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# Improvement of Some Hayashi–Ostrowski Type Inequalities with Applications in a Probability Setting

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**Abstract:** Different types of mathematical inequalities have been largely analyzed and employed. In this paper, we introduce improvements to some Ostrowski type inequalities and present their corresponding proofs. The presented proofs are based on applying the celebrated Hayashi inequality to certain functions. We provide examples that show these improvements. Illustrations of the obtained results are stated in a probability framework.

**Keywords:** Gauss and Simpson quadratures; Hayashi inequality; Ostrowski inequality

**MSC:** 26D15

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## 1. Introduction and Background

A number of inequalities have been widely studied and used in different contexts [1]. For instance, some integral inequalities involving the Taylor remainder were established in [2,3]. Sharp Hermite–Hadamard integral inequalities, sharp Ostrowski inequalities and generalized trapezoid type for Riemann–Stieltjes integrals, as well as a companion of this generalization, were introduced in [4–6], respectively. In addition, some authors provided Grüss type inequalities in one and several variables [7–9]. Specifically, Grüss type inequalities with multiple points for derivatives bounded by functions on time scales, Ostrowski–Grüss type inequalities of the Chebyshev functional with an application to one-point integrals, and Grüss type inequalities for vector-valued functions, were analyzed in [10–12], respectively. In [13], it was proved that an endpoint Kato–Ponce inequality holds and presented endpoint approximations for variants of this inequality.

In 1938, Ostrowski [14] established an interesting inequality for differentiable functions with bounded derivatives as follows. Let $I$ be an interval and $f\colon I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on $I^\circ$ such that $f' \in L[a,b]$, for $a, b \in I$ and $a < b$, where $I^\circ$ denotes the interior of $I$, $f'$ is the derivative of $f$, and $L[a,b]$ stands for an integrable function on $[a,b]$. If $|f'(x)| \le M$, then the inequality

$$\left|(b-a)f(x) - \int_a^b f(u)\mathrm{d}u\right| \le M\left(\frac{1}{4}(b-a)^2 + \left(x - \frac{(a+b)}{2}\right)^2\right) \tag{1}$$

holds for all $x \in [a,b]$. The constant $1/4$ defined in (1) is the best possible in the sense that it cannot be replaced by a smaller value. This inequality has been rewritten for arbitrary two-points in [15] and then generalized in [16]. For recent results and extensions concerning the Ostrowski inequality, we refer the reader to the comprehensive book [17] and the recent survey [18].

The celebrated Hayashi inequality [19] (see also [1], pp. 311–312), which is presented in the following theorem, is one of the most important inequalities that has been utilized to develop some Ostrowski type inequalities.

**Theorem 1.** *Let $q\colon [c,d] \to \mathbb{R}$ be a nonincreasing function on $[c,d]$ and $k\colon [c,d] \to \mathbb{R}$ an integrable function on $[c,d]$, with $0 \le k(s) \le B$, for all $s \in [c,d]$. Then, the inequality*

$$B\int_{d-\mu}^{d} q(s)\mathrm{d}s \le \int_{c}^{d} q(s)k(s)\mathrm{d}s \le B\int_{c}^{c+\mu} q(s)\mathrm{d}s \tag{2}$$

*holds, where $\mu := (1/B)\int_c^d k(s)\mathrm{d}s$ and $B$ is a positive real constant.*

The Hayashi inequality is a generalization of the Steffensen inequality [20], which holds under the same conditions with $B = 1$ in the expression given in (2). To observe the importance and applications of the Hayashi inequality, the expression stated in (2) was used to prove three Ostrowski type inequalities as presented in the following theorem.

**Theorem 2.** *Let $\psi\colon [c,d] \to \mathbb{R}$ be an absolutely continuous function on $[c,d]$, with $0 \le \psi'(s) \le (d-c)$, and $\psi'$ being integrable on $[c,d]$. Then, we have that:*

*(i)*

$$\left|\frac{1}{(d-c)}\int_c^d \psi(s)\mathrm{d}s - \frac{(y-c)\psi(c)+(d-y)\psi(d)}{(d-c)} - \mu\left(y - \frac{(c+d)}{2}\right)\right| \le \frac{\mu}{2}(d-c-\mu) \le \frac{(d-c)^2}{8}, \tag{3}$$

*for all $y \in [c,d]$, with the equality with respect to (3) being satisfied when $\psi(y) := y$, for $y \in [0,1]$;*

*(ii)*

$$\left|\frac{1}{(d-c)}\int_c^d \psi(s)\mathrm{d}s - \psi(y) + \mu\left(y - \frac{(c+d)}{2}\right)\right| \le \frac{\mu(d-c)}{2} - \mu^2 \le \frac{(d-c)^2}{16}, \tag{4}$$

*for all $y \in [c,d]$; and*

*(iii)*

$$\left|\frac{1}{(d-c)}\int_c^d \psi(s)\mathrm{d}s - \frac{\psi(y)+\psi(c+d-y)}{2}\right| \le \mu\left(\frac{(d-c)}{2} - \frac{3\mu}{2}\right) \le \frac{(d-c)^2}{24}, \tag{5}$$

*for all $y \in [c,(c+d)/2]$, where $\mu := (\psi(d)-\psi(c))/(d-c)$.*

Same versions of the inequalities given in (3)–(5) for general absolutely continuous functions $\psi$ on $[c,d]$, with $\zeta \le \psi'(s) \le Z$, were proved. Note that the upper bounds of the inequalities stated in (3)–(5) are better than the same inequalities presented in the literature [6,21–25]. In [26], it was proved a generalized Ostrowski inequality for differentiable functions defined on an interval $[c,d]$ expressed as

$$\left|\int_c^d \psi(s)\mathrm{d}s - (d-c)\left(\frac{\eta(\psi(c)+\psi(d))}{2} + (1-\eta)\psi(y)\right)\right| \le \left(\frac{(d-c)^2}{4}\left(\eta^2+(1-\eta)^2\right) + \left(y - \frac{(c+d)}{2}\right)^2\right)\|\psi'\|_\infty, \tag{6}$$

for all $\eta \in [0,1]$ and $y \in [c+\eta(d-c)/2, d-\eta(d-c)/2]$, with $\|\psi'\|_\infty := \sup_{s\in[c,d]} |\psi'(s)| < \infty$.

