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Double Sources Queuing-Inventory System with Hybrid Replenishment Policy

Agassi Melikov [1,*], Ramil Mirzayev [2] and Sajeev S. Nair [3]

[1] Institute of Control Systems, National Academy of Science, AZ 1141 Baku, Azerbaijan
[2] Faculty of Engineering, National Aviation Academy, AZ 1045 Baku, Azerbaijan; r_mirzayev@mail.ru
[3] Department of Mathematics, Government Engineering College, Thrissur 680009, India; saji72nair@gmail.com
* Correspondence: agassi.melikov@gmail.com

Abstract: A hybrid replenishment policy in double sources queuing-inventory system is proposed. If the inventory level drops to the reorder point s, then a regular order of the fixed volume $Q = S - s$ is generated to a slow and cheap source, where S denotes the maximum size of the system's warehouse. If the inventory level falls below a certain threshold value r, where $r < s$, then the system instantly cancels the regular order and generates an emergency order to a fast and expensive source where the replenishment quantity should be able to bring the inventory level back to S at the replenishment epoch. In addition to consuming customers, the system also receives destructive customers that do not require inventory but destroy them. The stability condition for the system under study is found, steady-state probabilities are calculated, and formulas for finding performance measures are proposed. The problem of minimizing the total cost of the system under the proposed hybrid replenishment policy is solved by choosing the appropriate values of the order point and the threshold value.

Keywords: queuing-inventory system; double sources; hybrid replenishment policy; matrix-geometric method

MSC: 60J28; 60K25; 90B05; 90B22

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1. Introduction

There are deep connections between models of queuing systems (QS) and inventory management systems (IMS), see [1–3]. Moreover, in some cases, QS models are transformed into IMS models and vice versa by a simple change of terms.

For a long time, QS and IMS models were studied separately. The study of logistic systems, which are hybrids of QS and IMS, began in the early 1990s. The first publications devoted to the study of these models are [4,5]. Subsequently, these systems were called queuing-inventory systems (QIS) [6,7]. The theory of QISs has been intensively developed over the past three decades, and its state of the art is described in detail in the review papers [8,9].

These days there are various important directions of development of QISs theory. Taking into account the features studied here QIS, the works devoted to systems with perishable inventory [10–15], systems with impatient customers [16–21], systems with multi-class customers [22–30], systems with retrial customers [31–42], and production-inventory systems [43–48] should be noted. Note that the classification of QIS models based on some specific feature (indicator) is not a simple problem since many of the above-indicated models simultaneously have many features.

In classical models of perishable QIS, it is assumed that items might be damaged after a random period of time (aging of items) with known cumulative distribution function (CDF), in other words, the models with non-instantaneous damaging of inventory are considered. However, in practice, items can be damaged instantly due to various reasons. For example, in a shop that sells breakable items, damage to items may be the result of the careless actions of an employee. Another example is the sale of electrical goods or computers, where damage to an item can occur even at the time of their sale as a result of a power surge in the electrical network. Perishable QIS models of this type have hardly been studied, see [49]. In this paper, the flow of inventory deterioration events is modeled as a flow of destructive customers (d-customers). Note that d-customers can be considered as an analog of negative customers in classical QS. However, it is necessary to take into account that in QS, negative customers force regular customers out of the system, but here d-customers destroy stocks (items).

The impatience of customers in QIS is taken into account similarly in classical models of queuing systems. However, in contrast to models of QS, a new feature appears in QIS for the classification of consuming customers (c-customers). Indeed, in QIS some customers may refuse to purchase the item after being served, i.e., they require only service and do not require items. In other words, here, the classification of customers is carried out not at the moment of their receipt, but after the completion of the service process. The multi-class QIS model of this type was first studied in [50]. Here, we also take into account this phenomenon.

An analysis of the available literature showed that the vast majority of studies are devoted to QIS with a single source of replenishment, while it is assumed that the rate of replenishment is a constant value. However, there are a few works devoted to production-inventory systems in which the inventory is replenished by an external supplier and an internal production [48]. In [48] it is assumed that an order for replenishment is sent to an external as soon as the inventory level drops to some value, and the items are delivered after the random lead-time. In addition to the external supplier, an internal replenishment (production) follows a Poisson process with a finite rate. The waiting room of the c-customers and the storage size are assumed to be unlimited. Note that the last assumption is unusual for real QIS.

Models of QIS with a single source but with different replenishment rates were first studied in [33,43]. In these works, authors propose models of QISs in which replenishment rates depend on inventory level, i.e., the lower the inventory level, the faster the replenishment rate.

Note that in order to increase the reliability of timely servicing of c-customers, improve performance measures, and minimize the total cost of the investigated systems, it is necessary to organize the supply of inventory from several sources. At the same time, the usage of multiple sources leads to solving a number of problems, including determining the moments of supply from various sources, as well as the distribution of order quantities between them. It is important to note that these tasks differ from the tasks of choosing the efficient supplier from a finite set of suppliers with different characteristics (order lead times and replenishment costs). Note that the task of optimal distribution of orders between a fast and expensive supplier and a slow but inexpensive supplier is solved in [51].

In addition, almost all works assume that a unified replenishment policy (RP) is used throughout the entire period of operating of the QIS. Nevertheless, the study of state-dependent RP (when RP depends on inventory level, number of customers in the system, etc.) is of great scientific and practical interest. A replenishment policy that simultaneously uses different (known) policies within a single system is called a hybrid policy.

Despite their importance, the models of multi-source QISs with hybrid RP have been minimally explored. To our best knowledge, the first work that studies QIS with a hybrid RP, but with a single source, is [10]. It considers a perishable QIS with a single server and a limited buffer for primary customers and an unlimited buffer for feedback customers. Primary customers form a MAP-flow (Markov Arrival Process, MAP), their service time, and inventory lifetime have exponential CDF, and the lead-time has phase-type CDF. It is assumed that, depending on the inventory level of the system, either a fixed or a variable supply RP is used.

