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Stability of a Bi-Jensen Functional Equation on Restricted Unbounded Domains and Some Asymptotic Behaviors

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Abstract: In this paper, we give some properties of the bi-Jensen functional equation and investigate its Hyers–Ulam stability and hyperstability. We construct a function which is bi-Jensen and is not continuous. Additionally, we investigate the Hyers–Ulam stability of the bi-Jensen functional equation on some restricted unbounded domains. Finally, we apply the obtained results to study some interesting asymptotic behaviors of bi-Jensen functions.

Keywords: Hyers–Ulam stability; functional equation; bi-Jensen function; ε-bi-Jensen function; asymptotic behavior

MSC: 39B82; 39B52

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1. Introduction and Preliminaries

Let $\mathcal{V}$ and $\mathcal{W}$ be linear spaces. A function $g : \mathcal{V} \to \mathcal{W}$ is called a Jensen function if

$$2g\left(\frac{x+y}{2}\right) = g(x) + g(y), \quad x, y \in \mathcal{V}.$$

It is well known that a continuous Jensen function $g : \mathbb{R} \to \mathbb{R}$ is of the form $g(x) = ax + b$ for some real constants a, b (see for example [1], Theorem 1.52).

For a given function $f : \mathcal{V} \times \mathcal{V} \to \mathcal{W}$, we define the function $\mathcal{J}f : \mathcal{V}^4 \to \mathcal{W}$ by

$$\mathcal{J}f(x,y,z,w) = 4f\left(\frac{x+y}{2}, \frac{z+w}{2}\right) - [f(x,z) + f(x,w) + f(y,z) + f(y,w)].$$

A function $f : \mathcal{V} \times \mathcal{V} \to \mathcal{W}$ is said to be a bi-Jensen function if $\mathcal{J}f(x,y,z,w) = 0$ for all $x, y, z, w \in \mathcal{V}$. It is clear that $\mathcal{J}f(0,0,0,0) = 0$ for every function $f : \mathcal{V} \times \mathcal{V} \to \mathcal{W}$. It is obvious that a function $f : \mathcal{V} \times \mathcal{V} \to \mathcal{W}$ is bi-Jensen if and only if

$$2f\left(\frac{x+y}{2}, z\right) = f(x,z) + f(y,z), \quad 2f\left(z, \frac{x+y}{2}\right) = f(z,x) + f(z,y) \tag{1}$$

for all $x, y, z \in \mathcal{V}$.

Bae and Park [2] obtained the general solution of the bi-Jensen functional equation. Indeed, they showed a function $f : \mathcal{V} \times \mathcal{V} \to \mathcal{W}$ is bi-Jensen if and only if there exist a bi-additive function $B : \mathcal{V} \times \mathcal{V} \to \mathcal{W}$ and two additive functions $A, A' : \mathcal{V} \to \mathcal{W}$ such that

$$f(x,y) = B(x,y) + A(x) + A(y) + f(0,0), \quad x, y \in \mathcal{V}.$$

For the case $\mathcal{V} = \mathcal{W} = \mathbb{R}$, it is easy to see that the function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $f(x, y) = axy + bx + cy + d$ is a bi-Jensen function. Of course, we will see that every continuous bi-Jensen function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ has this form.

Generally speaking, a functional equation is called stable in a class of functions $\mathfrak{F}$ if any function from $\mathfrak{F}$, satisfying the functional equation approximately (in some sense), then it is near to an exact solution of the functional equation. It should be noted that the stability problem of functional equations appeared from a question of Ulam [3] about the stability of group homomorphisms.

Bae and Park [2] investigated the generalized Hyers–Ulam stability of (1). Some stability results associated with the bi-Jensen functional equation can be found in [2,4–7].

In this paper, we deal with the bi-Jensen functional equation

$$4f\left(\frac{x+y}{2}, \frac{z+w}{2}\right) = f(x,z) + f(x,w) + f(y,z) + f(y,w), \quad x, y, z, w \in \mathcal{V}, \tag{2}$$

where $f : \mathcal{V} \times \mathcal{V} \to \mathcal{W}$ is the unknown function. We give the general continuous solutions of (2) when $\mathcal{V} = \mathcal{W} = \mathbb{R}$. We use a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$ (the field of rational numbers) in constructing a function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, which is a solution of (2) and is not continuous. We investigate the Hyers–Ulam stability and hyperstability of (2). Moreover, we investigate the Hyers–Ulam stability of the bi-Jensen functional equation on some restricted unbounded domains. This enables us to study some of interesting asymptotic behaviors of bi-Jensen functions.

In the past decades and recent years, various types of stability problems for different functional equations have been studied by many mathematicians (cf. e.g., [2,8–15] and the bibliography quoted there).

2. Some Properties of Bi-Jensen Functions

In this section, some properties of bi-Jensen functions are presented.

Proposition 1. *Let $\mathcal{V}$ and $\mathcal{W}$ denote linear spaces and $f : \mathcal{V} \times \mathcal{V} \to \mathcal{W}$ be a bi-Jensen function. Then,*

$$\begin{aligned} f(x+y, z+w) = {} & f(x,z) + f(x,w) + f(y,z) + f(y,w) \\ & - [f(x,0) + f(y,0) + f(0,z) + f(0,w)] + f(0,0), \end{aligned} \tag{3}$$

for all $x, y, z, w \in \mathcal{V}$.

Proof. Since $\mathcal{J}f(x, y, z, z) = 0$ and $\mathcal{J}f(x, x, z, w) = 0$, we have

$$2f\left(\frac{x+y}{2}, z\right) = f(x,z) + f(y,z), \quad 2f\left(x, \frac{z+w}{2}\right) = f(x,z) + f(x,w), \tag{4}$$

for all $x, y, z, w \in \mathcal{V}$. Letting $y = 0$ and $w = 0$ in (4), we obtain

$$2f\left(\frac{x}{2}, z\right) = f(x,z) + f(0,z), \quad 2f\left(x, \frac{z}{2}\right) = f(x,z) + f(x,0), \tag{5}$$

for all $x, z \in \mathcal{V}$. Applying (5) in (4), one gets

$$\begin{aligned} f(x+y, z) &= f(x,z) + f(y,z) - f(0,z), \\ f(x, z+w) &= f(x,z) + f(x,w) - f(x,0), \quad x, y, z, w \in \mathcal{V}. \end{aligned}$$

Hence we get the desired result (3). □

Proposition 2. *Let $\mathcal{V}$ and $\mathcal{W}$ be normed linear spaces and $f : \mathcal{V} \times \mathcal{V} \to \mathcal{W}$ be a bi-Jensen function. Take $a, b \in \mathcal{V}$. Then, f is continuous at $(0,0), (a,0), (0,b)$ if and only if f is continuous at $(a,b), (a,0), (0,b)$.*

Proof. Let $\{x_n\}$ and $\{y_n\}$ be sequences in $\mathcal{V}$ such that $x_n, y_n \to 0$ as $n \to \infty$. By Proposition 1, we have

$$\begin{aligned} f(x_n + a, y_n + b) &= f(x_n, y_n) + f(x_n, b) + f(a, y_n) + f(a, b) \\ &\quad - [f(x_n, 0) + f(a, 0) + f(0, y_n) + f(0, b)] + f(0, 0). \end{aligned}$$

This proves the proposition. □

Proposition 3. *Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous bi-Jensen function. Then,*

$$f(x, y) = axy + bx + cy + d, \quad x, y \in \mathbb{R}, \tag{6}$$

where a, b, c, d are real constants.

