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A Reliable Way to Deal with Fractional-Order Equations That Describe the Unsteady Flow of a Polytropic Gas

M. Mossa Al-Sawalha [1], Ravi P. Agarwal [2], Rasool Shah [3], Osama Y. Ababneh [4] and Wajaree Weera [5,*]

1 Department of Mathematics, Faculty of Science, University of Ha'il, Ha'il 2440, Saudi Arabia; m.alswalha@uoh.edu.sa
2 Department of Mathematics, Texas A & M University-Kingsville, Kingsville, TX 78363, USA; ravi.agarwal@tamuk.edu
3 Department of Mathematics, Abdul Wali Khan University, Mardan 23200, Pakistan; rasoolshah@awkum.edu.pk
4 Department of Mathematics, Faculty of Science, Zarqa University, Zarqa 13110, Jordan; osababneh@zu.edu.jo
5 Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand
* Correspondence: wajawe@kku.ac.th

Abstract: In this paper, fractional-order system gas dynamics equations are solved analytically using an appealing novel method known as the Laplace residual power series technique, which is based on the coupling of the residual power series approach with the Laplace transform operator to develop analytical and approximate solutions in quick convergent series types by utilizing the idea of the limit with less effort and time than the residual power series method. The given model is tested and simulated to confirm the proposed technique's simplicity, performance, and viability. The results show that the above-mentioned technique is simple, reliable, and appropriate for investigating nonlinear engineering and physical problems.

1. Introduction

Fractional calculus (FC) is a quickly growing branch of mathematics that studies the derivatives and integrals of any order of functions. Due to the outstanding findings gained when some tools of this calculus were employed to mimic real-world processes, it is gaining favour among scientists working in various disciplines. The idea of FC was discussed in the seventeenth century. FC was started almost 324 years ago, and it has been based on mathematical ideas ever since then. The fundamental and critical results of fractional differential equation solutions are contained in [1–8]. The nonlocal fractionals are derivatives, whereas the integer-order derivatives are local. We can use integer-order derivatives to look at changes near a point, but we can use fractional-order derivatives to look at changes across the whole interval [9]. Systems with an arbitrary order have recently gotten a lot of attention and acceptance as a generalization of the classical order system. Here, ref. [10] introduces a homotopy perturbation method for nonlinear transport equations. The review paper [11] contains a comprehensive amount of various modern fractional calculus applications. Papers [12–14] address the application of ADM to various fractional transport models. Finally, works [15–20] reflect some developments and reviews of various numerical approaches to transport problems.

Most complex phenomena are mathematically modelled by nonlinear fractional partial differential equations (FPDEs). The dynamical processes of these nonlinear FPDEs are very important for both production and scientific research and should be studied with a

suitable method to handle the nonlinear problems. Numerical and analytical solutions of the nonlinear FPDEs have fundamental importance. In recent decades, many researchers have used different analytical and numerical techniques to study various nonlinear FPDEs. Due to the complexity of the nonlinear equations, there is no united method to find every solution to the nonlinear FPDEs. Some of the numerical and analytical methods are the homotopy perturbation method (HPM) [21], Adomian decomposition method (ADM) [22], variational iteration method (VIM) [23], Jacobi spectral collocation method [24], G/G-expansion method [25], tau method [26], meshless method [27], the Haar wavelet method [28], Bernstein polynomials [29], the Legendre base method [30], the Laplace transform method [31], fractional complex transform method [32], Laplace variational iteration method [33], spectral Legendre–Gauss–Lobatto collocation method [34], and cylindrical-coordinate method [35] and so on [36–38].

Studies show that gases with many different properties are important to astrophysics and cosmology. Gases with many different properties can act like dark energy [39]. The set of equations forgas dynamics that describe the change in the flow of a perfect gas with fractional order [40]:

$$
\begin{aligned}
&D_t^\sigma u + u_\varphi u + v u_\xi + \frac{p_\varphi}{\rho} = 0, \quad 0 < \sigma \leq 1, \\
&D_t^\sigma v + v_\varphi u + v v_\xi + \frac{p_\xi}{\rho} = 0, \\
&D_t^\sigma \rho + \rho_\varphi u + v \rho_\xi + \rho u_\varphi + \rho v_\xi = 0, \\
&D_t^\sigma p + p_\varphi u + v p_\xi + \tau p u_\varphi + \tau p v_\xi = 0,
\end{aligned}
\tag{1}
$$

with initial conditions (IC's)

$$
\begin{aligned}
u(\varphi, \xi, 0) &= f_0(\varphi, \xi), \\
v(\varphi, \xi, 0) &= g_0(\varphi, \xi), \\
\rho(\varphi, \xi, 0) &= h_0(\varphi, \xi), \\
p(\varphi, \xi, 0) &= l_0(\varphi, \xi),
\end{aligned}
\tag{2}
$$

where D_t^σ denotes the Caputo's fractional derivative, $u(\varphi, \xi, t)$ and $u(\varphi, \xi, t)$ are the velocity components in the φ and ξ directions, $\rho(\varphi, \xi, t)$ is the density, $p(\varphi, \xi, t)$ is the pressure, and τ is the ratio of the specific heat, and it represents the adiabatic index. The solution for the considered system of equations was studied using different analytical methods, such as the q-homotopy analysis transform method (q-HATM) [40], variational iteration technique [41], fractional natural decomposition scheme [42], and Adomian decomposition technique [43].

The Laplace Residual power series method (LRPSM) [44–46] is a combination of the Laplace transform (LT) and RPSM [47–52]. In the LRPSM technique, the Laplace transform (LT) is used to simplify the focused problem into new algebraic equations. The series solution is then computed using the RPSM. Finally, to obtain the desired result, the inverse LT is applied. The LRPSM needs minimal calculations to be completed in less time and with greater accuracy.

The generalized LRPSM technique is described, and the LRPSM algorithm is applied to Equation (1). The outcomes and accuracy of the suggested technique are shown in tables and graphs. The fractional-order LRPSM solutions can be used to investigate the dynamics of the problems. LRPSM has the highest degree of accuracy, according to the tables.

The following is a summary of the current paper. Some required definitions and results were discussed from FC theory in Section 2. In Section 3, the basic methodology is discussed, and the effectiveness of LRPSM is confirmed by some test models in Section 4. In Section 5, results are discussed, and the conclusion is given in Section 6.

2. Preliminaries

In this section, we will discuss several basic definitions and conclusions relating to the Caputo fractional derivative, as well as the fractional Laplace transform.

