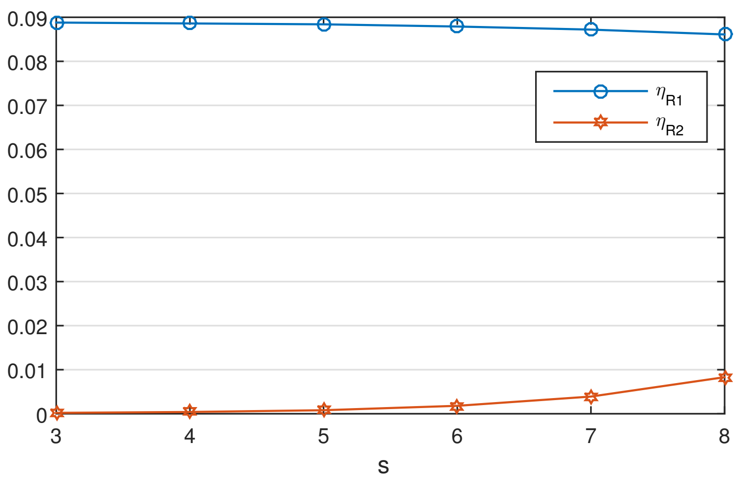

Figure 3. Mean reorder rate for two suppliers vs. s.

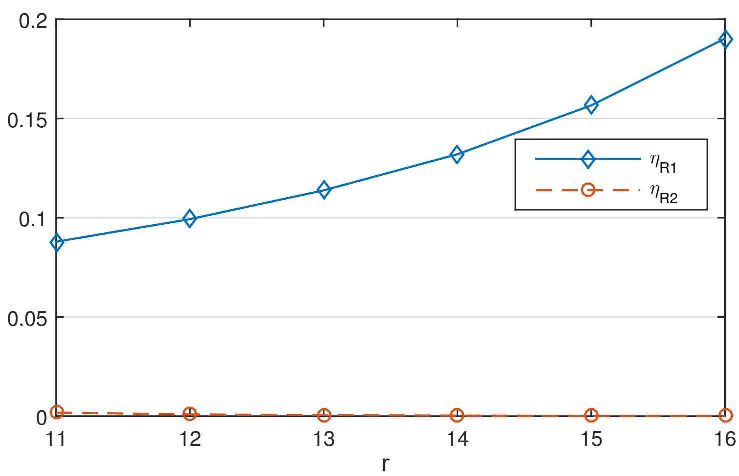

Figure 4. Mean reorder rate for two suppliers vs. r.

