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Regular Partial Residuated Lattices and Their Filters

Nan Sheng and Xiaohong Zhang *

School of Mathematics and Data Science, Shaanxi University of Science & Technology, Xi'an 710021, China; snsust@163.com
* Correspondence: zhangxiaohong@sust.edu.cn; Tel.: +86-29-86168320

Abstract: To express wider uncertainty, Běhounek and Daňková studied fuzzy partial logic and partial function. At the same time, Borzooei generalized t-norms and put forward the concept of partial t-norms when studying lattice valued quantum effect algebras. Based on partial t-norms, Zhang et al. studied partial residuated implications (PRIs) and proposed the concept of partial residuated lattices (PRLs). In this paper, we mainly study the related algebraic structure of fuzzy partial logic. First, we provide the definitions of regular partial t-norms and regular partial residuated implication (rPRI) through the general forms of partial t-norms and partial residuated implication. Second, we define regular partial residuated lattices (rPRLs) and study their corresponding properties. Third, we study the relations among commutative quasi-residuated lattices, commutative Q-residuated lattices, partial residuated lattices, and regular partial residuated lattices. Last, in order to obtain the filter theory of regular partial residuated lattices, we restrict certain conditions and then propose special regular partial residuated lattices (srPRLs) in order to finally construct the quotient structure of regular partial residuated lattices.

Keywords: fuzzy logic; partial t-norm; regular partial residuated lattice; Q-residuated lattice; filter

MSC: 06B10; 06B75; 08A55

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1. Introduction

Fuzzy logic and quantum logic are the research directions of many scholars. In fuzzy logic, it is common to use lattice theory to deal with the framework of fuzzy logic, especially research on the structure of residuated lattices related to t-norms and their residuated implications [1,2]. In quantum logic, lattice effect algebra is a representative kind of quantum structure, and much rich research has been carried out on it [3,4]. Now, more and more scholars study these two logical structures both alone and in terms of the relationship between them, including the relationship between residuated lattices and lattice effect algebras [5–7]. Recently, the research on various implications in lattice effect algebras has become increasingly active [8–10]. However, these studies are based on full operations. In practical application, people encounter many "undefined" situations, which are called partial structures. Effect algebra is a partial algebraic structure in quantum logic. Běhounek and Novák proposed fuzzy partial logic in [11], which well describes situations in which certain elements cannot be operated. In [9], Chajda and Länger provide the concept of quasiresiduated lattice, which is closely related to lattice effect algebras. In [12], Zhang et al. provide the concept of Q-residuated lattices. In [13], Zhang et al. continued to provide clear definitions of partial fuzzy implications, partial residuated implications, and partial algebraic structure (that is, partial residuated lattices), thereby making great contributions to the study of the relationship between the two logics. In terms of application, there are many examples of using fuzzy logic to model transition systems with uncertainty [14,15]. In [16], Běhounek and Daňková consider the case in which several inputs or outputs of the aggregate function are undefined, and apply this theory to a case study. Thus we ask, is it possible to put forward different concepts in order to further study partial algebraic structures? What is the relationship between these structures? This is what we seek to reveal in this paper. At the same time, whether it a residuated lattice or effect algebra, the filters represent an important research direction [17–20]. Therefore, we additionally study the filter theory and quotient structure of the newly proposed partial algebraic structure.

In this paper, based on various partial implications and the partial residuated lattices proposed by Zhang et al. in [13], we continue to study the structure of partial algebra, propose regular partial residuated lattices, analyze their properties, determine the conditions for constructing rPRLs in lattice effect algebra, and further reveal the relationship between it and PRLs, commutative quasiresiduated lattices, and commutative Q-residuated lattices. Finally, the concept of a special regular partial residuated lattice is proposed, the filter is defined, and the quotient structure of the regular partial residuated lattice is constructed.

2. Preliminaries

In this part, we list existing knowledge in order to provide a basis for follow-up research.

Definition 1 ([3])**.** *An effect algebra $(E, +, ', 0, 1)$ is a partial algebra where $+$ is a partial operation and $'$ is a unary operation such that for all $a, b, c \in E$, if:*

(E1) $a + b$ is defined iff $b + a$ is defined, and then $a + b = b + a$;
(E2) $a + b$ and $(a + b) + c$ are defined iff $b + c$ and $a + (b + c)$ are defined, and then $(a + b) + c = a + (b + c)$;
(E3) a unique $a' \in E$ with $a + a' = 1$;
(E4) if $1 + a$ is defined then $a = 0$.

Define a partial order $\leq$ on E by: $a \leq b$ iff there exists an element $c \in E$ such that $a + c = b$. For all $a \in E$, $0 \leq a \leq 1$, and if $(E; \leq)$ is a lattice, we say that E is a lattice effect algebra.

Definition 2 ([9])**.** *A commutative quasiresiduated lattice $C = (C, \vee, \wedge, \odot, \rightarrow, 0, 1)$ is a partial algebra where $(C, \vee, \wedge, 0, 1)$ is a bounded lattice, $\odot$ is a partial operation, and $\rightarrow$ is a full operation such that for all $a, b, c \in C$, if:*

(i) $(C, \odot, 1)$ is a commutative partial monoid, $a \odot b$ is defined iff $a' \leq b$;
(ii) $a'' = a$, if $a \leq b$ then $b' \leq a'$;
(iii) $(a \vee b') \odot b \leq b \wedge c$ iff $a \vee b' \leq b \rightarrow c$.

where $a' = a \rightarrow 0$.

Theorem 1 ([12])**.** *Let Q be a commutative quasiresiduated lattice. For any $a, b, c \in Q$, the following hold:*

(1) If $a' \leq b$, then $a \leq b \rightarrow (a \odot b)$;
(2) If $a \leq b$, then $(b \rightarrow a) \odot b = a$;
(3) If $a' \leq b$ and $c \leq b$, then $a \odot b \leq c$ iff $a \leq b \rightarrow c$.

Definition 3 ([12])**.** *A commutative Q-residuated lattice $Q = (Q, \vee, \wedge, \odot, \rightarrow, 0, 1)$ is a partial algebra where $(Q, \vee, \wedge, 0, 1)$ is a bounded lattice and $\odot$ and $\rightarrow$ are partial operations such that for all $a, b, c \in Q$, if:*

(Q1) $(Q, \odot, 1)$ is a partial monoid, $a \odot b$ is defined iff $a' \leq b$;
(Q2) $a'' = a$, $a \leq b$ implies $b' \leq a'$;
(Q3) if $a' \leq b$, then $a \odot b = b \odot a$;
(Q4) if $a \leq b$, then $b \rightarrow a$ is defined; If $b' \leq a$ and $c \leq b$, then $a \odot b \leq c$ iff $a \leq b \rightarrow c$.

where $a' = a \rightarrow 0$.

Theorem 2 ([12])**.** *Let Q be a commutative Q-residuated lattice. For any $a, b, c \in Q$, the following hold:*

(1) If $a' \leq c$, $b' \leq c$ and $a \leq b$, then $a \odot c \leq b \odot c$;
(2) If $a \leq c$, $b \leq c$ and $a \leq b$, then $c \rightarrow a \leq c \rightarrow b$.