Definition 1. *The fractional derivative of a function $u(\varphi, t)$ of order σ in Caputo sense is defined as [45]*

$$^{C}D_t^{\sigma} u(\varphi, t) = J_t^{m-\sigma} u^m(\varphi, t), \quad m - 1 < \sigma \leq m, \quad t > 0. \tag{3}$$

where $m \in N$ and J_t^{σ} is the Riemann-Liouville (RL) fractional integral (FI) of $u(\varphi, t)$ of fractional-order σ is defined as

$$J_t^{\sigma} u(\varphi, t) = \frac{1}{\Gamma(\sigma)} \int_0^t (t - \tau)^{\sigma-1} u(\varphi, \tau) d\tau \tag{4}$$

assuming that the given integral exists.

Definition 2. *The Laplace transform of a function $u(\varphi, t)$ is defined as [45]*

$$u(\varphi, s) = \mathcal{L}_t[u(\varphi, t)] = \int_0^\infty e^{-st} u(\varphi, t) dt, \quad s > \sigma, \tag{5}$$

where the inverse LT is given as

$$u(\varphi, t) = \mathcal{L}_t^{-1}[u(\varphi, s)] = \int_{l-i\infty}^{l+i\infty} e^{st} u(\varphi, s) ds, \quad l = Re(s) > l_0, \tag{6}$$

where l_0 is in the right half-plane of the Laplace integral's absolute convergence.

Lemma 1. *Assume that $u(\varphi, t)$ is a continuous piecewise function and of exponential order ζ and $U(\varphi, s) = \mathcal{L}_t[u(\varphi, t)]$, we have*

1. $\mathcal{L}_t[J_t^{\sigma} u(\varphi, t)] = \frac{U(\varphi, s)}{s^{\sigma}}, \quad \sigma > 0.$
2. $\mathcal{L}_t[D_t^{\sigma} u(\varphi, t)] = s^{\sigma} U(\varphi, s) - \sum_{k=0}^{m-1} s^{\sigma-k-1} u^k(\varphi, 0), \quad m - 1 < \sigma \leq m.$
3. $\mathcal{L}_t[D_t^{n\sigma} u(\varphi, t)] = s^{n\sigma} U(\varphi, s) - \sum_{k=0}^{n-1} s^{(n-k)\sigma-1} D_t^{k\sigma} u(\varphi, 0), \quad 0 < \sigma \leq 1.$

Proof. For proof, see Refs. [4–6,53]. □

Theorem 1. *Let $u(\varphi, t)$ be a piecewise continuous on $I \times [0, \infty)$ and of exponential order ζ. Suppose that the function $U(\varphi, s) = \mathcal{L}_t[u(\varphi, t)]$ has the following fractional expansion:*

$$U(\varphi, s) = \sum_{n=0}^{\infty} \frac{f_n(\varphi)}{s^{1+n\sigma}}, 0 < \sigma \leq 1, \varphi \in I, s > \zeta. \tag{7}$$

Then, $f_n(\varphi) = D_t^{n\sigma} u(\varphi, 0)$.

Proof. For proof, see Ref. [45]. □

Remark 1. *The inverse LT of Equation (7) is represented as [45]:*

$$u(\varphi, t) = \sum_{i=0}^{\infty} \frac{D_t^{\sigma} u(\varphi, 0)}{\Gamma(1 + i\sigma)} t^{i(\zeta)}, \quad 0 < \zeta \leq 1, \quad t \geq 0. \tag{8}$$

which is equivalent to the fractional Taylor's formula shown in [54].

The convergence of the FPS in Theorem (1) is explained in the following theorem.

Theorem 2. *Assume that $u(\varphi, t)$ is continuous piecewise on $\mathbf{I} \times [0, \infty)$ and of order ξ and as shown in Theorem (1), $u(\varphi, s) = \mathcal{L}_t[u(\varphi, t)]$ can be written as the new form of fractional Taylor's formula. If $\left| s\mathcal{L}_t[D_t^{i\sigma+1} u(\varphi, t)] \right| \leq M(\varphi)$, on $\mathbf{I} \times (\xi, \gamma]$ where $0 < \sigma \leq 1$, then $R_i(\varphi, s)$ the*

remainder of the new form of fractional Taylor's formula in Theorem (1) satisfies the following inequality

$$|R_i(\varphi, s)| \leq \frac{M(\varphi)}{S^{1+(i+1)\sigma}}, \quad \varphi \in \mathbf{I}, \ \xi < s \leq \gamma. \tag{9}$$

Proof. For proof, see Ref. [45]. □

3. LRPS Methodology

In this section, we will go through the LRPS methodology for the nonlinear system of fractional order partial differential equations

Applying LT of Equation (1), we get

$$
\begin{aligned}
U(\varphi, \xi, s) - \frac{f_0(\varphi, \xi)}{s} + \frac{1}{s^\sigma}\mathcal{L}_t\left[\mathcal{L}_t^{-1}[U_\varphi]\mathcal{L}_t^{-1}[U] + \mathcal{L}_t^{-1}[V]\mathcal{L}_t^{-1}[U_\xi] + \frac{\mathcal{L}_t^{-1}[P_\varphi]}{\mathcal{L}_t^{-1}[W]}\right] = 0, \\
V(\varphi, \xi, s) - \frac{g_0(\varphi, \xi)}{s} + \frac{1}{s^\sigma}\mathcal{L}_t\left[\mathcal{L}_t^{-1}[V_\varphi]\mathcal{L}_t^{-1}[U] + \mathcal{L}_t^{-1}[V]\mathcal{L}_t^{-1}[V_\xi] + \frac{\mathcal{L}_t^{-1}[P_\xi]}{\mathcal{L}_t^{-1}[W]}\right] = 0, \\
W(\varphi, \xi, s) - \frac{h_0(\varphi, \xi)}{s} + \frac{1}{s^\sigma}\mathcal{L}_t\left[\mathcal{L}_t^{-1}[W_\varphi]\mathcal{L}_t^{-1}[U] + \mathcal{L}_t^{-1}[V]\mathcal{L}_t^{-1}[W_\xi] + \mathcal{L}_t^{-1}[W]\mathcal{L}_t^{-1}[U_\varphi] + \mathcal{L}_t^{-1}[W]\mathcal{L}_t^{-1}[V_\xi]\right] = 0, \\
P(\varphi, \xi, s) - \frac{l_0(\varphi, \xi)}{s} + \frac{1}{s^\sigma}\mathcal{L}_t\left[\mathcal{L}_t^{-1}[P_\varphi]\mathcal{L}_t^{-1}[U] + \mathcal{L}_t^{-1}[V]\mathcal{L}_t^{-1}[P_\xi] + \tau\mathcal{L}_t^{-1}[P]\mathcal{L}_t^{-1}[U_\varphi] + \tau\mathcal{L}_t^{-1}[P]\mathcal{L}_t^{-1}[V_\xi]\right] = 0.
\end{aligned}
\tag{10}
$$