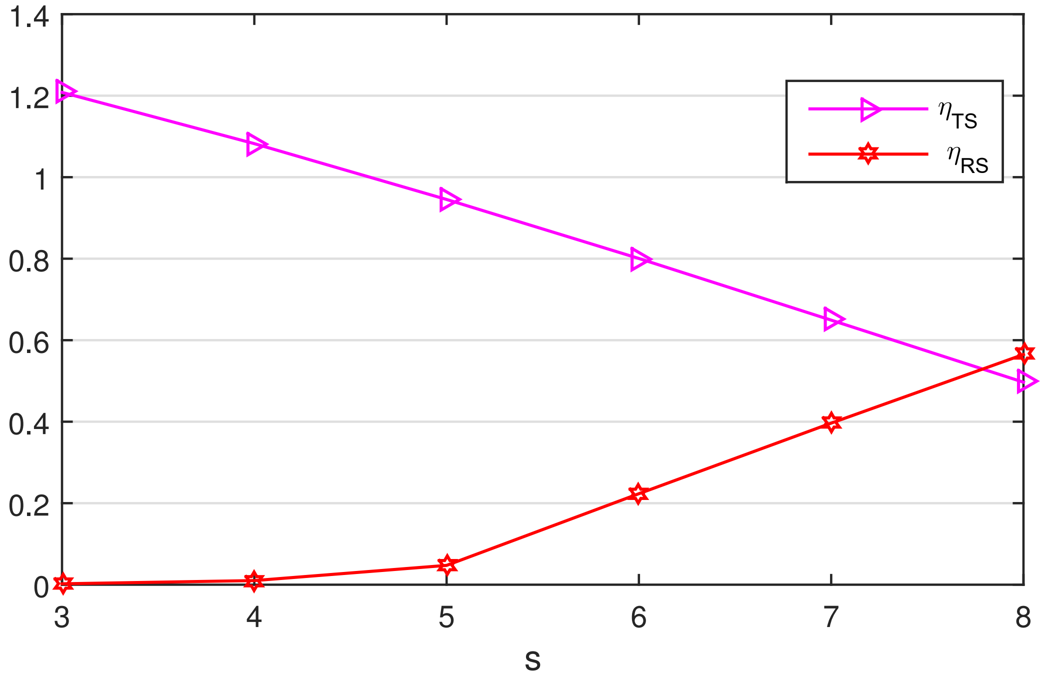

Figure 5. η_{TS} and η_{RS} vs. s.

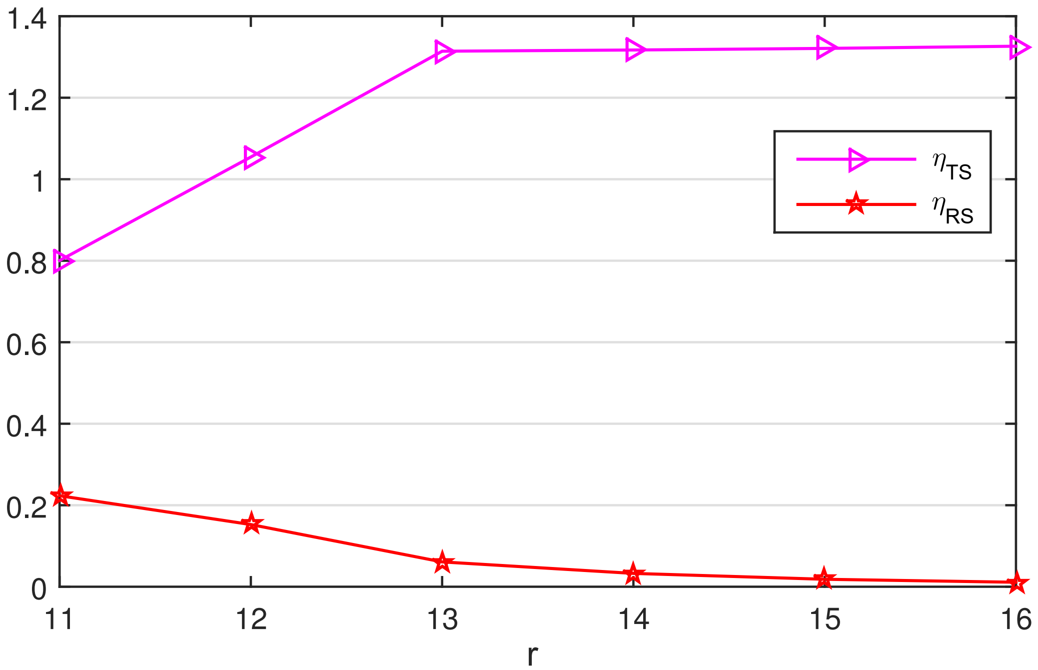

Figure 6. η_{TS} and η_{RS} vs. r.