Proof. Since f is bi-Jensen, f satisfies (4) for all $x, y, z, w \in \mathbb{R}$. Then, for each fixed $z \in \mathbb{R}$, the mappings $x \mapsto f(x, z)$ and $x \mapsto f(z, x)$ are continuous Jensen. Hence, there exist real constants c, d and a function $\varphi : \mathbb{R} \to \mathbb{R}$ such that

$$f(0, z) = cz + d \quad \text{and} \quad f(x, z) = \varphi(z)x + f(0, z), \quad x, z \in \mathbb{R}.$$

Therefore, $f(x, y) = \varphi(y)x + cy + d$ for all $x, y \in \mathbb{R}$. It is clear that φ is continuous. We show that φ is Jensen. By (4), we obtain

$$2f\left(x, \frac{z + w}{2}\right) = f(x, z) + f(x, w) = [\varphi(z) + \varphi(w)]x + c(z + w) + 2d, \quad x, z \in \mathbb{R}. \tag{7}$$

On the other hand, we have

$$2f\left(x, \frac{z + w}{2}\right) = 2\varphi\left(\frac{z + w}{2}\right)x + c(z + w) + 2d, \quad x, z \in \mathbb{R}. \tag{8}$$

By Equations (7) and (8), one concludes φ is Jensen. So, $\varphi(y) = ay + b$ for some $a, b \in \mathbb{R}$. Then,

$$f(x, y) = \varphi(y)x + cy + d = axy + bx + cy + d, \quad x, y \in \mathbb{R}.$$

□

In the following, we use Hamel bases in constructing a bi-Jensen function f, which is not of the form (6), and so is not continuous. First, we construct the most general bi-Jensen function. Then, we show the existence of a bi-Jensen function, which is not of the form (6).

Theorem 1. *Let $\mathcal{B}$ be a Hamel basis of $\mathbb{R}$ over the field of rational numbers $\mathbb{Q}$, and $g : \mathcal{B} \to \mathbb{R}$ be defined arbitrarily on $\mathcal{B}$. Then, there exists a bi-Jensen function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $f(0, 0) = 0$ and*

$$f(x, 0) = f(0, x) = g(x), \quad x \in \mathcal{B}.$$

Proof. All real numbers x and y can be represented uniquely as a rational linear combination

$$x = \sum_{i=1}^{n} r_i a_i, \quad y = \sum_{i=1}^{m} s_i b_i, \quad r_i, s_i \in \mathbb{Q},\ a_i, b_i \in \mathcal{B}.$$

Let $\{a_1, a_2, \cdots, a_n\} \cup \{b_1, b_2, \cdots, b_m\} = \{c_1, c_2, \cdots, c_k\}$. Then, $k \leqslant m + n$, and

$$x = \sum_{i=1}^{k} p_i c_i, \quad y = \sum_{i=1}^{k} q_i c_i, \quad p_i, q_i \in \mathbb{Q},\ c_i \in \mathcal{B},$$

where p_i, q_j may be zero for some i, j. We define

$$f(x,y) := \sum_{i=1}^{k}(p_i+q_i)g(c_i) + \sum_{i,j=1}^{k} p_iq_jg(c_i)g(c_j).$$

We show that f is bi-Jensen. Let $x, y, z, w \in \mathbb{R}$ be represented as follows

$$x = \sum_{i=1}^{n} r_ib_i, \quad y = \sum_{i=1}^{n} s_ib_i, \quad z = \sum_{i=1}^{n} p_ib_i, \quad w = \sum_{i=1}^{n} q_ib_i, \quad r_i, s_i, p_i, q_i \in \mathbb{Q},\ b_i \in \mathcal{B}.$$

Then

$$\begin{aligned}
4f\left(\frac{x+y}{2}, \frac{z+w}{2}\right) &= 2\sum_{i=1}^{n}(r_i+s_i+p_i+q_i)g(b_i) + \sum_{i,j=1}^{n}(r_i+s_i)(p_j+q_j)g(b_i)g(b_j) \\
&= \sum_{i=1}^{n}(r_i+p_i)g(b_i) + \sum_{i,j=1}^{n} r_ip_jg(b_i)g(b_j) \\
&\quad + \sum_{i=1}^{n}(r_i+q_i)g(b_i) + \sum_{i,j=1}^{n} r_iq_jg(b_i)g(b_j) \\
&\quad + \sum_{i=1}^{n}(s_i+p_i)g(b_i) + \sum_{i,j=1}^{n} s_ip_jg(b_i)g(b_j) \\
&\quad + \sum_{i=1}^{n}(s_i+q_i)g(b_i) + \sum_{i,j=1}^{n} s_iq_jg(b_i)g(b_j) \\
&= f(x,z) + f(x,w) + f(y,z) + f(y,w).
\end{aligned}$$

It is clear that $f(0,0) = 0$. If $x \in \mathcal{B}$, we have $x = 1x$ and $0 = 0x$. So, by the definition of f, we get

$$f(x,0) = g(x) \quad \text{and} \quad f(0,x) = g(x), \quad x \in \mathcal{B}.$$

□

Corollary 1. *There is a bi-Jensen function $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, which is not of the form* (6).

Proof. Let $\mathcal{B}$ be a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$ and $e, \gamma \in \mathcal{B}$ with $e \neq \gamma$. Define $g : \mathcal{B} \to \mathbb{R}$ by $g(e) = 1$ and $g(x) = 0$ for all $x \in \mathcal{B}\setminus\{e\}$. By Theorem (1), there exists a bi-Jensen function $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ with $f(0,0) = 0$ and $f(x,0) = f(0,x) = g(x)$ for all $x \in \mathcal{B}$. If $f(x,y) = axy + bx + cy + d$ for some real numbers a, b, c, d, then

$$1 = f(e,0) = be + d, \quad 1 = f(0,e) = ce + d, \quad 0 = f(\gamma,0) = b\gamma + d, \quad 0 = f(0,0) = d.$$

This yields $b = c = d = 0$. So, $1 = f(e,0) = 0$, which is a contradiction. □

3. Hyers-Ulam Stability

In this section, the stability problem is treated for the bi-Jensen function in the sense of Hyers–Ulam. Some basic properties of a bi-Jensen function were established by Jun et al. [4].