Definition 4 ([8]). *Let L be a bounded lattice. A binary operation $\odot$ is a partial t-norm on L such that for all $a, b, c \in L$, if:*

(1) $1 \odot a = a$;
(2) *if $a \odot b$ is defined, then $b \odot a$ is defined and $a \odot b = b \odot a$;*
(3) *if $b \odot c$ and $a \odot (b \odot c)$ are defined, then $a \odot b$ and $(a \odot b) \odot c$ are defined and $(a \odot b) \odot c = a \odot (b \odot c)$;*
(4) *if $a \leq b$, $u \leq v$ and $a \odot u$, $b \odot v$ are defined, then $a \odot u \leq b \odot v$.*

Definition 5 ([13]). *Let L be a bounded lattice and $\odot$ be a partial t-norm on L. A partial operation $\rightarrow_\odot$ induced by $\odot$ is called a partial residuated implication (PRI) such that for all $a, b \in L$, if:*

$$a \rightarrow_\bigodot b := \begin{cases} \sup\{u \mid a \bigodot u \text{ is defined and } a \bigodot u \leq b\} & \text{if the supremum of the } S \text{ exists} \\ \text{undefined} & \text{otherwise} \end{cases} \tag{1}$$

where $S = \{u \mid a \bigodot u$ is defined and $a \bigodot u \leq b\}$.

Definition 6 ([13]). *A pair $(\otimes, \rightarrow)$ on a poset $(P; \leq)$ is a partial adjoint pair (PAP) where $\otimes$ and $\rightarrow$ are two partial operations such that for all $x, y, z \in L$, if:*

(PAP1) *The operation $\otimes$ is isotone, if $x \leq y$, $x \otimes z$ and $y \otimes z$ are defined, then $x \otimes z \leq y \otimes z$; if $x \leq y$, $z \otimes x$ and $z \otimes y$ are defined, then $z \otimes x \leq z \otimes y$.*
(PAP2) *The operation $\rightarrow$ is antitone in the first argument, if $x \leq y$, $x \rightarrow z$ and $y \rightarrow z$ are defined, then $y \rightarrow z \leq x \rightarrow z$; $\rightarrow$ is isotone in the second argument, if $x \leq y$, $z \rightarrow x$ and $z \rightarrow y$ are defined, then $z \rightarrow x \leq z \rightarrow y$.*
(PAP3) *If $x \otimes y$ and $x \rightarrow z$ are defined, then $x \otimes y \leq z$ iff $y \leq x \rightarrow z$.*

Definition 7 ([13]). *A partial algebra $(L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ is a partial residuated lattice (PRL) where $(L, \vee, \wedge, 0, 1)$ is a bounded lattice, $\otimes$ and $\rightarrow$ are two partial operations, such that for all $x, y, z \in L$:*

(PRL1) *if $x \otimes y$ is defined, then $y \otimes x$ is defined, and then $x \otimes y = y \otimes x$;*
(PRL2) *if $y \otimes z$, $x \otimes (y \otimes z)$ are defined, then $x \otimes y$, $(x \otimes y) \otimes z$ are defined, and then $x \otimes (y \otimes z) = (x \otimes y) \otimes z$;*
(PRL3) *$x \otimes 1$ is defined and $x \otimes 1 = x$;*
(PRL4) *$(\otimes, \rightarrow)$ is a PAP on L.*

Definition 8 ([13]). *Let L be a bounded lattice. The function $PI : L \times L \rightarrow L$ is called a partial fuzzy implication (PFI)*

(PI1) *if $a_1 \leq a_2$, $PI(a_1, b)$ and $PI(a_2, b)$ are defined, then $PI(a_2, b) \leq PI(a_1, b)$;*
(PI2) *if $b_1 \leq b_2$, $PI(a, b_1)$ and $PI(a, b_2)$ are defined, then $PI(a, b_1) \leq PI(a, b_2)$;*
(PI3) *$PI(0, 0) = PI(1, 1) = 1$, $PI(1, 0) = 0$.*

By [13], we know that the PRI defined in Definition 5 is a partial fuzzy implication.

3. Regular Partial Residuated Implications (rPRIs) and Regular Partial Residuated Lattices (rPRLs)

In [13], Zhang et al. defined partial residuated implication. Based on this, we limit certain conditions, propose regular partial residuated implication, and study its relationship with partial residuated implication and partial fuzzy implication. Next, we define the regular partial residuated lattice, study its properties and the relationship with lattice effect algebra, and then propose special regular partial residuated lattice and normal regular partial residuated lattice, which paves the way for the study of the filter theory of regular partial residuated lattices.

Definition 9. *Let L be a bounded lattice, $\odot$ be a partial t-norm on L, and $\rightarrow_\odot$ be a PRI derived by $\odot$. If the following conditions hold, then we can say that $\odot$ is a regular partial t-norm on L and*

$\rightarrow_\odot$ *is a regular partial residuated implication (rPRI).*

$$\forall x, y \in L, x \le y \Rightarrow (y \rightarrow_\odot z \le x \rightarrow_\odot z \text{ and } z \rightarrow_\odot x \le z \rightarrow_\odot y)$$

Example 1. *Let* $L = \{0, a, b, 1\}$*. The Hasse-diagram of* $(L; \le)$ *is shown in Figure 1, and the operations* $\odot$ *and* $\rightarrow_\odot$ *are defined by Tables 1 and 2. Then,* $\odot$ *is a regular partial t-norm and* $\rightarrow_\odot$ *is an rPRI.*

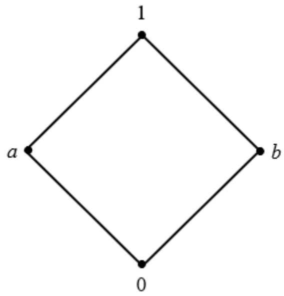

Figure 1. Hasse diagram of lattice L.

Table 1. The operation $\odot$.

$\odot$	**0**	***a***	***b***	**1**
0		0		0
a	0	a	0	a
b		0		b
1	0	a	b	1

Table 2. The operation $\rightarrow_\odot$.

$\rightarrow_\odot$	**0**	***a***	***b***	**1**
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

Theorem 3. *Let* L *be a bounded lattice,* $\odot$ *be a regular partial t-norm on* L*, and* $\rightarrow_\odot$ *be an rPRI induced by* $\odot$*. Then,* $\rightarrow_\odot$ *is a PFI.*

Proof. The proof follows from Theorem 4.11 in [13]. □

Through [13], we know that the algebraic structure corresponding to partial t-norms and their partial residuated implication is a partial residuated lattice. Next, we provide the corresponding algebraic structures of regular partial t-norms and regular partial residuated implication: regular partial residuated lattices.

Definition 10. *A pair* $(\otimes, \rightarrow)$ *on a poset* $(P; \le)$ *is a regular partial adjoint pair (rPAP) where* $\otimes$ *is a partial operation and* $\rightarrow$ *is a full operation such that for all* $x, y, z \in L$*, if*

(rPAP1) The operation $\otimes$ *is isotone, if* $x \le y$*,* $x \otimes z$ *and* $y \otimes z$ *are defined, then* $x \otimes z \le y \otimes z$*; if* $x \le y$*,* $z \otimes x$ *and* $z \otimes y$ *are defined, then* $z \otimes x \le z \otimes y$*.*

(rPAP2) The operation $\rightarrow$ *is antitone in the first argument and isotone in the second argument, if* $x \le y$*, then* $y \rightarrow z \le x \rightarrow z$*,* $z \rightarrow x \le z \rightarrow y$*.*

(rPAP3) If $x \otimes y$ *is defined, then* $x \otimes y \le z$ *iff* $y \le x \rightarrow z$*.*

Definition 11. *A partial algebra* $(L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ *is a regular partial residuated lattice (rPRL), where* $(L, \vee, \wedge, 0, 1)$ *is a bounded lattice,* $\otimes$ *is a partial operation, and* $\rightarrow$ *is a full operation, such that for all* $x, y, z \in L$*,*

(rPRL1) if $x \otimes y$ is defined, then $y \otimes x$ is defined, and then $x \otimes y = y \otimes x$;
(rPRL2) if $y \otimes z$, $x \otimes (y \otimes z)$ are defined, then $x \otimes y$, $(x \otimes y) \otimes z$ are defined, and then $x \otimes (y \otimes z) = (x \otimes y) \otimes z$;
(rPRL3) $x \otimes 1$ is defined and $x \otimes 1 = x$;
(rPRL4) $(\otimes, \to)$ is an rPAP on L.

