Article

Exponential Synchronization of Hyperbolic Complex Spatio-Temporal Networks with Multi-Weights

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Abstract: This paper deals with the leader-following synchronization of first-order, semi-linear, complex spatio-temporal networks. Firstly, two sorts of complex spatio-temporal networks based on hyperbolic partial differential equations (CSTNHPDEs) are built: one with a single weight and the other with multi-weights. Then, a new distributed controller is designed to address CSTNHPDE with a single weight. Sufficient conditions for the synchronization and exponential synchronization of CSTNHPDE are presented by showing the gain ranges. Thirdly, the proposed distributed controller addresses of CSTNHPDE with multi-weights, and gain ranges are obtained for synchronization and exponential synchronization, respectively. Finally, two examples show the effectiveness and good performance of the control methods.

Keywords: synchronization; complex networks; multi-weights; hyperbolic partial differential equations

MSC: 05C82

1. Introduction

The synchronization of complex networks, a group dynamical behavior, aims to drive nodes to perform a designated task synchronously. It has been applied to many engineering aspects, such as intelligent traffic [1,2], circuit systems [3], image processing [4–6], smart grids [7], secure communication [8,9], multi-agent systems [10], rumor propagation [11], data security [12], biological systems [13], etc.

A number of important works discuss the synchronization of complex networks [14–18]. This literature shows node dynamics depending only on time. In practice, the dynamics of all processes are spatio-temporal [19–21]. As a consequence, it is necessary to study complex spatio-temporal networks (CSTNs), which is with spatio-temporal characteristics [22]. Wu et al. studied the synchronization of CSTNs with space-independent coefficients and space-dependent coefficients, with or without spatio-temporal disturbance [23]. Huang et al. proposed a fuzzy synchronization method for nonlinear CSTNs with reaction—diffusion terms [24]. Luo et al. studied event-triggered control for the finite-time synchronization of reaction–diffusion CSTNs [25]. Yang et al. studied the boundary control of fractional-order CSTNs [26]. Zheng et al. researched synchronization analysis for fractional-order CSTNs with time delays [27]. Shen et al. studied the H_∞ synchronization of Markov jump CSTNs using an observer-based method [28]. Kabalan et al. studied boundary control for the synchronization of leader–follower CSTNs with in-domain coupling [29].

Most references are modeled by parabolic PDEs, while there are few methods studying hyperbolic PDEs. There are many hyperbolic PDEs systems in practice, including shallow-water systems [30], epidemic models [31], district heating networks [32], heat exchangers [33], and reactor models [34]. Therefore, it is important to study the synchronization of hyperbolic PDEs-based CSTNs (HPDECSTNs).

Chueshov presented invariant manifolds and nonlinear master–slave synchronization hyperbolic and parabolic CSTNs [35]. Li studied the synchronization, exact synchronization and approximate synchronization of HPDECSTNs [36]. Li and Lu researched exact-boundary synchronization for a kind of first-order hyperbolic system [37]. Lu proposed a local exact-boundary synchronization for a kind of first-order, quasi-linear hyperbolic system [38]. However, technical difficulties remain regarding the synchronization of a semi-linear, first-order HPDECSTNs when the convection coefficient is symmetric semi-negative definite or semi-positive definite, which motivate this paper. Multi-weights exist in many physical networks [39–43]. As a result, HPDECSTN with multi-weights is important and remains challenging.

This paper mainly studies the leader-following synchronization control of a semi-linear HPDECSTN with two sorts of boundary conditions in a one-dimensional space. This paper's contributions are as follows: (1) Two sorts of HPDECSTN models are built, one with a single weight and the other with multi-weights. (2) A new distributed controller is designed to address CSTNHPDE with a single weight. Sufficient conditions for the synchronization and exponential synchronization of CSTNHPDE are presented by providing the gain ranges. (3) The proposed distributed controller addresses CSTNHPDE with multi-weights and gain ranges, obtained for synchronization and exponential synchronization, respectively. (4) Two examples show the effectiveness and good performance of the control methods.

Notations: Let I_N denote the identity matrix with Nth order, $P > 0$ ($P < 0$) denote symmetric positive definite (negative definite), and $\lambda_{\max(\min)}(\cdot)$ denote the maximum (minimum) eigenvalue.