In [27], the following perturbed version of the inequality stated in (6), for differentiable functions satisfying $\zeta \le \psi'(s) \le Z$, for all $s \in [c,d]$, was proved:

$$\left|\int_c^d \psi(s)\mathrm{d}s - (d-c)\left((1-\eta)\psi(y) + \frac{\eta(\psi(c)+\psi(d))}{2} - (1-\eta)\frac{(Z+\zeta)}{2}\left(y - \frac{(c+d)}{2}\right)\right)\right| \le \frac{(Z-\zeta)}{2}\left(\frac{(d-c)^2}{4}\left(\eta^2 + (1-\eta)^2\right) + \left(y - \frac{(c+d)}{2}\right)^2\right), \quad (7)$$

for $y \in [c+\eta(d-c)/2, d-\eta(d-c)/2]$. The best constant of the inequality given in (6) is $1/8$ occurring when $y := (c+d)/2$. However, this constant becomes $1/16$ if one employs the inequality defined in (7), which means that inequality presented in (7) is better than the inequality expressed in (6). For more information about the above inequalities, the reader is referred to [22,27–29] and the references therein.

The objective of this work is to improve the inequalities stated in (3)–(7). Then, we obtain better bounds for such inequalities. Our proofs for these inequalities are based on applying the Hayashi inequality formulated in (2). We give some examples that show these improvements. Hence, an application of this inequality in a probability framework is performed, as well as its role in constructing and improving some old inequalities.

The remainder of this paper is organized as follows. In Section 2, we provide the main results of this investigation and examples that show the improvements that we obtain. Section 3 presents applications of our results in a probability setting. We end this paper with brief conclusions about our study in Section 4.

## 2. Main Results

We begin with a generalization of the inequality established in (3) in the following theorem.

**Theorem 3.** *Under the assumptions of Theorem 2, we reach*

$$\left|\frac{1}{(d-c)}\int_c^d \psi(s)\mathrm{d}s - \left(\frac{\eta(\psi(y)+\psi(c+d-y))}{2} + (1-\eta)\psi\left(\frac{c+d}{2}\right)\right)\right| \le \frac{\mu(d-c)}{2} - 2\mu^2 \le \frac{(d-c)^2}{32}, \quad (8)$$

*for all $y \in [c,(c+d)/2]$. In particular, if $y := c$ in the inequality given in (8), we get*

$$\left|\frac{1}{(d-c)}\int_c^d \psi(s)\mathrm{d}s - \left(\frac{\eta(\psi(c)+\psi(d))}{2} + (1-\eta)\psi\left(\frac{c+d}{2}\right)\right)\right| \le \frac{\mu(d-c)}{2} - 2\mu^2 \le \frac{(d-c)^2}{32},$$

*for all $\eta \in [0,1]$, where $\mu := (\psi(d)-\psi(c))/(d-c)$.*

**Proof.** Fix $y \in [c,(c+d)/2]$. Seeking simplicity to reach our proof, we divide it into four steps as follows.

Step 1: Let $g(s) := c - s$ and $s \in [c,y]$. Applying the Hayashi inequality stated in (2), setting $q(s) = g(s)$ and $k(s) = \psi'(s)$, we have

$$(d-c)\int_{y-\mu}^{y}(c-s)\mathrm{d}s \le \int_c^y (c-s)\psi'(s)\mathrm{d}s \le (d-c)\int_c^{c+\mu}(c-s)\mathrm{d}s, \quad (9)$$

where

$$\mu := \frac{1}{(d-c)} \int_c^d \psi'(s)\mathrm{d}s = \frac{\psi(d)-\psi(c)}{(d-c)}.$$

Now, we attain at

$$\int_{y-\mu}^{y} (c-s)\mathrm{d}s = -\mu(y-c) + \frac{\mu^2}{2}, \tag{10}$$

$$\int_c^y (c-s)\psi'(s)\mathrm{d}s = -(y-c)\psi(y) + \int_c^y \psi(s)\mathrm{d}s, \tag{11}$$

$$\int_c^{c+\mu} (c-s)\mathrm{d}s = -\frac{\mu^2}{2}. \tag{12}$$

Then, substituting Equations (10)–(12) in the inequality given in (9), we generate

$$(d-c)\left(\frac{\mu^2}{2} - \mu(y-c)\right) \le -(y-c)\psi(y) + \int_c^y \psi(s)\mathrm{d}s \le -\frac{\mu^2}{2}(d-c). \tag{13}$$

Step 2: Let $g(s) := (1-\eta\, c/2) + (\eta\, d/2) - s$ and $s \in (y, (c+d)/2]$, with $\eta \in [0,1]$. Employing the inequality presented in (2) again, we generate

$$(d-c)\int_{\frac{c+d}{2}-\mu}^{\frac{c+d}{2}} \left(\left(1-\frac{\eta}{2}\right)c + \frac{\eta\, d}{2} - s\right)\mathrm{d}s \le \int_y^{\frac{c+d}{2}} \left(\left(1-\frac{\eta\, c}{2}\right) + \frac{\eta\, d}{2} - s\right)\psi'(s)\mathrm{d}s$$
$$\le (d-c)\int_y^{y+\mu} \left(\left(1-\frac{\eta\, c}{2}\right) + \frac{\eta\, d}{2} - s\right)\mathrm{d}s. \tag{14}$$

Then, we obtain

$$\int_{\frac{c+d}{2}-\mu}^{\frac{c+d}{2}} \left(\left(1-\frac{\eta}{2}\right)c + \frac{\eta\, d}{2} - s\right)\mathrm{d}s = -(1-\eta)\frac{\mu(d-c)}{2} + \frac{\mu^2}{2}, \tag{15}$$

$$= \left(y - c - \frac{\eta(d-c)}{2}\right)\psi(y) \tag{16}$$
$$-(1-\eta)\frac{(d-c)}{2}\psi\left(\frac{c+d}{2}\right) + \int_y^{\frac{c+d}{2}} \psi(s)\mathrm{d}s,$$

$$\int_y^{y+\mu} \left(\left(1-\frac{\eta}{2}\right)c + \frac{\eta\, b}{2} - s\right)\mathrm{d}s = \frac{\eta\mu(d-c)}{2} - \mu(y-c) - \frac{\mu^2}{2}. \tag{17}$$