The following lemma extends the results of ([4], Lemma 1).

Lemma 1. *Let $\mathcal{V}$ and $\mathcal{W}$ be linear spaces and $f : \mathcal{V} \times \mathcal{V} \to \mathcal{W}$. Then, for all $n \in \mathbb{N}$ and all $x, y \in V$, we have*

$$f(2^n x, 0) = 2^n f(x,0) + (1-2^n) f(0,0) - \sum_{i=1}^{n} 2^{n-i-1} \mathcal{J}f(2^i x, 0, 0, 0), \tag{9}$$

$$f(0, 2^n y) = 2^n f(0,y) + (1-2^n) f(0,0) - \sum_{i=1}^{n} 2^{n-i-1} \mathcal{J}f(0, 0, 2^i y, 0), \tag{10}$$

$$\begin{aligned} f(2^n x, 2^n y) = {} & 4^n f(x,y) + (2^n - 4^n)\big[f(x,0) + f(0,y)\big] \\ & + (1-2^n)^2 f(0,0) - \sum_{i=1}^{n} 4^{n-i} \mathcal{J}f(2^i x, 0, 2^i y, 0) \\ & + \sum_{i=1}^{n} (4^{n-i} - 2^{n-i-1}) \Big[\mathcal{J}f(2^i x, 0, 0, 0) + \mathcal{J}f(0, 0, 2^i y, 0)\Big]. \end{aligned} \tag{11}$$

Proof. Let $n \in \mathbb{N}$ and $x, y \in \mathcal{V}$. Then,

$$\begin{aligned} \sum_{i=1}^{n} 2^{n-i-1} \mathcal{J}f(2^i x, 0, 0, 0) &= \sum_{i=1}^{n} 2^{n-i-1} \Big[4f(2^{i-1}x, 0) - 2f(2^i x, 0) - 2f(0,0)\Big] \\ &= \sum_{i=1}^{n} \Big[2^{n-i+1} f(2^{i-1}x, 0) - 2^{n-i} f(2^i x, 0)\Big] - \sum_{i=1}^{n} 2^{n-i} f(0,0) \\ &= 2^n f(x,0) - f(2^n x, 0) + (1-2^n) f(0,0). \end{aligned}$$

This proves (9). Similarly, (10) is also obtained. To prove (11), we have

$$\begin{aligned} & \sum_{i=1}^{n} 4^{n-i} \Big[\mathcal{J}f(2^i x, 0, 2^i y, 0) - \mathcal{J}f(2^i x, 0, 0, 0) - \mathcal{J}f(0, 0, 2^i y, 0)\Big] \\ & = \sum_{i=1}^{n} 4^{n-i} \Big[4f(2^{i-1}x, 2^{i-1}y) - f(2^i x, 2^i y)\Big] \\ & \quad + \sum_{i=1}^{n} 4^{n-i} \Big[f(2^i x, 0) - 4f(2^{i-1}x, 0)\Big] \\ & \quad + \sum_{i=1}^{n} 4^{n-i} \Big[f(0, 2^i y) - 4f(0, 2^{i-1}y)\Big] + 3\sum_{i=1}^{n} 4^{n-i} f(0,0) \\ & = 4^n f(x,y) - f(2^n x, 2^n y) + f(2^n x, 0) - 4^n f(x,0) + f(0, 2^n y) - 4^n f(0,y) + (4^n - 1) f(0,0). \end{aligned}$$

Now, using equalities (9) and (10), we obtain (11). This completes the proof. □

Corollary 2. *Let $\mathcal{V}$ and $\mathcal{W}$ be linear spaces and $f : \mathcal{V} \times \mathcal{V} \to \mathcal{W}$ be a bi-Jensen function. Then,*

(i) $f(2^n x, 0) = 2^n f(x,0) + (1-2^n) f(0,0)$;

(ii) $f(0, 2^n y) = 2^n f(0,y) + (1-2^n) f(0,0)$;

(iii) $f(2^n x, 2^n y) = 4^n f(x,y) + (1-2^n)[f(2^n x, 0) + f(0, 2^n y)] - (1-2^n)^2 f(0,0)$;

(iv) $f(2^n x, 2^n y) = 4^n f(x,y) + (2^n - 4^n)\big[f(x,0) + f(0,y)\big] + (1-2^n)^2 f(0,0)$,

for all $n \in \mathbb{N}$ and all $x, y \in \mathcal{V}$.

Let $\varepsilon \geqslant 0$, $\mathcal{V}$ be a linear space and $\mathcal{W}$ a linear normed space. A function $f : \mathcal{V} \times \mathcal{V} \to \mathcal{W}$ is said to be ε- bi-Jensen if $\|\mathcal{J}f(x,y,z,w)\| \leqslant \varepsilon$ for all $x, y, z, w \in \mathcal{V}$.

Lemma 2. *Let $\mathcal{V}$ be a linear space, $\mathcal{W}$ a normed linear space and $f : \mathcal{V} \times \mathcal{V} \to \mathcal{W}$ be an ε-bi-Jensen function. Then, $\{4^{-n} f(2^n x, 2^n y)\}_n$, $\{2^{-n} f(2^n x, 0)\}_n$ and $\{2^{-n} f(0, 2^n y)\}_n$ are Cauchy sequences for each $x, y \in \mathcal{V}$.*

Proof. We have

$$\mathcal{J}f(2^{i+1}x,0,0,0) = 4f(2^i x,0) - 2[f(2^{i+1}x,0) + f(0,0)],$$
$$\mathcal{J}f(0,0,2^{i+1}y,0) = 4f(0,2^i y) - 2[f(0,2^{i+1}y) + f(0,0)], \quad x,y \in \mathcal{V}.$$

Then

$$\frac{\mathcal{J}f(2^{i+1}x,0,0,0)}{2^{i+2}} = \frac{f(2^i x,0)}{2^i} - \frac{f(2^{i+1}x,0)}{2^{i+1}} - \frac{f(0,0)}{2^{i+1}},$$
$$\frac{\mathcal{J}f(0,0,2^{i+1}y,0)}{2^{i+2}} = \frac{f(0,2^i y)}{2^i} - \frac{f(0,2^{i+1}y)}{2^{i+1}} - \frac{f(0,0)}{2^{i+1}}, \quad x,y \in \mathcal{V}.$$