2. Problem Formulation

This paper first studies a class of leader-following, semi-linear, hyperbolic PDE-based, complex spatio-temporal networks (HPDECSTNs) with a single weight. The following node is assumed to be

$$
\begin{cases}
\dfrac{\partial z_i(\zeta,t)}{\partial t} = \dfrac{\partial z_i(\zeta,t)}{\partial \zeta} + Az_i(\zeta,t) + Bf(z_i(\zeta,t)) + c\sum_{j=1}^{N} g_{ij}\Gamma z_j(\zeta,t) + u_i(\zeta,t), \\
z_i(L,t) = 0, \\
z_i(\zeta,0) = z_i^0(\zeta), i \in \{1,2,\cdots,N\},
\end{cases}
\tag{1}
$$

where $(\zeta,t) \in [0,L] \times [0,\infty)$ are space and time, respectively. $z_i(\zeta,t)$ and $u_i(\zeta,t) \in \mathbb{R}^n$ are the state and control input, respectively. $0 < L \in \mathbb{R}$ is a constant. $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n}$, and $\Gamma \in \mathbb{R}^{n\times n}$ are constant matrices. $f(\cdot)$ is a nonlinear function. The coupling strength $c > 0$ is a constant. $G = (g_{ij})_{N\times N}$ satisfies $g_{ii} = -\sum_{j=1,j\neq i}^{N} g_{ij}$.

The leader node is assumed to be

$$
\begin{cases}
\dfrac{\partial s(\zeta,t)}{\partial t} = \dfrac{\partial s(\zeta,t)}{\partial \zeta} + As(\zeta,t) + Bf(s(\zeta,t)), \\
s(L,t) = 0, \\
s(\zeta,0) = s^0(\zeta),
\end{cases}
\tag{2}
$$

where $s(\zeta,t) \in \mathbb{R}^n$ is the state.

This paper aims to study a distributed controller $u_i(\zeta,t)$, driving HPDECSTN (1) to the leader node (2), designed as

$$
u_i(\zeta,t) = d_i(s(\zeta,t) - z_i(\zeta,t)),
\tag{3}
$$

where d_i are the control gains to be determined.

Definition 1. *HPDECSTN (1) reaches synchronization, if*

$$\lim_{t \to \infty} ||z_i(\zeta, t) - s(\zeta, t)|| = 0, i \in \{1, 2, \cdots, N\}. \tag{4}$$

Definition 2. *Given $\rho > 0$, HPDECSTN (1) reaches exponential synchronization, if there is a real number $\sigma > 0$ such that*

$$||z_i(\zeta, t) - s(\zeta, t)|| \leq \sigma \exp(-2\rho t) ||z_i^0(\zeta) - s^0(\zeta)||, i \in \{1, 2, \cdots, N\}. \tag{5}$$

Assumption 1. *For any $\zeta_1, \zeta_2 \in \mathbb{R}$, then $0 < \mathcal{X} \in \mathbb{R}$, satisfying*

$$|f(\zeta_1) - f(\zeta_2)| \leq \mathcal{X} |\zeta_1 - \zeta_2|. \tag{6}$$

3. Synchronization of HPDECSTNs with a Single Weight

Let the synchronization error be $e_i(\zeta, t) \triangleq z_i(\zeta, t) - s(\zeta, t)$. The error system of between HPDECSTN (1) and (2)) yields

$$
\begin{cases}
\dfrac{\partial e(\zeta, t)}{\partial t} = \dfrac{\partial e(\zeta, t)}{\partial \zeta} + (I_N \otimes A)e(\zeta, t) + (I_N \otimes B)F(e(\zeta, t)) + (G \otimes \Gamma)e(\zeta, t) + u(\zeta, t), \\
e(L, t) = 0, \\
e(\zeta, 0) = e^0(\zeta),
\end{cases} \tag{7}
$$

where $e_i^0(\zeta) \triangleq z_i^0(\zeta) - s^0(\zeta)$, $u(t) \triangleq [u_1^T(t), u_2^T(t), \cdots, u_N^T(t)]^T$, $e(\zeta, t) \triangleq [e_1^T(\zeta, t), e_2^T(\zeta, t), \cdots, e_N^T(\zeta, t)]^T$, $F(e_i(\zeta, t)) \triangleq f(z_i(\zeta, t)) - f(s(\zeta, t))$, and $F(e(\zeta, t)) \triangleq [F^T(e_1(\zeta, t)), F^T(e_2(\zeta, t)), \cdots, F^T(e_N(\zeta, t))]^T$.