Assuming that the solution of Equation (10) has the following expansion

$$
\begin{aligned}
U(\varphi, \xi, s) = \sum_{n=0}^{\infty} \frac{f_n(\varphi, \xi, s)}{s^{n\sigma+1}}, \quad V(\varphi, \xi, s) = \sum_{n=0}^{\infty} \frac{g_n(\varphi, \xi, s)}{s^{n\sigma+1}}, \\
W(\varphi, \xi, s) = \sum_{n=0}^{\infty} \frac{h_n(\varphi, \xi, s)}{s^{n\sigma+1}}, \quad P(\varphi, \xi, s) = \sum_{n=0}^{\infty} \frac{l_n(\varphi, \xi, s)}{s^{n\sigma+1}}.
\end{aligned}
\tag{11}
$$

The kth-truncated term series are

$$
\begin{aligned}
U(\varphi, \xi, s) = \frac{f_0(\varphi, \xi, s)}{s} + \sum_{n=1}^{k} \frac{f_n(\varphi, \xi, s)}{s^{n\sigma+1}}, \quad V(\varphi, \xi, s) = \frac{g_0(\varphi, \xi, s)}{s} + \sum_{n=1}^{k} \frac{g_n(\varphi, \xi, s)}{s^{n\sigma+1}}, \\
W(\varphi, \xi, s) = \frac{h_0(\varphi, \xi, s)}{s} + \sum_{n=1}^{k} \frac{h_n(\varphi, \xi, s)}{s^{n\sigma+1}}, \quad P(\varphi, \xi, s) = \frac{l_0(\varphi, \xi, s)}{s} + \sum_{n=1}^{k} \frac{l_n(\varphi, \xi, s)}{s^{n\sigma+1}}. \ k = 1, 2, 3, 4\cdots
\end{aligned}
\tag{12}
$$

Laplace residual functions (LRFs) [55] are

$$
\begin{aligned}
\mathcal{L}_t Res_u(\varphi, \xi, s) =& U(\varphi, \xi, s) - \frac{f_0(\varphi, \xi, s)}{s} + \frac{1}{s^\sigma}\mathcal{L}_t\left[\mathcal{L}_t^{-1}[U_\varphi]\mathcal{L}_t^{-1}[U] + \mathcal{L}_t^{-1}[V]\mathcal{L}_t^{-1}[U_\xi] + \frac{\mathcal{L}_t^{-1}[P_\varphi]}{\mathcal{L}_t^{-1}[W]}\right], \\
\mathcal{L}_t Res_v(\varphi, \xi, s) =& V(\varphi, \xi, s) - \frac{g_0(\varphi, \xi, s)}{s} + \frac{1}{s^\sigma}\mathcal{L}_t\left[\mathcal{L}_t^{-1}[V_\varphi]\mathcal{L}_t^{-1}[U] + \mathcal{L}_t^{-1}[V]\mathcal{L}_t^{-1}[V_\xi] + \frac{\mathcal{L}_t^{-1}[P_\xi]}{\mathcal{L}_t^{-1}[W]}\right], \\
\mathcal{L}_t Res_\rho(\varphi, \xi, s) =& W(\varphi, \xi, s) - \frac{h_0(\varphi, \xi, s)}{s} + \frac{1}{s^\sigma}\mathcal{L}_t\left[\mathcal{L}_t^{-1}[W_\varphi]\mathcal{L}_t^{-1}[U] + \mathcal{L}_t^{-1}[V]\mathcal{L}_t^{-1}[W_\xi] + \mathcal{L}_t^{-1}[W]\mathcal{L}_t^{-1}[U_\varphi]\right. \\
& \left. + \mathcal{L}_t^{-1}[W]\mathcal{L}_t^{-1}[V_\xi]\right], \\
\mathcal{L}_t Res_p(\varphi, \xi, s) =& P(\varphi, \xi, s) - \frac{l_0(\varphi, \xi, s)}{s} + \frac{1}{s^\sigma}\mathcal{L}_t\left[\mathcal{L}_t^{-1}[P_\varphi]\mathcal{L}_t^{-1}[U] + \mathcal{L}_t^{-1}[V]\mathcal{L}_t^{-1}[P_\xi] + \tau\mathcal{L}_t^{-1}[P]\mathcal{L}_t^{-1}[U_\varphi]\right. \\
& \left. + \tau\mathcal{L}_t^{-1}[P]\mathcal{L}_t^{-1}[V_\xi]\right].
\end{aligned}
\tag{13}
$$

Therefore, the kth-LRFs is:

$$\mathcal{L}_t Res_{u,k}(\varphi, \xi, s) = U_k(\varphi, \xi, s) - \frac{f_0(\varphi, \xi, s)}{s} + \frac{1}{s^\sigma} \mathcal{L}_t \left[\mathcal{L}_t^{-1}[U_{\varphi,k}] \mathcal{L}_t^{-1}[U_k] + \mathcal{L}_t^{-1}[V_k] \mathcal{L}_t^{-1}[U_{\xi,k}] + \frac{\mathcal{L}_t^{-1}[P_{\varphi,k}]}{\mathcal{L}_t^{-1}[W_k]} \right],$$

$$\mathcal{L}_t Res_{v,k}(\varphi, \xi, s) = V_k(\varphi, \xi, s) - \frac{g_0(\varphi, \xi, s)}{s} + \frac{1}{s^\sigma} \mathcal{L}_t \left[\mathcal{L}_t^{-1}[V_{\varphi,k}] \mathcal{L}_t^{-1}[U_k] + \mathcal{L}_t^{-1}[V_k] \mathcal{L}_t^{-1}[V_{\xi,k}] + \frac{\mathcal{L}_t^{-1}[P_{\xi,k}]}{\mathcal{L}_t^{-1}[W_k]} \right],$$

$$\mathcal{L}_t Res_{w,k}(\varphi, \xi, s) = W_k(\varphi, \xi, s) - \frac{h_0(\varphi, \xi, s)}{s} + \frac{1}{s^\sigma} \mathcal{L}_t \left[\mathcal{L}_t^{-1}[W_{\varphi,k}] \mathcal{L}_t^{-1}[U_k] + \mathcal{L}_t^{-1}[V_k] \mathcal{L}_t^{-1}[W_{\xi,k}] + \mathcal{L}_t^{-1}[W_k] \mathcal{L}_t^{-1}[U_{\varphi,k}] \quad (14)$$
$$+ \mathcal{L}_t^{-1}[\rho_k] \mathcal{L}_t^{-1}[V_{\xi,k}] \right],$$

$$\mathcal{L}_t Res_{p,k}(\varphi, \xi, s) = P_k(\varphi, \xi, s) - \frac{l_0(\varphi, \xi, s)}{s} + \frac{1}{s^\sigma} \mathcal{L}_t \left[\mathcal{L}_t^{-1}[P_\varphi] \mathcal{L}_t^{-1}[U_k] + \mathcal{L}_t^{-1}[V_k] \mathcal{L}_t^{-1}[P_{\xi,k}] + \tau \mathcal{L}_t^{-1}[P_k] \mathcal{L}_t^{-1}[U_{\varphi,k}] \right.$$
$$\left. + \tau \mathcal{L}_t^{-1}[P_k] \mathcal{L}_t^{-1}[V_{\xi,k}] \right].$$