Since f is ε-bi-Jensen, we infer that

$$\begin{aligned} &\left\| \frac{f(2^m x,0)}{2^m} - \frac{f(2^{n+1}x,0)}{2^{n+1}} - \sum_{i=m}^{n} \frac{f(0,0)}{2^{i+1}} \right\| \\ &= \left\| \sum_{i=m}^{n} \left[\frac{f(2^i x,0)}{2^i} - \frac{f(2^{i+1}x,0)}{2^{i+1}} \right] - \sum_{i=m}^{n} \frac{f(0,0)}{2^{i+1}} \right\| \\ &\leqslant \sum_{i=m}^{n} \left\| \frac{\mathcal{J}f(2^{i+1}x,0,0,0)}{2^{i+2}} \right\| \\ &\leqslant \sum_{i=m}^{n} \frac{\varepsilon}{2^{i+2}}, \end{aligned} \tag{12}$$

and similarly,

$$\left\| \frac{f(0,2^m y)}{2^m} - \frac{f(0,2^{n+1}y)}{2^{n+1}} - \sum_{i=m}^{n} \frac{f(0,0)}{2^{i+1}} \right\| \leqslant \sum_{i=m}^{n} \frac{\varepsilon}{2^{i+2}}, \tag{13}$$

for all $x, y \in \mathcal{V}$ and integers $n \geqslant m \geqslant 0$. Therefore, $\{2^{-n} f(2^n x, 0)\}_n$ and $\{2^{-n} f(0, 2^n y)\}_n$ are Cauchy sequences.

We now prove that $\{4^{-n} f(2^n x, 2^n y)\}_n$ is Cauchy. First, we have

$$\begin{aligned} &\frac{\mathcal{J}f(2^{i+1}x,0,2^{i+1}y,0) - \mathcal{J}f(2^{i+1}x,0,0,0) - \mathcal{J}f(0,0,2^{i+1}y,0)}{4^{i+1}} \\ &= \frac{f(2^i x,2^i y) - f(2^i x,0) - f(0,2^i y) + f(0,0)}{4^i} \\ &\quad - \frac{f(2^{i+1}x,2^{i+1}y) - f(2^{i+1}x,0) - f(0,2^{i+1}y) + f(0,0)}{4^{i+1}}, \quad x,y \in \mathcal{V}. \end{aligned} \tag{14}$$

For $n \in \mathbb{N}$ and $x, y \in \mathcal{V}$, we set

$$\mathcal{F}_n(x,y) := \frac{f(2^n x,2^n y) - f(2^n x,0) - f(0,2^n y) + f(0,0)}{4^n}.$$

Then (14) yields

$$\|\mathcal{F}_m(x,y) - \mathcal{F}_{n+1}(x,y)\| \leqslant \sum_{i=m}^{n} \frac{3\varepsilon}{4^{i+1}}, \quad x,y \in \mathcal{V},\ n \geqslant m \geqslant 0. \tag{15}$$

Hence, $\{\mathcal{F}_n(x,y)\}_n$ is a Cauchy sequence. Because $\{2^{-n}f(2^n x,0)\}_n$ and $\{2^{-n}f(0,2^n y)\}_n$ are Cauchy sequences, we infer that $\{4^{-n}f(2^n x,2^n y)\}_n$ is Cauchy. □

In the following theorem we investigate the Hyers–Ulam stability of a bi-Jensen function.

Theorem 2. *Let $\mathcal{Y}$ be a Banach space and $f:\mathcal{V}\times\mathcal{V}\to\mathcal{Y}$ be an ε-bi-Jensen function. Then there is a unique bi-Jensen function $g:\mathcal{V}\times\mathcal{V}\to\mathcal{Y}$ such that*

$$g(0,0)=f(0,0) \quad \text{and} \quad \|f(x,y)-g(x,y)\|\leqslant 2\varepsilon, \quad x,y\in\mathcal{V}. \tag{16}$$

Moreover, g is given by

$$g(x,y):=\lim_{n\to\infty}\Big[f(0,0)+2^{-n}f(2^n x,0)+2^{-n}f(0,2^n y)+4^{-n}f(2^n x,2^n y)\Big].$$

Proof. By Lemma 2, we can define the functions $\mathcal{P},\mathcal{Q}:\mathcal{V}\to\mathcal{Y}$ and $\mathcal{R}:\mathcal{V}\times\mathcal{V}\to\mathcal{Y}$ by

$$\begin{aligned}\mathcal{P}(x):&=\lim_{n\to\infty}2^{-n}f(2^n x,0),\\ \mathcal{Q}(y):&=\lim_{n\to\infty}2^{-n}f(0,2^n y),\\ \mathcal{R}(x,y):&=\lim_{n\to\infty}4^{-n}f(2^n x,2^n y), \quad x,y\in\mathcal{V}.\end{aligned}$$

Putting $m=0$ and taking $n\to\infty$ in (12), (13) and (15), we obtain

$$\begin{aligned}\|f(x,0)-\mathcal{P}(x)-f(0,0)\|&\leqslant\frac{\varepsilon}{2},\\ \|f(0,y)-\mathcal{Q}(y)-f(0,0)\|&\leqslant\frac{\varepsilon}{2},\\ \|f(x,y)-f(x,0)-f(0,y)+f(0,0)-\mathcal{R}(x,y)\|&\leqslant\varepsilon, \quad x,y\in\mathcal{V}.\end{aligned}$$

Adding these inequalities, we get

$$\|f(x,y)-\mathcal{P}(x)-\mathcal{Q}(y)-\mathcal{R}(x,y)-f(0,0)\|\leqslant 2\varepsilon, \quad x,y\in\mathcal{V}.$$

This means (16), where $g(x,y)=f(0,0)+\mathcal{P}(x)+\mathcal{Q}(y)+\mathcal{R}(x,y)$.

It is clear that $g(0,0)=f(0,0)$. Now, we show that g is bi-Jensen. It is easy to see that

$$\begin{aligned}&\Big\|\mathcal{J}g(x_1,y_1,,x_2,y_2)\Big\|\\ &=\lim_{n\to\infty}\Big\|4^{-n}\mathcal{J}f(2^n x_1,2^n y_1,2^n x_2,2^n y_2)+2^{-n}\mathcal{J}f(2^n x_1,2^n y_1,0,0)+2^{-n}\mathcal{J}f(0,0,2^n x_2,2^n y_2)\Big\|\\ &\leqslant\lim_{n\to\infty}(4^{-n}+2^{-n+1})\varepsilon=0,\end{aligned}$$

for all $x_1,x_2,y_1,y_2\in\mathcal{V}$. To prove the uniqueness of g, let $h:\mathcal{V}\times\mathcal{V}\to\mathcal{Y}$ be another bi-Jensen function satisfying (16). By Corollary 2 (*iii*), we have

$$\begin{aligned}g(x,y)&=4^{-n}g(2^n x,2^n y)+(2^{-n}-4^{-n})[g(2^n x,0)+g(0,2^n y)]+4^{-n}(2^n-1)^2f(0,0),\\ h(x,y)&=4^{-n}h(2^n x,2^n y)+(2^{-n}-4^{-n})[h(2^n x,0)+h(0,2^n y)]+4^{-n}(2^n-1)^2f(0,0),\end{aligned}$$

