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Nonlocal Integro-Multi-Point (k, ψ)-Hilfer Type Fractional Boundary Value Problems

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Abstract: In this paper we investigate the criteria for the existence of solutions for single-valued as well as multi-valued boundary value problems involving (k, ψ)-Hilfer fractional derivative operator of order in $(1, 2]$, equipped with nonlocal integral multi-point boundary conditions. For the single-valued case, we rely on fixed point theorems due to Banach and Krasnosel'skiĭ, and Leray–Schauder alternative to establish the desired results. The existence results for the multi-valued problem are obtained by applying the Leray–Schauder nonlinear alternative for multi-valued maps for convex-valued case, while the nonconvex-valued case is studied with the aid of Covit–Nadler's fixed point theorem for multi-valued contractions. Numerical examples are presented for the illustration of the obtained results.

Keywords: (k, ψ)-Hilfer derivative operator; integral multi-point boundary conditions; single-valued; multi-valued; existence; fixed point

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1. Introduction

Fractional-order integral and differential operators are found to be of great help to study many engineering and scientific phenomena occurring in mathematical biology, mechanics, and so forth, see the monographs [1–9]. Fractional derivative operators are usually defined in terms of fractional integral operators. Many kinds of fractional derivative operators, such as Rieman–Liouville, Caputo, Hadamard, Katugampola, Hilfer, etc., appear in the literature on fractional calculus. Recently, the authors of [10] discussed the existence of solutions for Riemann–Stieltjes integral boundary value problems involving mixed Riemann–Liouville and Caputo fractional derivatives. An existence result for a periodic boundary value problem of fractional semilinear differential equations in a Banach space was proved in [11]. The concept of generalized fractional derivative introduced by Katugampola in [12,13] includes both Riemann–Liouville and Hadamard fractional derivatives. The Hilfer fractional derivative operator [14] includes both Rieman–Liouville and Caputo fractional derivative operators. Another fractional derivative operator is the ψ-fractional derivative operator [15] which unifies Caputo, Caputo–Hadamard and Caputo–Erdélyi-Kober fractional derivative operators. It is imperative to note the (k, ψ)-Hilfer fractional derivative operator, introduced in [16], generalizes several known fractional derivative operators.

In [16], the authors studied the existence of solutions for a nonlinear initial value problem involving (k, ψ)-Hilfer fractional derivative operator. In a more recent work [17], Tariboon et al. investigated (k, ψ)-Hilfer boundary value problems for fractional differential equations and inclusions with nonlocal multipoint boundary conditions.

To enrich the literature in this new direction, which is very limited at the moment, we formulate and study a boundary value problem involving (k, ψ)-Hilfer fractional derivative operator of order in $(1, 2]$, and nonlocal integro-multi-point boundary conditions:

$$\begin{cases} {}^{k,H}\mathfrak{D}^{\overline{\alpha},\beta;\psi}\vartheta(w) = \mathfrak{f}(w, \vartheta(w)), & w \in (a, b], \\ \vartheta(a) = 0, \quad \displaystyle\int_a^b \psi'(s)\vartheta(s)ds = \sum_{j=1}^{m} \eta_j \vartheta(\xi_j), \end{cases} \tag{1}$$

where ${}^{k,H}\mathfrak{D}^{\overline{\alpha},\beta;\psi}$ denotes the (k, ψ)-Hilfer fractional derivative operator of order $\overline{\alpha}$, $1 < \overline{\alpha} < 2$ and parameter β, $0 \le \beta \le 1$, $k > 0$, $\mathfrak{f} : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\eta_j \in \mathbb{R}$, and $a < \xi_j < b$, $j = 1, 2, \ldots, m$. We make use of Banach's contraction mapping principle, Krasnosel'skiĭ's fixed point theorem, and Laray–Schauder nonlinear alternative to derive the existence and uniqueness results for the problem (1). In passing, we remark that the choice of $k = 1$, $\psi(t) = t$, reduces the problem (1) to the one studied in [18] with $\mu = 0$.

The multivalued analogue of the problem (1) is

$$\begin{cases} {}^{k,H}\mathfrak{D}^{\overline{\alpha},\beta;\psi}\vartheta(w) \in \mathfrak{F}(w, \vartheta(w)), & w \in (a, b], \\ \vartheta(a) = 0, \quad \displaystyle\int_a^b \psi'(s)\vartheta(s)ds = \sum_{j=1}^{m} \eta_j \vartheta(\xi_j), \end{cases} \tag{2}$$

where $\mathfrak{F} : [a, b] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, while the other quantities are the same as explained in problem (1). The convex and non-convex valued cases of the multi-valued map $\mathfrak{F}$ in (2) are studied with the aid of Laray–Schauder nonlinear alternative for multi-valued maps and the Covitz–Nadler fixed point theorem for multi-valued contractions, respectively. Here, we remark that the (k, ψ)-Hilfer fractional differential equations and inclusions were studied with nonlocal multipoint boundary conditions in [17], while the present work deals with nonlocal integro-multi-point boundary conditions.

We arrange the remainder of the paper as follows. Section 2 contains preliminary material related to the proposed problems. In Section 3, we prove a basic lemma which is used to convert the nonlinear problem (1) into an equivalent fixed point problem. We present the existence and uniqueness results for the problem (1) in Section 4, while Section 5 contains the existence results for the problem (2). Section 6 is dedicated to the illustration of the results obtained in the previous two sections with the aid of numerical examples. Finally, Section 7 includes the conclusions of the paper.

2. Preliminaries

We first recall the preliminary concepts of fractional calculus related to our work.

Definition 1 ([2])**.** *Suppose that $\mathfrak{f} \in L^1([a, b], \mathbb{R})$. Then the Riemann–Liouville fractional integral is defined by*

$$\mathfrak{I}^{\alpha}_{a+}\mathfrak{f}(w) = \frac{1}{\Gamma(\alpha)} \int_a^w (w - u)^{\alpha - 1}\mathfrak{f}(u)du, \quad w > a, \tag{3}$$

where $\Gamma(\cdot)$ denotes the classical Euler gamma function.

Definition 2 ([2])**.** *Let $\mathfrak{f} \in C([a, b], \mathbb{R})$. Then the Riemann–Liouville fractional derivative operator of order $\alpha \in (n - 1, n]$, $n \in \mathbb{N}$ is defined by*

$$ {}^{RL}\mathfrak{D}^{\alpha}_{a+}\mathfrak{f}(w) = \mathfrak{D}^n \mathfrak{I}^{n-\alpha}_{a+}\mathfrak{f}(w) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dw^n} \int_a^w (w - u)^{n-\alpha-1}\mathfrak{f}(u)du, \quad w > a. \tag{4}$$

Definition 3 ([2]). *Let $\mathfrak{f} \in C^n([a,b],\mathbb{R})$. Then the Caputo fractional derivative operator of order $\alpha \in (n-1,n], n \in \mathbb{N}$ is defined by*

$$^{C}\mathfrak{D}^{\alpha}_{a+}\mathfrak{f}(w) = \mathfrak{I}^{n-a}_{a+}\mathfrak{D}^{n}\mathfrak{f}(w) = \frac{1}{\Gamma(n-\alpha)}\int_a^w (w-u)^{n-\alpha-1}\mathfrak{f}^{(n)}(u)du, \ \ w > a. \tag{5}$$

Definition 4 ([19]). *For $k, \alpha \in \mathbb{R}^+$, the k-Riemann–Liouville fractional integral of order α for a function $\mathfrak{h} \in L^1([a,b],\mathbb{R})$ is defined by*

$$^{k}\mathfrak{I}^{\alpha}_{a+}\mathfrak{h}(w) = \frac{1}{k\Gamma_k(\alpha)}\int_a^w (w-u)^{\frac{\alpha}{k}-1}\mathfrak{h}(u)du, \tag{6}$$

where Γ_k is the k-Gamma function given by

$$\Gamma_k(z) = \int_0^\infty s^{z-1}e^{-\frac{s^k}{k}}ds,$$

Which satisfies the following properties:

$$\Gamma(\theta) = \lim_{k\to 1}\Gamma_k(\theta), \ \Gamma_k(\theta) = k^{\frac{\theta}{k}-1}\Gamma\left(\frac{\theta}{k}\right) \text{ and } \Gamma_k(\theta+k) = \theta\Gamma_k(\theta).$$

Definition 5 ([20]). *Let $\mathfrak{h} \in L^1([a,b],\mathbb{R})$ and $k, \alpha \in \mathbb{R}^+$. Then the k-Riemann–Liouville fractional derivative of order α for the function $\mathfrak{h}$ is given by*

$$^{k,RL}\mathfrak{D}^{\alpha}_{a+}\mathfrak{h}(w) = \left(k\frac{d}{dw}\right)^n {}^{k}\mathfrak{I}^{nk-\alpha}_{a+}\mathfrak{h}(w), \ \ n = \left\lceil \frac{\alpha}{k} \right\rceil, \tag{7}$$

where $\left\lceil \frac{\alpha}{k} \right\rceil$ is the ceiling function of $\frac{\alpha}{k}$.