Substituting Equations (15)–(17) in the inequality stated in (14), we arrive at

$$(d-c)\left(\frac{\mu^2}{2} - (1-\eta)\frac{\mu(d-c)}{2}\right)$$
$$\le \left(y - c - \frac{\eta(d-c)}{2}\right)\psi(y) - (1-\eta)\frac{(d-c)}{2}\psi\left(\frac{c+d}{2}\right) + \int_y^{\frac{c+d}{2}} \psi(s)\mathrm{d}s$$
$$\le (d-c)\left(\frac{\eta\mu(d-c)}{2} - \mu(y-c) - \frac{\mu^2}{2}\right). \tag{18}$$

Step 3: Let $g(s) := (\eta\, c/2) + (1-\eta/2)d - s$ and $s \in ((c+d)/2, c+d-y]$, with $\eta \in [0,1]$. Using the inequality defined in (2) again, we have

$$(d-c)\int_{c+d-y-\mu}^{c+d-y}\left(\frac{\eta\, c}{2}+\left(1-\frac{\eta\, d}{2}\right)-s\right)\mathrm{d}s \leq \int_{\frac{c+d}{2}}^{c+d-y}\left(\frac{\eta\, c}{2}+\left(1-\frac{\eta\, d}{2}\right)-s\right)\psi'(s)\mathrm{d}s$$
$$\leq (d-c)\int_{\frac{c+d}{2}}^{\frac{c+d}{2}+\mu}\left(\frac{\eta\, c}{2}+\left(1-\frac{\eta\, d}{2}\right)-s\right)\mathrm{d}s. \quad (19)$$

Then, we obtain

$$\int_{c+d-y-\mu}^{c+d-y}\left(\frac{\eta\, c}{2}+\left(1-\frac{\eta\, d}{2}\right)-s\right)\mathrm{d}s = -\frac{\mu\eta(d-c)}{2}-\mu(y-c)+\frac{\mu^2}{2}, \quad (20)$$

$$\int_{\frac{c+d}{2}}^{c+d-y}\left(\frac{\eta\, c}{2}+\left(1-\frac{\eta\, d}{2}\right)s\right)\psi'(s)\mathrm{d}s = \left(y-c-\frac{\eta(d-c)}{2}\right)\psi(c+d-y) \quad (21)$$
$$-(1-\eta)\frac{(d-c)}{2}\psi\left(\frac{c+d}{2}\right)+\int_{\frac{c+d}{2}}^{c+d-y}\psi(s)\mathrm{d}s,$$

$$\int_{\frac{c+d}{2}}^{\frac{c+d}{2}+\mu}\left(\frac{\eta\, c}{2}+\left(1-\frac{\eta\, d}{2}\right)-s\right)\mathrm{d}s = -(1-\eta)\frac{\mu(d-c)}{2}-\frac{\mu^2}{2}. \quad (22)$$

Substituting Equations (20)–(22) in the inequality given in (19), we attain at

$$(d-c)\left(\frac{\mu^2}{2}-\frac{\mu\eta(d-c)}{2}-\mu(y-c)\right)$$
$$\leq \left(y-c-\frac{\eta(d-c)}{2}\right)\psi(c+d-y)-(1-\eta)\frac{(d-c)}{2}\psi\left(\frac{c+d}{2}\right)+\int_{\frac{c+d}{2}}^{c+d-y}\psi(s)\mathrm{d}s$$
$$\leq (d-c)\left(-(1-\eta)\frac{\mu(d-c)}{2}-\frac{\mu^2}{2}\right). \quad (23)$$

Step 4: Let $g(s) := d-s$ and $s\in(c+d-y,d]$. Using the inequality given in (2), we obtain

$$(d-c)\int_{d-\mu}^{d}(d-s)\mathrm{d}s \leq \int_{c+d-y}^{d}(d-s)\psi'(s)\mathrm{d}s \leq (d-c)\int_{c+d-y}^{c+d-y+\mu}(d-s)\mathrm{d}s. \quad (24)$$

Now, we have

$$\int_{d-\mu}^{d}(d-s)\mathrm{d}s = \frac{\mu^2}{2}, \quad (25)$$

$$\int_{c+d-y}^{d}(d-s)\psi'(s)\mathrm{d}s = -(y-c)\psi(c+d-y)+\int_{c+d-y}^{d}\psi(s)\mathrm{d}s, \quad (26)$$

$$\int_{c+d-y}^{c+d-y+\mu}(d-s)\mathrm{d}s = \mu(y-c)-\frac{\mu^2}{2}. \quad (27)$$

Substituting Equations (25)–(27) in the inequality expressed in (24), we generate

$$\frac{\mu^2}{2}(d-c) \leq -(y-c)\psi(c+d-y)+\int_{c+d-y}^{d}\psi(s)\mathrm{d}s \leq (d-c)\left(\mu(y-c)-\frac{\mu^2}{2}\right). \quad (28)$$

Adding the inequalities formulated in (13), (18), (23) and (28), we reach an inequality of the form $-L \leq u \leq L$, that is, $|u| \leq L$. Thus, we arrive at

$$\begin{aligned}
&(c-d)\left(\frac{\mu(d-c)}{2}-2\mu^2\right) \\
&\leq -\frac{\eta(d-c)}{2}\psi(y) - 2(1-\eta)\frac{(d-c)}{2}\psi\left(\frac{c+d}{2}\right) - \frac{\eta(d-c)}{2}\psi(c+d-y) + \int_c^d \psi(s)\mathrm{d}s \\
&\qquad \leq (d-c)\left(\frac{\mu(d-c)}{2}-2\mu^2\right),
\end{aligned}$$

which gives the first inequality stated in (8). The second inequality defined below the expression stated in (8) can be proved by considering the function $\delta(s) := -2s^2 + (d-c)/2s$. Then, $\max\{\delta(s)\} := \delta((d-c)/8) = (d-c)^2/32$, so that $\delta(\mu) := -\mu^2 + (d-c)/2\mu \leq (d-c)^2/32$, which completes the proof of the theorem. □

A generalization of the inequality given in (8) is incorporated in the following corollary.