for all $x, y \in \mathcal{V}$ and $n \in \mathbb{N}$. Then,

$$\begin{aligned}
&\|g(x,y) - h(x,y)\| \\
&= \left\|4^{-n}(g-h)(2^n x, 2^n y) + (2^{-n} - 4^{-n})[(g-h)(2^n x, 0) + (g-h)(0, 2^n y)]\right\| \\
&\leqslant 4^{-n}\|(g-f)(2^n x, 2^n y)\| + 4^{-n}\|(f-h)(2^n x, 2^n y)\| \\
&\quad + (2^{-n} - 4^{-n})\|(g-f)(2^n x, 0)\| + (2^{-n} - 4^{-n})\|(f-h)(2^n x, 0)\| \\
&\quad + (2^{-n} - 4^{-n})\|(g-f)(0, 2^n y)\| + (2^{-n} - 4^{-n})\|(f-h)(0, 2^n y)\| \\
&\leqslant 4(2^{1-n} - 4^{-n})\varepsilon
\end{aligned}$$

for all $x, y \in \mathcal{V}$ and $n \in \mathbb{N}$. Letting $n \to \infty$, we infer that $g(x,y) = h(x,y)$ for all $x, y \in \mathcal{V}$. □

4. Hyperstability

We start with the following lemmas.

Lemma 3. *Let $\mathcal{V}$ and $\mathcal{W}$ be linear spaces and $f : \mathcal{V} \to \mathcal{W}$ satisfying*

$$2f\left(\frac{x+y}{2}\right) = f(x) + f(y), \quad x, y \in \mathcal{V} \setminus \{0\}. \tag{17}$$

Then, f is Jensen on $\mathcal{V}$.

Proof. Letting $y = -x$ in (17), we get

$$2f(0) = f(x) + f(-x), \quad x \in \mathcal{V}. \tag{18}$$

Letting $y = 3x$ and $y = -3x$ in (17), respectively, we obtain

$$2f(2x) = f(x) + f(3x), \quad 2f(-x) = f(x) + f(-3x), \quad x \in \mathcal{V}. \tag{19}$$

Adding equations in (19) and using (18), we conclude

$$f(2x) = 2f(x) - f(0), \quad x \in \mathcal{V}.$$

Then

$$2f\left(\frac{x}{2}\right) = f(x) + f(0), \quad x \in \mathcal{V}. \tag{20}$$

By (20), one infers that (17) holds for all $x, y \in \mathcal{V}$. This completes the proof. □

Lemma 4. *Let $\mathcal{V}$ and $\mathcal{W}$ be linear spaces and $f : \mathcal{V} \times \mathcal{V} \to \mathcal{W}$ satisfying*

$$4f\left(\frac{x+y}{2}, \frac{z+w}{2}\right) = f(x,z) + f(x,w) + f(y,z) + f(y,w), \quad x, y, z, w \in \mathcal{V} \setminus \{0\}. \tag{21}$$

Then, f is bi-Jensen on $\mathcal{V} \times \mathcal{V}$.

Proof. Letting $w = z$ in (21), we get

$$2f\left(\frac{x+y}{2}, z\right) = f(x,z) + f(y,z), \quad x, y, z \in \mathcal{V} \setminus \{0\}. \tag{22}$$

Letting $y = x$ and $w = -z$ in (21), we get

$$2f(x,0) = f(x,z) + f(x,-z), \quad x, z \in \mathcal{V} \setminus \{0\}. \tag{23}$$

Putting $w = -z$ in (21) and applying (23), we obtain

$$2f\left(\frac{x+y}{2},0\right) = f(x,0) + f(y,0), \quad x,y \in \mathcal{V} \setminus \{0\}.$$

So, (22) holds for all $x, y \in \mathcal{V} \setminus \{0\}$ and $z \in \mathcal{V}$. Let $z \in \mathcal{V}$ and define $g : \mathcal{V} \to \mathcal{W}$ by $g(x) = f(x,z)$. Then, g is Jensen on $\mathcal{V} \setminus \{0\}$. By Lemma 3, we get g is Jensen on $\mathcal{V}$. This means (22) holds for all $x, y, z \in \mathcal{V}$. Similarly, one can show that

$$2f\left(x,\frac{z+w}{2}\right) = f(x,z) + f(x,w), \quad x,y,z \in \mathcal{V}.$$

Therefore f is bi-Jensen on $\mathcal{V} \times \mathcal{V}$. □

Theorem 3. *Let $\mathcal{V}$ and $\mathcal{W}$ be normed spaces, and $E \subseteq \mathcal{V} \setminus \{0\}$ be a nonempty set. Take $\varepsilon \geqslant 0$ and let p, q, r, s be real numbers with $p + q < 0$ and $r + s < 0$. Assume that for each $x \in E$ there exists a positive integer m_x such that $\frac{nx}{2} \in E$ for all $n \in \mathbb{N}$ with $n \geqslant m_x$. Then, every function $f : \mathcal{V} \times \mathcal{V} \to \mathcal{W}$ satisfying the inequality*

$$\left\|\mathcal{J}f(x,y,z,w)\right\| \leqslant \varepsilon\|x\|^p\|y\|^q\|z\|^r\|w\|^s, \quad x,y,z,w \in E, \ \frac{x+y}{2}, \frac{z+w}{2} \in E, \tag{24}$$

is bi-Jensen on $E \times E$, that is

$$\mathcal{J}f(x,y,z,w) = 0, \quad x,y,z,w \in E, \ \frac{x+y}{2}, \frac{z+w}{2} \in E.$$

Proof. Without loss of generality, we may assume that $q < 0$. Let $x, y, z \in E$ with $\frac{x+y}{2} \in E$. By this assumption, there exists a positive integer m such that $\{\frac{nx}{2}, \frac{ny}{2}, \frac{n(x+y)}{4}\} \subseteq E$ for all $n \geqslant m$. Then, (24) yields

$$\left\|2f\left(\frac{x+nx}{2},z\right) - f(x,z) - f(nx,z)\right\| \leqslant \frac{\varepsilon}{2}n^q\|x\|^{p+q}\|z\|^{r+s},$$
$$\left\|2f\left(\frac{y+ny}{2},z\right) - f(y,z) - f(ny,z)\right\| \leqslant \frac{\varepsilon}{2}n^q\|y\|^{p+q}\|z\|^{r+s},$$
$$\left\|2f\left(\frac{x+y+n(x+y)}{4},z\right) - f\left(\frac{x+y}{2},z\right) - f\left(\frac{n(x+y)}{2},z\right)\right\| \leqslant \frac{\varepsilon}{2}n^q\left\|\frac{x+y}{2}\right\|^{p+q}\|z\|^{r+s}.$$