Definition 6 ([2]). *Let $\psi : [a,b] \to \mathbb{R}$ be an increasing function with $\psi'(\theta) \neq 0$ for all $\theta \in [a,b]$. Then the ψ-Riemann–Liouville fractional integral of the function $\mathfrak{h} \in L^1([a,b],\mathbb{R})$ is defined by*

$$\mathfrak{I}^{\bar{\alpha};\psi}\mathfrak{h}(w) = \frac{1}{\Gamma_k(\alpha)}\int_a^w \psi'(u)(\psi(w)-\psi(u))^{\alpha-1}\mathfrak{h}(u)du. \tag{8}$$

Definition 7 ([2]). *Let $\psi \in C^n([a,b],\mathbb{R})$, with $\psi'(\theta) \neq 0, \theta \in [a,b]$. Then the ψ-Riemann–Liouville fractional derivative of the function $\mathfrak{h} \in C([a,b],\mathbb{R})$ of order $\alpha \in (n-1,n]$, $n \in \mathbb{N}$ is given by*

$$^{RL}\mathfrak{D}^{\alpha;\psi}\mathfrak{h}(w) = \left(\frac{1}{\psi'(w)}\frac{d}{dw}\right)^n \mathfrak{I}^{n-\bar{\alpha};\psi}_{a+}\mathfrak{h}(w). \tag{9}$$

Definition 8 ([15]). *Let $\psi \in C^n([a,b],\mathbb{R})$, with $\psi'(\theta) \neq 0, \theta \in [a,b]$. Then the ψ-Caputo fractional derivative of the function $\mathfrak{h} \in C([a,b],\mathbb{R})$ of order $\alpha \in (n-1,n]$, $n \in \mathbb{N}$ is given by*

$$^{C}\mathfrak{D}^{\alpha;\psi}\mathfrak{h}(w) = \mathfrak{I}^{n-\bar{\alpha};\psi}_{a+}\left(\frac{1}{\psi'(w)}\frac{d}{dw}\right)^n \mathfrak{h}(w). \tag{10}$$

Definition 9 ([21]). *Let $\psi \in C^n([a,b],\mathbb{R})$, with $\psi'(\theta) \neq 0, \theta \in [a,b]$. Then the ψ-Hilfer fractional derivative of the function $\mathfrak{h} \in C([a,b],\mathbb{R})$ of order $\alpha \in (n-1,n]$, $n \in \mathbb{N}$ and type $\beta \in [0,1]$ is defined by*

$$^{H}\mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{h}(w) = \mathfrak{I}^{\beta(n-\bar{\alpha});\psi}_{a+}\left(\frac{1}{\psi'(w)}\frac{d}{dw}\right)^n \mathfrak{I}^{(1-\beta)(n-\bar{\alpha});\psi}_{a+}\mathfrak{h}(w). \tag{11}$$

Definition 10 ([22]). *The (k,ψ)-Riemann–Liouville fractional integral of order $\alpha > 0$ ($\alpha \in \mathbb{R}$) of the function $\mathfrak{h} \in L^1([a,b],\mathbb{R})$ is given by*

$$ {}^k\mathfrak{J}_{a+}^{\alpha;\psi}\mathfrak{h}(w) = \frac{1}{k\Gamma_k(\alpha)}\int_a^w \psi'(u)(\psi(w)-\psi(u))^{\frac{\alpha}{k}-1}\mathfrak{h}(u)du,\ k>0. \tag{12}$$

Definition 11 ([16]). *Let $\alpha, k \in \mathbb{R}^+ = (0,\infty)$, $\beta \in [0,1]$, and $\psi \in C^n([a,b],\mathbb{R})$ with $\psi'(\theta) \neq 0, \theta \in [a,b.]$ We define the (k,ψ)-Hilfer fractional derivative of the function $\mathfrak{h} \in C^n([a,b],\mathbb{R})$ of order α and type β, as*

$$ {}^{k,H}\mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{h}(w) = {}^k\mathfrak{J}_{a+}^{\beta(nk-\alpha);\psi}\left(\frac{k}{\psi'(w)}\frac{d}{dw}\right)^n {}^k\mathfrak{J}_{a+}^{(1-\beta)(nk-\alpha);\psi}\mathfrak{h}(w),\ \ n = \left\lceil\frac{\alpha}{k}\right\rceil. \tag{13}$$

Remark 1. *The (k,ψ)-Hilfer fractional derivative can be expressed in terms of (k,ψ)-Riemann–Liouville fractional integral as*

$$\begin{aligned} {}^{k,H}\mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{h}(w) &= {}^k\mathfrak{J}_{a+}^{\theta_k-\alpha;\psi}\left(\frac{k}{\psi'(w)}\frac{d}{dw}\right)^n {}^k\mathfrak{J}_{a+}^{nk-\theta_k;\psi}\mathfrak{h}(w) \\ &= {}^k\mathfrak{J}_{a+}^{\theta_k-\alpha;\psi}\left({}^{k,RL}\mathfrak{D}^{\theta_k;\psi}\mathfrak{h}\right)(w),\ (1-\beta)(nk-\alpha) = nk-\theta_k. \end{aligned}$$

Note that $n-1 < \frac{\theta_k}{k} \leq n$ when $\beta \in [0,1]$ and $n-1 < \frac{\alpha}{k} \leq n$.

Lemma 1 ([16]). *(a) Let $\mathfrak{h} \in C^n([a,b],\mathbb{R})$ and ${}^k\mathfrak{J}_{a+}^{nk-\mu;\psi}\mathfrak{h} \in C^n([a,b],\mathbb{R})$. Then*

$$ {}^k\mathfrak{J}^{\mu;\psi}\left({}^{k,RL}\mathfrak{D}^{\mu;\psi}\mathfrak{h}(w)\right) = \mathfrak{h}(w) - \sum_{j=1}^n \frac{(\psi(w)-\psi(a))^{\frac{\mu}{k}-j}}{\Gamma_k(\mu-jk+k)}\left[\left(\frac{k}{\psi'(w)}\frac{d}{dw}\right)^{n-j} {}^k\mathfrak{J}_{a+}^{nk-\mu;\psi}\mathfrak{h}(w)\right]_{z=a},$$

where $0<\mu, k<\infty$ and $n = \left\lceil\frac{\mu}{k}\right\rceil$.
(b) Let $0<\alpha,k<\infty$ with $\alpha<k$, $\beta\in[0,1]$ and $\theta_k = \alpha+\beta(k-\alpha)$. Then

$$ {}^k\mathfrak{J}^{\theta_k;\psi}\left({}^{k,RL}\mathfrak{D}^{\theta_k;\psi}\mathfrak{h}\right)(w) = {}^k\mathfrak{J}^{\alpha;\psi}\left({}^{k,H}\mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{h}\right)(w),\ \ \mathfrak{h} \in C^n([a,b],\mathbb{R}).$$

(c) Let $0<\zeta,k<\infty$ and $\eta\in\mathbb{R}$ such that $\frac{\eta}{k} > -1$. Then

$$ (i)\ \ {}^k\mathcal{I}^{\zeta,\psi}(\psi(t)-\psi(a))^{\frac{\eta}{k}} = \frac{\Gamma_k(\eta+k)}{\Gamma_k(\eta+k+\zeta)}(\psi(t)-\psi(a))^{\frac{\eta+\zeta}{k}};$$

$$ (ii)\ \ {}^k\mathcal{D}^{\zeta,\psi}(\psi(t)-\psi(a))^{\frac{\eta}{k}} = \frac{\Gamma_k(\eta+k)}{\Gamma_k(\eta+k-\zeta)}(\psi(t)-\psi(a))^{\frac{\eta-\zeta}{k}}.$$

3. An Auxiliary Result

In the following lemma, we solve a linear variant of (1).

Lemma 2. *Let $a<b$, $k>0$, $1<\overline{\alpha}\leq 2$, $\beta\in[0,1]$, $\theta_k = \overline{\alpha}+\beta(2k-\overline{\alpha})$, $\mathfrak{g}\in C^2([a,b],\mathbb{R})$ and*

$$\mathcal{H} := k\frac{(\psi(b)-\psi(a))^{\frac{\theta_k}{k}}}{\Gamma_k(\theta_k+k)} - \sum_{j=1}^m \eta_j\frac{(\psi(\xi_j)-\psi(a))^{\frac{\theta_k}{k}-1}}{\Gamma_k(\theta_k)} \neq 0. \tag{14}$$

Then the function $\vartheta \in C([a,b],\mathbb{R})$ is a solution to the (k,ψ)-Hilfer fractional nonlocal integro-multi-point boundary value problem

$$\begin{cases} {}^{k,H}\mathfrak{D}^{\bar{\alpha},\beta;\psi}\vartheta(w) = \mathfrak{g}(w), \quad w \in (a,b], \\ \vartheta(a) = 0, \quad \int_a^b \psi'(s)\vartheta(s)ds = \sum_{j=1}^{m} \eta_j \vartheta(\xi_j), \end{cases} \tag{15}$$

if and only if

$$\vartheta(w) = {}^{k}\mathfrak{J}^{\bar{\alpha};\psi}\mathfrak{g}(w) + \frac{(\psi(w)-\psi(a))^{\frac{\theta_k}{k}-1}}{\mathcal{H}\Gamma_k(\theta_k)}\left[\sum_{j=1}^{m}\eta_j\, {}^{k}\mathcal{I}^{\bar{\alpha};\psi}g(\xi_j) - \int_a^b \psi'(s)\, {}^{k}\mathcal{I}^{\bar{\alpha};\psi}g(s)ds\right]. \tag{16}$$

Proof. Suppose that ϑ is a solution to the problem (15). Applying the fractional integral operator ${}^{k}\mathfrak{J}^{\alpha;\psi}$ on both sides of equation in (15) and using Lemma 1 (a) and (b), we obtain

$$\begin{aligned} {}^{k}\mathfrak{J}^{\bar{\alpha};\psi}\left({}^{k,H}\mathfrak{D}^{\bar{\alpha},\beta;\psi}\vartheta\right)(w) &= {}^{k}\mathfrak{J}^{\theta_k;\psi}\left({}^{k,RL}\mathfrak{D}^{\theta_k;\psi}\vartheta\right)(w) \\ &= \vartheta(w) - \frac{(\psi(w)-\psi(a))^{\frac{\theta_k}{k}-1}}{\Gamma_k(\theta_k)}\left[\left(\frac{k}{\psi'(w)}\frac{d}{dw}\right){}^{k}\mathfrak{J}^{2k-\theta_k;\psi}\vartheta(w)\right]_{w=a} \\ &\quad - \frac{(\psi(w)-\psi(a))^{\frac{\theta_k}{k}-2}}{\Gamma_k(\theta_k-k)}\left[{}^{k}\mathfrak{J}^{2k-\theta_k;\psi}\vartheta(w)\right]_{w=a}, \end{aligned}$$

which can be written as

$$\vartheta(w) = {}^{k}\mathfrak{J}^{\bar{\alpha};\psi}\mathfrak{g}(w) + c_0\frac{(\psi(w)-\psi(a))^{\frac{\theta_k}{k}-1}}{\Gamma_k(\theta_k)} + c_1\frac{(\psi(w)-\psi(a))^{\frac{\theta_k}{k}-2}}{\Gamma_k(\theta_k-k)}, \tag{17}$$

where we have set

$$c_0 = \left[\left(\frac{k}{\psi'(w)}\frac{d}{dw}\right){}^{k}\mathfrak{J}^{2k-\theta_k;\psi}\vartheta(w)\right]_{w=a}, \quad c_1 = \left[{}^{k}\mathfrak{J}^{2k-\theta_k;\psi}\vartheta(w)\right]_{w=a}.$$

We will find the values for c_0 and c_1 by using the given boundary data. By the condition $\vartheta(a) = 0$, we get $c_1 = 0$ as $\frac{\theta_k}{k} - 2 < 0$ by Remark 1.