**Corollary 1.** *Let $\psi$: $[c,d] \to \mathbb{R}$ be an absolutely continuous function on $[c,d]$, with $\zeta \leq \psi'(s) \leq Z$, and $\psi'$ being integrable on $[c,d]$. Then, we have*

$$\begin{aligned}
&\left|\frac{1}{(d-c)}\int_c^d \psi(s)\mathrm{d}s - \left(\frac{\eta(\psi(y)+\psi(c+d-y))}{2} + (1-\eta)\psi\left(\frac{c+d}{2}\right)\right)\right| \\
&\qquad \leq \frac{(Z-\zeta)}{2}\frac{\mu(d-c-2\mu)}{(d-c)} \leq \frac{(Z-\zeta)(d-c)}{32}, \qquad (29)
\end{aligned}$$

*for all $y \in [c,(c+d)/2]$. In particular, for $y := c$, we obtain*

$$\begin{aligned}
\left|\frac{1}{(d-c)}\int_c^d \psi(s)\mathrm{d}s - \left(\frac{\eta(\psi(c)+\psi(d))}{2} + (1-\eta)\psi\left(\frac{c+d}{2}\right)\right)\right| &\leq \frac{(Z-\zeta)}{2}\frac{\mu(d-c-2\mu)}{(d-c)} \\
&\leq \frac{(Z-\zeta)(d-c)}{32},
\end{aligned}$$

*for all $\eta \in [0,1]$, where $\mu := (\psi(d)-\psi(c)-\zeta(d-c))/(Z-\zeta)$.*

**Proof.** It may be established by repeating the proof of Theorem 3, with $k(s) := \psi'(s) - \zeta$, for $s \in [c,d]$. □

Some cases of the inequality stated in (29) are deduced in the following remarks.

**Remark 1.** *As noticed, the inequalities presented in (6) and (7) give the best upper bounds when $y := (c+d)/2$. Therefore, the inequality expressed in (29) improves the upper bound of the inequality established in (6) by $1/4$, and of the inequality formulated in (7) by $1/2$, which is better than the Ujević improvement [27].*

**Remark 2.** *If we set $\eta = 1$ in the inequality stated in (29), then this inequality is better than the inequality given in (5).*

**Remark 3.** *If we fix $\eta = 0$ in the inequality defined in (29), then we recapture the inequality presented in (4). However, if one chooses $y := (c+d)/2$, then the inequality formulated in (29) is better than the inequality expressed in (4).*

**Remark 4.** *Let $[c,d] \equiv [-1,1]$ in the inequality established in (29). Then, we reach*

$$\left|\frac{1}{2}\int_{-1}^1 \psi(s)\mathrm{d}s - \left(\frac{\eta(\psi(-y)+\psi(y))}{2} + (1-\eta)\psi(0)\right)\right| \leq \frac{(Z-\zeta)}{2}\mu(1-\mu) \leq \frac{(Z-\zeta)}{16}, \qquad (30)$$

*for all $y \in [-1, 0]$. In particular, if $y = -\sqrt{3/5}$ in the inequality presented in (30), then we arrive at*

$$\begin{aligned}\left|\frac{1}{2}\int_{-1}^{1}\psi(s)\mathrm{d}s - \left(\frac{\eta\left(\psi(-\sqrt{3/5}) + \psi(\sqrt{3/5})\right)}{2} + (1-\eta)\psi(0)\right)\right| &\leq \frac{(Z-\zeta)}{2}\mu(1-\mu)\\ &\leq \frac{(Z-\zeta)}{16},\end{aligned}$$

*where $\mu := (\psi(1) - \psi(-1) - 2\zeta)/(Z - \zeta)$. Thus, the inequality stated in (30) can be re-written for $\eta = 5/9$ as*

$$\left|\int_{-1}^{1}\psi(s)\mathrm{d}s - \left(\frac{5}{9}\psi\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9}\psi(0) + \frac{5}{9}\psi\left(\sqrt{\frac{3}{5}}\right)\right)\right| \leq (Z-\zeta)\mu(1-\mu) \leq \frac{(Z-\zeta)}{8},$$

*which gives an approximation error for the Gauss–Legendre quadrature rule of 3rd order; that is,*

$$\int_{-1}^{1}\psi(t)\mathrm{d}t \approx \frac{5}{9}\psi\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9}\psi(0) + \frac{5}{9}\psi\left(\sqrt{\frac{3}{5}}\right).$$

*It is helpful to remember that the error term of the Gauss–Legendre quadrature rule involves a 5th degree derivative, so that this rule can precisely integrate polynomials of 5th degree. However, if the 5th derivative of a given function is either unbounded or does not exist, then we cannot apply it. Using our most recent approximations, we can apply the Gauss–Legendre quadrature of 3rd order to continuous differentiable functions (that is, with only a first bounded derivative) and an absolute error less than $(Z - \zeta)/8$, where $\zeta \leq \psi'(s) \leq Z$. This is an elegant advantage of our result.*

**Example 1.** *Let $\psi(t) := t^5 \sin(1/t)$, for $t \in [-1, 1]$. Clearly, $\psi^{(5)}(t)$ does not exist (since the 5th derivative is unbounded). Thus, we cannot apply the Gauss–Legendre quadrature of 3rd order. However, as noted in Remark 4, one can find $7.97443 \times 10^{-6} \leq \psi'(t) \leq 3.66705$, and then*

$$\int_{-1}^{1}\psi(t)\mathrm{d}t \approx \frac{5}{9}\psi\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9}\psi(0) + \frac{5}{9}\psi\left(\sqrt{\frac{3}{5}}\right) = 0.29779. \tag{31}$$

*Note that the exact value of the integral stated in Equation (31) is equal to 0.30159. Moreover, the absolute error is equal to 0.00379, which is less than $(Z - \zeta)/8 = 0.45838$.*

An improvement of the inequality defined in (6) can be deduced by applying the Hayashi inequality as presented in the following theorem.