Letting $n \to \infty$ in the above inequalities, we get

$$f(x,z) = \lim_{n\to\infty}\left[2f\left(\frac{x+nx}{2},z\right) - f(nx,z)\right],$$
$$f(y,z) = \lim_{n\to\infty}\left[2f\left(\frac{y+ny}{2},z\right) - f(ny,z)\right],$$
$$f\left(\frac{x+y}{2},z\right) = \lim_{n\to\infty}\left[2f\left(\frac{x+y+n(x+y)}{4},z\right) - f\left(\frac{n(x+y)}{2},z\right)\right].$$

Then,

$$\begin{aligned}
&\left\|2f\left(\frac{x+y}{2},z\right)-f(x,z)-f(y,z)\right\| \\
&=\lim_{n\to\infty}\left\|\left[4f\left(\frac{x+y+n(x+y)}{4},z\right)-2f\left(\frac{n(x+y)}{2},z\right)\right]-\left[2f\left(\frac{x+nx}{2},z\right)-f(nx,z)\right]\right. \\
&\qquad\left.-\left[2f\left(\frac{y+ny}{2},z\right)-f(ny,z)\right]\right\| \\
&\leqslant 2\limsup_{n\to\infty}\left\|2f\left(\frac{x+y+n(x+y)}{4},z\right)-f\left(\frac{x+nx}{2},z\right)-f\left(\frac{y+ny}{2},z\right)\right\| \\
&\quad+\limsup_{n\to\infty}\left\|2f\left(\frac{n(x+y)}{2},z\right)-f(nx,z)-f(ny,z)\right\| \qquad \text{(by (24))} \\
&\leqslant \limsup_{n\to\infty}\frac{\varepsilon}{2}\left[2\left(\frac{n+1}{2}\right)^{p+q}+n^{p+q}\right]\|x\|^p\|y\|^q\|z\|^{r+s}=0.
\end{aligned}$$

Therefore, $2f\left(\frac{x+y}{2},z\right)=f(x,z)+f(y,z)$ for all $x,y,z\in E$ with $\frac{x+y}{2}\in E$. Similarly, one can show

$$2f\left(x,\frac{z+w}{2}\right)=f(x,z)+f(x,w),\quad x,z,w\in E,\quad \frac{z+w}{2}\in E.$$

This ends the proof. □

Theorem 4. *Suppose $\varepsilon\geqslant 0$ and p,q,r,s be real numbers with $p+q<0$ and $r+s<0$. Let $\mathcal{V}$ and $\mathcal{W}$ be normed linear spaces and $f:\mathcal{V}\times\mathcal{V}\to\mathcal{W}$ be a function satisfying*

$$\|\mathcal{J}f(x,y,z,w)\|\leqslant\varepsilon\|x\|^p\|y\|^q\|z\|^r\|w\|^s,\quad x,y,z,w\in\mathcal{V}\setminus\{0\}. \tag{25}$$

Then, f is bi-Jensen on $\mathcal{V}\times\mathcal{V}$.

Proof. By (25), we get

$$f(x,z)=\lim_{n\to\infty}\left[2f\left(\frac{x+nx}{2},z\right)-f(nx,z)\right],\quad x,z\in\mathcal{V}\setminus\{0\}. \tag{26}$$

It is clear that (26) is also true for $x=0$. By the same argument presented in the proof of Theorem 3, we conclude

$$2f\left(\frac{x+y}{2},z\right)=f(x,z)+f(y,z),\quad x,y,z\in\mathcal{V}\setminus\{0\}. \tag{27}$$

On the other hand, (25) yields

$$\begin{aligned}
&\|2f(x,0)-f(x,nz)-f(x,-nz)\|\leqslant\frac{\varepsilon}{2}n^{r+s}\|x\|^{p+q}\|z\|^{r+s}, \\
&\left\|4f\left(\frac{x+y}{2},0\right)-f(x,nz)-f(x,-nz)-f(y,nz)-f(y,-nz)\right\|\leqslant\varepsilon n^{r+s}\|x\|^p\|y\|^q\|z\|^{r+s},
\end{aligned}$$

for all $x,z\in\mathcal{V}\setminus\{0\}$. Letting $n\to\infty$ in the above inequalities, we get

$$\begin{aligned}
2f(x,0)&=\lim_{n\to\infty}[f(x,nz)+f(x,-nz)],\quad x\in\mathcal{V}\setminus\{0\},\ z\in\mathcal{V}, \\
2f\left(\frac{x+y}{2},0\right)&=f(x,0)+f(y,0),\quad x,y\in\mathcal{V}\setminus\{0\}.
\end{aligned}$$

So (27) implies

$$2f\left(\frac{x+y}{2},z\right)=f(x,z)+f(y,z),\quad x,y\in\mathcal{V}\setminus\{0\},\ z\in\mathcal{V}.$$

Similarly, one can show

$$2f\left(x,\frac{z+w}{2}\right)=f(x,z)+f(x,w),\quad z,w\in\mathcal{V}\setminus\{0\},\ x\in\mathcal{V}.$$

Therefore, $\mathcal{J}f(x,y,z,w)=0$ for all $x,y,z,w\in\mathcal{V}\setminus\{0\}$. By Lemma 4, we infer that f is bi-Jensen on $\mathcal{V}\times\mathcal{V}$. □

5. Hyers–Ulam Stability on Restricted Domains

In this section, the Hyers–Ulam stability of the bi-Jensen functional equation on some restricted domains is presented. We apply the obtained results to the study of an interesting asymptotic behavior of bi-Jensen functions.

Theorem 5. *Let $\mathcal{V}$ and $\mathcal{W}$ are normed linear spaces and $\varepsilon\geqslant 0, d>0$. Suppose that $f:\mathcal{V}\times\mathcal{V}\to\mathcal{W}$ is a function satisfying*

$$\|\mathcal{J}f(a_1,b_1,a_2,b_2)\|\leqslant\varepsilon \tag{28}$$

for all $a_1,a_2,b_1,b_2\in V$ with $\|a_1-b_1\|+\|a_2-b_2\|\geqslant d$. Then, f is a $\frac{11}{2}\varepsilon$-bi-Jensen function.