By Lemma 1 (c), we have

$$\int_a^b \psi'(s)\frac{(\psi(s)-\psi(a))^{\frac{\theta_k}{k}-1}}{\Gamma_k(\theta_k)}ds = k\frac{(\psi(b)-\psi(a))^{\frac{\theta_k}{k}}}{\Gamma_k(\theta_k+k)}. \tag{18}$$

From (18) and the boundary condition $\int_a^b \psi'(s)\vartheta(s)ds = \sum_{j=1}^{m}\eta_j\vartheta(\xi_j)$, we find that

$$c_0 = \frac{1}{\mathcal{H}}\left[\sum_{j=1}^{m}\eta_j\, {}^{k}\mathcal{I}^{\bar{\alpha};\psi}g(\xi_j) - \int_a^b \psi'(s)\, {}^{k}\mathcal{I}^{\bar{\alpha};\psi}g(s)ds\right].$$

Inserting the values of c_0 and c_1 in (17) leads to the solution (16). Conversely, using the result: ${}^{k,RL}\mathfrak{D}^{\mu;\psi}\, {}^{k}\mathfrak{J}^{\mu;\psi}\mathfrak{h}(w) = \mathfrak{h}(w)$ from [16] together with Lemma 1 (a) and (b), one can obtain the (k,ψ)-Hilfer fractional differential equation in (15) after applying the operator ${}^{k,H}\mathfrak{D}^{\bar{\alpha},\beta;\psi}$ on (16). On the other hand, it is straightforward to verify that $\vartheta(w)$ given by (16) satisfies the boundary conditions in (15). The proof is finished. □

4. The Single Valued Problem

In view of Lemma 2, we introduce an operator $\mathcal{A} : C([a,b],\mathbb{R}) \to C([a,b],\mathbb{R})$ by

$$(\mathcal{A}\vartheta)(w) = \frac{(\psi(w)-\psi(a))^{\frac{\theta_k}{k}-1}}{\mathcal{H}\Gamma_k(\theta_k)}\left[\sum_{j=1}^{m}\eta_j\,{}^k\mathcal{I}^{\overline{\alpha};\psi}\mathfrak{f}(\xi_j,\vartheta(\xi_j)) - \int_a^b \psi'(s)\,{}^k\mathcal{I}^{\overline{\alpha};\psi}\mathfrak{f}(s,\vartheta(s))ds\right]$$
$$+{}^k\mathfrak{I}^{\overline{\alpha};\psi}\mathfrak{f}(w,\vartheta(w)),\ \ w\in[a,b], \tag{19}$$

where $C([a,b],\mathbb{R})$ is the Banach space of all continuous real valued functions defined on $[a,b]$ equipped with the sup-norm $\|\vartheta\| = \sup_{w\in[a,b]}|\vartheta(w)|$.

Notice that the fixed points of the operator $\mathcal{A}$ are the solutions to the nonlocal (k,ψ)-Hilfer fractional boundary value problem (1).

4.1. Existence of a Unique Solution via Banach's Contraction Mapping Principle

Here we prove the existence of a unique solution to the nonlocal integro-multi-point boundary value problem (1) by applying the Banach's contraction mapping principle [23].

Theorem 1. *Let $\mathfrak{f}:[a,b]\times\mathbb{R}\to\mathbb{R}$ be such that*

(H_1) $|\mathfrak{f}(w,\vartheta)-\mathfrak{f}(w,y)| \le \mathfrak{L}|\vartheta-y|,$ $\mathfrak{L}>0$ *for each* $w\in[a,b]$ *and* $\vartheta,y\in\mathbb{R}$.

Then there exists a unique solution to the nonlocal integro-multi-point (k,ψ)-Hilfer fractional boundary value problem (1) on $[a,b]$, provided that

$$\mathfrak{L}\mathfrak{G} < 1, \tag{20}$$

where

$$\mathfrak{G} = \frac{(\psi(b)-\psi(a))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)} + \frac{(\psi(b)-\psi(a))^{\frac{\theta_k}{k}-1}}{|\mathcal{H}|\Gamma_k(\theta_k)}\left[\sum_{j=1}^{m}|\eta_j|\frac{(\psi(\xi_j)-\psi(a))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)}\right.$$
$$\left.+\frac{(\psi(b)-\psi(a))^{\frac{\overline{\alpha}}{k}+1}}{\Gamma_k(\overline{\alpha}+k)}\right]. \tag{21}$$

Proof. We shall show in the first step that the operator $\mathcal{A}$ defined in (19), maps $B_r=\{\vartheta\in C([a,b],\mathbb{R}):\|\vartheta\|\le r\}$ into itself, that is, $\mathcal{A}B_r\subset B_r$, where

$$r \ge \frac{\mathfrak{M}\mathfrak{G}}{1-\mathfrak{L}\mathfrak{G}},\quad \sup_{w\in[a,b]}|\mathfrak{f}(w,0)| = \mathfrak{M} < \infty. \tag{22}$$

By (H_1) we have

$$|\mathfrak{f}(w,\vartheta(w))| \le |\mathfrak{f}(w,\vartheta(w))-\mathfrak{f}(w,0)| + |\mathfrak{f}(w,0)| \le \mathfrak{L}r+\mathfrak{M}. \tag{23}$$

Then, using (22) and (23) in (19) together with the notation (21), we obtain

$$\begin{aligned}
|(\mathcal{A}\vartheta)(w)| \le{}& \sup_{w\in[a,b]}\left\{\frac{(\psi(w)-\psi(a))^{\frac{\theta_k}{k}-1}}{|\mathcal{H}|\Gamma_k(\theta_k)}\left[\sum_{j=1}^{m}|\eta_j|\,{}^k\mathcal{I}^{\overline{\alpha};\psi}\mathfrak{f}(\xi_j,\vartheta(\xi_j))\right.\right.\\
&\left.\left.+\int_a^b\psi'(s)\,{}^k\mathcal{I}^{\overline{\alpha};\psi}|\mathfrak{f}(s,\vartheta(s))|ds\right] + {}^k\mathfrak{I}^{\overline{\alpha};\psi}|\mathfrak{f}(w,\vartheta(w))|\right\}\\
\le{}& {}^k\mathfrak{I}^{\overline{\alpha};\psi}(|\mathfrak{f}(w,\vartheta(w))-\mathfrak{f}(w,0)|+|\mathfrak{f}(w,0)|)\\
&+\frac{(\psi(b)-\psi(a))^{\frac{\theta_k}{k}-1}}{|\mathcal{H}|\Gamma_k(\theta_k)}\left(\sum_{j=1}^{m}|\eta_j|\,{}^k\mathfrak{I}^{\overline{\alpha};\psi}(|\mathfrak{f}(\xi,\vartheta(\xi))-\mathfrak{f}(\xi,0)|+|\mathfrak{f}(\xi,0)|)\right.\\
&\left.+\int_a^b\psi'(s)\,{}^k\mathcal{I}^{\overline{\alpha};\psi}(|\mathfrak{f}(b,\vartheta(b))-\mathfrak{f}(b,0)|+|\mathfrak{f}(b,0)|)\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \left\{ \frac{(\psi(b)-\psi(a))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)} + \frac{(\psi(b)-\psi(a))^{\frac{\theta_k}{k}-1}}{|\mathcal{H}|\Gamma_k(\theta_k)} \left[\sum_{j=1}^{m} |\eta_j| \frac{(\psi(\xi_j)-\psi(a))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)} \right.\right. \\
&\quad \left.\left. + \frac{(\psi(b)-\psi(a))^{\frac{\overline{\alpha}}{k}+1}}{\Gamma_k(\overline{\alpha}+k)} \right] \right\} (\mathfrak{L}\|\vartheta\| + \mathfrak{M}) \\
&\leq (\mathfrak{L}r + \mathfrak{M})\mathfrak{G} \leq r.
\end{aligned}
$$

In consequence we obtain $\|\mathcal{A}\vartheta\| \leq r$ and hence $\mathcal{A}B_r \subset B_r$.