Example 2. *Let $L = \{0, a, b, c, d, 1\}$. The Hasse-diagram of L is shown in Figure 2, and the operations $\otimes$ and $\to$ are defined by Tables 3 and 4. Then L is an rPRL.*

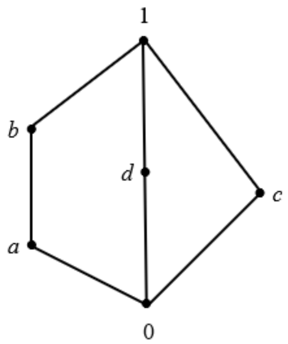

Figure 2. Hasse diagram of lattice L.

Table 3. The operation $\otimes$.

$\otimes$	**0**	***a***	***b***	***c***	***d***	**1**
0				0		0
a						a
b						b
c	0			c		c
d					d	d
1	0	a	b	c	d	1

Table 4. The operation $\to$.

$\to$	**0**	***a***	***b***	***c***	***d***	**1**
0	1	1	1	1	1	1
a	0	1	1	c	d	1
b	0	a	1	c	d	1
c	0	a	b	1	d	1
d	0	b	b	c	1	1
1	0	a	b	c	d	1

Theorem 4. *Let $(L, \vee, \wedge, \otimes, \to, 0, 1)$ be an rPRL, then L is a PRL.*

Proof. We can prove it easily by Definitions 7 and 11. □

Theorem 5. *Let $(L, \vee, \wedge, \otimes, \to, 0, 1)$ be an rPRL. Then, for all $a, b, c \in L$,*

(1) $a \to a = 1$;
(2) $a \to 1 = 1$;
(3) $1 \to a = a$;
(4) $a \to b = 1$ iff $a \leq b$;
(5) If $a \otimes b$ is defined, then $a \leq b \to (a \otimes b)$;
(6) If $a \otimes b$ is defined, then $a \otimes b \leq a$, $a \otimes b \leq b$ and $a \otimes b \leq a \wedge b$;
(7) If $a \otimes b$ is defined, then $a \leq b \to a$.

Proof. (1)–(4) can be obtained from Theorem 4.16 in [13].

(5) Suppose that $a \otimes b$ is defined; we know $a \otimes b \leq a \otimes b$, hence, $a \leq b \to (a \otimes b)$.
(6) Suppose that $a \otimes b$ is defined; then, it follows from (rPAP1) that $a \otimes b \leq a \otimes 1 = a$, $a \otimes b \leq 1 \otimes b = b$. Hence, $a \otimes b \leq (a \otimes 1) \wedge (1 \otimes b) = a \wedge b$.
(7) Suppose that $a \otimes b$ is defined; from (6), $a \otimes b \leq a$, thus, $a \leq b \to a$.

□

Theorem 6. *Let $\odot$ be a regular partial t-norm on L and let $\to_\odot$ be an rPRI induced by $\odot$. Then, $(L, \vee, \wedge, \odot, \to_\odot, 0, 1)$ is an rPRL.*

Proof. By Definitions 4 and 9–11, we can easily come to this conclusion. □

Through this theorem, we can find that in Example 1, the lattice structure $(L; \leq, \odot, \to_\odot, 0, 1)$ composed of $\odot$ and $\to_\odot$ is an rPRL.

Definition 12. *An rPRL $(L, \vee, \wedge, \otimes, \to, 0, 1)$ is a special regular partial residuated lattice (srPRL) if:*

(1) for all $a, b \in L$, $a \otimes (a \to b)$ is defined;
(2) for all $a, b, c \in L$, if $a \otimes b$ and $a \otimes c$ are defined, then $a \otimes (b \wedge c)$ is defined.

Theorem 7. *Let $(L, \vee, \wedge, \otimes, \to, 0, 1)$ be an srPRL. Then, for all $a, b, c \in L$:*

(1) $a \otimes (a \to b) \leq b$;
(2) If $(b \to c) \otimes (a \to b) \otimes a$ is defined, then $(b \to c) \otimes (a \to b) \leq a \to c$;
(3) If $a \otimes b$ and $a \otimes c$ are defined, then $(a \otimes b) \vee (a \otimes c) \leq a \otimes (b \vee c)$;
(4) If $a \leq b \to c$, then $b \leq a \to c$;
(5) $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$;
(6) $(a \vee b) \to c = (a \to c) \wedge (b \to c)$.

Proof.

(1) Because $a \to b \leq a \to b$, $a \otimes (a \to b) \leq b$.
(2) By (1), we know $a \otimes (a \to b) \leq b$; $b \otimes (b \to c) \leq c$. From $b \otimes (b \to c) \leq c$, we have $b \leq (b \to c) \to c$. Thus, $a \otimes (a \to b) \leq b \leq (b \to c) \to c$. Further, $(b \to c) \otimes (a \to b) \otimes a \leq c$, then we get $(b \to c) \otimes (a \to b) \leq a \to c$.
(3) If $a \otimes b$ and $a \otimes c$ are defined, then $a \otimes (b \vee c)$ is defined. From $b \leq b \vee c$, $c \leq b \vee c$, we have $a \otimes b \leq a \otimes (b \vee c)$, $a \otimes c \leq a \otimes (b \vee c)$. Hence, $(a \otimes b) \vee (a \otimes c) \leq a \otimes (b \vee c)$.
(4) Because $b \otimes (b \to c) \leq b \otimes (b \to c)$, then $b \leq (b \to c) \to (b \otimes (b \to c))$. From $a \leq b \to c$, we have $(b \to c) \to (b \otimes (b \to c)) \leq a \to (b \otimes (b \to c))$, and we know $b \otimes (b \to c) \leq c$, so, $a \to (b \otimes (b \to c)) \leq a \to c$. Hence, $b \leq (b \to c) \to (b \otimes (b \to c)) \leq a \to (b \otimes (b \to c)) \leq a \to c$, i.e., $b \leq a \to c$.
(5) Because $b \wedge c \leq b$, $b \wedge c \leq c$, then $a \to (b \wedge c) \leq a \to b$, $a \to (b \wedge c) \leq a \to c$; hence, $a \to (b \wedge c) \leq (a \to b) \wedge (a \to c)$. By Definition 12 (1) and Theorem 7 (1), $a \otimes (a \to b)$, $a \otimes (a \to c)$ are defined and $a \otimes (a \to b) \leq b$, $a \otimes (a \to c) \leq c$, thus, by Definition 12 (2), $a \otimes ((a \to b) \wedge (a \to c))$ is defined and $a \otimes ((a \to b) \wedge (a \to c)) \leq b \wedge c$, so, $(a \to b) \wedge (a \to c) \leq a \to (b \wedge c)$. Hence, $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$.
(6) Because $a \leq a \vee b$, $b \leq a \vee b$, then $(a \vee b) \to c \leq a \to c$, $(a \vee b) \to c \leq b \to c$, thus, $(a \vee b) \to c \leq (a \to c) \wedge (b \to c)$. Then, suppose for any $t \in L$, $t \leq a \to c$, $t \leq b \to c$; by Theorem 7 (4), $a \leq t \to c$, $b \leq t \to c$, thus, $a \vee b \leq t \to c$, using Theorem 7 (4) again, $t \leq (a \vee b) \to c$. If we make $t = (a \to c) \wedge (b \to c)$, it is obviously that $t \leq a \to c$, $t \leq b \to c$; then, $t \leq (a \vee b) \to c$, i.e., $(a \to c) \wedge (b \to c) \leq (a \vee b) \to c$. Hence, $(a \vee b) \to c = (a \to c) \wedge (b \to c)$.