Proof. Let $a_1,a_2,b_1,b_2\in\mathcal{V}$ be arbitrary. Choose $x_1,x_2\in\mathcal{V}\setminus\{0\}$ such that

$$\min\{\|x_1\|,\|x_2\|,\|a_1-x_1\|,\|b_1+x_1\|,\|a_1-b_1+2x_1\|\}\geqslant d.$$

From (28), we have

$$\|\mathcal{J}f(2a_1,2x_1,2a_2,2x_2)\|\leqslant\varepsilon, \tag{29}$$
$$\|\mathcal{J}f(2b_1,-2x_1,2b_2,-2x_2)\|\leqslant\varepsilon, \tag{30}$$
$$\|\mathcal{J}f(2a_1,2x_1,2b_2,-2x_2)\|\leqslant\varepsilon, \tag{31}$$
$$\|\mathcal{J}f(2b_1,-2x_1,2a_2,2x_2)\|\leqslant\varepsilon. \tag{32}$$

Adding (29) and (30), we get

$$\begin{aligned}\Big\|&4f(a_1+x_1,a_2+x_2)+4f(b_1-x_1,b_2-x_2)-\big[f(2a_1,2a_2)+f(2a_1,2x_2)\\&+f(2x_1,2a_2)+f(2x_1,2x_2)+f(2b_1,2b_2)+f(2b_1,-2x_2)\\&+f(-2x_1,2b_2)+f(-2x_1,-2x_2)\big]\Big\|\leqslant 2\varepsilon.\end{aligned} \tag{33}$$

Adding (31) and (32), we get

$$\begin{aligned}\Big\|&4f(a_1+x_1,b_2-x_2)+4f(b_1-x_1,a_2+x_2)-\big[f(2a_1,2b_2)+f(2a_1,-2x_2)\\&+f(2x_1,2b_2)+f(2x_1,-2x_2)+f(2b_1,2a_2)+f(2b_1,2x_2)\\&+f(-2x_1,2a_2)+f(-2x_1,2x_2)\big]\Big\|\leqslant 2\varepsilon.\end{aligned} \tag{34}$$

By (28), we obtain $\|\mathcal{J}f(a_1+x_1,b_1-x_1,a_2+x_2,b_2-x_2)\|\leqslant\varepsilon$. This means

$$\begin{aligned}\bigg\|&4f\left(\frac{a_1+b_1}{2},\frac{a_2+b_2}{2}\right)-\big[f(a_1+x_1,a_2+x_2)+f(a_1+x_1,b_2-x_2)\\&+f(b_1-x_1,a_2+x_2)+f(b_1-x_1,b_2-x_2)\big]\bigg\|\leqslant\varepsilon.\end{aligned} \tag{35}$$

Multiplying (33) and (34) by $\frac{1}{4}$ and then adding the resultant inequalities to (35), we obtain

$$\begin{aligned} &\bigg\| 4f\left(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}\right) - \frac{1}{4}\Big[f(2a_1,2a_2)+f(2a_1,2x_2)+f(2x_1,2a_2) \\ &+f(2x_1,2x_2)+f(2b_1,2b_2)+f(2b_1,-2x_2)+f(-2x_1,2b_2) \\ &+f(-2x_1,-2x_2)+f(2a_1,2b_2)+f(2a_1,-2x_2)+f(2x_1,2b_2) \\ &+f(2x_1,-2x_2)+f(2b_1,2a_2)+f(2b_1,2x_2)+f(-2x_1,2a_2) \\ &+f(-2x_1,2x_2)\Big]\bigg\| \leqslant 2\varepsilon. \end{aligned} \tag{36}$$

On the other hand, by (28), we have

$$\begin{aligned} &\frac{1}{8}\mathcal{J}f(2a_1,2a_1,2x_2,-2x_2) \\ &= \bigg\| -\frac{1}{2}f(2a_1,0)+\frac{1}{4}\Big[f(2a_1,2x_2)+f(2a_1,-2x_2)\Big]\bigg\| \leqslant \frac{1}{8}\varepsilon, \end{aligned} \tag{37}$$

$$\begin{aligned} &\frac{1}{8}\mathcal{J}f(2x_1,-2x_1,2a_2,2a_2) \\ &= \bigg\| -\frac{1}{2}f(0,2a_2)+\frac{1}{4}\Big[f(2x_1,2a_2)+f(-2x_1,2a_2)\Big]\bigg\| \leqslant \frac{1}{8}\varepsilon, \end{aligned} \tag{38}$$

$$\begin{aligned} &\frac{1}{8}\mathcal{J}f(2b_1,2b_1,2x_2,-2x_2) \\ &= \bigg\| -\frac{1}{2}f(2b_1,0)+\frac{1}{4}\Big[f(2b_1,2x_2)+f(2b_1,-2x_2)\Big]\bigg\| \leqslant \frac{1}{8}\varepsilon, \end{aligned} \tag{39}$$

$$\begin{aligned} &\frac{1}{8}\mathcal{J}f(2x_1,-2x_1,2b_2,2b_2) \\ &= \bigg\| -\frac{1}{2}f(0,2b_2)+\frac{1}{4}\Big[f(2x_1,2b_2)+f(-2x_1,2b_2)\Big]\bigg\| \leqslant \frac{1}{8}\varepsilon, \end{aligned} \tag{40}$$

$$\begin{aligned} &\frac{1}{4}\mathcal{J}f(2x_1,-2x_1,2x_2,-2x_2) \\ &= \bigg\| -f(0,0)+\frac{1}{4}\Big[f(2x_1,2x_2)+f(2x_1,-2x_2)+f(-2x_1,2x_2) \\ &\quad +f(-2x_1,-2x_2)\Big]\bigg\| \leqslant \frac{1}{4}\varepsilon. \end{aligned} \tag{41}$$

Adding (36), (37), (38), (39), (40) and (41), we obtain

$$\begin{aligned} &\bigg\| 4f\left(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}\right) - \frac{1}{4}\Big[f(2a_1,2a_2)+f(2b_1,2b_2)+f(2a_1,2b_2)+f(2b_1,2a_2)\Big] \\ &-\frac{1}{2}\Big[f(2a_1,0)+f(0,2a_2)+f(2b_1,0)+f(0,2b_2)+2f(0,0)\Big]\bigg\| \leqslant \frac{11}{4}\varepsilon, \end{aligned} \tag{42}$$

for all $a_1, a_2, b_1, b_2 \in \mathcal{V}$. Replacing b_1 by a_1 and b_2 by a_2 in (42), we obtain

$$\|-4f(a_1,a_2)+f(2a_1,2a_2)+f(2a_1,0)+f(0,2a_2)+f(0,0)\| \leqslant \frac{11}{4}\varepsilon, \tag{43}$$

for all $a_1, a_2 \in V$. Then, (43) yields

$$\|-4f(b_1,b_2)+f(2b_1,2b_2)+f(2b_1,0)+f(0,2b_2)+f(0,0)\| \leqslant \frac{11}{4}\varepsilon, \tag{44}$$

$$\|-4f(b_1,a_2)+f(2b_1,2a_2)+f(2b_1,0)+f(0,2a_2)+f(0,0)\| \leqslant \frac{11}{4}\varepsilon, \tag{45}$$

$$\|-4f(a_1,b_2)+f(2a_1,2b_2)+f(2a_1,0)+f(0,2b_2)+f(0,0)\| \leqslant \frac{11}{4}\varepsilon, \tag{46}$$

for all $a_1, a_2, b_1, b_2 \in \mathcal{V}$. Multiplying (43)–(46) by $\frac{1}{4}$, and then adding the resultant inequalities to (42), one concludes

$$\left\|4f\left(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}\right) - \left[f(a_1,a_2) + f(a_1,b_2) + f(b_1,a_2) + f(b_1,b_2)\right]\right\| \leqslant \frac{11}{2}\varepsilon,$$

for all $a_1, a_2, b_1, b_2 \in \mathcal{V}$. So, f is $\frac{11}{2}\varepsilon$-bi-Jensen. □