In the second step, it will be shown that $\mathcal{A}$ is a contraction. For $w \in [a,b]$ and $\vartheta, y \in C([a,b],\mathbb{R})$, we obtain

$$
\begin{aligned}
& |(\mathcal{A}\vartheta)(w) - (\mathcal{A}y)(w)| \\
\leq\ & {}^k\mathfrak{I}^{\overline{\alpha};\psi}|\mathfrak{f}(w,\vartheta(w)) - \mathfrak{f}(w,y(w))| \\
& + \frac{(\psi(b)-\psi(a))^{\frac{\theta_k}{k}-1}}{|\mathcal{H}|\Gamma_k(\theta_k)} \left[\sum_{j=1}^{m} |\eta_j|\, {}^k\mathcal{I}^{\overline{\alpha};\psi}|\mathfrak{f}(\xi_j,\vartheta(\xi_j)) - \mathfrak{f}(\xi_j,y(\xi_j))| \right. \\
& \left. + \int_a^b \psi'(s)\, {}^k\mathcal{I}^{\overline{\alpha};\psi}|\mathfrak{f}(s,\vartheta(s)) - \mathfrak{f}(s,y(s))|ds \right] \\
\leq\ & \left\{ \frac{(\psi(b)-\psi(a))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)} + \frac{(\psi(b)-\psi(a))^{\frac{\theta_k}{k}-1}}{|\mathcal{H}|\Gamma_k(\theta_k)} \left[\sum_{j=1}^{m} |\eta_j| \frac{(\psi(\xi_j)-\psi(a))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)} \right.\right. \\
& \left.\left. + \frac{(\psi(b)-\psi(a))^{\frac{\overline{\alpha}}{k}+1}}{\Gamma_k(\overline{\alpha}+k)} \right] \right\} \mathfrak{L}\|x-y\| \\
=\ & \mathfrak{L}\mathfrak{G}\|x-y\|,
\end{aligned}
$$

which, on taking the norm for $w \in [a,b]$, yields $\|\mathcal{A}x - \mathcal{A}y\| \leq \mathfrak{L}\mathfrak{G}\|x-y\|$. Thus, it follows by (20) that $\mathcal{A}$ is a contraction. Consequently, an immediate consequence of Banach's fixed point theorem implies that the operator $\mathcal{A}$ has a unique fixed point. Hence, the problem (1) has a unique solution on $[a,b]$. The proof is finished. □

4.2. Existence Result via Krasnosel'skiĭ's Fixed Point Theorem

In this subsection we apply Krasnosel'skiĭ's fixed point theorem [24] to obtain an existence result for the problem (1).

Theorem 2. *Suppose that (H_1) and the following condition hold:*
(H_2) $|\mathfrak{f}(w,\vartheta)| \leq \varpi(w)$, $\quad \forall (w,\vartheta) \in [a,b] \times \mathbb{R}$, *and* $\varpi \in C([a,b],\mathbb{R}^+)$.
Then there exists at least one solution for the problem (1) on $[a,b]$, if

$$
\frac{\mathfrak{L}(\psi(b)-\psi(a))^{\frac{\theta_k}{k}-1}}{|\mathcal{H}|\Gamma_k(\theta_k)} \left[\sum_{j=1}^{m} |\eta_j| \frac{(\psi(\xi_j)-\psi(a))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)} + \frac{(\psi(b)-\psi(a))^{\frac{\overline{\alpha}}{k}+1}}{\Gamma_k(\overline{\alpha}+k)} \right] < 1. \tag{24}
$$

Proof. Let $B_\rho = \{\vartheta \in C([a,b],\mathbb{R}) : \|\vartheta\| \leq \rho\}$, with $\rho \geq \|\varpi\|\mathfrak{G}$ and $\sup_{w\in[a,b]} \varpi(w) = \|\varpi\|$, where $\mathfrak{G}$ is given by (21). We decompose the the operator $\mathcal{A}$ on B_ρ into two operators $\mathcal{A}_1$, $\mathcal{A}_2$ as

$$
(\mathcal{A}_1\vartheta)(w) = {}^k\mathfrak{I}^{\overline{\alpha};\psi}\mathfrak{f}(w,\vartheta(w)),
$$

$$
(\mathcal{A}_2\vartheta)(w) = \frac{(\psi(w)-\psi(a))^{\frac{\theta_k}{k}-1}}{\mathcal{H}\Gamma_k(\theta_k)} \left[\sum_{j=1}^{m} \eta_j\, {}^k\mathcal{I}^{\overline{\alpha};\psi}\mathfrak{f}(\xi_j,\vartheta(\xi_j)) - \int_a^b \psi'(s)\, {}^k\mathcal{I}^{\overline{\alpha};\psi}\mathfrak{f}(s,\vartheta(s))ds \right],
$$

$w \in [a,b]$. For any $\vartheta, y \in B_\rho$, we have

$$
\begin{aligned}
&|(\mathcal{A}_1\vartheta)(w) + (\mathcal{A}_2 y)(w)| \\
\le\ & \sup_{w\in[a,b]} \left\{ \frac{(\psi(w)-\psi(a))^{\frac{\theta_k}{k}-1}}{|\mathcal{H}|\Gamma_k(\theta_k)} \left[\sum_{j=1}^{m} |\eta_j|\ {}^k\mathcal{I}^{\overline{\alpha};\psi} \mathfrak{f}(\xi_j, y(\xi_j)) + \int_a^b \psi'(s)\ {}^k\mathcal{I}^{\overline{\alpha};\psi} |\mathfrak{f}(s,y(s))| ds \right] \right. \\
& \left. + {}^k\mathfrak{J}^{\overline{\alpha};\psi} |\mathfrak{f}(w,\vartheta(w))| \right\} \\
\le\ & \left\{ \frac{(\psi(b)-\psi(a))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)} + \frac{(\psi(b)-\psi(a))^{\frac{\theta_k}{k}-1}}{|\mathcal{H}|\Gamma_k(\theta_k)} \left[\sum_{j=1}^{m} |\eta_j|\ \frac{(\psi(\xi_j)-\psi(a))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)} \right.\right. \\
& \left.\left. + \frac{(\psi(b)-\psi(a))^{\frac{\overline{\alpha}}{k}+1}}{\Gamma_k(\overline{\alpha}+k)} \right] \right\} \|\varpi\| \\
=\ & \mathfrak{G}\|\varpi\| \le \rho.
\end{aligned}
$$

Therefore $\|(\mathcal{A}_1\vartheta) + (\mathcal{A}_2 y)\| \le \rho$, which shows that $\mathcal{A}_1\vartheta + \mathcal{A}_2 y \in B_\rho$.
Next, it will be shown that $\mathcal{A}_2$ is a contraction mapping. We have

$$
\begin{aligned}
&|(\mathcal{A}_2\vartheta)(w) - (\mathcal{A}_2 y)(w)| \\
\le\ & \frac{(\psi(b)-\psi(a))^{\frac{\theta_k}{k}-1}}{|\mathcal{H}|\Gamma_k(\theta_k)} \Big[\sum_{j=1}^{m} |\eta_j|\ {}^k\mathcal{I}^{\overline{\alpha};\psi} |\mathfrak{f}(\xi_j, \vartheta(\xi_j)) - \mathfrak{f}(\xi_j, y(\xi_j)| \\
& + \int_a^b \psi'(s)\ {}^k\mathcal{I}^{\overline{\alpha};\psi} |\mathfrak{f}(s,\vartheta(s)) - \mathfrak{f}(s,y(s))| ds \Big] \\
\le\ & \frac{(\psi(b)-\psi(a))^{\frac{\theta_k}{k}-1}}{|\mathcal{H}|\Gamma_k(\theta_k)} \left[\sum_{j=1}^{m} |\eta_j|\ \frac{(\psi(\xi_j)-\psi(a))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)} \right. \\
& \left. + \frac{(\psi(b)-\psi(a))^{\frac{\overline{\alpha}}{k}+1}}{\Gamma_k(\overline{\alpha}+k)} \right] \mathfrak{L}\|x-y\| \\
=\ & \mathfrak{L} \frac{(\psi(b)-\psi(a))^{\frac{\theta_k}{k}-1}}{|\mathcal{H}|\Gamma_k(\theta_k)} \left[\sum_{j=1}^{m} |\eta_j|\ \frac{(\psi(\xi_j)-\psi(a))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)} \right. \\
& \left. + \frac{(\psi(b)-\psi(a))^{\frac{\overline{\alpha}}{k}+1}}{\Gamma_k(\overline{\alpha}+k)} \right] \|x-y\|,
\end{aligned}
$$

which implies that $\mathcal{A}_2$ is a contraction mapping by (24).

Observe that continuity of $\mathfrak{f}$ implies that of the operator $\mathcal{A}_1$. Moreover, $\mathcal{A}_1$ is uniformly bounded on B_ρ as

$$
\|\mathcal{A}_1\vartheta\| \le \frac{(\psi(b)-\psi(a))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)} \|\varpi\|.
$$

In order to show that the operator $\mathcal{A}_1$ is compact, let $w_1, w_2 \in [a,b]$ with $w_1 < w_2$. Then we have

$$
\begin{aligned}
&|(\mathcal{A}_1\vartheta)(w_2) - (\mathcal{A}_1\vartheta)(w_1)| \\
\le\ & \frac{1}{\Gamma_k(\overline{\alpha})} \left| \int_a^{w_1} \psi'(s)[(\psi(w_2)-\psi(s))^{\frac{\overline{\alpha}}{k}-1} - (\psi(w_1)-\psi(s))^{\frac{\overline{\alpha}}{k}-1}] \mathfrak{f}(s,\vartheta(s)) ds \right. \\
& \left. + \int_{w_1}^{w_2} \psi'(s)(\psi(w_2)-\psi(s))^{\frac{\overline{\alpha}}{k}-1} \mathfrak{f}(s,\vartheta(s)) ds \right|
\end{aligned}
$$

$$\leq \frac{\|\varpi\|}{\Gamma_k(\overline{\alpha}+k)}[2(\psi(w_2)-\psi(w_1))^{\frac{\overline{\alpha}}{k}} + |(\psi(w_2)-\psi(a))^{\frac{\overline{\alpha}}{k}} - (\psi(w_1)-\psi(a))^{\frac{\overline{\alpha}}{k}}|],$$

which tends to zero as $w_2 - w_1 \to 0$, independently of $\vartheta \in B_\rho$. So $\mathcal{A}_1$ is equicontinuous. Therefore, $\mathcal{A}_1$ is completely continuous by an application of the Arzelá–Ascoli theorem. In view of the foregoing steps, we deduce that the hypotheses of Krasnosel'skiĭ's fixed point theorem are verified. Hence, it follows that there exists at least one solution for the problem (1) on $[a,b]$. This completes the proof. □

4.3. Existence Result via Leray–Schauder's Nonlinear Alternative

Our second existence result for the problem (1) is based on Leray–Schauder's nonlinear alternative [25].