Here are some properties that arise in the LRPSM [55]:

- $\mathcal{L}_t Res(\varphi, \xi, s) = 0$ and $\lim_{j \to \infty} \mathcal{L}_t Res_{u,k}(\varphi, \xi, s) = \mathcal{L}_t Res_u(\varphi, \xi, s)$ for each $s > 0$.
- $\lim_{s \to \infty} s \mathcal{L}_t Res_u(\varphi, \xi, s) = 0 \Rightarrow \lim_{s \to \infty} s \mathcal{L}_t Res_{u,k}(\varphi, \xi, s) = 0$.
- $\lim_{s \to \infty} s^{k\sigma+1} \mathcal{L}_t Res_{u,k}(\varphi, \xi, s) = \lim_{s \to \infty} s^{k\sigma+1} \mathcal{L}_t Res_{u,k}(\varphi, \xi, s) = 0$, $0 < \sigma \le 1$, $k = 1, 2, 3, \cdots$.

To find the coefficients $f_n(\varphi, \xi, s)$, $g_n(\varphi, \xi, s)$, $h_n(\varphi, \xi, s)$, and $l_n(\varphi, \xi, s)$, we recursively solve the following system

$$
\begin{aligned}
\lim_{s \to \infty} s^{k\sigma+1} \mathcal{L}_t Res_{u,k}(\varphi, \xi, s) &= 0, \quad k = 1, 2, \cdots, \\
\lim_{s \to \infty} s^{k\sigma+1} \mathcal{L}_t Res_{v,k}(\varphi, \xi, s) &= 0, \quad k = 1, 2, \cdots, \\
\lim_{s \to \infty} s^{k\sigma+1} \mathcal{L}_t Res_{\rho,k}(\varphi, \xi, s) &= 0, \quad k = 1, 2, \cdots, \\
\lim_{s \to \infty} s^{k\sigma+1} \mathcal{L}_t Res_{p,k}(\varphi, \xi, s) &= 0, \quad k = 1, 2, \cdots.
\end{aligned}
\tag{15}
$$

In the last one, we use inverse LT from Equation (12) to get the kth approximate solutions of $u_k(\varphi, \xi, t)$, $v_k(\varphi, \xi, t)$, $\rho_k(\varphi, \xi, t)$, and $p_k(\varphi, \xi, t)$

4. Numerical Problem

In this section, we examine a system of fractional-order equations that models the unsteady flow of a polytropic gas in order to validate the applicability and precision of the suggested methodology.

Problem

Consider a system of equations of unsteady flow of a polytropic gas of fractional order of the form as [40,41,43]:

$$
\begin{aligned}
D_t^\sigma u + u_\varphi u + v u_\xi + \frac{p_\varphi}{\rho} &= 0, \\
D_t^\sigma v + v_\varphi u + v v_\xi + \frac{p_\xi}{\rho} &= 0, \\
D_t^\sigma \rho + \rho_\varphi u + v \rho_\xi + \rho u_\varphi + \rho v_\xi &= 0, \\
D_t^\sigma p + p_\varphi u + v p_\xi + \tau p u_\varphi + \tau p v_\xi &= 0.
\end{aligned}
\tag{16}
$$

Consider Equation (16), with the following IC's [47]:

$$
\begin{aligned}
u(\varphi, \xi, 0) &= e^{\varphi + \xi}, \\
v(\varphi, \xi, 0) &= -1 - e^{\varphi + \xi}, \\
\rho(\varphi, \xi, 0) &= e^{\varphi + \xi}, \\
p(\varphi, \xi, 0) &= \eta.
\end{aligned}
\tag{17}
$$

Applying LT to Equation (16) and making use of Equation (17), we get

$$
U(\varphi, \xi, s) - \frac{e^{\varphi + \xi}}{s} + \frac{1}{s^{\sigma}} \mathcal{L}_t \left[\mathcal{L}_t^{-1}[U_\varphi] \mathcal{L}_t^{-1}[U] + \mathcal{L}_t^{-1}[V] \mathcal{L}_t^{-1}[U_\xi] + \frac{\mathcal{L}_t^{-1}[P_\varphi]}{\mathcal{L}_t^{-1}[W]} \right] = 0,
$$

$$
V(\varphi, \xi, s) - \frac{-1 - e^{\varphi + \xi}}{s} + \frac{1}{s^{\sigma}} \mathcal{L}_t \left[\mathcal{L}_t^{-1}[V_\varphi] \mathcal{L}_t^{-1}[U] + \mathcal{L}_t^{-1}[V] \mathcal{L}_t^{-1}[V_\xi] + \frac{\mathcal{L}_t^{-1}[P_\xi]}{\mathcal{L}_t^{-1}[W]} \right] = 0,
\tag{18}
$$

$$
W(\varphi, \xi, s) - \frac{e^{\varphi + \xi}}{s} + \frac{1}{s^{\sigma}} \mathcal{L}_t \left[\mathcal{L}_t^{-1}[W_\varphi] \mathcal{L}_t^{-1}[U] + \mathcal{L}_t^{-1}[V] \mathcal{L}_t^{-1}[W_\xi] + \mathcal{L}_t^{-1}[W] \mathcal{L}_t^{-1}[U_\varphi] + \mathcal{L}_t^{-1}[W] \mathcal{L}_t^{-1}[V_\xi] \right] = 0,
$$

$$
P(\varphi, \xi, s) - \frac{\eta}{s} + \frac{1}{s^{\sigma}} \mathcal{L}_t \left[\mathcal{L}_t^{-1}[P_\varphi] \mathcal{L}_t^{-1}[U] + \mathcal{L}_t^{-1}[V] \mathcal{L}_t^{-1}[P_\xi] + \tau \mathcal{L}_t^{-1}[P] \mathcal{L}_t^{-1}[U_\varphi] + \tau \mathcal{L}_t^{-1}[P] \mathcal{L}_t^{-1}[V_\xi] \right] = 0,
$$