□

Example 3. *Let $L = \{0, a, b, 1\}$. The Hasse-diagram of L is shown in Figure 1, and the operations $\otimes$ and $\to$ are defined by Tables 5 and 6. Then, although L is an rPRL, it is not an srPRL, because $a \otimes (a \to 0) = a \otimes 0$ is undefined.*

Table 5. The operation $\otimes$.

$\otimes$	**0**	***a***	***b***	**1**
0				0
a		a		a
b				b
1	0	a	b	1

Table 6. The operation $\rightarrow$.

$\rightarrow$	**0**	***a***	***b***	**1**
0	1	1	1	1
a	0	1	b	1
b	a	a	1	1
1	0	a	b	1

Example 4. *Let $L = \{0, a, b, c, 1\}$. The Hasse-diagram of L is shown in Figure 3, and the operations $\otimes$ and $\rightarrow$ are defined by Tables 7 and 8. Then, L is both an rPRL and an srPRL.*

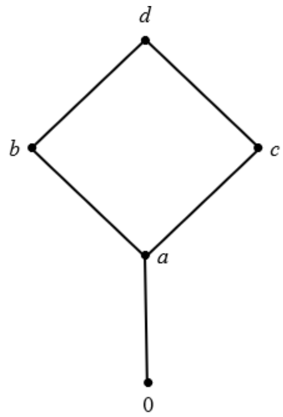

Figure 3. Hasse diagram of lattice L.

Table 7. The operation $\otimes$.

$\otimes$	**0**	***a***	***b***	***c***	**1**
0		0	0	0	0
a	0	a	a	a	a
b	0	a	b	a	b
c	0	a	a	c	c
1	0	a	b	c	1

Table 8. The operation $\rightarrow$.

$\rightarrow$	**0**	***a***	***b***	***c***	**1**
0	1	1	1	1	1
a	0	1	1	1	1
b	0	c	1	c	1
c	0	b	b	1	1
1	0	a	b	c	1

Next, we will provide a special regular partial residuated lattice: normal regular partial residuated lattice, and discuss its related properties in connection with [13].

Definition 13. *A pair $(\otimes, \rightarrow)$ on a poset $(P; \leq)$ is a normal regular partial adjoint pair (nrPAP) where $\otimes$ is a partial operation and $\rightarrow$ is a full operation such that for all $x, y, z \in L$, if:*

(nrPAP1) the operation $\otimes$ is isotone, if $x \leq y$ and $x \otimes z$ is defined, then $y \otimes z$ is defined, $x \otimes z \leq y \otimes z$; if $x \leq y$ and $z \otimes x$ is defined, then $z \otimes y$ is defined and $z \otimes x \leq z \otimes y$.

(nrPAP2) The operation $\to$ is antitone in the first argument and isotone in the second argument, if $x \leq y$, then $y \to z \leq x \to z$, $z \to x \leq z \to y$.

(nrPAP3) If $x \otimes y$ is defined and $x \otimes y \leq z$ iff $y \leq x \to z$.

Definition 14. *A partial algebra $(L, \vee, \wedge, \otimes, \to, 0, 1)$ is a normal regular partial residuated lattice (nrPRL) where $(L, \vee, \wedge, 0, 1)$ is a bounded lattice, $\otimes$ is a partial operation, and $\to$ is a full operation such that for all $x, y, z \in L$:*

(nrPRL1) if $x \otimes y$ is defined, then $y \otimes x$ is defined and $x \otimes y = y \otimes x$;

(nrPRL2) if $y \otimes z$, $x \otimes (y \otimes z)$ are defined, then $x \otimes y$, $(x \otimes y) \otimes z$ are defined and $x \otimes (y \otimes z) = (x \otimes y) \otimes z$;

(nrPRL3) $1 \otimes x$ is defined and $1 \otimes x = x$;

(nrPRL4) $(\otimes, \to)$ is an nrPAP on L.

Theorem 8. *Let $(L; \leq, \otimes, \to, 0, 1)$ be an nrPRL; then, it is a residuated lattice.*

Proof. Through Theorem 4.18 in [13], we conclude that the above statement is true. □

Theorem 9. *Let L be a bounded lattice, $\odot$ be a regular partial t-norm, and $\to_\odot$ be an rPRI derived from $\odot$. Then, $(L, \vee, \wedge, \odot, \to_\odot, 0, 1)$ is an nrPRL.*

Proof. By Definitions 9, 13 and 14, we know that $(L; \leq, \odot, \to_\odot, 0, 1)$ is an nrPRL. □

Corollary 1. *Let L be a bounded lattice, $\odot$ be a regular partial t-norm, and $\to_\odot$ be an rPRI derived from $\odot$. Then, $(L, \vee, \wedge, \odot, \to_\odot, 0, 1)$ is a residuated lattice.*

Proof. The proof can be obtained from Theorem 8. □

4. Commutative Quasiresiduated Lattices (cqRLs), Commutative Q-Residuated Lattices (cQRLs) and rPRLs

This section mainly provides the relationship between regular partial residuated lattices and commutative quasiresiduated lattices, commutative Q-residuated lattices, and the partial residuated lattices mentioned in [9,12,13].

Theorem 10. *Let $\mathbb{Q} = (Q, \vee, \wedge, \odot, \to, 0, 1)$ be a cQRL. If it satisfies (for any $a, b, c \in Q$) that $c \leq a$, $c \leq b$ and $a \leq b$, then $b \to c \leq a \to c$. Then, $\mathbb{L}(Q) := (Q, \vee, \wedge, \odot, \to, 0, 1)$ is a PRL.*

Proof. Applying Definitions 3, 6 and 7, we can easily find that (PAP3), (PRL1), (PRL2) and (PRL3) hold. By Theorem 2 (1), (PAP1) holds, while by Theorem 2 (2) and the content of the above theorem, (PAP2) holds. Hence, $\mathbb{L}(Q)$ is a PRL. □

Remark 1. *We call the commutative Q-residuated lattice satisfying Theorem 10 a perfect commutative Q-residuated lattice (PCQR).*

Theorem 11. *Let $\mathbb{L} = (L, \vee, \wedge, \to, 0, 1)$ be a PRL. If it satisfies (for any $a, b, c \in L$),*

(1) If $a \leq b$, then $b \to a$ is defined;

(2) $a \otimes b$ is defined iff $a \to 0 \leq b$;

(3) $(a \to 0) \to 0 = a$.

Then $\mathbb{Q}(L) := (L, \vee, \wedge, \otimes, \to, 0, 1)$ is a cQRL.