Theorem 3. *Assume that:*

(H_3) *there exist a continuous, nondecreasing function $v : [0,\infty) \to (0,\infty)$ and a positive continuous function ϱ satisfying*

$$|\mathfrak{f}(w,u)| \leq \varrho(w)v(|u|) \quad \textit{for each} \quad (w,u) \in [a,b] \times \mathbb{R},$$

where $\mathfrak{f} : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function;

(H_4) *there exists a constant $\mathfrak{K} > 0$ such that*

$$\frac{\mathfrak{K}}{v(\mathfrak{K})\|\varrho\|\mathfrak{G}} > 1.$$

Then the (k,ψ)-Hilfer nonlocal boundary value problem (1) has at least one solution on $[a,b]$.

Proof. Consider the operator $\mathcal{A}$ defined by (19). For $r > 0$, let us define $B_r = \{\vartheta \in C([a,b],\mathbb{R}) : \|\vartheta\| \leq r\}$. Taking $w \in [a,b]$, we obtain

$$\begin{aligned}
&|(\mathcal{A}\vartheta)(w)| \\
\leq\ & \sup_{w\in[a,b]} \Bigg\{ \frac{(\psi(w)-\psi(a))^{\frac{\theta_k}{k}-1}}{|\mathcal{H}|\Gamma_k(\theta_k)} \Bigg[\sum_{j=1}^{m} |\eta_j|\ {}^k\mathcal{I}^{\overline{\alpha};\psi} \mathfrak{f}(\xi_j, y(\xi_j)) + \int_a^b \psi'(s)\ {}^k\mathcal{I}^{\overline{\alpha};\psi}|\mathfrak{f}(s,y(s))|ds \Bigg] \\
& + {}^k\mathfrak{I}^{\overline{\alpha};\psi}|\mathfrak{f}(w,\vartheta(w))| \Bigg\} \\
\leq\ & \Bigg\{ \frac{(\psi(b)-\psi(a))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)} + \frac{(\psi(b)-\psi(a))^{\frac{\theta_k}{k}-1}}{|\mathcal{H}|\Gamma_k(\theta_k)} \Bigg[\sum_{j=1}^{m} |\eta_j| \frac{(\psi(\xi_j)-\psi(a))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)} \\
& + \frac{(\psi(b)-\psi(a))^{\frac{\overline{\alpha}}{k}+1}}{\Gamma_k(\overline{\alpha}+k)} \Bigg] \Bigg\} \|\varrho\| v(\|\vartheta\|),
\end{aligned}$$

and consequently,

$$\|\mathcal{A}x\| \leq v(r)\|\varrho\|\mathfrak{G},$$

which shows that the operator $\mathcal{A}$ maps bounded sets into bounded set in $C([a,b],\mathbb{R})$.

Next it will be established that $\mathcal{A}$ maps bounded sets into equicontinuous sets of $C([a,b],\mathbb{R})$. As in the proof of the previous theorem, for $w_1, w_2 \in [a,b]$ with $w_1 < w_2$ and $\vartheta \in B_r$, we obtain

$$\begin{aligned}
&|(\mathcal{A}\vartheta)(w_2) - (\mathcal{A}\vartheta)(w_1)| \\
\leq\ & \frac{\|\varrho\|v(r)}{\Gamma_k(\overline{\alpha}+k)}[2(\psi(w_2)-\psi(w_1))^{\frac{\overline{\alpha}}{k}} + |(\psi(w_2)-\psi(a))^{\frac{\overline{\alpha}}{k}} - (\psi(w_1)-\psi(a))^{\frac{\overline{\alpha}}{k}}|]
\end{aligned}$$

$$
+\frac{(\psi(w_2)-\psi(a))^{\frac{\theta_k}{k}-1}-(\psi(w_1)-\psi(a))^{\frac{\theta_k}{k}-1}}{|\mathcal{H}|\Gamma_k(\theta_k)}\left[\sum_{j=1}^{m}|\eta_j|\,\frac{(\psi(\xi_j)-\psi(a))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)}\right.
$$

$$
\left.+\frac{(\psi(b)-\psi(a))^{\frac{\overline{\alpha}}{k}+1}}{\Gamma_k(\overline{\alpha}+k)}\right]\|\varrho\|v(r)\to 0 \text{ as } w_2-w_1\to 0,
$$

independently of $\vartheta\in B_r$. Hence, the operator $\mathcal{A}$ defined by (19) is completely continuous by the application of the Arzelá–Ascoli theorem.

Lastly, it will be shown that the set of all solutions to the equation $\vartheta=\lambda\mathcal{A}\vartheta$, $\lambda\in(0,1)$ is bounded. As in the first step, for $w\in[a,b]$, one can find that

$$
\frac{\|\vartheta\|}{v(\|\vartheta\|)\|\varrho\|\mathfrak{G}}\leq 1.
$$

By the assumption (H_4), we can find a positive constant $\mathfrak{K}$ satisfying $\|\vartheta\|\neq\mathfrak{K}$. Notice that the operator $\mathcal{A}:\overline{U}\to C([a,b],\mathbb{R})$ is continuous and completely continuous, where $U=\{\vartheta\in C([a,b],\mathbb{R}):\|\vartheta\|\leq\mathfrak{K}\}$ and $\overline{U}$ denotes the closure of U. Clearly the choice of U does not imply the existence of any $\vartheta\in\partial U$ (boundary of U) satisfying $\vartheta=\lambda\mathcal{A}\vartheta$ for some $\lambda\in(0,1)$. Hence, there exists a fixed point $\vartheta\in\overline{U}$ for the operator $\mathcal{A}$ as an immediate application of the nonlinear alternative of Leray–Schauder type [25]. Therefore, the problem (1) has a solution on $[a,b]$, which finishes the proof. □

5. The Multivalued Problem

Definition 12. *A function $\vartheta\in C([a,b],\mathbb{R})$ is called a solution of the (k,ψ)-Hilfer nonlocal multi-valued problem (2) if ϑ satisfies the differential equation ${}^{k,H}\mathfrak{D}^{\alpha,\beta;\psi}\vartheta(w)=f(w)$ on $[a,b]$ and the boundary conditions $\vartheta(a)=0$, $\int_a^b\psi'(s)\vartheta(s)ds=\sum_{j=1}^{m}\eta_j\vartheta(\xi_j)$, where $f\in L^1([a,b],\mathbb{R})$ with $f(w)\in\mathfrak{F}(w,\vartheta)$ for a.e. $w\in[a,b]$.*

Our first existence result, dealing with the convex-valued multi-valued map $\mathfrak{F}$, relies on nonlinear alternative of Leray–Schauder type [25].

Theorem 4. *Assume that:*

(G_1) *$\mathfrak{F}:[a,b]\times\mathbb{R}\to\mathcal{P}_{cp,c}(\mathbb{R})$ is L^1-Carathéodory, where*

$$
\mathcal{P}_{cp,c}(\mathbb{R})=\{\mathfrak{A}\in\mathcal{P}(\mathbb{R}):\mathfrak{A}\ is\ compact\ and\ convex\};
$$

(G_2) *there exist a continuous nondecreasing function $z:[0,\infty)\to(0,\infty)$ and a continuous positive function q such that*

$$
\|\mathfrak{F}(w,\vartheta)\|_{\mathcal{P}}:=\sup\{|f|:f\in\mathfrak{F}(w,\vartheta)\}\leq q(w)z(\|\vartheta\|)\ \textit{for each}\ (w,\vartheta)\in[a,b]\times\mathbb{R};
$$

(G_3) *there exists a constant $\mathfrak{K}>0$ satisfying $\mathfrak{K}>\|q\|z(\mathfrak{K})\mathfrak{G}$.*

Then there exists at least one solution for the problem (2) on $[a,b]$.

Proof. Introduce an operator $\mathcal{F}:C([a,b],\mathbb{R})\longrightarrow\mathcal{P}(C([a,b],\mathbb{R}))$ as

$$
\mathcal{F}(\vartheta)=\left\{\begin{array}{l}\mathfrak{h}\in C([a,b],\mathbb{R}):\\ \mathfrak{h}(w)=\left\{\begin{array}{l}\dfrac{(\psi(w)-\psi(a))^{\frac{\theta_k}{k}-1}}{\mathcal{H}\Gamma_k(\theta_k)}\left[\displaystyle\sum_{j=1}^{m}\eta_j\,{}^{k}\mathcal{I}^{\overline{\alpha};\psi}f(\xi_j)-\int_a^b\psi'(s)\,{}^{k}\mathcal{I}^{\overline{\alpha};\psi}f(s)ds\right]\\ +{}^{k}\mathfrak{J}^{\overline{\alpha};\psi}f(w),\ \ w\in[a,b],\end{array}\right.\end{array}\right\}
$$

where $f\in S_{\mathfrak{F},x}:=\{\widehat{f}\in L^1([a,b],\mathbb{R}):\widehat{f}(w)\in\mathfrak{F}(w,x(w))\text{ on }[a,b]\}$.