As far as the authors are aware, in [52] the first attempt was made to develop hybrid RP in Markovian double sources QIS with perishable inventory and negative customers. In that paper, the following RP is proposed. If the inventory level drops to a fixed value $r, r > (S/2)$, order of quantity $Q_1 = S - r$ is placed from the first source; if it drops to a prefixed level $s, s < Q_1$, order of quantity $Q_2 = S - s > s + 1$ is placed from the second source. The lead times from the first and second sources have an exponential CDF with paramerets η_1 and η_2, respectively, $\eta_2 > \eta_1$. Unfortunately, as proposed in [52], hybrid RP is not the correct one. Indeed, in accordance with the proposed RP, it is expected that if the current inventory level is, say s, then two replenishments are possible: one of them might arrive from the first source (with size $S - r$) and another replenishment can be done by the second source (with size $S - s$). As a result, total size of replenishments is more than stock capacity of the system. Note that the indicated inaccuracy is also clearly visible from the proposed formulas (see p. 296), which determine the elements of the corresponding matrices for applying the matrix-geometric method [53].

In [54], separate models of double-sources QIS with instantly damaging inventory that use a single RP were proposed. In the referenced paper, one model uses the (s, S)-policy, and the other model uses the (s, Q)-policy during the entire period of the system's operation. Note that (s, S)-policy (sometimes it is called "Up to S" policy) is an inventory policy in which the order volume is as much as it is able to bring the level back to S at the replenishment epoch; in (s, Q)-policy the order volume is fixed and is equal to Q = S-s. It is assumed that when using each RP, the choice of the source is made depending on the current inventory level. This paper is a continuation of the investigation that began in [54]. Here, new hybrid RP with cancellation is proposed in double-sources QIS, instantly damaging the inventory. This assumes that no stock order is placed until the inventory level is above the reorder point $s, s < (S/2)$. As soon as the inventory level drops to the value s, then the (s, Q)-policy is used. Since the order is fulfilled with a random delay, before the order arrives, the inventory level may decrease as a result of their release to c-customers, as well as the result of their destruction. To increase the chances of c-customers being serviced, it is allowed to order inventory from a fast source if the inventory level of the system falls to a dangerous (critical) level $r, 0 \leq r < s$. Such an order is called an emergency order, and the "Up to S" policy is used here. Regular and emergency orders may arrive at the same time (or within a short period), and as a result, the total volume of these orders may be greater than the system's maximum warehouse size. To prevent such cases, at the time of placing an emergency order, the regular order is canceled immediately.

QIS models with a procedure for canceling a previously booked inventory by c-customers have been proposed in the papers [34,55,56]. Note that in the indicated papers canceling is related to booked inventory, but not to orders for replenishment of inventory.

Analysis of available literature shows that in the known models of QIS with double sources, it is assumed that the system uses a single replenishment policy. Here we propose a new hybrid RP in the QIS with double sources where RP is state-dependent, i.e., it is changed depending on the current inventory level. Moreover, we take into account the instant damage to the items due to technical reasons.

Note that the proposed model of QIS with hybrid RP is realistic and it is applicable in practice. The practical application of this model allows managers to more reliably organize the work of a particular QIS with double sources by choosing an effective RP. To our best knowledge, this model is not studied in the literature.

In this paper, we propose a new hybrid RP in QIS with double sources and develop a method for calculating and optimizing its performance measures by selecting optimal values of parameters of the proposed RP.

The paper is organized as follows. In Section 2, we describe the QIS model with a new hybrid replenishment policy and formulate the problem. In Section 3, the calculation of steady-state probabilities is carried out using the matrix-geometric method, and the stability condition under the proposed replenishment policy is derived. Formulas for calculations of the main performance measures are presented in Section 4. A numerical illustration, including the minimization of the total cost, is provided in Section 5. Section 6 concludes the study and indicates future directions of investigation.

2. Model and Proposed Hybrid RP

Consider single server QIS with double sources. Consumer customers form Poisson flow with rate λ, and for simplicity let us assume that each c-customer requires an item of the same size. Besides the c-customers, Poisson flow of d-customers with rate κ arrives in the system. Upon the arrival of d-customers, the inventory level instantly decreases by one item. It is assumed that the d-customers may even destroy the stock that is in the release status to the c-customer. If the stock level is equal to zero, then the received d-customer does not affect the operation of the system.

If upon the arrival of c-customer the server is idle and the inventory level is positive, then it is immediately accepted for servicing; if at this moment the inventory level is positive and the server is busy, then the c-customer enters an infinite buffer for waiting.

If upon arrival of the c-customer the inventory level is equal to zero, then it either enters the queue with a probability (w.p.) ϕ_1 or leaves the system w.p. $\phi_2 = 1 - \phi_1$.

A customer at the head of the queue becomes impatient if the inventory level drops to zero before it can be serviced. In other words, in such cases, the c-customer at the head of the queue waits for some random time, which has an exponential CDF with average τ^{-1}, and after this time it leaves the system as an unsatisfied demand.

After the completion of the service, the c-customer either does not buy the goods w.p. σ_1 or w.p. $\sigma_2 = 1 - \sigma_1$ it buys the goods. In both cases, the service time of c-customers has exponential CDF, but its mean value is different, i.e., if the c-customer refuses to buy the goods, then the average time of its service is μ_1^{-1}, otherwise this time is μ_2^{-1}.

It is assumed that the system has the ability to change both the lead-time and order size according to the current inventory level in order to increase the chances of c-customers to acquire inventory. This means that replenishments are made from two sources: slow Source-1 and fast Source-2. The lead time from each source has an exponential CDF, but their average values are different, i.e., if an order is sent to Source-i, then the average delay for the supply of stock is $\nu_i^{-1}, i = 1, 2$, where $\nu_2 > \nu_1$.

The following hybrid RP is proposed. If the stock level drops to the value $s, 0 < s < (S/2)$, then an order of size $Q = S - s$ is made to Source-1. When the inventory level drops to the threshold (critical) value $r, 0 \leq r < s$, then the order from Source-1 is canceled instantly and the "Up to S" policy is used to send the order to Source-2. A fast source is assumed to be expensive compared to a slow source and cancellations from a slow source associated with certain penalties.

The problem is to find the joint distribution of the number of c-customers in the system and the inventory level of the system. In addition, it is required to calculate the main performance measures of the system and solve the problem of choosing the optimal values of the reorder point and the threshold value in order to minimize the total cost (TC) when using the proposed replenishment policy. The TC include variable and fixed costs for each type of order and cancellation of the order, as well as the cost of holding inventory, penalties due to the loss of c-customers as a result of their impatience, and penalties for their sojourn time in the system.

3. Model Analysis

This section may be divided by subheadings. It should provide a concise and precise description of the experimental results, their interpretation, and the experimental conclusions that can be drawn.