Theorem 1. *Suppose Assumption 1 holds. HPDECSTN (1) reaches synchronization under the controller (2), if*

$$d_i > \lambda_{\max}(\Psi), \tag{8}$$

where $\Psi \triangleq I_N \otimes \dfrac{A + A^T}{2} + 0.5 I_N \otimes BB^T + 0.5\chi^2 I_{Nn} + 0.5c(G \otimes \Gamma + G^T \otimes \Gamma^T)$.

Proof. Choose the Lyapunov functional candidate as follows:

$$V(t) = 0.5 \int_0^L e^T(\zeta, t) e(\zeta, t) d\zeta. \tag{9}$$

One has

$$
\begin{aligned}
\dot{V}(t) &= \int_0^L e^T(\zeta, t) \frac{\partial e(\zeta, t)}{\partial t} d\zeta \\
&= \int_0^L e^T(\zeta, t) \frac{\partial e(\zeta, t)}{\partial \zeta} d\zeta \\
&\quad + \int_0^L e^T(\zeta, t)(I_N \otimes A + cG \otimes \Gamma)e(\zeta, t)d\zeta \\
&\quad + \int_0^L e^T(\zeta, t)F(e(\zeta, t))d\zeta - \int_0^L e^T(\zeta, t)(D \otimes I_n)e(\zeta, t)d\zeta,
\end{aligned} \tag{10}
$$

where $D \triangleq diag\{d_1, d_2, \cdots, d_N\}$.

By integrating by parts,

$$
\int_0^L e^T(\zeta,t)\frac{\partial e(\zeta,t)}{\partial\zeta}d\zeta
$$
$$
= e^T(\zeta,t)e(\zeta,t)\big|_{\zeta=0}^{\zeta=L}
$$
$$
-\int_0^L \frac{\partial e^T(\zeta,t)}{\partial\zeta}e(\zeta,t)
$$
$$
= -e^T(0,t)e(0,t)
$$
$$
-\int_0^L e^T(\zeta,t)\frac{\partial e(\zeta,t)}{\partial\zeta}d\zeta
$$
$$
\leq -\int_0^L e^T(\zeta,t)\frac{\partial e(\zeta,t)}{\partial\zeta}d\zeta,
$$

(11)

which implies

$$
\int_0^L e^T(\zeta,t)\frac{\partial e(\zeta,t)}{\partial\zeta}d\zeta \leq -0.5e^T(0,t)e(0,t).
$$

(12)

Under Assumption 1,

$$
\int_0^L e^T(\zeta,t)BF(e(\zeta,t))d\zeta
$$
$$
\leq 0.5\int_0^L e^T(\zeta,t)BB^T e(\zeta,t)d\zeta + 0.5\int_0^L F^T(\zeta,t)F(\zeta,t)d\zeta
$$
$$
= \int_0^L e^T(\zeta,t)(0.5I_N \otimes BB^T + 0.5\chi^2 I_{Nn})e(\zeta,t)d\zeta.
$$

(13)

The substitution of (11)–(13) into (10) yields,

$$
\dot{V}(t) \leqslant \int_0^L e^T(\zeta,t)(\Psi - D \otimes I_n)e(\zeta,t)d\zeta,
$$

(14)

where $\Psi \triangleq I_N \otimes \frac{A+A^T}{2} + 0.5I_N \otimes BB^T + 0.5\chi^2 I_{Nn} + cG \otimes \Gamma$ and $D = diag\{d_1,d_2,\cdots,d_N\}$. It is obvious that (8) implies

$$
\Psi - D \otimes I_n < 0.
$$

(15)

The substitution of (15) into (14) yields, $\dot{V}(t) \leqslant -\lambda_{\min}(D \otimes I_n - \Psi)||e(\cdot,t)||$, for all non-zero $e(\zeta,t)$, implying synchronization of HPDECSTN (1). $\square$

Theorem 2. *Suppose Assumption 1 holds. Given $\rho > 0$, HPDECSTN (1) reaches exponential synchronization under the controller (2), if*

$$
d_i > \lambda_{\max}(\Psi + \rho I_{Nn}),
$$

(16)

where $\Psi \triangleq I_N \otimes \frac{A+A^T}{2} + 0.5I_N \otimes BB^T + 0.5\chi^2 I_{Nn} + 0.5c(G \otimes \Gamma + G^T \otimes \Gamma^T)$.