Note that the existence of fixed points of the operator $\mathcal{F}$ implies the existence of solutions to the problem (2).

We split the proof into several parts.

Step 1. $\mathcal{F}(\vartheta)$ *is convex for each* $\vartheta \in C([a,b],\mathbb{R})$.

It is easy to show that $S_{F,\vartheta}$ is convex as $\mathfrak{F}$ has convex values.

Step 2. $\mathcal{F}$ *maps bounded sets in* $C([a,b],\mathbb{R})$ into bounded sets.

For $r > 0$, let $B_r = \{\theta \in C([a,b],\mathbb{R}) : \|\theta\| \le r\}$. Then, for each $\mathfrak{h} \in \mathcal{F}(\theta), \theta \in B_r$, there exists $f \in S_{\mathfrak{F},x}$ such that

$$\mathfrak{h}(w) = \frac{(\psi(w)-\psi(a))^{\frac{\theta_k}{k}-1}}{\mathcal{H}\Gamma_k(\theta_k)}\left[\sum_{j=1}^{m}\eta_j\,{}^{k}\mathcal{I}^{\overline{\alpha};\psi}f(\xi_j) - \int_a^b \psi'(s)\,{}^{k}\mathcal{I}^{\overline{\alpha};\psi}f(s)ds\right] + {}^{k}\mathfrak{I}^{\overline{\alpha};\psi}f(w).$$

Then, for $w \in [a,b]$, we have

$$\begin{aligned}
|\mathfrak{h}(w)| &\le \sup_{w\in[a,b]}\left\{\frac{(\psi(w)-\psi(a))^{\frac{\theta_k}{k}-1}}{|\mathcal{H}|\Gamma_k(\theta_k)}\left[\sum_{j=1}^{m}|\eta_j|\,{}^{k}\mathcal{I}^{\overline{\alpha};\psi}f(\xi_j) + \int_a^b \psi'(s)\,{}^{k}\mathcal{I}^{\overline{\alpha};\psi}f(s)ds\right]\right. \\
&\qquad\left. + {}^{k}\mathfrak{I}^{\overline{\alpha};\psi}|f(w)|\right\} \\
&\le \left\{\frac{(\psi(b)-\psi(a))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)} + \frac{(\psi(b)-\psi(a))^{\frac{\theta_k}{k}-1}}{|\mathcal{H}|\Gamma_k(\theta_k)}\left[\sum_{j=1}^{m}|\eta_j|\,\frac{(\psi(\xi_j)-\psi(a))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)}\right.\right. \\
&\qquad\left.\left. + \frac{(\psi(b)-\psi(a))^{\frac{\overline{\alpha}}{k}+1}}{\Gamma_k(\overline{\alpha}+k)}\right]\right\}\|q\|z(\|\vartheta\|),
\end{aligned}$$

which implies that $\|\mathfrak{h}\| \le z(r)\|q\|\mathfrak{G}$.

Step 3. $\mathcal{F}$ *maps bounded sets into equicontinuous sets of* $C([a,b],\mathbb{R})$.

For each $\mathfrak{h} \in \mathcal{F}(\theta), \theta \in B_r$, and $w_1, w_2 \in [a,b]$ with $w_1 < w_2$, we obtain

$$\begin{aligned}
&|\mathfrak{h}(w_2) - \mathfrak{h}(w_1)| \\
\le\ & \frac{1}{\Gamma_k(\alpha)}\left|\int_a^{w_1}\psi'(s)[(\psi(w_2)-\psi(s))^{\frac{\overline{\alpha}}{k}-1} - (\psi(w_1)-\psi(s))^{\frac{\overline{\alpha}}{k}-1}]f(s)ds\right. \\
&\left. + \int_{w_1}^{w_2}\psi'(s)(\psi(w_2)-\psi(s))^{\frac{\overline{\alpha}}{k}-1}f(s)ds\right| \\
& + \frac{\left((\psi(w_2)-\psi(a))^{\frac{\theta_k}{k}-1} - (\psi(w_1)-\psi(a))^{\frac{\theta_k}{k}-1}\right)}{|\mathcal{H}|\Gamma_k(\theta_k)} \\
& \times\left[\sum_{j=1}^{m}|\eta_j|\,{}^{k}\mathcal{I}^{\overline{\alpha};\psi}f(\xi_j) + \int_a^b\psi'(s)\,{}^{k}\mathcal{I}^{\overline{\alpha};\psi}f(s)ds\right] \\
\le\ & \frac{\|q\|z(r)}{\Gamma_k(\overline{\alpha}+k)}[2(\psi(w_2)-\psi(w_1))^{\frac{\overline{\alpha}}{k}} + |(\psi(w_2)-\psi(a))^{\frac{\overline{\alpha}}{k}} - (\psi(w_1)-\psi(a))^{\frac{\overline{\alpha}}{k}}|], \\
& + \frac{(\psi(w_2)-\psi(a))^{\frac{\theta_k}{k}-1} - (\psi(w_1)-\psi(a))^{\frac{\theta_k}{k}-1}}{|\mathcal{H}|\Gamma_k(\theta_k)}\left[\sum_{j=1}^{m}|\eta_j|\,\frac{(\psi(\xi_j)-\psi(a))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha}+k)}\right. \\
&\left. + \frac{(\psi(b)-\psi(a))^{\frac{\overline{\alpha}}{k}+1}}{\Gamma_k(\overline{\alpha}+k)}\right]\|q\|z(r) \to 0 \text{ as } w_2 - w_1 \to 0,
\end{aligned}$$

independently of $\theta \in B_r$. Clearly the conclusion of the Arzelá–Ascoli theorem applies and hence $\mathcal{F} : C([a,b],\mathbb{R}) \to \mathcal{P}(C([a,b],\mathbb{R}))$ is completely continuous.

In view of Proposition 1.2 of [26], we need to establish that the operator $\mathcal{F}$ has a closed graph, which is equivalent to the fact that $\mathcal{F}$ is upper semi-continuous multi-valued map.

Step 4. $\mathcal{F}$ has a closed graph.

Assuming $\vartheta_n \to \vartheta_*, h_n \in \mathcal{F}(\vartheta_n)$ and $\mathfrak{h}_n \to \mathfrak{h}_*$, it will be shown that $\mathfrak{h}_* \in \mathcal{F}(\vartheta_*)$. Related to $\mathfrak{h}_n \in \mathcal{F}(\vartheta_n)$, we can find $v_n \in S_{\mathfrak{F},\vartheta_n}$ such that, for each $w \in [a,b]$, we have

$$\mathfrak{h}_n(w) = \frac{(\psi(w)-\psi(a))^{\frac{\theta_k}{k}-1}}{\mathcal{H}\Gamma_k(\theta_k)}\left[\sum_{j=1}^{m}\eta_j\,{}^k\mathcal{I}^{\overline{\alpha};\psi}f_n(\xi_j) - \int_a^b \psi'(s)\,{}^k\mathcal{I}^{\overline{\alpha};\psi}f_n(s)ds\right] + {}^k\mathfrak{J}^{\overline{\alpha};\psi}f_n(w).$$

Then it suffices to establish that there exists $v_* \in S_{\mathfrak{F},\vartheta_*}$ satisfying

$$\mathfrak{h}_*(w) = \frac{(\psi(w)-\psi(a))^{\frac{\theta_k}{k}-1}}{\mathcal{H}\Gamma_k(\theta_k)}\left[\sum_{j=1}^{m}\eta_j\,{}^k\mathcal{I}^{\overline{\alpha};\psi}f_*(\xi_j) - \int_a^b \psi'(s)\,{}^k\mathcal{I}^{\overline{\alpha};\psi}f_*(s)ds\right] + {}^k\mathfrak{J}^{\overline{\alpha};\psi}f_*(w),$$

for each $w \in [a,b]$.

Define a linear operator $\Theta : L^1([a,b],\mathbb{R}) \to C([a,b],\mathbb{R})$ as

$$v \mapsto \Theta(v)(w) = \frac{(\psi(w)-\psi(a))^{\frac{\theta_k}{k}-1}}{\mathcal{H}\Gamma_k(\theta_k)}\left[\sum_{j=1}^{m}\eta_j\,{}^k\mathcal{I}^{\overline{\alpha};\psi}f(\xi_j) - \int_a^b \psi'(s)\,{}^k\mathcal{I}^{\overline{\alpha};\psi}f(s)ds\right] + {}^k\mathfrak{J}^{\overline{\alpha};\psi}f(w).$$

Note that

$$\|\mathfrak{h}_n - \mathfrak{h}_*\| \to 0, \text{ as } n \to \infty.$$

Therefore, owing to a result by Lazota–Opial [27], we deduce that $\Theta \circ S_{\mathfrak{F}}$ is a closed graph operator. Furthermore, $\mathfrak{h}_n(w) \in \Theta(S_{\mathfrak{F},\vartheta_n})$. As $\vartheta_n \to \vartheta_*$, for some $v_* \in S_{\mathfrak{F},\vartheta_*}$, we obtain

$$\mathfrak{h}_*(w) = \frac{(\psi(w)-\psi(a))^{\frac{\theta_k}{k}-1}}{\mathcal{H}\Gamma_k(\theta_k)}\left[\sum_{j=1}^{m}\eta_j\,{}^k\mathcal{I}^{\overline{\alpha};\psi}f_*(\xi_j) - \int_a^b \psi'(s)\,{}^k\mathcal{I}^{\overline{\alpha};\psi}f_*(s)ds\right] + {}^k\mathfrak{J}^{\overline{\alpha};\psi}f_*(w).$$

Step 5. $\vartheta \notin \nu\mathcal{F}(\vartheta)$ for any $\nu \in (0,1)$ and all $\vartheta \in \partial\mathcal{U}$, where $\mathcal{U} \subseteq C([a,b],\mathbb{R})$ is an open set.