Proof. For any $a, b, c \in L$, by (2), $a' = a \to 0 \leq b \Leftrightarrow a \otimes b$ is defined. Hence, (Q1) holds. From this and by (PRL1), (Q3) holds. By (3), $a'' = (a \to 0)' = (a \to 0) \to 0 = a$, and if

$a \le b$, by (PAP2), $b \to 0 \le a \to 0 \Leftrightarrow b' \le a'$. This means (Q2) holds. By (1), (2) and (PAP3), (Q4) holds. □

Example 5. *Let $Q = \{0, a, b, c, d, 1\}$. The Hasse-diagram of L is shown in Figure 4, the operations $\odot$, $\to_1$ and $\to_2$ are defined by Tables 9–11.*

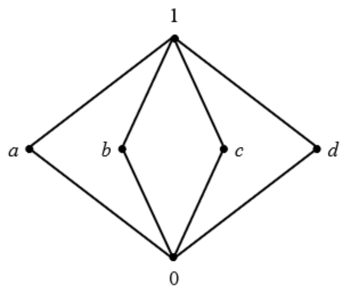

Figure 4. Hasse diagram of lattice *L*.

Table 9. The operation $\odot$.

$\odot$	**0**	***a***	***b***	***c***	***d***	**1**
0						0
a			0			*a*
b		0				*b*
c					0	*c*
d				0		*d*
1	0	*a*	*b*	*c*	*d*	1

Table 10. The operation $\to_1$.

$\to_1$	**0**	***a***	***b***	***c***	***d***	**1**
0	1	1	1	1	1	1
a	*b*	1		*b*		
b	*a*		1			
c	*d*	*d*		1		
d	*c*		*c*	*c*	1	1
1	0	*a*	*b*	*c*	*d*	1

Table 11. The operation $\to_2$.

$\to_2$	**0**	***a***	***b***	***c***	***d***	**1**
0	1					
a	*b*	1				
b	*a*		1			1
c	*d*			1	*d*	1
d	*c*			*c*	1	
1	0	*a*	*b*	*c*	*d*	1

Then $(Q, \vee, \wedge, \odot, \to_1, 0, 1)$ is a cQRL, but it is not a PRL (because $\to_1$ is not antitone in first argument: $a \le 1$, but it is not true that $1 \to c \le a \to c$, i.e., it does not satisfy Theorem 10), while $(Q, \vee, \wedge, \odot, \to_2, 0, 1)$ is a cQRL and a PRL (i.e., it satisfies Theorems 10 and 11).

Theorem 12. *Let $\mathbb{C} = (C, \vee, \wedge, \odot, \to, 0, 1)$ be a cqRL. If it satisfies (for any $a, b, c \in C$),*

(1) If $a' \le b$ and $a \odot b \le c$, then $a \odot b \le b \wedge c$;
(2) If $c \le a$, $c \le b$ and $a \le b$, then $b \to c \le a \to c$.

Then $\mathbb{L}(C) := (C; \le, \odot, \to, 0, 1)$ is an rPRL.

Proof. By Definitions 2, 10 and 11 we know that we only need to prove the following conditions:

(1) For any $a, b, c \in C$, if $a' \leq b$ and $a \odot b \leq c$, then $a \odot b \leq c \Leftrightarrow a \odot b \leq b \wedge c \Leftrightarrow (a \vee b') \odot b \leq b \wedge c \Leftrightarrow a \vee b' \leq b \to c \Leftrightarrow a \leq b \to c$.

(2) By Theorem 4.8 in [12] and Theorem 2, we can easily find that if $a' \leq c$, $b' \leq c$ and $a \leq b$, then $a \odot c \leq b \odot c$; if $a \leq c$, $b \leq c$ and $a \leq b$, then $c \to a \leq c \to b$.

Hence, $\mathbb{L}(C)$ is an rPRL. □

Example 6. *Let $Q = \{0, a, b, c, d, 1\}$. The Hasse-diagram of L is shown in Figure 4, and the operations $\odot$ and $\to_3$ are defined by Tables 9 and 12. Then $(Q, \vee, \wedge, \odot, \to_3, 0, 1)$ is a cqRL, but it is not an rPRL (because $\to_3$ is not antitone in first argument; $a \leq 1$, but it is not true that $1 \to c \leq a \to c$).*

Table 12. The operation $\to_3$.

$\to_3$	**0**	***a***	***b***	***c***	***d***	**1**
0	1	1	1	1	1	1
a	b	1	b	b	b	1
b	a	a	1	a	a	1
c	d	d	d	1	d	1
d	c	c	c	c	1	1
1	0	a	b	c	d	1

Example 7. *Let $Q = \{0, a, b, 1\}$. The Hasse-diagram of L is shown in Figure 1, the operations $\odot$ and $\to$ are defined by Tables 13 and 14. Then, $(Q, \vee, \wedge, \odot, \to, 0, 1)$ is a cqRL and an rPRL.*

Table 13. The operation $\odot$.

$\odot$	**0**	***a***	***b***	**1**
0				0
a			0	a
b		0		b
1	0	a	b	1

Table 14. The operation $\to$.

$\to$	**0**	***a***	***b***	**1**
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

5. Filters in Special Regular Partial Residuated Lattices (srPRLs)

In this section, we first define filters on regular partial residuated lattices, and then define filters and congruence relations on special regular partial residuated lattices. Furthermore, the filter theory of rPRL is established and its quotient structure is constructed.

Definition 15. *Let $(L, \vee, \wedge, \otimes, \to, 0, 1)$ be an srPRL. A nonempty subset $F \subseteq L$ is called a filter if:*

(F1) $1 \in F$;

(F2) *if $a \in F$, $b \in L$ and $a \leq b$, then $b \in F$;*

(F3) *if $a \in F$, $b \in F$ and $a \otimes b$ is defined, then $a \otimes b \in F$.*

A filter is called proper if $F \neq L$.

Example 8. *Let $L = \{0, a, b, c, 1\}$ be an srPRL in Example 4. Then, the proper filters in L are: $\{1\}$, $\{b, 1\}$, $\{c, 1\}$ and $\{a, b, c, 1\}$.*

Theorem 13. *Let* $(L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ *be an srPRL. A subset* F *is a filter in* L *iff:*

(1) $1 \in F$;

(2) *if* $a \in F$, $a \rightarrow b \in F$, *then* $b \in F$.

Proof. $(F2) + (F3) \Rightarrow (2)$: Because $a \rightarrow b \leq a \rightarrow b$, we have $a \otimes (a \rightarrow b) \leq b$, thus, $a \otimes (a \rightarrow b) \in F$ and then $b \in F$.

$(2) \Rightarrow (F2) + (F3)$: (F2) is clearly established. We only need to prove (F3). Because $a \otimes b \leq a \otimes b$, we have $b \leq a \rightarrow (a \otimes b)$, and $b \in F$; hence, $a \rightarrow (a \otimes b) \in F$, and then $a \otimes b \in F$. □

Next, we consider the congruence relation on L, and further make the quotient structure $L/\sim_F$. For this purpose, we provide the following concept of a good filter.

Definition 16. *Let* $(L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ *be an srPRL, where* F *is a filter in* L*. We call* F *good if it satisfies (for any* $a, b, c \in L$*):*

(g1) *if* $a \in F$, $b \in F$, *then* $a \wedge b \in F$;

(g2) *if* $a \otimes b$ *is defined,* $a \rightarrow (b \rightarrow c) \in F$, *then* $(a \otimes b) \rightarrow c \in F$.

Proposition 1. *Let* $(L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ *be an srPRL and let* F *be a good filter in* L*. Then, for all* $a, b, c \in L$*:*

(1) *if* $a \otimes b$ *is defined,* $(a \otimes b) \rightarrow c \in F$, *then* $a \rightarrow (b \rightarrow c) \in F$;

(2) *if* $a \rightarrow b \in F$, *then* $(c \rightarrow a) \rightarrow (c \rightarrow b) \in F$;

(3) *if* $a \rightarrow b \in F$, *then* $(b \rightarrow c) \rightarrow (a \rightarrow c) \in F$;

(4) *if* $a \otimes c$, $b \otimes c$ *are defined and* $a \rightarrow b \in F$, *then* $(a \otimes c) \rightarrow (b \otimes c) \in F$.