The operation of the system is described by a two-dimensional Markov chain (2D MC) with states of the form (n, m), where n is number of c-customers in the system, $n \geq 0$, and m is inventory level, $m = 0, 1, I, S$. State space of this 2D MC is given as

$$E = \bigcup_{n=0}^{\infty} L(n),$$

where $L(n) = \{(n, 0), I \ldots, (n, S)\}$ is n-th level, $n = 0, 1, 2, \ldots$.

3.1. The Stability Condition

Transition rate from state $(n_1, m_1) \in E$ to state $(n_2, m_2) \in E$ is denoted by $q((n_1, m_1), (n_2, m_2))$. Transitions between states in space E occur due to the following events: (i) the arrival of c-customers, (ii) the completion of servicing of c-customers, (iii) the departure of c-customers from the queue due to its impatience, (iv) the arrival of d-customers, and (v) restocking.

As a result of the analysis of events (i)–(v), we find that the positive elements of the generator of the 2D MC study are defined as follows:

$$q((n_1, m_1), (n_2, m_2)) = \begin{cases} \lambda\phi_1, & \text{if } n_2 = n_1 + 1, m_2 = m_1 = 0, \\ \lambda, & \text{if } n_2 = n_1 + 1, m_2 = m_1 > 0, \\ \mu_1\sigma_1, & \text{if } n_2 = n_1 - 1, m_2 = m_1 > 0, \\ \mu_2\sigma_2, & \text{if } n_2 = n_1 - 1, m_2 = m_1 - 1, \\ \kappa, & \text{if } n_2 = n_1, m_1 > 0, m_2 = m_1 - 1, \\ \tau, & \text{if } n_1 > 0, n_2 = n_1 - 1, m_2 = m_1 = 0, \\ \nu_1, & \text{if } n_2 = n_1, r < m_1 \leq s, m_2 = m_1 + S - s, \\ \nu_2, & \text{if } n_2 = n_1, 0 \leq m_1 \leq r, m_2 = S. \end{cases} \quad (1)$$

From relations (1), we conclude that constructed 2D MC represents Level Independent Quasi-Birth-Death (LIQBD) process. After re-numbering of states in a lexicographic way (i.e., the states are numbered according to the order $(0, 0), (0, 1), \ldots, (0, S), (1, 0), (1, 1), \ldots,$ $(1, S), \ldots$), we find that the generator of this LIQBD has the following form:

$$G = \begin{pmatrix} B & A_0 & O & \ldots & O & \ldots \\ A_2 & A_1 & A_0 & O & O & \ldots \\ O & A_2 & A_1 & A_0 & O & \ldots \\ O & O & A_2 & A_1 & A_0 & \ldots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \end{pmatrix}$$

where O denotes zero square matrix with dimension $S + 1$, and block matrices $B = \|b_{ij}\|$, $A_k = \|a_{ij}^{(k)}\|, k = \overline{0, 2}, i, j = \overline{0, S}$, are square matrix with the same dimension, where non-zero entities are determined as follows:

$$b_{ij} = \begin{cases} \nu_2, & \text{if } 0 \leq i \leq r, j = S, \\ \nu_1, & \text{if } r < i \leq s, j = i + S - s, \\ \kappa, & \text{if } 0 < i \leq S, j = i - 1, \\ -(\nu_2 + \lambda\phi_1), & \text{if } i = j = 0, \\ -(\nu_2 + \kappa + \lambda), & \text{if } 0 < i \leq r, i = j, \\ -(\nu_1 + \kappa + \lambda), & \text{if } r < i \leq s, i = j, \\ -(\kappa + \lambda), & \text{if } s < i \leq S, i = j; \end{cases} \quad (2)$$

$$a_{ij}^{(0)} = \begin{cases} \lambda\phi_1, & \text{if } i = j = 0, \\ \lambda, & \text{if } i > 0, i = j; \end{cases} \quad (3)$$

$$a_{ij}^{(1)} = \begin{cases} \nu_2, & \text{if } 0 \le i \le r, j = S, \\ \nu_1, & \text{if } r < i \le s, j = i + S - s, \\ \kappa, & \text{if } 0 < i \le S, j = i - 1, \\ -(\tau + \nu_2 + \lambda\phi_1), & \text{if } i = j = 0, \\ -(\nu_2 + \kappa + \lambda + \mu_1\sigma_1 + \mu_2\sigma_2), & \text{if } 0 < i \le r, i = j, \\ -(\nu_1 + \kappa + \lambda + \mu_1\sigma_1 + \mu_2\sigma_2), & \text{if } r < i \le s, i = j, \\ -(\kappa + \lambda + \mu_1\sigma_1 + \mu_2\sigma_2), & \text{if } s < i \le S, i = j; \end{cases} \quad (4)$$

$$a_{ij}^{(2)} = \begin{cases} \tau, & \text{if } i = j = 0, \\ \mu_1\sigma_1, & \text{if } i > 0, i = j, \\ \mu_2\sigma_2, & \text{if } i > 0, j = i - 1. \end{cases} \quad (5)$$

Theorem 1. The system is stable if, and only if, the following the condition is satisfied:

$$\lambda(1 - \phi_2\pi(0)) < \tau\pi(0) + (\mu_1\sigma_1 + \mu_2\sigma_2)(1 - \pi(0)), \quad (6)$$

where

$$\pi(0) = \left(\sum_{m=0}^{r+1} \alpha_m + \sum_{m=r+2}^{s+1} \beta_m + (S - 2s + r - 1)\beta_{s+1} + \sum_{m=S-s+r+1}^{S} \gamma_m \right)^{-1};$$

$$\alpha_m = \begin{cases} 1, & \text{if } m = 0, \\ \theta_2(1 + \theta_2)^{m-1}, & \text{if } 1 \le m \le r + 1; \end{cases}$$

$$\beta_m = \theta_2 \left(\frac{1 + \theta_2}{1 + \theta_1} \right)^r (1 + \theta_1)^{m-1}, r + 1 < m \le s + 1;$$

$$\gamma_m = \theta_2 \sum_{k=0}^{r} \alpha_k + \theta_1 \sum_{k=m-(S-s)}^{s} \beta_k, S - s + r + 1 < m \le S;\ \theta_i\nu_i/(\mu_2\sigma_2 + \kappa), i = 1, 2.$$

Proof of Theorem 1. Steady-state distribution corresponding to generator

$$A = A_0 + A_1 + A_2$$

is denoted by $\pi = (\pi(0), \pi(1), \ldots, \pi(S))$. Quantities $\pi(m)$ are the probabilities that the inventory level is equal to m, and they are satisfied with the following system of equilibrium equations (SEE):

$$\pi A = 0, \pi e = 1 \quad (7)$$

where 0 is zero row vector with dimension $S + 1$ and e is a column vector of the same dimension in which all entities are one.