On the contrary, suppose that $\vartheta \in \nu\mathcal{F}(\vartheta)$ for $\nu \in (0,1)$. Then, there exists $f \in L^1([a,b],\mathbb{R})$ with $f \in S_{\mathfrak{F},\vartheta}$ satisfying

$$\vartheta(w) = \nu\frac{(\psi(w)-\psi(a))^{\frac{\theta_k}{k}-1}}{\mathcal{H}\Gamma_k(\theta_k)}\left[\sum_{j=1}^{m}\eta_j\,{}^k\mathcal{I}^{\overline{\alpha};\psi}f(\xi_j) - \int_a^b \psi'(s)\,{}^k\mathcal{I}^{\overline{\alpha};\psi}f(s)ds\right] + \nu\,{}^k\mathfrak{J}^{\overline{\alpha};\psi}f(w),$$

for $w \in [a,b]$.

As argued in the second step, one can obtain $\|\vartheta\| \leq \|q\|z(\|\vartheta\|)\mathfrak{G}$. By the assumption (H_3), we can find $\mathfrak{K}$ such that $\|\vartheta\| \neq \mathfrak{K}$. Introduce a set

$$\mathcal{U} = \{\vartheta \in C([a,b],\mathbb{R}) : \|\vartheta\| < \mathfrak{K}\},$$

and define a compact, upper semi-continuous, convex and closed multivalued map $\mathcal{F} : \overline{\mathcal{U}} \to \mathcal{P}(C([a,b],\mathbb{R}))$, where $\overline{\mathcal{U}}$ denotes the closure of U. By the definition of $\mathcal{U}$, there does not exist any $\vartheta \in \partial\mathcal{U}$ (boundary of $\mathcal{U}$) satisfying $\vartheta \in \nu\mathcal{F}(\vartheta)$ for some $\nu \in (0,1)$. Therefore, the operator $\mathcal{F}$ has a fixed point $\vartheta \in \overline{\mathcal{U}}$ by the application of the nonlinear alternative of Leray–Schauder type [25], which is indeed a solution of the problem (2). This ends the proof. □

In the following Theorem, we apply Covitz and Nadler's fixed point theorem for multivalued contractive maps [28] to obtain an existence result for the problem (2) when the multi-valued map F in the problem is not necessarily convex valued.

Theorem 5. *Suppose that*

(A_1) $\mathfrak{F}:[a,b]\times\mathbb{R}\to\mathcal{P}_{cp}(\mathbb{R})$ *is such that* $\mathfrak{F}(\cdot,\vartheta):[a,b]\to\mathcal{P}_{cp}(\mathbb{R})$ *is measurable for each* $\vartheta\in\mathbb{R}$, *where* $\mathcal{P}_{cp}(\mathbb{R})=\{\mathfrak{Q}\in\mathcal{P}(\mathbb{R}):\mathfrak{Q}$ *is compact*$\}$;

(A_2) *There exists* $q\in C([a,b],\mathbb{R}^+)$ *such that*

$$H_d(\mathfrak{F}(w,\vartheta),\mathfrak{F}(w,\widehat{\vartheta}))\le q(w)|\vartheta-\widehat{\vartheta}|,$$

with $d(0,\mathfrak{F}(w,0))\le q(w)$, *where* H_d *is the generalized metric [29],* $\vartheta,\widehat{\vartheta}\in\mathbb{R}$ *for almost all* $w\in[a,b]$.

Then there exists at least one solution for the multi-valued problem (2) on $[a,b]$ *if*

$$\delta:=\mathfrak{G}\|q\|<1, \tag{25}$$

where $\mathfrak{G}$ *is given by (21).*

Proof. Observe that the set $S_{\mathfrak{F},\vartheta}$ is nonempty for each $\vartheta\in C([a,b],\mathbb{R})$ by the condition (A_1). Hence, by Theorem III.6 of [30], $\mathfrak{F}$ has a measurable selection. For each $\vartheta\in C([a,b],\mathbb{R})$, let us now verify that $\mathcal{F}(\vartheta)\in\mathcal{P}_{cl}(C([a,b],\mathbb{R}))$, where $\mathcal{P}_{cl}(\mathbb{R})=\{\mathfrak{Q}\in\mathcal{P}(\mathbb{R}):\mathfrak{Q}$ is closed$\}$. For that, let $\{u_n\}_{n\ge0}\in\mathcal{F}(\vartheta)$ with $u_n\to u$ $(n\to\infty)$ in $C([a,b],\mathbb{R})$. Then $u\in C([a,b],\mathbb{R})$ and we can find $v_n\in S_{\mathfrak{F},\vartheta_n}$ such that, for each $w\in[a,b]$,

$$u_n(w)=\frac{(\psi(w)-\psi(a))^{\frac{\theta_k}{k}-1}}{\mathcal{H}\Gamma_k(\theta_k)}\left[\sum_{j=1}^m\eta_j\,{}^k\mathcal{I}^{\bar{\alpha};\psi}v_n(\xi_j)-\int_a^b\psi'(s)\,{}^k\mathcal{I}^{\bar{\alpha};\psi}v_n(s)ds\right]+{}^k\mathfrak{J}^{\bar{\alpha};\psi}v_n(w).$$

As $\mathfrak{F}$ is compact valued, we pass onto a subsequence (if necessary) to obtain that v_n converges to v in $L^1([a,b],\mathbb{R})$. Thus, $v\in S_{\mathfrak{F},\vartheta}$ and for each $w\in[a,b]$, we have

$$\begin{aligned}u_n(w)\to u(w)&=\frac{(\psi(w)-\psi(a))^{\frac{\theta_k}{k}-1}}{\mathcal{H}\Gamma_k(\theta_k)}\left[\sum_{j=1}^m\eta_j\,{}^k\mathcal{I}^{\bar{\alpha};\psi}v(\xi_j)-\int_a^b\psi'(s)\,{}^k\mathcal{I}^{\bar{\alpha};\psi}v(s)ds\right]\\&\quad+\ {}^k\mathfrak{J}^{\bar{\alpha};\psi}v(w),\end{aligned}$$

which shows that $u\in\mathcal{F}(\vartheta)$.

Next we establish that there exists $0<\delta<1$ (defined by (25)) satisfying

$$H_d(\mathcal{F}(\vartheta),\mathcal{F}(\widehat{\vartheta}))\le\delta\|\vartheta-\widehat{\vartheta}\|,\ \text{for each}\ \ \vartheta,\bar{\vartheta}\in C^2([a,b],\mathbb{R}).$$

Letting $\vartheta,\widehat{\vartheta}\in C^2([a,b],\mathbb{R})$ and $\mathfrak{h}_1\in\mathcal{F}(x)$, $\exists\ v_1(w)\in\mathfrak{F}(w,\vartheta(w))$ for each $w\in[a,b]$, such that

$$\mathfrak{h}_1(w)=\frac{(\psi(w)-\psi(a))^{\frac{\theta_k}{k}-1}}{\mathcal{H}\Gamma_k(\theta_k)}\left[\sum_{j=1}^m\eta_j\,{}^k\mathcal{I}^{\bar{\alpha};\psi}v_1(\xi_j)-\int_a^b\psi'(s)\,{}^k\mathcal{I}^{\bar{\alpha};\psi}v_1(s)ds\right]+{}^k\mathfrak{J}^{\bar{\alpha};\psi}v_1(w).$$

In view of (A_2), we have

$$H_d(\mathfrak{F}(w,\vartheta),\mathfrak{F}(w,\widehat{\vartheta}))\le q(w)|\vartheta(w)-\widehat{\vartheta}(w)|.$$

So we can find $\chi\in\mathfrak{F}(w,\bar{x}(w))$ satisfying

$$|v_1(w)-\chi|\le q(w)|\vartheta(w)-\widehat{\vartheta}(w)|,\ \ w\in[a,b].$$

Let $U:[a,b]\to\mathcal{P}(\mathbb{R})$ be defined by

$$U(w)=\{w\in\mathbb{R}:|v_1(w)-\chi|\le q(w)|\vartheta(w)-\widehat{\vartheta}(w)|\}.$$

By Proposition III.4 of [30], the multi-valued operator $U(w) \cap \mathfrak{F}(w, \widehat{\vartheta}(w))$ is measurable. Therefore, we can find a measurable selection $v_2(w)$ for U such that $v_2(w) \in \mathfrak{F}(w, \widehat{\vartheta}(w))$ satisfying $|v_1(w) - v_2(w)| \leq q(w)|\vartheta(w) - \widehat{\vartheta}(w)|$ for each $w \in [a, b]$.