Proof.

(1) According to the hypothesis, we know $(a \otimes b) \otimes ((a \otimes b) \rightarrow c)$ is defined, by $(a \otimes b) \rightarrow c \leq (a \otimes b) \rightarrow c$, so $((a \otimes b) \rightarrow c) \otimes (a \otimes b) \leq c$, we have $((a \otimes b) \rightarrow c) \otimes a \leq b \rightarrow c$, hence $(a \otimes b) \rightarrow c \leq a \rightarrow (b \rightarrow c)$. Because $(a \otimes b) \rightarrow c \in F$, $a \rightarrow (b \rightarrow c) \in F$.

(2) By Definition 12 (1), $c \otimes (c \rightarrow a)$ is defined, and by Theorem 7 (1), $c \otimes (c \rightarrow a) \leq a$, then $a \rightarrow b \leq (c \otimes (c \rightarrow a)) \rightarrow b$, and $a \rightarrow b \in F$, thus, $(c \otimes (c \rightarrow a)) \rightarrow b \in F$, by Proposition 1 (1), $(c \rightarrow a) \rightarrow (c \rightarrow b) \in F$.

(3) By Definition 12 (1), $b \otimes (b \rightarrow c)$ is defined, and by Theorem 7 (1), $b \otimes (b \rightarrow c) \leq c$, then $b \leq (b \rightarrow c) \rightarrow c$, so, $a \rightarrow b \leq a \rightarrow ((b \rightarrow c) \rightarrow c)$, and $a \rightarrow b \in F$, so, $a \rightarrow ((b \rightarrow c) \rightarrow c) \in F$, by Proposition 1 (1), $(b \rightarrow c) \rightarrow (a \rightarrow c) \in F$.

(4) According to the hypothesis, we know $a \rightarrow (c \rightarrow (b \otimes c))$ is defined, and $(b \otimes c) \rightarrow (b \otimes c) \in F$, then by Proposition 1 (1), $b \rightarrow (c \rightarrow (b \otimes c)) \in F$, use Proposition 1 (2), $(a \rightarrow b) \rightarrow [a \rightarrow (c \rightarrow (b \otimes c))] \in F$, hence, $a \rightarrow (c \rightarrow (b \otimes c)) \in F$, applying Definition 16 (g2), we obtain $(a \otimes c) \rightarrow (b \otimes c) \in F$.

□

Definition 17. *Let* $(L, \vee, \wedge, \rightarrow, 0, 1)$ *be an rPRL and let* F *be a filter in* L*. For all* $a, b \in L$*, define a binary relation* $\sim_F$*:*

$$a \sim_F b \text{ iff } a \rightarrow b \in F \text{ and } b \rightarrow a \in F$$

Theorem 14. *Let* $(L, \vee, \wedge, \rightarrow, 0, 1)$ *be an srPRL,* F *be a good filter in* L*, and* $\sim_F$ *be the binary relation in Definition 17. Then,* $\sim_F$ *is an equivalence relation on* L*.*

Proof.

(1) $a \rightarrow a = 1 \in F$, so $a \sim_F a$.

(2) Applying Definition 17, $\sim_F$ is symmetric.

(3) If $a \sim_F b$ and $b \sim_F c$, then for some $a, b, c \in L$. For one thing, $a \rightarrow b \in F$, by Proposition 1 (3), $(b \rightarrow c) \rightarrow (a \rightarrow c) \in F$, so $a \rightarrow c \in F$. For another, $c \rightarrow b \in F$, similarly, $(b \rightarrow a) \rightarrow (c \rightarrow a) \in F$, so $c \rightarrow a \in F$. Hence, $a \sim_F c$.

□

Example 9. *Let $L = \{0, a, b, c, 1\}$ be an srPRL in Example 4. Then, the proper filters in L are:$\{1\}$, $\{b, 1\}$, $\{c, 1\}$ and $\{a, b, c, 1\}$; they are all good filters.*

Definition 18. *Let $(L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ be an srPRL. A binary relation $\sim$ is called a congruence such that for all $a, b, a_1, b_1 \in L$, if:*

(C1) $\sim$ is an equivalence relation;
(C2) if $a \sim a_1$, $b \sim b_1$, then $(a \vee b) \sim (a_1 \vee b_1)$;
(C3) if $a \sim a_1$, $b \sim b_1$, then $(a \wedge b) \sim (a_1 \wedge b_1)$;
(C4) if $a \sim a_1$, $b \sim b_1$, $a \otimes b$ and $a_1 \otimes b_1$ are defined, then $a \otimes b \sim a_1 \otimes b_1$;
(C5) if $a \sim a_1$, $b \sim b_1$, then $(a \rightarrow b) \sim (a_1 \rightarrow b_1)$.

Theorem 15. *Let $(L, \vee, \wedge, \rightarrow, 0, 1)$ be an srPRL, F be a good filter in L, and $\sim_F$ be the binary relation in Definition 17. Then, $\sim_F$ is a congruence relation on L.*

Proof. By Theorem 14, $\sim_F$ is an equivalence relation.

(C2) Suppose that $a \sim_F a_1, b \sim_F b_1$; then, $a \rightarrow a_1 \in F, b \rightarrow b_1 \in F$. Applying $a_1 \leq a_1 \vee b_1$, $b_1 \leq a_1 \vee b_1$, we have $a \rightarrow a_1 \leq a \rightarrow (a_1 \vee b_1)$, $b \rightarrow b_1 \leq b \rightarrow (a_1 \vee b_1)$. From $a \rightarrow a_1 \in F$, $b \rightarrow b_1 \in F$, we have $a \rightarrow (a_1 \vee b_1) \in F$, $b \rightarrow (a_1 \vee b_1) \in F$. By Definition 16 (g1), $(a \rightarrow (a_1 \vee b_1)) \wedge (b \rightarrow (a_1 \vee b_1)) \in F$, from this and Theorem 7 (6), $(a \vee b) \rightarrow (a_1 \vee b_1) = (a \rightarrow (a_1 \vee b_1)) \wedge (b \rightarrow (a_1 \vee b_1))$, so, $(a \vee b) \rightarrow (a_1 \vee b_1) \in F$. Similarly, we can prove that $(a_1 \vee b_1) \rightarrow (a \vee b) \in F$. Thus, $(a \vee b) \sim_F (a_1 \vee b_1)$.

(C3) Suppose that $a \sim_F a_1$, $b \sim_F b$, then $a \rightarrow a_1 \in F$, $b \rightarrow b_1 \in F$. Applying $a \wedge b \leq a$, $a \wedge b \leq b$, we have $a \rightarrow a_1 \leq (a \wedge b) \rightarrow a_1$, $b \rightarrow b_1 \leq (a \wedge b) \rightarrow b_1$. From $a \rightarrow a_1 \in F$, $b \rightarrow b_1 \in F$, we have $(a \wedge b) \rightarrow a_1 \in F$, $(a \wedge b) \rightarrow b_1 \in F$. By Definition 16 (g1), $((a \wedge b) \rightarrow a_1) \wedge ((a \wedge b) \rightarrow b_1) \in F$. By Theorem 7 (5), $((a \wedge b) \rightarrow a_1) \wedge ((a \wedge b) \rightarrow b_1) = (a \wedge b) \rightarrow (a_1 \wedge b_1)$. Thus, $(a \wedge b) \rightarrow (a_1 \wedge b_1) \in F$. Similarly, we can prove that $(a_1 \wedge b_1) \rightarrow (a \wedge b) \in F$. Thus, $(a \wedge b) \sim_F (a_1 \wedge b_1)$.