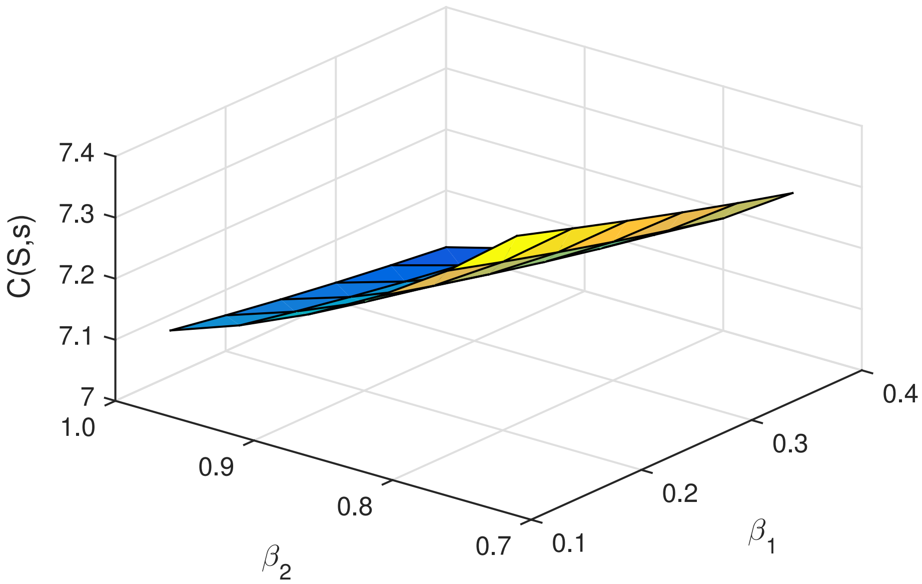

Figure 7. The total expected cost value for different values of β_1 and β_2.

Table 1. Total expected cost rate as a function of s and S.

S/s	**2**	**3**	**4**	**5**	**6**	**7**	**8**
24	5.8251	5.7887	5.7691	<u>5.7673</u>	5.7848	5.8239	5.8884
26	5.7800	5.7462	5.7260	<u>5.7200</u>	5.7290	5.7544	5.7985
28	**5.7609**	**5.7302**	**5.7106**	<u>**5.7025**</u>	**5.7063**	5.7230	5.7540
30	5.7634	5.7357	5.7173	<u>5.7083</u>	5.7090	**5.7200**	**5.7422**
32	5.7838	5.7590	5.7421	5.7329	<u>5.7317</u>	5.7389	5.7551
34	5.8194	5.7973	5.7818	5.7729	<u>5.7707</u>	5.7753	5.7872

Table 2. The optimal cost value for different values of lead times.

S = 14				S = 16				S = 18				S = 20			
β_1	$C^*(S,s)$	β_2	$C^*(S,s)$	β_1	$C^*(S,s)$	β_2	$C^*(S,s)$	β_1	$C^*(S,s)$	β_2	$C^*(S,s)$	β_1	$C^*(S,s)$	β_2	$C^*(S,s)$
0.1	6.6730	0.7	6.8189	0.1	6.3526	0.7	6.4803	0.1	6.1278	0.7	6.2380	0.1	5.9690	0.7	6.0646
0.2	6.7730	0.8	6.7955	0.2	6.4317	0.8	6.4552	0.2	6.1905	0.8	6.2132	0.2	6.0202	0.8	6.0414
0.3	6.8334	0.9	6.7730	0.3	6.4740	0.9	6.4317	0.3	6.2207	0.9	6.1905	0.3	6.0434	0.9	6.0202
0.4	6.8689	1.0	6.7516	0.4	6.4952	1.0	6.4099	0.4	6.2336	1.0	6.1697	0.4	6.0527	1.0	6.0009
0.5	6.8892	1.1	6.7315	0.5	6.5048	1.1	6.3897	0.5	6.2381	1.1	6.1505	0.5	6.0558	1.1	5.9832
0.6	6.9002	1.2	6.7125	0.6	6.5081	1.2	6.3710	0.6	6.2385	1.2	6.1330	0.6	6.0562	1.2	5.9671
S = 22				S = 24				S = 26				S = 28			
β_1	$C^*(S,s)$	β_2	$C^*(S,s)$	β_1	$C^*(S,s)$	β_2	$C^*(S,s)$	β_1	$C^*(S,s)$	β_2	$C^*(S,s)$	β_1	$C^*(S,s)$	β_2	$C^*(S,s)$
0.1	5.8583	0.7	5.9431	0.1	5.7839	0.7	5.8619	0.1	5.7377	0.7	5.8129	0.1	5.7142	0.7	5.7903
0.2	5.9024	0.8	5.9217	0.2	5.8251	0.8	5.8426	0.2	5.7800	0.8	5.7956	0.2	5.7609	0.8	5.7748
0.3	5.9230	0.9	5.9024	0.3	5.8464	0.9	5.8251	0.3	5.8044	0.9	5.7800	0.3	5.7902	0.9	5.7609
0.4	5.9320	1.0	5.8848	0.4	5.8573	1.0	5.8093	0.4	5.8185	1.0	5.7659	0.4	5.8083	1.0	5.7484
0.5	5.9360	1.1	5.8689	0.5	5.8635	1.1	5.7950	0.5	5.8277	1.1	5.7531	0.5	5.8207	1.1	5.7371
0.6	5.9379	1.2	5.8543	0.6	5.8678	1.2	5.7820	0.6	5.8345	1.2	5.7415	0.6	5.8300	1.2	5.7268