Then, for each $w \in [a, b]$, we can define

$$\mathfrak{h}_2(w) = \frac{(\psi(w) - \psi(a))^{\frac{\theta_k}{k} - 1}}{\mathcal{H}\Gamma_k(\theta_k)} \left[\sum_{j=1}^{m} \eta_j \, {}^k\mathcal{I}^{\overline{\alpha};\psi} v_2(\xi_j) - \int_a^b \psi'(s) \, {}^k\mathcal{I}^{\overline{\alpha};\psi} v_2(s) ds \right] + {}^k\mathfrak{J}^{\overline{\alpha};\psi} v_2(w).$$

Thus,

$$\begin{aligned}
& |\mathfrak{h}_1(w) - \mathfrak{h}_2(w)| \\
\leq \; & \frac{(\psi(w) - \psi(a))^{\frac{\theta_k}{k} - 1}}{\mathcal{H}\Gamma_k(\theta_k)} \Big[\sum_{j=1}^{m} |\eta_j| \, {}^k\mathcal{I}^{\overline{\alpha};\psi} (|v_1(s) - v_2(s)|)(\xi_j) \\
& + \int_a^b \psi'(s) \, {}^k\mathcal{I}^{\overline{\alpha};\psi} (|v_1(s) - v_2(s)|) ds \Big] + {}^k\mathfrak{J}^{\overline{\alpha};\psi} (|v_1(s) - v_2(s)|)(w) \\
\leq \; & \left\{ \frac{(\psi(b) - \psi(a))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha} + k)} + \frac{(\psi(b) - \psi(a))^{\frac{\theta_k}{k} - 1}}{|\mathcal{H}|\Gamma_k(\theta_k)} \left[\sum_{j=1}^{m} |\eta_j| \frac{(\psi(\xi_j) - \psi(a))^{\frac{\overline{\alpha}}{k}}}{\Gamma_k(\overline{\alpha} + k)} \right.\right. \\
& \left.\left. + \frac{(\psi(b) - \psi(a))^{\frac{\overline{\alpha}}{k} + 1}}{\Gamma_k(\overline{\alpha} + k)} \right] \right\} \|q\| \|\vartheta - \widehat{\vartheta}\| \\
= \; & \mathfrak{G} \|q\| \|\vartheta - \widehat{\vartheta}\|.
\end{aligned}$$

In consequence, we obtain

$$\|\mathfrak{h}_1 - \mathfrak{h}_2\| \leq \mathfrak{G} \|q\| \|\vartheta - \widehat{\vartheta}\|.$$

Analogously, we can interchange the roles of ϑ and $\widehat{\vartheta}$ to find that

$$H_d(\mathcal{F}(\vartheta), \mathcal{F}(\widehat{\vartheta})) \leq \mathfrak{G} \|q\| \|\vartheta - \widehat{\vartheta}\|,$$

which shows that $\mathcal{F}$ is a contraction. Thus the conclusion of Covitz and Nadler's fixed point theorem [28] applies and hence $\mathcal{F}$ has a fixed point ϑ, which is a solution of the multi-valued problem (2). The proof is complete. □

6. Examples

This section is devoted to some examples showing the applicability of our results.

Example 1. *Consider the following nonlocal (k, ψ)-Hilfer type fractional integro-multi-point boundary value problem:*

$$\begin{cases}
{}^{\frac{5}{4},H}\mathfrak{D}^{\frac{3}{2},\frac{1}{4};\log(w^2+1)} \vartheta(w) = \mathfrak{f}(w, \vartheta(w)), \quad \dfrac{2}{5} < w < \dfrac{9}{5}, \\
\vartheta\left(\dfrac{2}{5}\right) = 0, \quad 2\displaystyle\int_{\frac{2}{5}}^{\frac{9}{5}} \left(\frac{s}{s^2 + 1}\right) \vartheta(s) ds = \dfrac{1}{13}\vartheta\left(\dfrac{3}{5}\right) + \dfrac{3}{23}\vartheta\left(\dfrac{4}{5}\right) \\
\qquad\qquad + \dfrac{5}{33}\vartheta\left(\dfrac{6}{5}\right) + \dfrac{7}{43}\vartheta\left(\dfrac{7}{5}\right) + \dfrac{9}{53}\vartheta\left(\dfrac{8}{5}\right).
\end{cases} \tag{26}$$

Here $\overline{\alpha} = 3/2$, $\beta = 1/4$, $\psi(w) = \log(w^2 + 1)$, $k = 5/4$, $a = 2/5$, $b = 9/5$, $m = 5$, $\eta_1 = 1/13$, $\eta_2 = 3/23$, $\eta_3 = 5/33$, $\eta_4 = 7/43$, $\eta_5 = 9/53$, $\xi_1 = 3/5$, $\xi_2 = 4/5$, $\xi_3 = 6/5$, $\xi_4 = 7/5$, $\xi_5 = 8/5$. From this set of values, we get $\theta_{\frac{5}{4}} = 7/4$, $\Gamma_{\frac{5}{4}}(\theta_{\frac{5}{4}}) \approx 0.9701006072$, $\Gamma_{\frac{5}{4}}\left(\theta_{\frac{5}{4}} + \frac{5}{4}\right) \approx 1.697676063$, $\Gamma_{\frac{5}{4}}(\overline{\alpha} + \frac{5}{4}) \approx 1.440110329$, $\mathcal{H} \approx 0.3205592914$, $\mathfrak{G} \approx 6.630909065$, $\mathfrak{G}_1 \approx 5.682953800$.

(i) Let the nonlinear unbounded Lipschitzian function $\mathfrak{f}(w,\vartheta)$ be given by

$$\mathfrak{f}(w,\vartheta)=\frac{5}{2(5w+33)}\left(\frac{\vartheta^2+2|\vartheta|}{1+|\vartheta|}\right)+\frac{3}{4}w^2+\frac{1}{2}w+\frac{1}{4}. \tag{27}$$

Observe that $\lim_{\vartheta\to\infty,w\in[2/5,9/5]}|\mathfrak{f}(w,\vartheta)|=\infty$, which is an unbounded function. Clearly $\mathfrak{L}=1/7$ as

$$|\mathfrak{f}(w,\vartheta)-\mathfrak{f}(w,\theta)|\leq\frac{1}{7}|\vartheta-\theta|,$$

for all $w\in[2/5,9/5]$ and $\vartheta,\theta\in\mathbb{R}$. Moreover, $\mathfrak{L}\mathfrak{G}\approx 0.9472727236<1$. Therefore, by Theorem 1, the nonlocal (k,ψ)-Hilfer integro-multi-point boundary value problem (26) with a nonlinear function given in (27) has a unique solution on the interval $[2/5,9/5]$.

(ii) Consider the nonlinear Lipschitzian function $\mathfrak{f}(w,\vartheta)$ given by

$$\mathfrak{f}(w,\vartheta)=\frac{10}{\pi(5w+28)}\tan^{-1}\vartheta+\frac{1}{3}\sin^2 w+\frac{1}{7}, \tag{28}$$

and note that it is bounded as

$$|\mathfrak{f}(w,\vartheta)|\leq\frac{5}{(5w+28)}+\frac{1}{3}\sin^2 w+\frac{1}{7}:=\varpi(w),$$

for all $w\in[1/5,8/5]$ and $\vartheta\in\mathbb{R}$. Further, the function $\mathfrak{f}$ satisfies the Lipschitz condition in (H_1) with Lipschitz constant $\mathfrak{L}=1/6$ and $\mathfrak{L}\mathfrak{G}_1\approx 0.9471589667<1$. Therefore, the nonlocal (k,ψ)-Hilfer fractional integro-multi-point boundary value problem (26), with $\mathfrak{f}$ given by (28), has at least one solution on $[2/5,9/5]$ by Theorem 2. Here, one can notice that the unique solution is not possible as $\mathfrak{L}\mathfrak{G}\approx 1.105151511>1$.

(iii) Let the nonlinear unbounded non-Lipschitzian function $\mathfrak{f}(w,\vartheta)$ be expressed by

$$\mathfrak{f}(w,\vartheta)=\frac{1}{5w+1}\left(\frac{\vartheta^{2022}}{4(1+\vartheta^{2020})}+\frac{1}{5}\right). \tag{29}$$

Note that we can find the following quadratic relation in terms of an unknown function ϑ:

$$|\mathfrak{f}(w,\vartheta)|\leq\frac{1}{5w+1}\left(\frac{1}{4}\vartheta^2+\frac{1}{5}\right).$$

Choosing $\varrho(w)=1/(5w+1)$ and $v(y)=(1/4)y^2+(1/5)$, we have $\|\varrho\|=1/3$ and hence we get a constant $\mathfrak{K}\in(0.7678879533,1.041818662)$ satisfying condition (H_4) in Theorem 3. Thus, By the conclusion of Theorem 3, the nonlocal (k,ψ)-Hilfer fractional integro-multi-point boundary value problem (26) with $\mathfrak{f}$ given by (29) has at least one solution on $[2/5,9/5]$.

(iv) Replace the first equation of (26) by

$$^{\frac{5}{4},H}\mathfrak{D}^{\frac{3}{2},\frac{1}{4};\log(w^2+1)}\vartheta(w)\in\mathfrak{F}(w,\vartheta(w)),\quad \frac{2}{5}<w<\frac{9}{5}, \tag{30}$$

where

$$\mathfrak{F}(w,\vartheta)=\left[0,\frac{1}{4(5w+2)}\Big(|\tan^{-1}\vartheta|+\sin\vartheta+1\Big)\right]. \tag{31}$$

It is obvious that $\mathfrak{F}(w,\vartheta)$ is a measurable set. From (31), we have a relation

$$H_d\big(\mathfrak{F}(w,\vartheta),\mathfrak{F}(w,\overline{\vartheta})\big)\leq\frac{1}{2(5w+2)}|\vartheta-\overline{\vartheta}|.$$

Setting $q(w) = 1/(2(5w+2))$, we obtain $\|q\| = 1/8$. In addition, for almost all $w \in [2/5, 9/5]$, we find that

$$d(0, \mathfrak{F}(w,0)) \leq 1/(4(5w+2)) \leq 1/(2(5w+2)) = q(w)$$

and $\delta = \mathfrak{G}\|q\| \approx 0.8288636331 < 1$. As the hypothesis of Theorem 5 is satisfied, so the (k,ψ)-Hilfer fractional differential inclusion (30) with boundary conditions given in (26), has at least one solution on $[2/5, 9/5]$.

7. Conclusions

In the present research, we have investigated existence criteria for the solutions of (k,ψ)-Hilfer type fractional nonlocal integro-multi-point single valued and multi-valued boundary value problems. The fixed-point approach is employed to derive the desired results for the given problems by applying the standard fixed point theorems for single valued and multi-valued maps. We have discussed both convex and non-convex multi-valued cases for the inclusion problem. Numerical examples are given for demonstrating the application of the main results. Our results in the given configuration are new and contribute significantly to the literature on this new topic of research.

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