(C4) Suppose that $a \sim_F a_1$, $b \sim_F b$, $a \otimes b$ and $a_1 \otimes b$ are defined, and $a \rightarrow a_1 \in F$, applying Proposition 1 (4), we have $(a \otimes b) \rightarrow (a_1 \otimes b) \in F$. Similarly, we can prove that $(a_1 \otimes b) \rightarrow (a \otimes b) \in F$. Thus, $a \otimes b \sim_F a_1 \otimes b$. For the same reason, $a_1 \otimes b \sim_F a_1 \otimes b_1$. In conclusion, $a \otimes b \sim_F a_1 \otimes b_1$.

(C5) From $a \sim_F a$, $b \sim_F b_1$, and $b \rightarrow b_1 \in F$, applying Proposition 1 (2), $(a \rightarrow b) \rightarrow (a \rightarrow b_1) \in F$, similarly, $(a \rightarrow b_1) \rightarrow (a \rightarrow b) \in F$. Hence, $(a \rightarrow b) \sim_F (a \rightarrow b_1)$. Similarly, applying Proposition 1 (3), we can obtain $(a \rightarrow b_1) \sim_F (a_1 \rightarrow b_1)$. Thus, $(a \rightarrow b) \sim_F (a_1 \rightarrow b_1)$.

□

In the following content, $[a]_F$ represents the equivalence class of a for the equivalence relation $\sim_F$ for any $a \in L$, and we denote the set of all equivalence classes as $L/\sim_F$. Next, we construct the quotient structure of rPRLs.

Definition 19. *Let $(L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ be an rPRL, F be a good filter, and $\sim_F$ be the congruence relation in Theorem 15. Define the following binary relation and binary operations on $L/\sim_F$ (for all $m, n \in L$):*

$$[m]_F \leq [n]_F \ iff \ [m]_F \rightarrow [n]_F = [1]_F \tag{2}$$

$$[m]_F \vee [n]_F := [m \vee n]_F \tag{3}$$

$$[m]_F \wedge [n]_F := [m \wedge n]_F \tag{4}$$

$$[m]_F \otimes [n]_F := \begin{cases} [m \otimes n]_F, & \forall u \in [m]_F, v \in [n]_F \text{ and } u \otimes v \text{ is defined} \\ undefined, & \exists u \in [m]_F, v \in [n]_F \text{ and } u \otimes v \text{ is undefined} \end{cases} \tag{5}$$

$$[m]_F \rightarrow [n]_F := [m \rightarrow n]_F \tag{6}$$

According to Theorem 15, it is easy to verify that the binary relation and binary operations defined in Definition 19 are good.

Theorem 16. *Let* $(L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ *be an rPRL and let* F *be a good filter. Then,* $(L/\sim_F, \vee, \wedge, \otimes, \rightarrow, [0]_F, [1]_F)$ *is an rPRL.*

Proof. Through the congruence relation defined in Definition 18, we know that it is reasonable to define $\leq$ on $L/\sim_F$. Next, we prove that $(L/\sim_F, \vee, \wedge, \otimes, \rightarrow, [0]_F, [1]_F)$ is an rPRL. First, we prove that $\leq$ is a partial order relation on $L/\sim_F$.

(1) Reflexivity is clearly established;
(2) If $[m]_F \leq [n]_F$ and $[n]_F \leq [m]_F$, then $[m]_F \rightarrow [n]_F = [m \rightarrow n]_F = [1]_F$, so, $1 \rightarrow (m \rightarrow n) = m \rightarrow n \in F$, $[n]_F \rightarrow [m]_F = [n \rightarrow m]_F = [1]_F$, so, $1 \rightarrow (n \rightarrow m) = n \rightarrow m \in F$. Hence, $[m]_F = [n]_F$. Antisymmetry holds.
(3) If $[m]_F \leq [n]_F$ and $[n]_F \leq [l]_F$, then $[m]_F \rightarrow [n]_F = [m \rightarrow n]_F = [1]_F$, $[n]_F \rightarrow [l]_F = [n \rightarrow l]_F = [1]_F$. By (2), we know $m \rightarrow n \in F$; for the same reason, $n \rightarrow l \in F$. Applying Proposition 1 (2), $(m \rightarrow n) \rightarrow (m \rightarrow l) \in F$, thus, $(m \rightarrow l) \in F$. Hence, $(m \rightarrow l) \rightarrow 1 = 1 \in F$, and $1 \rightarrow (m \rightarrow l) = m \rightarrow l \in F$, that is, $[m]_F \rightarrow [l]_F = [m \rightarrow l]_F = [1]_F$, hence, $[m]_F \leq [l]_F$. Transitivity holds.

Secondly, we prove that $(L/\sim_F, \vee, \wedge, [0]_F, [1]_F)$ is a bounded lattice.

For any $m, n \in L$, $m \wedge n \leq m, n$, then $([m]_F \wedge [n]_F) \rightarrow [m]_F = [(m \wedge n) \rightarrow m]_F = [1]_F$, $([m]_F \wedge [n]_F) \rightarrow [n]_F = [(m \wedge n) \rightarrow n]_F = [1]_F$, thus, $[m]_F \wedge [n]_F \leq [m]_F, [n]_F$. If $[a]_F \leq [m]_F, [n]_F$, we have $[a]_F \rightarrow [m]_F = [a \rightarrow m]_F = [1]_F$, $[a]_F \rightarrow [n]_F = [a \rightarrow n]_F = [1]_F$. That is, $a \rightarrow m \in F$, $a \rightarrow n \in F$, then, by Definition 16 (g1), $(a \rightarrow m) \wedge (a \rightarrow n) \in F$. By Theorem 7 (5), $a \rightarrow (m \wedge n) = (a \rightarrow m) \wedge (a \rightarrow n)$. Hence, $a \rightarrow (m \wedge n) \in F$, then we have $[a]_F \rightarrow [m \wedge n]_F = [a \rightarrow (m \wedge n)]_F = [1]_F$, i.e., $[a]_F \leq [m \wedge n]_F$. Hence, $[m \wedge n]_F = [m]_F \wedge [n]_F$.

For any $m, n \in L$, $m, n \leq (m \vee n)$, then $[m]_F \rightarrow ([m]_F \vee [n]_F) = [m \rightarrow (m \vee n)]_F = [1]_F$, $[n]_F \rightarrow ([m]_F \vee [n]_F) = [n \rightarrow (m \vee n)]_F = [1]_F$, so, $[m]_F, [n]_F \leq [m]_F \vee [n]_F$. If $[m]_F, [n]_F \leq [b]_F$, we have $[m]_F \rightarrow [b]_F = [m \rightarrow b]_F = [1]_F$, $[n]_F \rightarrow [b]_F = [n \rightarrow b]_F = [1]_F$. That is, $m \rightarrow b \in F$, $n \rightarrow b \in F$, then, by Definition 16 (g1), $(m \rightarrow b) \wedge (n \rightarrow b) \in F$. By Theorem 7 (6), $(m \vee n) \rightarrow b = (m \rightarrow b) \wedge (n \rightarrow b)$. Hence, $(m \vee n) \rightarrow b \in F$, then we have $[m \vee n]_F \rightarrow [b]_F = [(m \vee n) \rightarrow b]_F = [1]_F$, i.e., $[m \vee n]_F \leq [b]_F$. So, $[m \vee n]_F = [m]_F \vee [n]_F$.