From relations (3)–(5), we conclude that non-zero entities of the generator $A = \|a_{ij}\|, i, j = \overline{0, S}$, are calculated as follows:

$$a_{ij} = \begin{cases} -\nu_2, & \text{if } i = j = 0, \\ \nu_2, & \text{if } 0 \le i \le r, j = S, \\ \nu_1, & \text{if } r < i \le s, j = i + S - s, \\ \mu_2\sigma_2 + \kappa, & \text{if } i > 0, j = i - 1, \\ -(\mu_2\sigma_2 + \kappa + \nu_2), & \text{if } 0 < i \le r, j = i, \\ -(\mu_2\sigma_2 + \kappa + \nu_1), & \text{if } r < i \le s, j = i, \\ -(\mu_2\sigma_2 + \kappa), & \text{if } i > s, j = i. \end{cases} \quad (8)$$

When constructing the SEE (7), taking into account the specific structure of the matrix A, the following four cases should be distinguished.

Case $0 \leq m \leq r$:

$$(\nu_2 + (\mu_2\sigma_2 + \kappa)(1 - \delta_{m,0}))\pi(m) = (\mu_2\sigma_2 + \kappa)\pi(m+1); \tag{9}$$

Case $r + 1 \leq m \leq s$:

$$(\nu_1 + (\mu_2\sigma_2 + \kappa))\pi(m) = (\mu_2\sigma_2 + \kappa)\pi(m+1); \tag{10}$$

Case $s + 1 \leq m \leq S - s + r$:

$$(\mu_2\sigma_2 + \kappa)\pi(m) = (\mu_2\sigma_2 + \kappa)\pi(m+1); \tag{11}$$

Case $S - s + r + 1 \leq m \leq S$:

$$(\mu_2\sigma_2 + \kappa)\pi(m) = (\mu_2\sigma_2 + \kappa)\pi(m+1)(1 - \delta_{m,S}) + \nu_1 \sum_{k=0}^{r} \pi(k) + \nu_2 \sum_{k=m-(S-s)}^{s} \pi(k) \tag{12}$$

Hereinafter, $\delta_{x,y}$ are Kronecker symbols. Recurrent methods might be used for solving the SEE (9)–(11). Indeed, from Equation (9) we have $\pi(m) = \alpha_m\pi(0), 0 \leq m \leq r + 1$; by taking into account Equation (10) we obtain $\pi(m) = \beta_m\pi(0), r + 2 \leq m \leq s + 1$. From Equation (11) we conclude that $\pi(m) = \pi(m+1), s + 1 \leq m \leq S - s + r - 1$. Equation for state $m = S$ (see a system of Equation (12)) is taken into account in the previous equation for state $m = S - 1$, and so on, i.e., this procedure is executed until equation for state $m = S - s + r + 1$. Then we obtain $\pi(m) = \gamma_m\pi(0), S - s + r + 1 \leq m \leq S$. In other words, we obtain solution of SEE (9)–(12) in the explicit form:

$$\pi(m) = \begin{cases} \alpha_m\pi(0), & \text{if } 1 \leq m \leq r + 1, \\ \beta_m\pi(0), & \text{if } r + 1 < m \leq s + 1, \\ \beta_{s+1}\pi(0), & \text{if } s + 1 < m \leq S - s + r, \\ \gamma_m\pi(0), & \text{if } S - s + r < m \leq S. \end{cases} \tag{13}$$

The unknown probability $\pi(0)$ in Formula (13) is determined from normalizing condition, i.e., $\pi(0) + \pi(1) + \ldots + \pi(S) = 1$. It means that $\pi(0)$ is calculated by the formula that is indicated in (6). From [33] (pp. 81–83) we conclude that the given LIQBD is ergodic if, and only if, the following condition is satisfied:

$$\pi A_0 e < \pi A_2 e \tag{14}$$

Therefore, by taking into account relations (3), (5), and (13), after some algebra from (14), we conclude that relation (6) is true. □

Remark 1. *The stability condition (6) has the following probabilistic meaning: when there is no inventory, and because of their impatience, the rate of c-customers admitted to the queue should be less than the weighted total rate of customers leaving the system as a result of service completion. Condition (6) becomes simpler if we assume that $\mu_1 = \mu_2 = \mu$, i.e., in this case condition (6) is written as $\lambda(1 - \phi_2\pi(0)) < \tau\pi(0) + \mu(1 - \pi(0))$. Last condition can be replaced by a simple condition $\lambda < max(\mu, \tau)$.*

Remark 2. *In special cases when $\phi_1 = 0$ and $\tau = 0$ from (6) at $\sigma_2 = 0$, we obtain classical stability condition of single server queue with infinity buffer, i.e., $\lambda < \mu_1$. This fact is expected one since under these assumptions initial QIS model becomes classical queuing model M/M/1/∞. Another interesting fact is the following result: if we set $\sigma_2 = 1$, $\phi_1 = 0$ and $\tau = 0$ then we obtain condition $\lambda < \mu_2$. In other words, under last assumptions stability, the condition does not depend on inventory capacity (S), rate of d-customers (κ), and replenishment rates from different sources (ν_1, ν_2). Similar results for single source QIS models are established in papers* [50,57,58].

3.2. The Steady-State Probabilities

In accordance with the algorithm for LIQBD (see [53], pp. 81–83), we conclude that under ergodicity condition (6) steady state probabilities $p = (p_0, p_1, p_2, \ldots)$, where $p_n = (p(n,0), p(n,1), \ldots, p(n,S))$ are determined as

$$p_n = p_0 R^n, n \geq 1,$$

where R is non-negative and minimal solution of the matrix-quadratic equation:

$$R^2 A_2 + R A_1 + A_0 = 0$$

Probabilities of boundary states, p_0, are calculated from the following system of equations:

$$p_0(B + RA_2) = 0,$$

$$p_0(I - R)^{-1} e = 1,$$

where I is the identity square matrix with dimension $S + 1$.