Corollary 3. *Suppose that $f : \mathcal{V} \times \mathcal{V} \to \mathcal{W}$ is a function satisfying $\|\mathcal{J}f(a_1, b_1, a_2, b_2)\| \leqslant \varepsilon$ for all $a_1, a_2, b_1, b_2 \in \mathcal{V}$ with $\|a_1\| + \|a_2\| + \|b_1\| + \|b_2\| \geqslant d$. Then, f is $\frac{11}{2}\varepsilon$-bi-Jensen.*

Theorem 6. *Let $\mathcal{V}$ and $\mathcal{W}$ be normed linear spaces and let $\varepsilon \geqslant 0, d > 0$. Suppose that $f : \mathcal{V} \times \mathcal{V} \to \mathcal{W}$ is a function satisfying (28) for all $a_1, a_2, b_1, b_2 \in \mathcal{V}$ with $\min\{\|a_1\| + \|a_2\|, \|b_1\| + \|b_2\|\} \geqslant d$. Then f is $\frac{11}{2}\varepsilon$-bi-Jensen.*

Proof. Let $a_1, a_2, b_1, b_2 \in \mathcal{V}$ be arbitrary and let $x_1, x_2 \in \mathcal{V} \setminus \{0\}$ such that

$$\min\{\|x_1\|, \|x_2\|, \|a_1 + x_1\|, \|b_1 - x_1\|\} \geqslant d.$$

It follows from (28) that

$$\begin{aligned} \|\mathcal{J}f(2x_1, 2a_1, 2a_2, 2x_2)\| &\leqslant \varepsilon, \\ \|\mathcal{J}f(-2x_1, 2b_1, 2b_2, -2x_2)\| &\leqslant \varepsilon, \\ \|\mathcal{J}f(2x_1, 2a_1, 2b_2, -2x_2)\| &\leqslant \varepsilon, \\ \|\mathcal{J}f(-2x_1, 2b_1, 2a_2, 2x_2)\| &\leqslant \varepsilon. \end{aligned}$$

The rest of the proof is similar to the proof of Theorem 5. □

Theorem 7. *Let $\mathcal{V}$ be a linear normed space and $\mathcal{W}$ be a Banach space. Take $\varepsilon \geqslant 0$ and $d > 0$. Suppose that $f : \mathcal{V} \times \mathcal{V} \to \mathcal{W}$ is a function satisfying one of the following conditions:*

(i) $\|\mathcal{J}f(a_1, b_1, a_2, b_2)\| \leqslant \varepsilon, \quad \|a_1 - b_1\| + \|a_2 - b_2)\| \geqslant d;$
(ii) $\|\mathcal{J}f(a_1, b_1, a_2, b_2)\| \leqslant \varepsilon, \quad \|a_1\| + \|a_2\| + \|b_1\| + \|b_2\| \geqslant d;$
(iii) $\|\mathcal{J}f(a_1, b_1, a_2, b_2)\| \leqslant \varepsilon, \quad \min\{\|a_1\| + \|a_2\|, \|b_1\| + \|b_2\|\} \geqslant d.$

Then there exists a unique bi-Jensen function $g : \mathcal{V} \times \mathcal{V} \to \mathcal{W}$ such that

$$\|f(x,y) - g(x,y)\| \leqslant 11\varepsilon, \quad x, y \in \mathcal{V}.$$

Proof. By Theorems 5 and 6, we infer that f is $\frac{11}{2}\varepsilon$-bi-Jensen function. Then, by Theorem 2, we get the desired result. □

Corollary 4. *Let $\mathcal{V}$ and $\mathcal{W}$ be normed linear spaces. Take $\varepsilon \geqslant 0$ and suppose that $f : \mathcal{V} \times \mathcal{V} \to \mathcal{W}$ and $\varphi : \mathcal{V}^4 \to \mathcal{W}$ are functions such that $\|\varphi(a_1, b_1, a_2, b_2)\| \leqslant \varepsilon$ for all $a_1, a_2, b_1, b_2 \in \mathcal{V}$. Then, f is $\frac{11}{2}\varepsilon$-bi-Jensen function if one of the following conditions holds:*

(i) $\lim\limits_{\|a_1 - b_1\| + \|a_2 - b_2\| \to \infty} \|\mathcal{J}f(a_1, b_1, a_2, b_2) - \varphi(a_1, b_1, a_2, b_2)\| = 0;$
(ii) $\lim\limits_{\|a_1\| + \|a_2\| + \|b_1\| + \|b_2\| \to \infty} \|\mathcal{J}f(a_1, b_1, a_2, b_2) - \varphi(a_1, b_1, a_2, b_2)\| = 0;$
(iii) $\lim\limits_{\min\{\|a_1\| + \|a_2\|, \|b_1\| + \|b_2\|\} \to \infty} \|\mathcal{J}f(a_1, b_1, a_2, b_2) - \varphi(a_1, b_1, a_2, b_2)\| = 0.$

Corollary 5. *Let $\mathcal{V}$ and $\mathcal{W}$ be normed linear spaces. A function $f : \mathcal{V} \times \mathcal{V} \to \mathcal{W}$ is bi-Jensen if one of the following conditions holds:*

(i) $\lim\limits_{\|a_1 - b_1\| + \|a_2 - b_2\| \to \infty} \|\mathcal{J}f(a_1, b_1, a_2, b_2)\| = 0;$
(ii) $\lim\limits_{\|a_1\| + \|a_2\| + \|b_1\| + \|b_2\| \to \infty} \|\mathcal{J}f(a_1, b_1, a_2, b_2)\| = 0;$

$$(iii) \quad \lim_{\min\{\|a_1\|+\|a_2\|,\|b_1\|+\|b_2\|\}\to\infty} \|\mathcal{J}f(a_1, b_1, a_2, b_2)\| = 0.$$

6. Conclusions

We studied some properties of the bi-Jensen functional equation

$$4f\left(\frac{x+y}{2}, \frac{z+w}{2}\right) = f(x,z) + f(x,w) + f(y,z) + f(y,w),$$

and obtained the form of continuous bi-Jensen functions $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. We constructed a function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, which is bi-Jensen and is not continuous. The Hyers–Ulam stability and hyperstability of the bi-Jensen functional equation have been investigated. Additionally, we investigated the Hyers–Ulam stability of the bi-Jensen functional equation on some restricted unbounded domains and used the obtained results to study some of interesting asymptotic behaviors of bi-Jensen functions.

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