and so the kth-truncated term series are

$$
U(\varphi, \xi, s) = \frac{e^{\varphi + \xi}}{s} + \sum_{n=1}^{k} \frac{f_n(\varphi, \xi, s)}{s^{n\sigma + 1}}, \quad V(\varphi, \xi, s) = \frac{-1 - e^{\varphi + \xi}}{s} + \sum_{n=1}^{k} \frac{g_n(\varphi, \xi, s)}{s^{n\sigma + 1}},
$$

$$
W(\varphi, \xi, s) = \frac{e^{\varphi + \xi}}{s} + \sum_{n=1}^{k} \frac{h_n(\varphi, \xi, s)}{s^{n\sigma + 1}}, \quad P(\varphi, \xi, s) = \frac{\eta}{s} + \sum_{n=1}^{k} \frac{l_n(\varphi, \xi, s)}{s^{n\sigma + 1}}, \quad k = 1, 2, 3, 4 \cdots
\tag{19}
$$

and the kth-LRFs as:

$$
\mathcal{L}_t Res_{u,k}(\varphi, \xi, s) = U_k(\varphi, \xi, s) - \frac{e^{\varphi + \xi}}{s} + \frac{1}{s^{\sigma}} \mathcal{L}_t \left[\mathcal{L}_t^{-1}[U_{\varphi,k}] \mathcal{L}_t^{-1}[U_k] + \mathcal{L}_t^{-1}[V_k] \mathcal{L}_t^{-1}[U_{\xi,k}] + \frac{\mathcal{L}_t^{-1}[P_{\varphi,k}]}{\mathcal{L}_t^{-1}[W_k]} \right],
$$

$$
\mathcal{L}_t Res_{v,k}(\varphi, \xi, s) = V_k(\varphi, \xi, s) - \frac{-1 - e^{\varphi + \xi}}{s} + \frac{1}{s^{\sigma}} \mathcal{L}_t \left[\mathcal{L}_t^{-1}[V_{\varphi,k}] \mathcal{L}_t^{-1}[U_k] + \mathcal{L}_t^{-1}[V_k] \mathcal{L}_t^{-1}[V_{\xi,k}] + \frac{\mathcal{L}_t^{-1}[P_{\xi,k}]}{\mathcal{L}_t^{-1}[W_k]} \right],
$$

$$
\mathcal{L}_t Res_{w,k}(\varphi, \xi, s) = W_k(\varphi, \xi, s) - \frac{e^{\varphi + \xi}}{s} + \frac{1}{s^{\sigma}} \mathcal{L}_t \left[\mathcal{L}_t^{-1}[W_{\varphi,k}] \mathcal{L}_t^{-1}[U_k] + \mathcal{L}_t^{-1}[V_k] \mathcal{L}_t^{-1}[W_{\xi,k}] + \mathcal{L}_t^{-1}[W_k] \mathcal{L}_t^{-1}[U_{\varphi,k}] \right.
\tag{20}
$$
$$
\left. + \mathcal{L}_t^{-1}[W_k] \mathcal{L}_t^{-1}[V_{\xi,k}] \right],
$$

$$
\mathcal{L}_t Res_{p,k}(\varphi, \xi, s) = P_k(\varphi, \xi, s) - \frac{\eta}{s} + \frac{1}{s^{\sigma}} \mathcal{L}_t \left[\mathcal{L}_t^{-1}[P_\varphi] \mathcal{L}_t^{-1}[U_k] + \mathcal{L}_t^{-1}[V_k] \mathcal{L}_t^{-1}[P_{\xi,k}] + \tau \mathcal{L}_t^{-1}[P_k] \mathcal{L}_t^{-1}[U_{\varphi,k}] \right.
$$
$$
\left. + \tau \mathcal{L}_t^{-1}[P_k] \mathcal{L}_t^{-1}[V_{\xi,k}] \right].
$$

Now, to determine $f_k(\varphi, \xi, s)$, $g_k(\varphi, \xi, s)$, $h_k(\varphi, \xi, s)$, and $l_k(\varphi, \xi, s)$, $k = 1, 2, 3, \cdots$, we substitute the kth-truncated series Equation (19) into the kth-Laplace residual function Equation (20), multiply the resulting equation by $s^{k\sigma + 1}$, and then recursively solve the relationship $\lim_{s \to \infty} (s^{k\sigma + 1} \mathcal{L}_t Res_{u,k}(\varphi, \xi, s)) = 0$, $\lim_{s \to \infty} (s^{k\sigma + 1} \mathcal{L}_t Res_{v,k}(\varphi, \xi, s)) = 0$, $\lim_{s \to \infty} (s^{k\sigma + 1}$

$\mathcal{L}_t Res_{w,k}(\varphi, \xi, s)) = 0$, and $\lim_{s \to \infty} (s^{k\sigma+1} \mathcal{L}_t Res_{p,k}(\varphi, \xi, s)) = 0$, $k = 1, 2, 3, \cdots$. Following are the first few terms:

$$
\begin{aligned}
f_1(\varphi, \xi, s) &= e^{\varphi+\xi}, \quad g_1(\varphi, \xi, s) = e^{\varphi+\xi}, \quad h_1(\varphi, \xi, s) = e^{\varphi+\xi}, \quad l_1(\varphi, \xi, s) = 0 \\
f_2(\varphi, \xi, s) &= e^{\varphi+\xi}, \quad g_2(\varphi, \xi, s) = e^{\varphi+\xi}, \quad h_2(\varphi, \xi, s) = e^{\varphi+\xi}, \quad l_2(\varphi, \xi, s) = 0 \\
f_3(\varphi, \xi, s) &= e^{\varphi+\xi}, \quad g_3(\varphi, \xi, s) = e^{\varphi+\xi}, \quad h_3(\varphi, \xi, s) = e^{\varphi+\xi}, \quad l_3(\varphi, \xi, s) = 0 \\
f_4(\varphi, \xi, s) &= e^{\varphi+\xi}, \quad g_4(\varphi, \xi, s) = e^{\varphi+\xi}, \quad h_4(\varphi, \xi, s) = e^{\varphi+\xi}, \quad l_4(\varphi, \xi, s) = 0 \\
f_5(\varphi, \xi, s) &= e^{\varphi+\xi}, \quad g_5(\varphi, \xi, s) = e^{\varphi+\xi}, \quad h_5(\varphi, \xi, s) = e^{\varphi+\xi}, \quad l_5(\varphi, \xi, s) = 0 \\
f_6(\varphi, \xi, s) &= e^{\varphi+\xi}, \quad g_6(\varphi, \xi, s) = e^{\varphi+\xi}, \quad h_6(\varphi, \xi, s) = e^{\varphi+\xi}, \quad l_6(\varphi, \xi, s) = 0.
\end{aligned}
\tag{21}
$$

and so on.