Table 3. The optimal cost value for different setup cost values of two suppliers.

S = 14				S = 16				S = 18				S = 20			
c_{ts}	$C^*(S,s)$	c_{rs}	$C^*(S,s)$	c_{ts}	$C^*(S,s)$	c_{rs}	$C^*(S,s)$	c_{ts}	$C^*(S,s)$	c_{rs}	$C^*(S,s)$	c_{ts}	$C^*(S,s)$	c_{rs}	$C^*(S,s)$
25	6.4712	10	6.5893	25	6.1603	10	6.2936	25	5.9435	10	6.0855	25	5.7932	10	5.9397
26	6.5718	11	6.6260	26	6.2508	11	6.3212	26	6.0258	11	6.1065	26	5.8688	11	5.9558
27	6.6724	12	6.6628	27	6.3412	12	6.3488	27	6.1082	12	6.1275	27	5.9445	12	5.9719
28	6.7730	13	6.6995	28	6.4317	13	6.3765	28	6.1905	13	6.1485	28	6.0202	13	5.9880
29	6.8736	14	6.7363	29	6.5221	14	6.4041	29	6.2729	14	6.1695	29	6.0958	14	6.0041
30	6.9742	15	6.7730	30	6.6126	15	6.4317	30	6.3552	15	6.1905	30	6.1715	15	6.0202
31	7.0749	16	6.8098	31	6.7030	16	6.4593	31	6.4376	16	6.2115	31	6.2472	16	6.0363
32	7.1755	17	6.8465	32	6.7935	17	6.4869	32	6.5199	17	6.2325	32	6.3228	17	6.0524
33	7.2761	18	6.8832	33	6.8839	18	6.5145	33	6.6023	18	6.2535	33	6.3985	18	6.0685
34	7.3767	19	6.9200	34	6.9744	19	6.5421	34	6.6846	19	6.2745	34	6.4742	19	6.0846
35	7.4773	20	6.9567	35	7.0648	20	6.5697	35	6.7670	20	6.2955	35	6.5498	20	6.1007
S = 22				S = 24				S = 26				S = 28			
c_{ts}	$C^*(S,s)$	c_{rs}	$C^*(S,s)$	c_{ts}	$C^*(S,s)$	c_{rs}	$C^*(S,s)$	c_{ts}	$C^*(S,s)$	c_{rs}	$C^*(S,s)$	c_{ts}	$C^*(S,s)$	c_{rs}	$C^*(S,s)$
25	5.6923	10	5.8403	25	5.6297	10	5.7769	25	5.5974	10	5.7425	25	5.5896	10	5.7316
26	5.7624	11	5.8527	26	5.6948	11	5.7866	26	5.6582	11	5.7500	26	5.6467	11	5.7375
27	5.8324	12	5.8651	27	5.7600	12	5.7962	27	5.7191	12	5.7575	27	5.7038	12	5.7433
28	5.9024	13	5.8775	28	5.8251	13	5.8058	28	5.7800	13	5.7650	28	5.7609	13	5.7492
29	5.9724	14	5.8900	29	5.8903	14	5.8155	29	5.8409	14	5.7725	29	5.8180	14	5.7551
30	6.0424	15	5.9024	30	5.9554	15	5.8251	30	5.9018	15	5.7800	30	5.8751	15	5.7609
31	6.1124	16	5.9148	31	6.0206	16	5.8348	31	5.9626	16	5.7875	31	5.9323	16	5.7668
32	6.1825	17	5.9273	32	6.0857	17	5.8444	32	6.0235	17	5.7950	32	5.9894	17	5.7727
33	6.2525	18	5.9397	33	6.1508	18	5.8540	33	6.0844	18	5.8025	33	6.0465	18	5.7785
34	6.3225	19	5.9521	34	6.2160	19	5.8637	34	6.1453	19	5.8100	34	6.1036	19	5.7844
35	6.3925	20	5.9645	35	6.2811	20	5.8733	35	6.2062	20	5.8175	35	6.1607	20	5.7903

Table 4. Total expected cost values for different cost values.