4. Performance Measures and Total Cost

Performance measures are calculated via steady state probabilities of the system as follows:

Average inventory level (S_{av})

$$S_{av} = \sum_{m=1}^{S} m \sum_{n=0}^{\infty} p(n,m); \tag{15}$$

Average supply from Source-i, $i = 1, 2$, $(V_{av}(i))$

$$V_{av}(1) = (S - s) \sum_{m=r+1}^{s} \sum_{n=0}^{\infty} p(n,m); \; V_{av}(2) = \sum_{m=0}^{r} (S - m) \sum_{n=0}^{\infty} p(n,m); \tag{16}$$

Average number of c-customers in system (L_{av})

$$L_{av} = \sum_{n=1}^{\infty} n \sum_{m=0}^{S} p(n,m); \tag{17}$$

Average damaging rate of stocks (DRS):

$$DRS = \kappa \left(1 - \sum_{n=0}^{\infty} p(n,0) \right); \tag{18}$$

Average reorder rate from Source-1 (RR_1):

$$RR_1 = \kappa p(0, s+1) + (\mu_2 \sigma_2 + \kappa) \sum_{n=1}^{\infty} p(n, s+1); \tag{19}$$

Average reorder rate from Source-2 (RR_2):

$$RR_2 = \kappa p(0, r+1) + (\mu_2 \sigma_2 + \kappa) \sum_{n=1}^{\infty} p(n, r+1). \tag{20}$$

Remark 3. *The average rate of emergency orders is equal to the average rate of cancellations (RC) of regular orders.*

Loss probability of c-customers (PL):

$$PL = \phi_2 \sum_{n=0}^{\infty} p(n,0) \; + \frac{\tau}{\tau + \lambda\phi_1 + \nu_2} \sum_{n=1}^{\infty} p(n, \rightleftarrows 0). \tag{21}$$

The developed Formulas (15)–(21) allow us to calculate the expected total costs (TC) as follows:

$$TC(s,r) = \sum_{i=1}^{2}(K_i + c_r(i)V_{av}(i))RR_i + c_c RR_2 + c_h S_{av} + c_d DRS + c_l \lambda PL + c_w L_{av}, \tag{22}$$

where K_i are the fixed costs of one order from Source-i; $c_r(i)$ are the procurement costs per unit inventory from Source-i, $i = 1,2$; c_c is the cost due to order cancellation from Source-1; c_h is the holding cost of an inventory unit per unit time; c_d is the cost due to damage per unit inventory; c_l is the cost of losing a c-customer; c_w is the waiting cost of a c-customer per unit time.

Remark 4. *For brevity, only on the left side of (22) are the arguments of the functional $TC(s,r)$ explicitly indicated, although it is obvious that all terms of the sum on the right side of (22) are also functions of these arguments.*

5. Numerical Results

The behavior of the system performance measures (15)–(21) concerning changes in the values of the initial system parameters when using the proposed RP can be studied using Tables 1–9. Let us briefly analyze the contents of these tables. Note that in all numerical experiments it is assumed that $\phi_1 = 0.7, \sigma_1 = 0.3$. The values of other parameters are indicated in the title of the tables. Note that the analysis of these tables showed that almost all results are theoretically expected.

Table 1. Dependence of the performance measures on parameter λ; $\mu_1 = 5$, $\mu_2 = 6$, $\kappa = 2$, $\tau = 2$, $\nu_1 = 3$, $\nu_2 = 5$, $r = 3$, $s = 8$, $S = 18$.

λ	S_{av}	V_{av} (1)	V_{av} (2)	RR_1	RR_2	DRS	PL	L_{av}
2	12.4415	1.0791	0.0472	0.3388	0.0151	1.9996	10^{-4}	0.5406
2.2	12.4227	1.0732	0.0972	0.351	0.016	1.9943	10^{-3}	0.6287
2.4	12.3999	1.0729	0.1435	0.3639	0.0172	1.9895	1.8×10^{-3}	0.7327
2.6	12.3769	1.0774	0.1805	0.3757	0.0186	1.9858	2.5×10^{-3}	0.8390
2.8	12.3509	1.0864	0.2148	0.3881	0.0202	1.9825	3.1×10^{-3}	0.9659
3.0	12.3231	1.0996	0.2446	0.4006	0.0221	1.9797	3.6×10^{-3}	1.1116
3.2	12.2940	1.1168	0.2701	0.413	0.0243	1.9775	4.0×10^{-3}	1.2806
3.4	12.2620	1.1389	0.2925	0.4262	0.0269	1.9757	4.4×10^{-3}	1.4899

Table 2. Dependence of the performance measures on parameter μ_1; $\lambda = 2$, $\mu_2 = 6$, $\kappa = 2$, $\tau = 2$, $\nu_1 = 3$, $\nu_2 = 5$, $r = 3$, $s = 8$, $S = 18$.

μ_1	S_{av}	V_{av} (1)	V_{av} (2)	RR_1	RR_2	DRS	PL	L_{av}
4.0	12.4207	1.1001	0.0509	0.3463	0.0163	1.9995	10^{-4}	0.5883
4.2	12.4250	1.0957	0.0501	0.3447	0.0160	1.9995	9.1×10^{-5}	0.5781
4.4	12.4294	1.0912	0.0493	0.3431	0.0157	1.9995	8.9×10^{-5}	0.5677
4.6	12.4334	1.0873	0.0486	0.3417	0.0155	1.9996	8.7×10^{-5}	0.5587
4.8	12.4374	1.0831	0.0478	0.3402	0.0153	1.9996	8.5×10^{-5}	0.5495
5.0	12.4414	1.0790	0.0471	0.3388	0.0151	1.9996	8.3×10^{-5}	0.5406
5.2	12.4454	1.0751	0.0464	0.3374	0.0149	1.9996	8.1×10^{-5}	0.5319
5.4	12.4492	1.0712	0.0458	0.3360	0.0146	1.9996	7.9×10^{-5}	0.5236

Table 3. Dependence of the performance measures on parameter μ_2; $\lambda = 2$, $\mu_1 = 5$, $\kappa = 2$, $\tau = 2$, $\nu_1 = 3$, $\nu_2 = 5$, $r = 3$, $s = 8$, $S = 18$.