**Theorem 4.** *Under the assumptions of Theorem 3, we have*

$$\begin{aligned}&\left|\frac{1}{(d-c)}\int_{c}^{d}\psi(s)\mathrm{d}s - (1-\beta)\psi(y) - \frac{\beta(\psi(c)+\psi(d))}{2} + \mu\left(y - \frac{(c+d)}{2}\right)\right|\\ &\qquad\leq \frac{\mu(d-c)}{2} - \mu^2 \leq \frac{(d-c)^2}{16},\end{aligned} \tag{32}$$

*for all $y \in [c + \beta(d - c)/2, d - \beta(d - c)/2]$, where $\mu := (\psi(d) - \psi(c))/(d - c)$, and for all $\beta \in [0, 1]$. As a particular case, a Simpson type inequality is deduced once we choose $y := (c + d)/2$ and $\beta = 1/3$. Hence, we obtain*

$$\left|\frac{1}{(d-c)}\int_{c}^{d}\psi(s)\mathrm{d}s - \frac{1}{3}\left(2\psi\left(\frac{c+d}{2}\right) + \frac{\psi(c)+\psi(d)}{2}\right)\right| \leq \frac{\mu(d-c)}{2} - \mu^2 \leq \frac{(d-c)^2}{16}.$$

**Proof.** Fix $y \in [c + \beta(d-c)/2, d - \beta(d-c)/2]$. Seeking simplicity to reach our proof, we divide it into four steps as follows.
Step 1: Let $g(s) := c + \beta(d-c)/2 - s$ and $s \in [c, y]$. Applying the Hayashi inequality stated in (2), setting $q(s) := g(s)$ and $k(s) := \psi'(s)$, we reach

$$\begin{aligned}(d-c)\int_{y-\mu}^{y}\left(c+\frac{\beta(d-c)}{2}-s\right)ds &\le \int_{c}^{y}\left(c+\frac{\beta(d-c)}{2}-s\right)\psi'(s)ds \\ &\le (d-c)\int_{c}^{c+\mu}\left(c+\frac{\beta(d-c)}{2}-s\right)ds, \qquad (33)\end{aligned}$$

where $\mu := (1/(d-c))\int_c^d \psi'(s)ds = (\psi(d)-\psi(c))/(d-c)$. Moreover, we have

$$\int_{y-\mu}^{y}\left(c+\frac{\beta(d-c)}{2}-s\right)ds = -\mu\left(y-c-\frac{\beta(d-c)}{2}\right)+\frac{\mu^2}{2}, \qquad (34)$$

$$\int_{c}^{y}\left(c+\frac{\beta(d-c)}{2}-s\right)\psi'(s)ds = \int_{c}^{y}\psi(s)ds - \left(y-c-\frac{\beta(d-c)}{2}\right)\psi(y) - \frac{\beta(d-c)}{2}\psi(c), \qquad (35)$$

$$\int_{c}^{c+\mu}\left(c+\frac{\beta(d-c)}{2}-s\right)ds = \mu\beta\left(\frac{d-c}{2}\right)-\frac{\mu^2}{2}. \qquad (36)$$

Substituting Equations (34)–(36) in the inequality established in (33), we arrive at

$$\begin{aligned}&(d-c)\left(-\mu\left(y-c-\frac{\beta(d-c)}{2}\right)+\frac{\mu^2}{2}\right) \\ &\le -\left(y-c-\frac{\beta(d-c)}{2}\right)\psi(y) - \frac{\beta(d-c)}{2}\psi(c) + \int_c^y \psi(s)ds \\ &\le (d-c)\left(\left(\frac{\mu\beta(d-c)}{2}\right)-\frac{\mu^2}{2}\right). \qquad (37)\end{aligned}$$

Step 2: Let $g(s) := d - \beta(d-c)/2 - s$ and $s \in [y, d]$. Employing the Hayashi inequality expressed in (2) again, we obtain

$$\begin{aligned}(d-c)\int_{d-\mu}^{d}\left(d-\frac{\beta(d-c)}{2}-s\right)ds &\le \int_{y}^{d}\left(d-\frac{\beta(d-c)}{2}-s\right)\psi'(s)ds \\ &\le (d-c)\int_{y}^{y+\mu}\left(d-\frac{\beta(d-c)}{2}-s\right)ds. \qquad (38)\end{aligned}$$

Note that

$$\int_{d-\mu}^{d}\left(d-\frac{\beta(d-c)}{2}-s\right)ds = -\frac{\mu\beta(d-c)}{2}+\frac{\mu^2}{2}, \qquad (39)$$

$$\int_{y}^{d}\left(d-\frac{\beta(d-c)}{2}-s\right)\psi'(s)ds = -\left(d-y-\frac{\beta(d-c)}{2}\right)\psi(y) - \frac{\beta(d-c)}{2}\psi(d) + \int_y^d \psi(s)ds, \qquad (40)$$

$$\int_{y}^{y+\mu}\left(d-\frac{\beta(d-c)}{2}-d\right)ds = \mu\left(d-y-\frac{\beta(d-c)}{2}\right)-\frac{\mu^2}{2}. \qquad (41)$$

Substituting Equations (39)–(41) in the inequality given in (38), we reach

$$\begin{aligned}(d-c)\left(\frac{\mu^2}{2}-\frac{\mu\beta(d-c)}{2}\right) \\ \le -\left(d-y-\frac{\beta(d-c)}{2}\right)\psi(y)-\frac{\beta(d-c)}{2}\psi(d)+\int_y^d\psi(s)\mathrm{d}s \\ \le (d-c)\left(\mu\left(d-y-\frac{\beta(d-c)}{2}\right)-\frac{\mu^2}{2}\right).\end{aligned} \quad (42)$$

Considering the expressions stated in (37) and (42), we obtain

$$\begin{aligned}(d-c)\left(\mu^2-\mu(y-c)\right) &\le -(1-\beta)(d-c)\psi(y)-\beta(d-c)\frac{\psi(c)+\psi(d)}{2}+\int_c^d\psi(s)\mathrm{d}s \\ &\le (d-c)\left(\mu(d-y)-\mu^2\right).\end{aligned}$$

Now, defining

$$J:=\frac{1}{(d-c)}\int_c^d\psi(s)\mathrm{d}s-(1-\beta)\psi(y)-\frac{\beta(\psi(c)+\psi(d))}{2},$$