Putting the values of $f_k(\varphi, \xi, s)$, $g_k(\varphi, \xi, s)$, $h_k(\varphi, \xi, s)$, and $l_k(\varphi, \xi, s)$, $k = 1, 2, 3, \cdots$, in Equation (19), we get

$$
\begin{aligned}
U(\varphi, \xi, s) &= \frac{e^{\varphi+\xi}}{s} + \frac{e^{\varphi+\xi}}{s^{\sigma+1}} + \frac{e^{\varphi+\xi}}{s^{2\sigma+1}} + \frac{e^{\varphi+\xi}}{s^{3\sigma+1}} + \frac{e^{\varphi+\xi}}{s^{4\sigma+1}} + \frac{e^{\varphi+\xi}}{s^{5\sigma+1}} + \frac{e^{\varphi+\xi}}{s^{6\sigma+1}} + \cdots, \\
V(\varphi, \xi, s) &= \frac{-1 - e^{\varphi+\xi}}{s} + \frac{e^{\varphi+\xi}}{s^{\sigma+1}} + \frac{e^{\varphi+\xi}}{s^{2\sigma+1}} + \frac{e^{\varphi+\xi}}{s^{3\sigma+1}} + \frac{e^{\varphi+\xi}}{s^{4\sigma+1}} + \frac{e^{\varphi+\xi}}{s^{5\sigma+1}} + \frac{e^{\varphi+\xi}}{s^{6\sigma+1}} + \cdots, \\
W(\varphi, \xi, s) &= \frac{e^{\varphi+\xi}}{s} + \frac{e^{\varphi+\xi}}{s^{\sigma+1}} + \frac{e^{\varphi+\xi}}{s^{2\sigma+1}} + \frac{e^{\varphi+\xi}}{s^{3\sigma+1}} + \frac{e^{\varphi+\xi}}{s^{4\sigma+1}} + \frac{e^{\varphi+\xi}}{s^{5\sigma+1}} + \frac{e^{\varphi+\xi}}{s^{6\sigma+1}} + \cdots, \\
P(\varphi, \xi, s) &= \frac{\eta}{s}.
\end{aligned}
\tag{22}
$$

Using inverse LT, we get

$$
\begin{aligned}
u(\varphi, \xi, t) &= e^{\varphi+\xi}\left(1 + \frac{t^\sigma}{\Gamma(\sigma+1)} + \frac{t^{2\sigma}}{\Gamma(2\sigma+1)} + \frac{t^{3\sigma}}{\Gamma(3\sigma+1)} + \frac{t^{4\sigma}}{\Gamma(4\sigma+1)} + \frac{t^{5\sigma}}{\Gamma(5\sigma+1)} + \cdots\right), \\
v(\varphi, \xi, t) &= -1 - e^{\varphi+\xi}\left(1 + \frac{t^\sigma}{\Gamma(\sigma+1)} + \frac{t^{2\sigma}}{\Gamma(2\sigma+1)} + \frac{t^{3\sigma}}{\Gamma(3\sigma+1)} + \frac{t^{4\sigma}}{\Gamma(4\sigma+1)} + \frac{t^{5\sigma}}{\Gamma(5\sigma+1)} + \cdots\right), \\
\rho(\varphi, \xi, t) &= e^{\varphi+\xi}\left(1 + \frac{t^\sigma}{\Gamma(\sigma+1)} + \frac{t^{2\sigma}}{\Gamma(2\sigma+1)} + \frac{t^{3\sigma}}{\Gamma(3\sigma+1)} + \frac{t^{4\sigma}}{\Gamma(4\sigma+1)} + \frac{t^{5\sigma}}{\Gamma(5\sigma+1)} + \cdots\right), \\
p(\varphi, \xi, t) &= \eta.
\end{aligned}
$$

Putting $\sigma = 1$

$$
\begin{aligned}
u(\varphi, \xi, t) &= e^{\varphi+\xi}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \cdots\right), \\
v(\varphi, \xi, t) &= -1 - e^{\varphi+\xi}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \cdots\right), \\
\rho(\varphi, \xi, t) &= e^{\varphi+\xi}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \cdots\right), \\
p(\varphi, \xi, t) &= \eta.
\end{aligned}
\tag{23}
$$

The exact solutions are

$$
\begin{aligned}
u(\varphi, \xi, t) &= e^{\varphi+\xi+t}, \\
v(\varphi, \xi, t) &= -1 - e^{\varphi+\xi+t}, \\
\rho(\varphi, \xi, t) &= e^{\varphi+\xi+t}, \\
p(\varphi, \xi, t) &= \eta.
\end{aligned}
\tag{24}
$$

5. Results and Disscusion

The numerical simulation of the obtained solution for four systems of differential equations describing the unsteady flow of a polytropic gas of arbitrary order is presented in this paper. The sixth-order LRPSM solutions are discovered in this study, which are given in Table 1. The proposed method is more accurate, as shown in the table. For a variety of parameter values, we present 2D and 3D plots to describe the behaviour of the LRPSM solutions. The natures of the LRPSM solutions for $u(\varphi, \xi, t)$ and $v(\varphi, \xi, t)$ at different orders are represented in Figure 1 as 2D plots, and Figures 2 and 3 represent surfaces of $u(\varphi, \xi, t)$ and $v(\varphi, \xi, t)$ respectively, and LRPSM solutions for $\rho(\varphi, \xi, t)$ and $w(\varphi, \xi, t)$ at different orders are represented in Figure 4 as 2D plots and in Figure 5 as 3D plots for $w(\varphi, \xi, t)$. By simulating and displaying the physical properties of nonlinear phenomena that arise in technology and science, we can study and analyse their physical behaviour. In the analysis of complex coupled fractional-order problems, the suggested technique is more appropriate and efficient.