μ_2	S_{av}	V_{av} (1)	V_{av} (2)	RR_1	RR_2	DRS	PL	L_{av}
5.0	12.4603	1.0599	0.0439	0.3320	0.0140	1.99962	7.6×10^{-5}	0.6667
5.2	12.4561	1.0641	0.0446	0.3335	0.0142	1.99961	7.7×10^{-5}	0.6370
5.4	12.4522	1.0682	0.0453	0.3349	0.0145	1.99961	7.9×10^{-5}	0.6098
5.6	12.4484	1.0720	0.0459	0.3363	0.0147	1.99960	8.0×10^{-5}	0.5848
5.8	12.4448	1.0756	0.0465	0.3375	0.0149	1.99959	8.2×10^{-5}	0.5618
6.0	12.4414	1.079	0.0471	0.3387	0.0150	1.99958	8.3×10^{-5}	0.5406
6.2	12.4382	1.0823	0.0477	0.3399	0.0152	1.99957	8.4×10^{-5}	0.5209
6.4	12.4351	1.0854	0.0482	0.3410	0.0154	1.99957	8.5×10^{-5}	0.5025

Table 4. Dependence of the performance measures on parameter κ; $\lambda = 2$, $\mu_1 = 5$, $\mu_2 = 6$, $\tau = 2$, $\nu_1 = 3$, $\nu_2 = 5$, $r = 3$, $s = 8$, $S = 18$.

κ	S_{av}	V_{av} (1)	V_{av} (2)	RR_1	RR_2	DRS	PL	L_{av}
2.0	12.4414	1.0790	0.0471	0.3387	0.0151	1.9995	8.3×10^{-5}	0.54062
2.2	12.4105	1.0836	0.1024	0.3557	0.0167	2.1933	1.054×10^{-3}	0.54068
2.4	12.3788	1.0881	0.1573	0.3724	0.0183	2.3859	2.018×10^{-3}	0.54074
2.6	12.3463	1.0923	0.2120	0.3890	0.0199	2.5775	2.976×10^{-3}	0.54081
2.8	12.3132	1.0963	0.2663	0.4055	0.0217	2.7680	3.927×10^{-3}	0.54088
3.0	12.2796	1.1002	0.3202	0.4218	0.0235	2.9574	4.87×10^{-3}	0.54095
3.2	12.2456	1.1040	0.3739	0.4380	0.0253	3.1487	5.811×10^{-3}	0.54102
3.4	12.2113	1.1076	0.4272	0.4541	0.0271	3.3332	6.743×10^{-3}	0.54110

Table 5. Dependence of the performance measures on parameter τ; $\lambda = 2$, $\mu_1 = 5$, $\mu_2 = 6$, $\kappa = 2$, $\nu_1 = 3$, $\nu_2 = 5$, $r = 3$, $s = 8$, $S = 18$.

τ	S_{av}	V_{av} (1)	V_{av} (2)	RR_1	RR_2	DRS	PL	L_{av}
2.0	12.4414	1.079	0.0471	0.3387	0.01507	1.9995	8.3×10^{-5}	0.54062
2.2	12.4414	1.079	0.0471	0.3387	0.01507	1.9995	8.4×10^{-5}	0.54061
2.4	12.4415	1.079	0.0471	0.3387	0.01507	1.9995	8.5×10^{-5}	0.54060
2.6	12.4415	1.079	0.0471	0.3387	0.01507	1.9995	8.6×10^{-5}	0.54059
2.8	12.4415	1.079	0.0471	0.3387	0.01507	1.9995	8.7×10^{-5}	0.54058
3.0	12.4415	1.079	0.0471	0.3387	0.01507	1.9995	8.8×10^{-5}	0.54057
3.2	12.4415	1.079	0.0471	0.3387	0.01507	1.9995	8.8×10^{-5}	0.54056
3.4	12.4415	1.079	0.0471	0.3387	0.01507	1.9995	8.9×10^{-5}	0.54056

Table 6. Dependence of the performance measures on parameter ν_1; $\lambda = 2$, $\mu_1 = 5$, $\mu_2 = 6$, $\kappa = 2$, $\nu_2 = 5$, $\tau = 2$, $r = 3$, $s = 8$, $S = 18$.

ν_1	S_{av}	V_{av} (1)	V_{av} (2)	RR_1	RR_2	DRS	PL	L_{av}
3	12.4414	1.0790	0.0471	0.3387	0.01501	1.9995	8.3×10^{-5}	0.54062
3.2	12.4958	1.0218	0.0406	0.3399	0.0129	1.9996	7.1×10^{-5}	0.54061
3.4	11.0000	0.9698	0.0351	0.3409	0.0112	1.9996	6.2×10^{-5}	0.54060
3.6	12.5909	0.9223	0.0305	0.3418	0.0098	1.9997	5.4×10^{-5}	0.54059
3.8	12.6326	0.8789	0.0266	0.3425	0.0085	1.9997	4.7×10^{-5}	0.54059
4	12.6709	0.8392	0.0232	0.3431	0.0074	1.9997	4.1×10^{-5}	0.54058
4.2	12.7079	0.8009	0.0203	0.3436	0.0065	1.9998	3.6×10^{-5}	0.54057
4.4	12.7388	0.7689	0.0179	0.3441	0.0058	1.9998	3.2×10^{-5}	0.54057

Table 7. Dependence of the performance measures on parameter ν_2; $\lambda = 2$, $\mu_1 = 5$, $\mu_2 = 6$, $\kappa = 2$, $\nu_1 = 3$, $\tau = 2$, $r = 3$, $s = 8$, $S = 18$.