Finally, according to [13], we can find that (rPRL1), (rPRL2), (rPRL3) and (rPAP1) are true. Next, we just prove that (rPAP2) and (rPAP3) are true:

(rPAP2) For any $[m]_F, [n]_F, [l]_F \in L/\sim_F$. By Definition 19, we have $[m]_F \leq [n]_F$, then $[m]_F \rightarrow [n]_F = [m \rightarrow n]_F = [1]_F$, i.e., $m \rightarrow n \in F$. For the first variable, $[n]_F \rightarrow [l]_F = [n \rightarrow l]_F$, $[m]_F \rightarrow [l]_F = [m \rightarrow l]_F$, applying Proposition 1 (3), we know $(n \rightarrow l) \rightarrow (m \rightarrow l) \in F = [1]_F$, then, $[(n \rightarrow l) \rightarrow (m \rightarrow l)]_F = [n \rightarrow l]_F \rightarrow [m \rightarrow l]_F = [1]_F$, i.e., $[n \rightarrow l]_F \leq [m \rightarrow l]_F$. For the second variable, $[l]_F \rightarrow [m]_F = [l \rightarrow m]_F$, $[l]_F \rightarrow [n]_F = [l \rightarrow n]_F$, applying Proposition 1 (2), we have $(l \rightarrow m) \rightarrow (l \rightarrow n) \in F = [1]_F$, then, $[(l \rightarrow m) \rightarrow (l \rightarrow n)]_F = [l \rightarrow m]_F \rightarrow [l \rightarrow n]_F = [1]_F$, i.e., $[l \rightarrow m]_F \leq [l \rightarrow n]_F$.

(rPAP3) For any $[m]_F, [n]_F, [l]_F \in L/\sim_F$. By Definition 19, if $[m]_F \otimes [n]_F$ is defined, then: $(\Rightarrow)$ when $[m]_F \otimes [n]_F \leq [l]_F$,

(1) If $\forall u \in [m]_F, v \in [n]_F$ and $u \otimes v$ is defined, then $[m]_F \otimes [n]_F = [m \otimes n]_F$. $[m]_F \otimes [n]_F \leq [l]_F \Leftrightarrow [m \otimes n]_F \leq [l]_F \Leftrightarrow [(m \otimes n) \rightarrow l]_F = [1]_F$, i.e., $(m \otimes n) \rightarrow l \in F$, applying Proposition 1 (1), we have $n \rightarrow (m \rightarrow l) \in F = [1]_F$, further, $[n \rightarrow (m \rightarrow l)]_F = [1]_F$, and $[n]_F \rightarrow [m \rightarrow l]_F = [1]_F \Leftrightarrow [n]_F \leq [m \rightarrow l]_F \Leftrightarrow [n]_F \leq [m]_F \rightarrow [l]_F$.
(2) Suppose $[n]_F = [1]_F$, $[m]_F \otimes [n]_F = [m]_F$. Thus, we have $[m]_F \otimes [n]_F \leq [l]_F \Leftrightarrow [m]_F \leq [l]_F \Leftrightarrow [m \rightarrow l]_F = [1]_F$, and we can obtain $[1]_F \leq [m \rightarrow l]_F \Leftrightarrow [n]_F \leq [m \rightarrow l]_F \Leftrightarrow [n]_F \leq [m]_F \rightarrow [l]_F$.

$(\Leftarrow)$ When $[n]_F \leq [m]_F \rightarrow [l]_F$, we know $[n]_F \leq [m \rightarrow l]_F \Leftrightarrow [n]_F \rightarrow [m \rightarrow l]_F = [1]_F$, hence, $n \rightarrow (m \rightarrow l) \in F = [1]_F$, applying Definition 16 (g2), we have $(n \otimes m) \rightarrow l \in F$, i.e., $(m \otimes n) \to l \in F = [1]_F$, thus, $[(m \otimes n) \to l]_F = [1]_F \Leftrightarrow [m \otimes n]_F \leq [l]_F \Leftrightarrow [m]_F \otimes [n]_F \leq [l]_F$.

Therefore, $(L/\sim_F, \vee, \wedge, \otimes, \to, [0]_F, [1]_F)$ is an rPRL. □

Example 10. *Let* $L = \{0, a, b, c, 1\}$, $(L, \vee, \wedge, \otimes, \to, 0, 1)$ *be an srPRL in Example 4.* $L/\sim_F = \{\{0\}, \{a, b\}, \{c, 1\}\}$*, where F is a good filter and* $F = \{c, 1\}$*. The Hasse-diagram of* $L/\sim_F$ *is shown in Figure 5, and the operations* $\odot$ *and* $\to$ *are defined by Tables 15 and 16. Then,* $L/\sim_F$ *is an rPRL.*

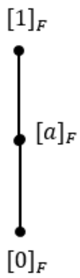

Figure 5. Hasse diagram of lattice $L/\sim_F$.

Table 15. The operation $\otimes$ on $L/\sim_F$.

$\otimes$	$[0]_F$	$[a]_F$	$[1]_F$
$[0]_F$		$[0]_F$	$[0]_F$
$[a]_F$	$[0]_F$	$[a]_F$	$[a]_F$
$[1]_F$	$[0]_F$	$[a]_F$	$[1]_F$

Table 16. The operation $\to$ on $L/\sim_F$.

$\to$	$[0]_F$	$[a]_F$	$[1]_F$
$[0]_F$	$[1]_F$	$[1]_F$	$[1]_F$
$[a]_F$	$[0]_F$	$[1]_F$	$[1]_F$
$[1]_F$	$[0]_F$	$[a]_F$	$[1]_F$

6. Conclusions

By constraining the implication operation in partial residuated lattices, we obtained regular partial residuated lattices and performed a series of tests on them. First, the regular partial residuated implication was defined, and the relationship between rPRI and PFI is revealed by Theorem 3. Second, according to the concept of regular partial residuated lattices, the properties were studied. Then, from Theorems 10 and 11, we obtained the transformation relationship between a commutative Q-residuated lattice and partial residuated lattice, and Theorem 12 shows the condition in which a commutative quasiresiduated lattice becomes a regular partial residuated lattice. This can be intuitively displayed with Figure 6.

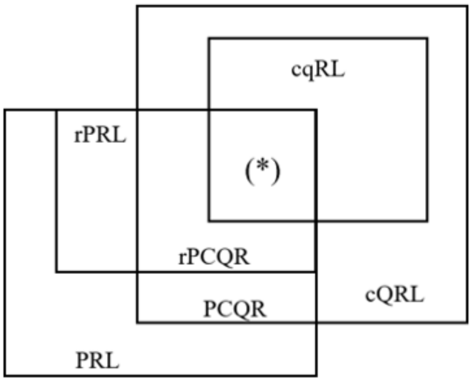

Figure 6. Relationship between some algebraic structures.

What we want to explain here is the following: if the implication in PCQR is a full operation, we call it rPCQR.

(*) represents that it is an rPCQR and that the commutative quasiresiduated lattice satisfies Theorem 12 (1) (Example 7).

Finally, after the concept of special regular partial residuated lattices is obtained, filters and good filters are defined, the quotient structure theory is provided by Definition 19, and it is proven that it is a regular partial residuated lattice using Theorem 16.

In the future, we will continue to study partial residuated lattices and their special subclasses, and reveal the internal relationship between partial residuated lattices (regular partial residuated lattices) and other logical algebras [21–23]. In addition, we will study fuzzy reasoning and fuzzy rough sets based on partial residuated lattices, as well as other applications [24–26].

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