Proof.

$$
\dot{V}(t) + 2\rho V(t)
$$
$$
\leq \int_0^L e^T(\zeta,t)(\Psi + \rho I_{Nn} - D \otimes I_n)e(\zeta,t)d\zeta
$$
$$
\leq 0,
$$

(17)

which implies

$$
V(t) \leq V(0)\exp(-2\rho t).
$$

(18)

It follows from (18) that

$$||e_i(\zeta, t)||_2^2 \leq \sigma \exp(-2\rho t),\tag{19}$$

where $\sigma = ||e_i^0(\zeta)||_2^2$. Therefore, exponential synchronization is obtained. □

4. Synchronization of HPDECSTNs with Multi-Weights

This section studies a class of semi-linear HPDECSTNs with multi-weights, where the following node is as follows:

$$\begin{cases} \dfrac{\partial z_i(\zeta, t)}{\partial t} = \dfrac{\partial z_i(\zeta, t)}{\partial \zeta} + A z_i(\zeta, t) + B f(z_i(\zeta, t)) + c_1 \sum\limits_{j=1}^{N} g_{ij}^1 \Gamma_1 z_j(\zeta, t) \\ \qquad\qquad + c_2 \sum\limits_{j=1}^{N} g_{ij}^2 \Gamma_2 z_j(\zeta, t) + \cdots + c_l \sum\limits_{j=1}^{N} g_{ij}^l \Gamma_l z_j(\zeta, t) + u_i(\zeta, t), \\ z_i(0, t) = 0, \\ z_i(\zeta, t) = z_i^0(\zeta, t), \end{cases}\tag{20}$$

where $\Gamma_1 \in \mathbb{R}^{n \times n}$, $\Gamma_2 \in \mathbb{R}^{n \times n}, \cdots, \Gamma_l \in \mathbb{R}^{n \times n}$ are constant matrices. $G_k = (g_{ij}^k)_{N \times N}$ satisfies $g_{ii}^k = -\sum\limits_{j=1, j \neq i}^{N} g_{ij}^k$.

The error system of between HPDECSTN (20) and (2) with multi-weights can be obtained as

$$\begin{cases} \dfrac{\partial e(\zeta, t)}{\partial t} = \Theta \dfrac{\partial e(\zeta, t)}{\partial \zeta} + (I_N \otimes A) e(\zeta, t) + F(e(\zeta, t)) + c_1 (G_1 \otimes \Gamma_1) e(\zeta, t) \\ \qquad\qquad + c_2 (G_2 \otimes \Gamma_2) e(\zeta, t) + \cdots + c_l (G_l \otimes \Gamma_l) e(\zeta, t) + u_i(\zeta, t), \\ \dfrac{\partial e(0, t)}{\partial \zeta} = 0, \\ e(\zeta, 0) = e^0(\zeta). \end{cases}\tag{21}$$

Theorem 3. *Suppose that Assumption 1 holds. HPDECSTN (20) reaches synchronization under the controller (2), if*

$$d_i > \lambda_{\max}(\Xi),\tag{22}$$

where $\Xi \triangleq I_N \otimes \frac{A + A^T}{2} + 0.5 I_N \otimes BB^T + 0.5 \chi^2 I_{Nn} + 0.5 c_1 (G_1 \otimes \Gamma_1 + G_1^T \otimes \Gamma_1^T) + 0.5 c_2 (G_2 \otimes \Gamma_2 + G_2^T \otimes \Gamma_2^T) + \cdots + 0.5 c_l (G_l \otimes \Gamma_l + G_l^T \otimes \Gamma_l^T).$

Proof. The proof is similar to that of Theorem 1, and so it is omitted. □

Theorem 4. *Suppose that Assumption 1 holds. Given $\rho > 0$, HPDECSTN (20) reaches exponential synchronization under the controller (2), if*

$$d_i > \lambda_{\max}(\Xi + \rho I_{Nn}),\tag{23}$$

where $\Xi \triangleq I_N \otimes \frac{A + A^T}{2} + 0.5 I_N \otimes BB^T + 0.5 \chi^2 I_{Nn} + 0.5 c_1 (G_1 \otimes \Gamma_1 + G_1^T \otimes \Gamma_1^T) + 0.5 c_2 (G_2 \otimes \Gamma_2 + G_2^T \otimes \Gamma_2^T) + \cdots + 0.5 c_l (G_l \otimes \Gamma_l + G_l^T \otimes \Gamma_l^T).$

Proof. The proof is similar to that of Theorem 2, and so it is omitted. □

Remark 1. *This paper addresses not only the synchronization of HPDECSTNs, but also the exponential synchronization. Moreover, this paper addresses HPDECSTNs not only with a single weight, but also with multi-weights.*