		$c_h = 0.1$					$c_h = 0.2$				
	c_w	8	9	10	11	12	8	9	10	11	12
c_{il}	5	4.6560	4.6566	4.6572	4.6578	4.6584	6.3673	6.3679	6.3685	6.3691	6.3697
	6	4.8806	4.8812	4.8818	4.8824	4.8830	6.5919	6.5925	6.5931	6.5936	6.5942
	7	5.1051	5.1057	5.1063	5.1069	5.1075	6.8164	6.8170	6.8176	6.8182	6.8188
	8	5.3297	5.3303	5.3309	5.3315	5.3321	7.0410	7.0416	7.0422	7.0428	7.0433
	9	5.5543	5.5549	5.5554	5.5560	5.5566	7.2655	7.2661	7.2667	7.2673	7.2679
	10	5.7788	5.7794	5.7800	5.7806	5.7812	7.4901	7.4907	7.4913	7.4919	7.4925
	11	6.0034	6.0040	6.0046	6.0051	6.0057	7.7146	7.7152	7.7158	7.7164	7.7170
	12	6.2279	6.2285	6.2291	6.2297	6.2303	7.9392	7.9398	7.9404	7.9410	7.9416
	13	6.4525	6.4531	6.4537	6.4543	6.4548	8.1638	8.1644	8.1649	8.1655	8.1661
	14	6.6770	6.6776	6.6782	6.6788	6.6794	8.3883	8.3889	8.3895	8.3901	8.3907
	15	6.9016	6.9022	6.9028	6.9034	6.9040	8.6129	8.6135	8.6141	8.6146	8.6152
		$c_h = 0.3$					$c_h = 0.4$				
	c_w	8	9	10	11	12	8	9	10	11	12
c_{il}	5	8.0786	8.0792	8.0798	8.0804	8.0810	9.7899	9.7905	9.7911	9.7916	9.7922
	6	8.3032	8.3037	8.3043	8.3049	8.3055	10.0144	10.0150	10.0156	10.0162	10.0168
	7	8.5277	8.5283	8.5289	8.5295	8.5301	10.2390	10.2396	10.2402	10.2408	10.2414
	8	8.7523	8.7529	8.7534	8.7540	8.7546	10.4635	10.4641	10.4647	10.4653	10.4659
	9	8.9768	8.9774	8.9780	8.9786	8.9792	10.6881	10.6887	10.6893	10.6899	10.6905
	10	9.2014	9.2020	9.2026	9.2031	9.2037	10.9127	10.9132	10.9138	10.9144	10.9150
	11	9.4259	9.4265	9.4271	9.4277	9.4283	11.1372	11.1378	11.1384	11.1390	11.1396
	12	9.6505	9.6511	9.6517	9.6523	9.6528	11.3618	11.3624	11.3629	11.3635	11.3641
	13	9.8750	9.8756	9.8762	9.8768	9.8774	11.5863	11.5869	11.5875	11.5881	11.5887
	14	10.0996	10.1002	10.1008	10.1014	10.1020	11.8109	11.8115	11.8121	11.8126	11.8132
	15	10.3241	10.3247	10.3253	10.3259	10.3265	12.0354	12.0360	12.0366	12.0372	12.0378

Table 5. Ramification of some measures.