$\nu_1(y):=-\mu(y-c)+\mu^2$, and $\nu_2(y):=\mu(d-y)-\mu^2$, we attain at

$$\begin{aligned}\left|J-\frac{\nu_1(y)+\nu_2(y)}{2}\right| &= \left|\frac{1}{(d-c)}\int_c^d\psi(s)\mathrm{d}s-(1-\beta)\psi(y)-\frac{\beta(\psi(c)+\psi(d))}{2}+\mu y-\frac{\mu(c+d)}{2}\right| \\ &\le \frac{\nu_2(y)-\nu_1(y)}{2}=\frac{\mu(d-c)}{2}-\mu^2,\end{aligned}$$

which proves the first inequality given in (32). To prove the second inequality expressed below the formula stated in (32), define the mapping $\delta(s):=-s^2+(d-c)/2s$, getting $\max\{\delta(s)\}:=\delta((d-c)/4)=((d-c)/4)^2$, so that $\delta(\mu):=-\mu^2+(d-c)/2\mu\le((d-c)/4)^2$, which completes the proof of the theorem. □

**Corollary 2.** *Under the assumptions of Corollary 1, we have*

$$\begin{aligned}\left|\frac{1}{(d-c)}\int_c^d\psi(s)\mathrm{d}s-(1-\beta)\psi(y)-\frac{\beta(\psi(c)+\psi(d))}{2}+\mu\left(y-\frac{(c+d)}{2}\right)\right| \\ \le\frac{(Z-\zeta)}{(d-c)}\left(\frac{\mu(d-c)}{2}-\mu^2\right)\le\frac{(d-c)(Z-\zeta)}{16},\end{aligned} \quad (43)$$

*where* $\mu:=(\psi(d)-\psi(c)-\zeta(d-c))/(Z-\zeta)$ *and* $y\in[c+\beta(d-c)/2,d-\beta(d-c)/2]$, *for all* $\beta\in[0,1]$. *In a particular case, a Simpson type inequality is deduced once we choose* $y:=(c+d)/2$ *and* $\beta=1/3$. *Thus, we obtain*

$$\begin{aligned}\left|\frac{1}{(d-c)}\int_c^d\psi(t)\mathrm{d}s-\frac{1}{3}\left(2\psi\left(\frac{c+d}{2}\right)+\frac{\psi(c)+\psi(d)}{2}\right)\right| &\le\frac{(Z-\zeta)}{(d-c)}\left(\frac{\mu(d-c)}{2}-\mu^2\right) \\ &\le\frac{(d-c)(Z-\zeta)}{16}.\end{aligned}$$

**Proof.** It can be established by repeating the proof of Theorem 4, with $k(s):=\psi'(s)-\zeta$ and $s\in[c,d]$. □

**Remark 5.** *Let $[c, d] \equiv [0, 1]$ in the inequality stated in (43). Then, we obtain*

$$\left| \int_0^1 \psi(s) \mathrm{d}s - (1-\beta)\psi(y) - \frac{\beta(\psi(0)+\psi(1))}{2} + \mu\left(y - \frac{1}{2}\right) \right| \\ \leq (Z-\zeta)\left(\frac{\mu}{2} - \mu^2\right) \leq \frac{Z-\zeta}{16},$$

*for all $y \in [\beta/2, 1-\beta/2]$, where $\mu := (\psi(1) - \psi(0) - \zeta)/(Z-\zeta)$. In a particular case, a Simpson type inequality is deduced once we choose $y = 1/2$ and $\beta = 1/3$. Thus, we reach*

$$\left| \int_0^1 \psi(t) \mathrm{d}s - \frac{1}{3}\left(2\psi\left(\frac{1}{2}\right) + \frac{\psi(0)+\psi(1)}{2}\right) \right| \leq (Z-\zeta)\left(\frac{\mu}{2} - \mu^2\right) \leq \frac{Z-\zeta}{16},$$

*which gives an approximation error for the Simpson quadrature rule; that is,*

$$\int_0^1 \psi(t) \mathrm{d}s \approx \frac{1}{3}\left(2\psi\left(\frac{1}{2}\right) + \frac{\psi(0)+\psi(1)}{2}\right). \tag{44}$$

*Note that the error term of the Simpson quadrature rule involves a 4th degree derivative, so that this rule allows us to integrate polynomials of 4th degree exactly. However, if the 4th degree derivative of a given function is either unbounded or does not exist, then we cannot apply it. Hence, the approximation given in Equation (44) permits us to apply the Simpson quadrature rule for continuous differentiable functions (that is, has a first bounded derivative only) with absolute error less than $(Z-\zeta)/16$, where $\zeta \leq \psi'(s) \leq Z$. Note that this is another elegant advantage of our result.*

**Example 2.** *Let $I$ be a real interval such that $[0, 1] \subset I$. Consider the function $\psi(t) := t^5 \sin(1/t)$, for $t \in I$. Clearly, $\psi^{(4)}(t)$ does not exist (since the 4th degree derivative is unbounded). Thus, we cannot apply the Simpson quadrature rule. However, as noted in Remark 5, one can find that $0 \leq \psi'(t) \leq 3.66705$ and*

$$\int_0^1 \psi(t) \mathrm{d}t \approx \frac{1}{3}\left(2\psi(0) + \frac{\psi(0)+\psi(1)}{2}\right) = 0.15919. \tag{45}$$

*Note that the exact value of the integral presented in Equation (45) is equal to 0.15079. Moreover, the absolute error is equal to 0.00839, which is less than $(Z-\zeta)/16 = 0.22919$.*

An improvement of the inequalities presented in (6) and (7) is considered in the following theorem.

**Theorem 5.** *Under the assumptions of Theorem 3, we have*

$$\left| \frac{1}{(d-c)} \int_c^d \psi(s) \mathrm{d}s - \frac{\beta(\psi(c)+\psi(d))}{2} - \frac{(1-\beta)(\psi(y)+\psi(c+d-y))}{2} \right| \\ \leq \mu\left(\frac{(b-a)}{2} - \frac{3\mu}{2}\right) \leq \frac{(d-c)^2}{24}, \tag{46}$$

*for all $\beta \in [0, 1]$ and $y \in (c + \beta(d-c)/2, (c+d)/2)$, where $\mu := (\psi(d) - \psi(c))/(d-c)$.*

**Proof.** First, note that

$$\mu := \frac{1}{(d-c)} \int_c^d \psi'(s) \mathrm{d}s = \frac{\psi(d) - \psi(c)}{d-c}.$$

Then, fix $y \in (c + \beta(d-c)/2, (c+d)/2)$. Seeking simplicity to reach our proof, we divide it into four steps as follows.