Table 1. Error analysis for the LRPSM solution for the proposed problem with various values of φ and t when $\xi = 1, \sigma = 1$.

t	φ	$\lvert u_{exact} - u_{LRPSM} \rvert$	$\lvert v_{exact} - v_{LRPSM} \rvert$	$\lvert \rho_{exact} - \rho_{LRPSM} \rvert$	$\lvert p_{exact} - p_{LRPSM} \rvert$
0.2	0.2	7.0×10^{-9}	7.0×10^{-9}	7.0×10^{-9}	0
	0.4	8.0×10^{-9}	8.0×10^{-9}	8.0×10^{-9}	0
	0.6	1.1×10^{-8}	1.1×10^{-8}	1.1×10^{-8}	0
	0.8	1.4×10^{-8}	1.4×10^{-8}	1.4×10^{-8}	0
	1	1.5×10^{-8}	1.5×10^{-8}	1.5×10^{-8}	0
0.4	0.2	1.134×10^{-6}	1.134×10^{-6}	1.134×10^{-6}	0
	0.4	1.385×10^{-6}	1.385×10^{-6}	1.385×10^{-6}	0
	0.6	1.693×10^{-6}	1.693×10^{-6}	1.693×10^{-6}	0
	0.8	2.067×10^{-6}	2.067×10^{-6}	2.067×10^{-6}	0
	1	2.52×10^{-6}	2.52×10^{-6}	2.52×10^{-6}	0
0.6	0.2	1.9921×10^{-5}	1.9921×10^{-5}	1.9921×10^{-5}	0
	0.4	2.4333×10^{-5}	2.4333×10^{-5}	2.4333×10^{-5}	0
	0.6	2.9720×10^{-5}	2.9720×10^{-5}	2.9720×10^{-5}	0
	0.8	3.630×10^{-5}	3.630×10^{-5}	3.630×10^{-5}	0
	1	4.434×10^{-5}	4.434×10^{-5}	4.434×10^{-5}	0
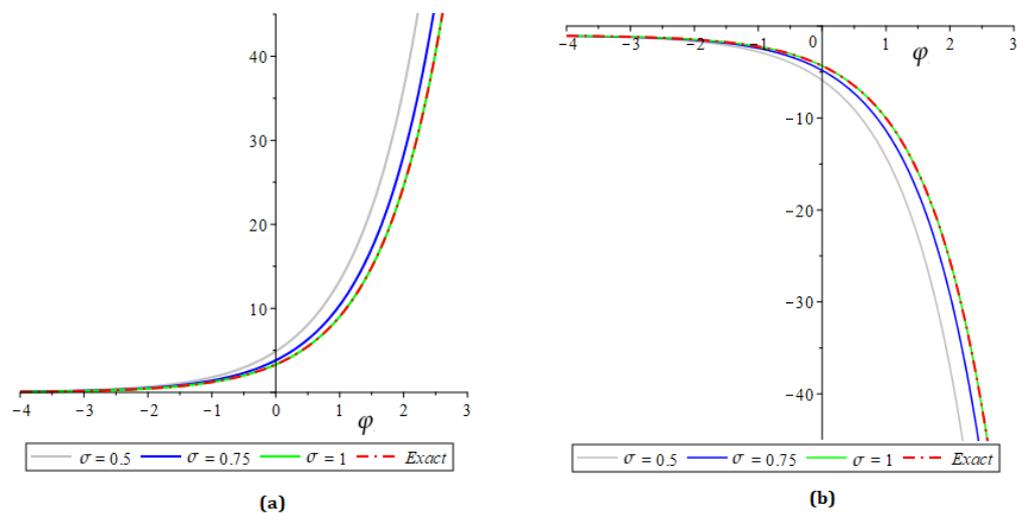
Figure 1. Exact and LRPSM solutions for (**a**) $u(\varphi, \xi, t)$ and (**b**) $v(\varphi, \xi, t)$ at distinct values of σ at ξ and $t = 1$.
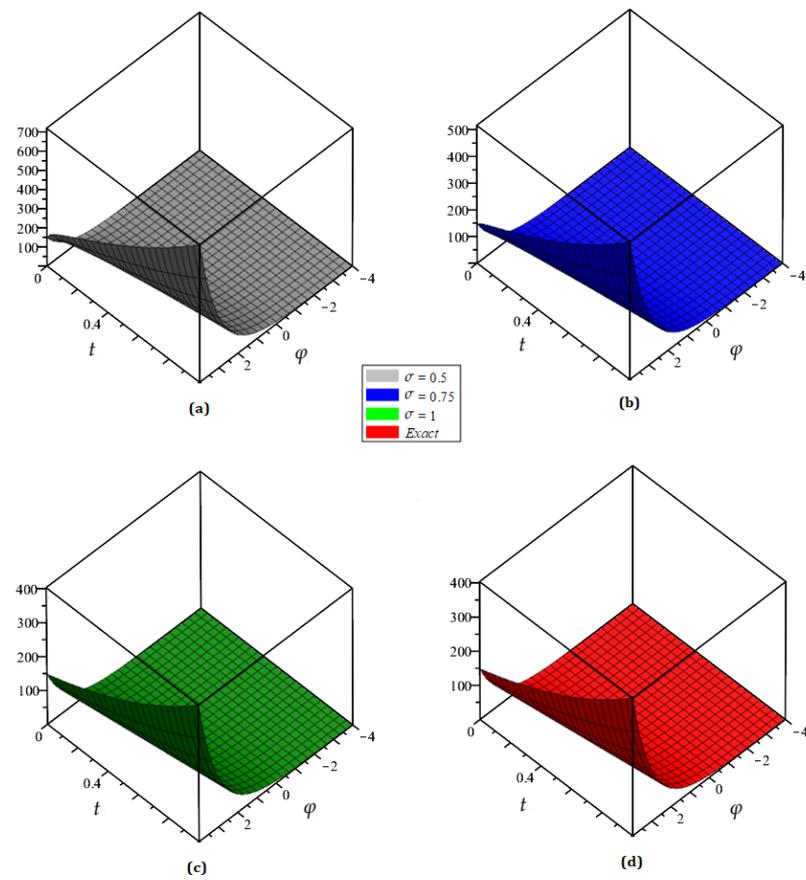
Figure 2. Three-dimensional plots of LRPSM solutions for $u(\varphi, \xi, t)$ at distinct values of σ at $\xi = 1$. (**a**) $\sigma = 0.5$ (**b**) $\sigma = 0.75$ (**c**) $\sigma = 1$ (**d**) Exact.