S = 20				S = 22				S = 24			
s	η_I	η_{OC}	η_{LC}	s	η_I	η_{OC}	η_{LC}	s	η_I	η_{OC}	η_{LC}
2	13.0196	0.0013	0.2345	2	14.3643	0.0010	0.2309	2	15.7293	0.0008	0.2276
3	13.2335	0.0009	0.2285	3	14.5581	0.0007	0.2256	3	15.9029	0.0005	0.2229
4	13.4608	0.0006	0.2232	4	14.7655	0.0004	0.2210	4	16.0901	0.0003	0.2188
5	13.7039	0.0004	0.2187	5	14.9896	0.0003	0.2170	5	16.2942	0.0002	0.2153
6	13.9615	0.0003	0.2150	6	15.2305	0.0002	0.2137	6	16.5161	0.0002	0.2124
7	14.2286	0.0002	0.2120	7	15.4855	0.0001	0.2109	7	16.7548	0.0001	0.2100
8	14.4968	0.0001	0.2095	8	15.7494	0.0001	0.2087	8	17.0071	0.0001	0.2080
S = 26				S = 28				S = 30			
s	η_I	η_{OC}	η_{LC}	s	η_I	η_{OC}	η_{LC}	s	η_I	η_{OC}	η_{LC}
2	17.1128	0.0006	0.2246	2	18.5131	0.0005	0.2218	2	19.9282	0.0004	0.2193
3	17.2671	0.0004	0.2204	3	18.6490	0.0003	0.2182	3	20.0472	0.0002	0.2161
4	17.4343	0.0003	0.2168	4	18.7972	0.0002	0.2150	4	20.1776	0.0002	0.2133
5	17.6181	0.0002	0.2137	5	18.9612	0.0001	0.2123	5	20.3228	0.0001	0.2109
6	17.8201	0.0001	0.2112	6	19.1431	0.0001	0.2100	6	20.4850	0.0001	0.2089
7	18.0401	0.0001	0.2090	7	19.3434	0.0001	0.2081	7	20.6654	0.0000	0.2073
8	18.2766	0.0001	0.2072	8	19.5616	0.0000	0.2065	8	20.8642	0.0000	0.2059

7. Conclusions

In this paper, we discussed two classes of suppliers in the stochastic inventory system on the RE. These two classes of suppliers have a huge impact on the management of the inventory system and also reduce replenishment difficulties. We showed the minimised total expected cost rate (refer to Table 1 and Figure 1) and showed it via a contour plot (refer Figure 2). When the lead time rate of both suppliers is smaller, the total expected cost value of the regular supplier is greater than the total expected cost value of the temporary supplier. Moreover, when the lead time rate of both suppliers goes up, the total expected cost value of the regular supplier is smaller than the total expected cost value of the temporary supplier (refer Table 2). Similarly, when the setup cost value is small, the total expected cost value of the temporary supplier is smaller than the total expected cost value of the regular supplier. Furthermore, when the setup cost value goes up, the total expected cost value of the temporary supplier is greater than the total expected cost value of the regular supplier (refer Table 3). Finally, we showed that both the value of the setup cost and the lead time rate are small, the best supplier is a temporary supplier, and both have higher value, making the best supplier a regular supplier.

7.1. Limitations

- This study deals with impulse customers and the probability of customers who may buy an item, p^{j_2} and the complementary probability q^{j_2}. Assuming the probability q^{j_2} value is zero, in this case, not all incoming customers will purchase the item, which does not always happen in real-life situations.
- The sum of the fixed probability distribution values must be one.

7.2. Future Directions

- We will discuss multi-server with phase-type distribution for service.
- We plan to investigate multi-suppliers.
- We plan to study RE use with payment mode.

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Notations and Abbreviations

The following Notations and Abbreviations are used in this manuscript:

$[A]_{ij}$	The element of submatrix at (i,j) the position of A.
$\mathbf{0}$	Zero matrix.
$\mathbf{I}$	Identity matrix of appropriate dimension.
$\mathbf{I_m}$	Identity matrix of dimension m.
$\mathbf{e}$	A column vector of 1's appropriate dimension.
$A \otimes B$	Kronecker product of matrices A and B.
$A \oplus B$	Kronecker sum of matrices A and B.
RE	Random Envirnonment.
MAP	Markovian Arrival Process.
$\bar{\delta}_{ij}$	$1 - \delta_{ij}$.
k_1	$N(S+1)m$
k_2	$(S+1)m$

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