Step 1: Let $g(s) := c + \beta(d-c)/2 - s$ and $s \in [c, y]$. Applying the Hayashi inequality stated in (2), setting $q(s) := g(s)$ and $k(s) := \psi'(s)$, the inequality given in (37) holds.
Step 2: Let $g(s) := (c+d)/2 - s$ and $s \in (y, c+d-y)$. Employing the Hayashi inequality expressed in (2) again, we obtain

$$(d-c)\int_{c+d-y-\mu}^{c+d-y}\left(\frac{(c+d)}{2}-s\right)\mathrm{d}s \le \int_{y}^{c+d-y}\left(\frac{(c+d)}{2}-t\right)\psi'(s)\mathrm{d}s \le (d-c)\int_{y}^{y+\mu}\left(\frac{(c+d)}{2}-s\right)\mathrm{d}s. \tag{47}$$

Observe that

$$\int_{c+d-y-\mu}^{c+d-y}\left(\frac{(c+d)}{2}-s\right)\mathrm{d}s = -\mu\left(\frac{(c+d)}{2}-y\right)+\frac{\mu^2}{2}, \tag{48}$$

$$\int_{y}^{c+d-y}\left(\frac{(c+d)}{2}-s\right)\psi'(s)\mathrm{d}s = \int_{y}^{c+d-y}\psi(s)\mathrm{d}s - \left(\frac{(a+b)}{2}-y\right)(\psi(y)+\psi(a+b-y)), \tag{49}$$

$$\int_{y}^{y+\mu}\left(\frac{(c+d)}{2}-s\right)\mathrm{d}s = \mu\left(\frac{(c+d)}{2}-y\right)-\frac{\mu^2}{2}. \tag{50}$$

Substituting Equations (48)–(50) in the inequality given in (47), we arrive at

$$(c-d)\left(\mu\left(\frac{(c+d)}{2}-y\right)-\frac{\mu^2}{2}\right) \le -\left(\frac{(c+d)}{2}-y\right)(\psi(y)+\psi(c+d-y))+\int_{y}^{c+d-y}\psi(s)\mathrm{d}s \le (d-c)\left(\mu\left(\frac{(c+d)}{2}-y\right)-\frac{\mu^2}{2}\right). \tag{51}$$

Step 3: Let $g(s) := d - \beta(d-c)/2 - s$ and $s \in [c+d-y, d]$. Assuming the Hayashi inequality defined in (2) again, we obtain

$$(d-c)\int_{d-\mu}^{d}\left(d-\frac{\beta(d-c)}{2}-s\right)\mathrm{d}s \le \int_{c+d-y}^{d}\left(d-\frac{\beta(d-c)}{2}-s\right)\psi'(s)\mathrm{d}s \le (d-c)\int_{c+d-y}^{c+d-y+\mu}\left(d-\frac{\beta(d-c)}{2}-s\right)\mathrm{d}s. \tag{52}$$

Now, we reach

$$\int_{d-\mu}^{d}\left(d-\frac{\beta(d-c)}{2}-s\right)\mathrm{d}s = \frac{\mu\beta(d-c)}{2}+\frac{\mu^2}{2}, \tag{53}$$

$$\int_{c+d-y}^{d}\left(d-\frac{\beta(d-c)}{2}-s\right)\psi'(s)\mathrm{d}s = -\frac{\beta(d-c)}{2}\psi(d)-\left(y-c-\frac{\beta(d-c)}{2}\right)\psi(c+d-y) + \int_{c+d-y}^{d}\psi(s)\mathrm{d}s, \tag{54}$$

$$\int_{c+d-y}^{c+d-y+\mu}\left(d-\frac{\beta(d-c)}{2}-s\right)\mathrm{d}s = \mu\left(y-c-\frac{\beta(d-c)}{2}\right)-\frac{\mu^2}{2}. \tag{55}$$

Substituting Equations (53)–(55) in the inequality established in (52), we have

$$(d-c)\left(\frac{\mu\beta(d-c)}{2}+\frac{\mu^2}{2}\right) \\ \leq -\frac{\beta(d-c)}{2}\psi(d)-\left(y-c-\frac{\beta(d-c)}{2}\right)\psi(c+d-y)+\int_{c+d-y}^{d}\psi(s)\mathrm{d}s \\ \leq (d-c)\left(\mu\left(y-c-\frac{\beta(d-c)}{2}\right)-\frac{\mu^2}{2}\right). \quad (56)$$

Considering the inequalities given in (37), (51) and (56), we obtain

$$-(d-c)\left(\frac{\mu(d-c)}{2}-\frac{3\mu^2}{2}\right) \\ \leq \int_c^d \psi(s)\mathrm{d}s-\frac{\beta(d-c)(\psi(c)+\psi(d))}{2}-\frac{(1-\beta)(d-c)(\psi(y)+\psi(c+d-y))}{2} \\ \leq (d-c)\left(\frac{\mu(d-c)}{2}-\frac{3\mu^2}{2}\right),$$

which is equivalent to the first inequality stated in (46). To prove the inequality given below the expression presented in (46), define the mapping $\delta(s) := -(3/2)(d-c)s^2 + ((d-c)^2/2)s$ and then $\max\{\delta(s)\} := \delta((d-c)/6) = (d-c)^2/24$, so that $\delta(\mu) := -(3/2)(d-c)\mu^2 + ((d-c)^2/2)\mu \leq (d-c)^2/24$, which completes the proof of the theorem. □

A generalization of the inequality considered in (46) is incorporated in the following corollary.

**Corollary 3.** *Under the assumptions of Theorem 5, we have*

$$\left|\frac{1}{(d-c)}\int_c^d \psi(s)\mathrm{d}s-\frac{\beta(\psi(c)+\psi(d))}{2}-\frac{(1-\beta)(\psi(y)+\psi(c+d-y))}{2}\right| \\ \leq \frac{\mu}{2}\frac{(Z-\zeta)}{(d-c)}(d-c-3\mu) \leq \frac{(Z-\zeta)(d-c)}{24}, \quad (57)$$

*where* $\mu := (\psi(d)-\psi(c)-\zeta(d-c))/(Z-\zeta)$ *and* $y \in (c+\beta(d-c)/2, (c+d)/2)$, *for all* $\beta \in [0,1]$.