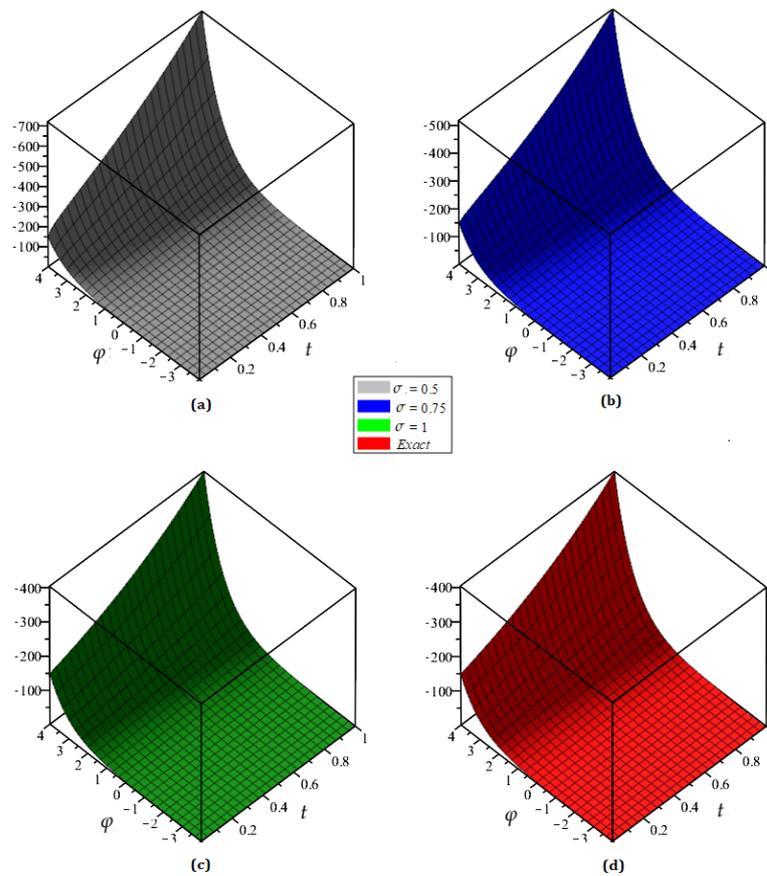
Figure 3. Three-dimensional plots of LRPSM solutions for $v(\varphi, \xi, t)$ at distinct values of σ at $\xi = 1$. (**a**) $\sigma = 0.5$ (**b**) $\sigma = 0.75$ (**c**) $\sigma = 1$ (**d**) Exact.
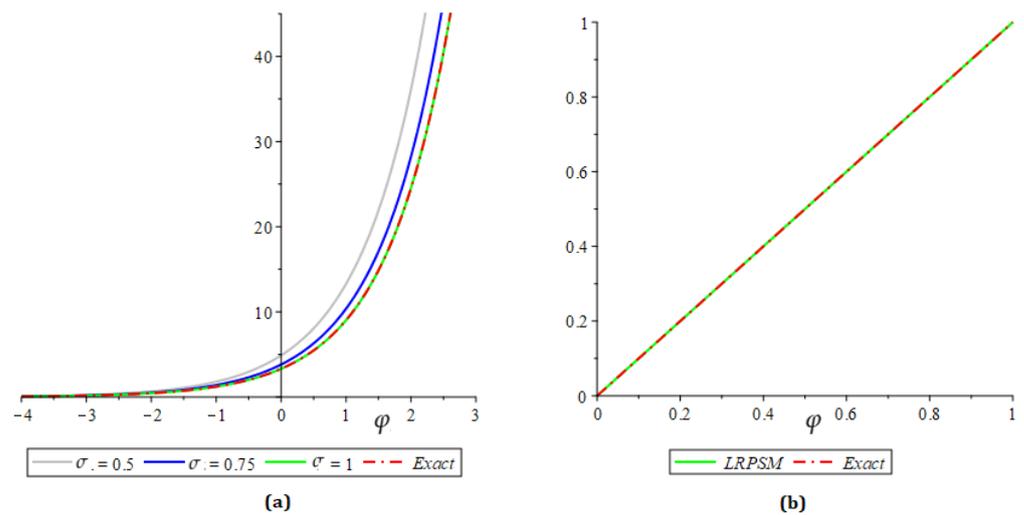
Figure 4. Exact and LRPSM solutions for (**a**) $\rho(\varphi, \xi, t)$ and (**b**) $p(\varphi, \xi, t)$ at distinct values of σ at ξ and $t = 1$.
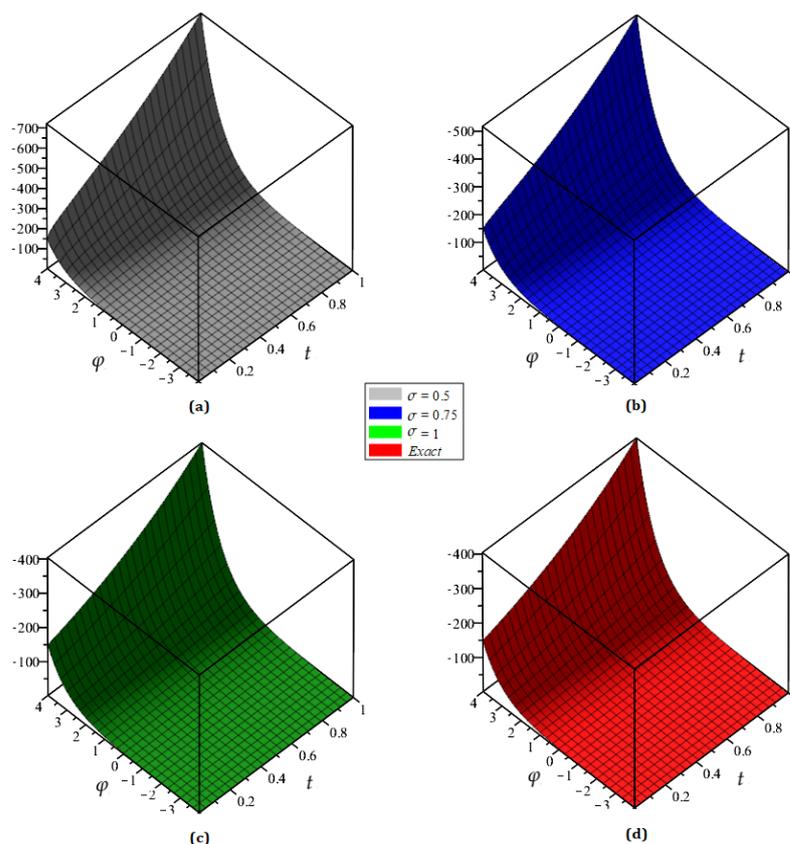
Figure 5. Three-dimensional plots of LRPSM solutions for $\rho(\varphi, \xi, t)$ at distinct values of σ at $\xi = 1$. (**a**) $\sigma = 0.5$ (**b**) $\sigma = 0.75$ (**c**) $\sigma = 1$ (**d**) Exact.

6. Conclusions

This article presents a combination of the residual power series and Laplace transform to solve the gas dynamics model. The advantage of the new method is that it reduces the amount of computational effort required to obtain a solution in a power series form whose coefficients must be computed in successive algebraic steps. The suggested method is employed to solve three distinct physical models, and its capacity to address fractional nonlinear equations with high precision and simple computation steps has been demonstrated. In future work, we intend to extend the Laplace-transform residual power series technique to physical applications with higher dimensions.

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