ν_2	S_{av}	V_{av} (1)	V_{av} (2)	RR_1	RR_2	DRS	PL	L_{av}
5.0	12.4414	1.0790	0.0471	0.3387	0.01507	1.9995	8.3×10^{-5}	0.540621
5.2	12.4427	1.0792	0.0452	0.3388	0.01507	1.9996	7.4×10^{-5}	0.540612
5.4	12.4438	1.0793	0.0435	0.3388	0.01507	1.9996	6.6×10^{-5}	0.540604
5.6	12.4449	1.0794	0.0419	0.3389	0.01507	1.9996	5.9×10^{-5}	0.540598
5.8	12.4459	1.0795	0.0404	0.3389	0.01507	1.9997	5.3×10^{-5}	0.540592
6.0	12.4468	1.0796	0.0390	0.3389	0.01507	1.9997	4.8×10^{-5}	0.540587
6.2	12.4476	1.0796	0.0377	0.3389	0.01507	1.9997	4.3×10^{-5}	0.540582
6.4	12.4484	1.0797	0.0365	0.3390	0.01508	1.9997	3.9×10^{-5}	0.540578

Table 8. Dependence of the performance measures on parameter s; $\lambda = 2$, $\mu_1 = 5$, $\mu_2 = 6$, $\kappa = 2$, $\nu_1 = 3$, $\nu_2 = 5$, $\tau = 2$, $r = 3$, $S = 21$.

s	S_{av}	V_{av} (1)	V_{av} (2)	RR_1	RR_2	DRS	PL	L_{av}
5	12.8203	0.7865	0.2224	0.2071	0.05962	1.9984	3.27×10^{-4}	0.540858
6	13.1548	0.9432	0.1285	0.2231	0.03446	1.999	1.89×10^{-4}	0.540724
7	13.5332	1.0332	0.0746	0.2414	0.02001	1.9994	1.1×10^{-4}	0.540647
8	13.9522	1.0852	0.0434	0.2621	0.01160	1.9996	6.4×10^{-5}	0.540603
9	14.4019	1.1154	0.0254	0.2856	0.00681	1.9998	3.7×10^{-5}	0.540577

Table 9. Dependence of the performance measures on parameter r; $\lambda = 2$, $\mu_1 = 5$, $\mu_2 = 6$, $\kappa = 2$, $\nu_1 = 3$, $\nu_2 = 5$, $\tau = 2$, $s = 8$, $S = 18$.

r	S_{av}	V_{av} (1)	V_{av} (2)	RR_1	RR_2	DRS	PL	L_{av}
2	12.3958	1.1125	0.0271	0.3419	0.0081	1.9994	1.1×10^{-4}	0.540647
3	12.4414	1.0791	0.0472	0.3387	0.0151	1.9995	8.3×10^{-5}	0.540621
4	12.5161	1.0217	0.0840	0.3350	0.0278	1.9995	6.9×10^{-5}	0.54060
5	12.6339	0.9261	0.1390	0.3286	0.0508	1.9997	4.7×10^{-5}	0.540586
6	12.7965	0.7652	0.2355	0.3223	0.0928	1.9998	3.5×10^{-5}	0.540575

From Table 1, we conclude that with an increase in the intensity of c-customers, the average inventory level decreases at a very low rate, i.e., with an increase in the intensity of c-customers by 70%, the inventory level decreases by 1.5%. Average supply volume from Source-2 increases at a fairly high rate with an increase in the intensity of c-customers. However, an average supply volume from Source-1 decreases first and then increases at a slow rate with an increase in the intensity of c-customers. Note that the last fact is a single counterintuitive result in all numerical experiments. The re-order rate for each source increases at a fairly high rate, with an increase in the intensity of c-customers. The last facts are explained as follows: with the increase in the intensity of c-customers, the inventory level frequently drops to the points of regular and emergency orders. The average damaging rate of stocks decreases at a low rate with increasing intensity of c-customers, since with an increase in the intensity of c-customers, the probability that the inventory level is greater than zero decreases (see Formula (18)). An increase in the intensity of c-customers leads to an increase in the probability of their loss and the average number of orders in the system also increases, while the growth rates of these measures are quite high.

With an increase in the service intensity of c-customers that do not require items, the average inventory level remains almost unchanged (it grows at a negligible rate), i.e., with an increase in the intensity of c-customers by 35%, the inventory level will increase by 0.1% (see Table 2). Table 2 also shows that the average supply volume from each source, as well as the reorder rates for each source, decrease at a very low rate, with an increase in the intensity of servicing c-customers that do not require items. Such behavior of indicated measures is explained by the fact that here the average inventory level is higher than the regular order point. The average damaging rate of stocks is almost constant since the probability that the inventory level is greater than zero is very close to 1. An increase in the intensity of service of c-customers that do not require items leads to a decrease in the loss probability of customers, and the average number of c-customers in the system decreases as well. The last facts coincide with well-known facts from the classical theory of queuing systems.

From Table 3, we conclude that with an increase in the service intensity of c-customers that require items, the average inventory level decreases at an insignificant rate (almost does not change). This is in contrast to Table 2, where the average supply volume from each source, as well as the reorder rate for each source, increase (albeit at a very slow rate). These facts are explained as follows: with an increase in the service intensity of c-customers that require items, the average inventory level decreases. The average damaging rate of stocks, as in Table 2, is almost constant. An increase in the service intensity of c-customers that require items leads to an increase in the loss probability of customers since in this case the probability that the inventory level drops to zero will increase (see Formula (21)). As expected, an increase in the service intensity of c-customers that require items decreases the average number of c-customers in the system.

From Table 4, we conclude that with an increase in the intensity of d-customers, the average inventory level decreases at a negligible rate. Here, the average supply volume from each source, as well as the reorder rates for each source, increase, because the average inventory level decreases. The average damaging rate of stocks, as expected, is also on the rise. An increase in the intensity of d-customers leads to an increase in the loss probability of c-customers since in this case the probability that the inventory level drops to zero increases. As expected, with an increase in the intensity of d-customers, the average number of c-customers in the system, although at an insignificant rate, increases.

Note that an increase in the intensity of c-customer's impatience has almost no effect on the performance measures of the system (see Table 5).

With an increase in the intensity of the supply of stocks from a slow source, the average inventory level grows at a low rate (see Table 6). From this table, it can be seen that the supply volumes from each source are decreasing, while the volumes of supplies from the fast source are decreasing at a high rate. An interesting result is that the intensity of orders to a slow source increase, while the intensity of orders to a fast source decrease. The average damaging rate of stocks is almost independent of the supply intensity from a slow source. Decreasing the lead-time from a slow source reduces the loss probability of c-customers because it reduces the probability that the inventory level is zero. The average number of c-customers in the system is almost constant.

As above mentioned, with an increase in the intensity of the supply of inventory from a fast source, the average inventory level grows at a low rate (see Table 7). Table 7 shows that the supply volume from the fast source decreases, and the supply volume from the slow source, as well as the reorder rates to both sources, is almost constant. In addition to them, the average damaging rate of stocks is a constant value. As above (see Table 6), reducing the lead-time from a fast source leads to a decrease in the loss probability of c-customers, and the average number of c-customers in the system is almost constant.