**Proof.** It can be obtained by repeating the proof of Theorem 5, with $k(s) := \psi'(s)-\zeta$ and $s \in [c,d]$. □

**Remark 6.** *Clearly, by choosing* $y := (c+d)/2$ *in the inequality established in (57), the upper bound in this inequality is better than both upper bounds in the inequalities defined in (6) and (7).*

**Remark 7.** *Let* $[c,d] \equiv [-1,1]$ *in the inequality formulated in (57). Then, we obtain*

$$\left|\frac{1}{2}\int_{-1}^{1}\psi(s)\mathrm{d}s-\frac{\beta(\psi(-1)+\psi(1))}{2}-\frac{(1-\beta)(\psi(y)+\psi(-y))}{2}\right| \\ \leq \frac{\mu}{4}(Z-\zeta)(2-3\mu) \leq \frac{(Z-\zeta)}{12}, \quad (58)$$

*for all* $y \in [-1, 0]$*. In particular, if* $y = -\sqrt{3/5}$ *in the inequality given in (58), we arrive at*

$$\left| \int_{-1}^{1} \psi(s)\mathrm{d}s - \beta(\psi(-1) + \psi(1)) - (1-\beta)\left(\psi\left(-\sqrt{\frac{3}{5}}\right) + \psi\left(\sqrt{\frac{3}{5}}\right)\right) \right| \\ \leq \frac{\mu}{2}(Z-\zeta)(2-3\mu) \leq \frac{(Z-\zeta)}{6}, \quad (59)$$

*where* $\mu := (\psi(1) - \psi(-1) - 2\zeta)/(Z-\zeta)$*. The inequality formulated in (59) gives an approximation error for the Gauss–Legendre quadrature rule of* 2nd *order; that is,*

$$\int_{-1}^{1} \psi(t)\mathrm{d}t \approx \psi\left(-\frac{1}{\sqrt{3}}\right) + \psi\left(\frac{1}{\sqrt{3}}\right).$$

*Observe that the error term of the Gauss–Legendre quadrature rule involves a* 3rd *degree derivative, so that this rule enables us to integrate polynomials of* 3rd *degree exactly. However, if the* 3rd *derivative of a given function is either unbounded or does not exist, then we cannot apply it. Thus, our last approximation permits us to apply the Gauss–Legendre quadrature rule of* 3rd *order for continuous differentiable functions (that is, has a first bounded derivative only) with absolute error less than* $(Z-\zeta)/6$*, where* $\zeta \leq \psi'(s) \leq Z$*. This is another elegant advantage of our result.*

**Example 3.** *We consider* $\psi(t)$ *as given in Example 1. As noted in Remark 7, one can find that* $7.97443 \times 10^{-6} \leq \psi'(t) \leq 3.66705$ *and*

$$\int_{-1}^{1} \psi(t)\mathrm{d}t \approx \beta(\psi(-1) + \psi(1)) + (1-\beta)\left(\psi\left(-\frac{1}{\sqrt{3}}\right) + \psi\left(\frac{1}{\sqrt{3}}\right)\right).$$

*Then, choosing* $\beta = 0.11240$*, we obtain*

$$\int_{-1}^{1} \psi(t)\mathrm{d}t \approx 0.11240(\psi(-1) + \psi(1)) + 0.88760\left(\psi\left(-\frac{1}{\sqrt{3}}\right) + \psi\left(\frac{1}{\sqrt{3}}\right)\right) = 0.30156,$$

*which is very close to the exact value* 0.30158*. Moreover, the absolute error is equal to* 0.00002*, which is less than* $(Z-\zeta)/6 = 0.61117$*. Observe that this gives a better approximation than Example 1.*

## 3. Applications in a Probability Setting

Let $Y$ be a random variable taking values in the interval $[c, d]$, with cumulative distribution function $G(y) := P(Y \leq y)$, for $y \in (c, d)$. Then, we have the following theorem.

**Theorem 6.** *With the assumptions of Corollary 1 for* $\psi := G$*, we have*

$$\left| \frac{(d - \mathrm{E}(Y))}{(d-c)} - \left( \frac{\eta(G(y) + G(c+d-y))}{2} + (1-\eta)G\left(\frac{c+d}{2}\right)\right) \right| \\ \leq \frac{(Z-\zeta)}{2} \frac{\mu(d-c-2\mu)}{(d-c)} \leq \frac{(Z-\zeta)(d-c)}{32},$$

*for all* $y \in (c, (c+d)/2)$*, where* $\mu := (G(d) - G(c) - \zeta(d-c))/(Z-\zeta)$*, and* $\mathrm{E}(Y)$ *is the expected value of* $Y$*.*

**Proof.** In the proof of Corollary 1, let $\psi := G$ and consider that

$$\mathrm{E}(Y) := \int_{c}^{d} s\,\mathrm{d}G(s) = d - \int_{c}^{d} G(s)\mathrm{d}s,$$

that is,

$$\int_c^d G(s)\mathrm{d}s = \int_c^d s\mathrm{d}G(s) = d - \mathrm{E}(Y).$$

Therefore, the required inequality follows and the proof is completed. □

**Theorem 7.** *With the assumptions of Corollary 2 for $\psi := G$, we obtain*

$$\left| \frac{(d-\mathrm{E}(Y))}{(d-c)} - (1-\beta)G(y) - \frac{\beta(G(c)+G(d))}{2} + \mu\left(y - \frac{(c+d)}{2}\right) \right| \\ \leq \frac{(Z-\zeta)}{(d-c)}\left(\frac{\mu(d-c)}{2} - \mu^2\right) \leq \frac{(d-c)(Z-\zeta)}{16},$$

*for all $y \in (c+\beta(d-c)/2, d-\beta(d-c)/2)$, where $\mu := (G(d) - G(c) - \zeta(d-c))/(Z-\zeta)$.*

**Proof.** We must apply Corollary 2 to $\psi := G$ and the rest of the proof is similar to that of Theorem 6. □

## 4. Concluding Remarks

In this paper, we have stated and proved refinements as well as introduced improvements to some Ostrowski type inequalities. The presented proofs are based on employing the celebrated Hayashi inequality with certain functions. We have provided examples that show these improvements. Applications in a probability framework of the obtained results were considered as well. These results may be helpful for other different contexts and applications, such as, for example, in the treatment of errors in numerical approximations.

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