From Table 8, we conclude that with an increase in the value of the regular order point, the average inventory level increases at a significant rate. An interesting result is that with an increase in the value of the regular order point, the supply volume and reorder rate from a slow source increase. At the same time, the supply volume and reorder rate from a fast source are decreasing. These facts are explained as follows: as the value of the regular reorder point increases, and the probability that the inventory level drops below the emergency reorder point decreases. The inventor deterioration rate increases at a negligible rate, as the probability that the inventory level is greater than zero increases. For the same reason, the loss probability of c-customers is reduced. The average number of c-customers in the system is almost constant.

Table 9 shows that with an increase in the value of the emergency order point, the average inventory level increases at an insignificant rate. Contrary to Table 8, here, with an increase in the value of the emergency order point, the supply volume and reorder rate from a fast source increase. At the same time, the supply volume and reorder rate from a slow source are decreasing. As above (see Table 8), the rate of inventory deterioration increases at an insignificant rate, the loss probability of c-customers decreases, and the average number of c-customers in the system is almost constant.

Now, consider the problem of minimizing the expected *TC* (22) by choosing the appropriate values of the parameters *s* (regular re-order point) and *r* (emergency re-order point). Here, we assume that the remaining parameters of the system (structural and load) are fixed.

The values of the initial parameters of the model and the coefficients in the functional (22) are chosen as follows [54]:

$$\lambda = 20, \kappa = 10, \ \mu_1 = 35, \mu_2 = 25, \tau = 20, \nu_1 = 5, \nu_2 = 10, \sigma_1 = 0.4, \phi_1 = 0.6;$$

$$K_1 = 100, K_2 = 200, \ c_r(1) = 50, c_r(2) = 100, c_c = 50, c_h = 35, c_d = 75, c_l = 200, c_w = 50.$$

Optimization problem is formulated as follows: it is required to find such pairs of optimal values (s^*, r^*) in order to minimize the functional (22). This problem is formally written as

$$(s^*, r^*) = arg \min_{(s,r)\in X} TC(s, r). \quad (23)$$

The problem (23) always has a solution since the domain of feasible solutions $X = \{(s,r) : 0 < s < (S/2), 0 \le r < s\}$ is a discrete and finite set. Results of problem are shown in Table 10, where the minimal value of TC is denoted by bold. For selected values of initial parameters, the optimal solution of the problem (23) is $(s^*, r^*) = (3, 0)$. Note that points $(4, 0)$ and $(5, 0)$ are very close to the optimal solution. In addition, we note the following properties of the TC for selected values of the initial parameters: (1) for fixed values of the argument *s*, it is increasing with respect to the argument *r*; (2) for fixed values of the argument *r*, it is a convex function with respect to the argument *s*; (3) a simultaneous increase in the values of both parameters *s* and *r* in the feasible set leads to an increase in the value of the TC. Based on the complexity of the form of the TC, it is not possible to draw general theoretical conclusions regarding its behavior, as well as the location of the optimal point (s^*, r^*).

At the end of this section, we note that, in contrast to single-source QISs, here managers have more freedom to achieve desired economic goals. At the same time, in real systems, managers cannot control the values of the load parameters of the system. Therefore, in order to achieve the goals indicated above, it is necessary to find the optimal values for the parameters *s* and *r* of the proposed hybrid policy. However, it should be noted that despite the fact that the performance measures change smoothly relative to the values of the initial parameters (both load and structural), finding the optimal values of these parameters is not a trivial task (moreover, in some cases the behavior of the performance measures can be unexpected). In other words, for each concrete QIS it is necessary to solve the specified problem. Here we also note that other formulations of the optimization problem for this QIS are also possible, i.e., it is possible to solve conditional optimization problems. In particular, it is possible to consider the problem of maximizing the revenue of the given QIS subject with existence of some restrictions on performance measures. For instance, it is possible to impose restrictions on the probability of loss of customers or on the re-order rates from various sources. Such tasks require special investigations.

Table 10. Results of problem (23); $\lambda = 20$, $\mu_1 = 35$, $\mu_2 = 25$, $\kappa = 2$, $\nu_1 = 5$, $\nu_2 = 10$, $\tau = 20$, $\phi_1 = 0.6$, $\sigma_1 = 0.4$, $\sigma_2 = 0.6$, $S = 25$.

	TC											
r \ s	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
0	1537	1505	1495	1496	1503	1513	1526	1542	1560	1580	1601	1623
1		1571	1534	1523	1522	1528	1538	1551	1567	1580	1605	1627
2			1604	1564	1550	1548	1553	1563	1576	1592	1609	1629
3				1637	1594	1578	1574	1578	1588	1601	1617	1636
4					1672	1624	1606	1601	1604	1614	1627	1642
5						1706	1656	1635	1629	1632	1641	1654
6							1742	1687	1665	1658	1660	1669
7								1779	1722	1697	1686	1690
8									1817	1798	1757	1730
9										1861	1795	1766
10											1907	1836
11												1956

6. Conclusions and Future Works

A hybrid replenishment policy is proposed in double sources QIS with an infinite queue, where a fast source is more expensive than a slow source. It is based on the combined use of two known RPs with a fixed supply volume and with a variable supply volume. If the inventory level drops to the reorder point, then regular replenishment of stocks is carried out from a slow source according to the RP with a fixed supply volume. When the stock level drops to a critical value, less than the reorder point, then emergency replenishment of stocks is carried out from a fast source according to the RP with a variable supply volume. Another feature of the model under study is that inventory damage occurs instantly upon the arrival of destructive customers.

It is shown that the mathematical model of the investigated system using the proposed hybrid RP is a 2D MC, which has a three-diagonal generator. The ergodicity condition for the constructed 2D MC is found and the matrix-geometric method is used to find its stationary distribution. Formulas for calculating the performance measures of the investigated system are proposed and the problem of its optimization is solved.

The direction of future work should be the investigation of the similar QISs with MAP flows of c-customers and/or d-customers as well as with PH distribution of service times for c-customers. Another direction is the investigation of QISs in which the size of damaged items is a random variable with a known CDF including models with catastrophes (i.e., when all stocks are destroyed). Finally, an important direction is the study of QISs, in which the order volumes depend not only on the level of inventory but also depend on the current number of c-customers in the system. These problems are the objects of our further research.

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