Remark 2. *Compared with the results modeled by ordinary differential equations with multi-weights [39–43], this paper addresses spatio-temporal models with multi-weights.*

Remark 3. *Different from the control design for synchronization of parabolic PDEs-based CSTNs [44,45], this paper deals with the synchronization of hyperbolic PDEs-based CSTNs.*

Remark 4. *Only a few important results discussed the synchronization, exact synchronization and approximate synchronization of HPDECSTNs [36–38]. Different from those with a single weight, this paper addresses the case with multi-weights.*

5. Numerical Simulation

Example 1. *Consider a single weighted HPDECSTN (1) with random initial conditions and*

$$A = \begin{bmatrix} 5.1 & 2.7 \\ -1.1 & 4.2 \end{bmatrix}, B = \begin{bmatrix} 0.5 & -0.2 \\ 0.2 & 1.5 \end{bmatrix}, \Gamma = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}, L = 1, c = 0.2, f(\cdot) = tanh(\cdot). \tag{24}$$

The single weight takes

$$G = \begin{bmatrix} -5 & 1 & 2 & 2 \\ -1 & 4 & 3 & 0 \\ 1 & 1 & -3 & 1 \\ -3 & -2 & -3 & 8 \end{bmatrix}. \tag{25}$$

Figure 1 shows that HPDECSTN (1) cannot reach synchronization without control. It is obvious that $\chi = 1$. With Theorem 1, solve (16) by Matlab, the feedback gains $d_i = 12.04$ are obtained. Figure 2 shows that HPDECSTN (1) reaches exponential synchronization under the controller (2) with $d_i = 12.04$. The controller (2) with the feedback gains $d_i = 12.04$ is shown in Figure 3.
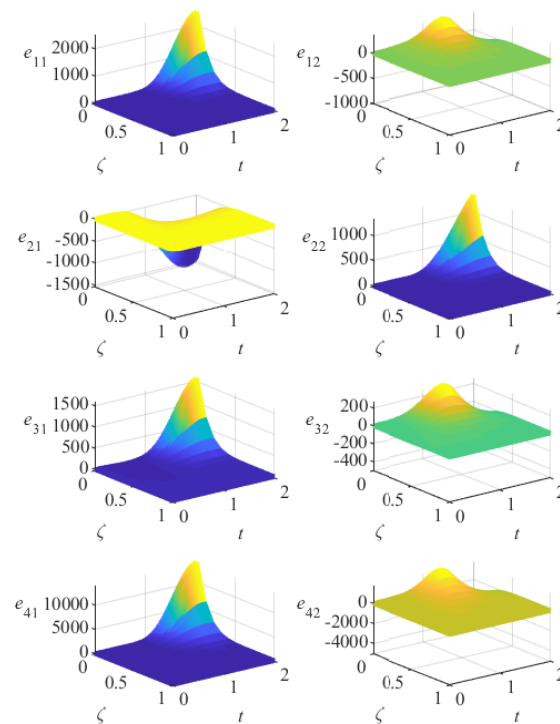

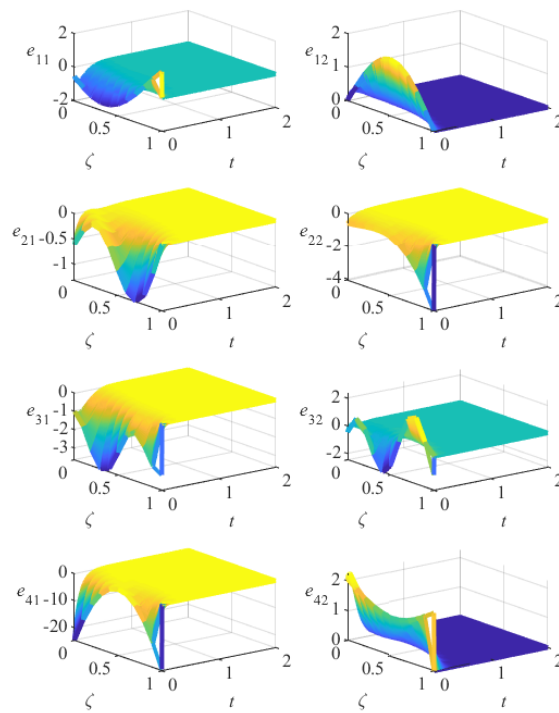**Figure 1.** $e(\zeta, t)$ of HPDECSTN (1) without control.

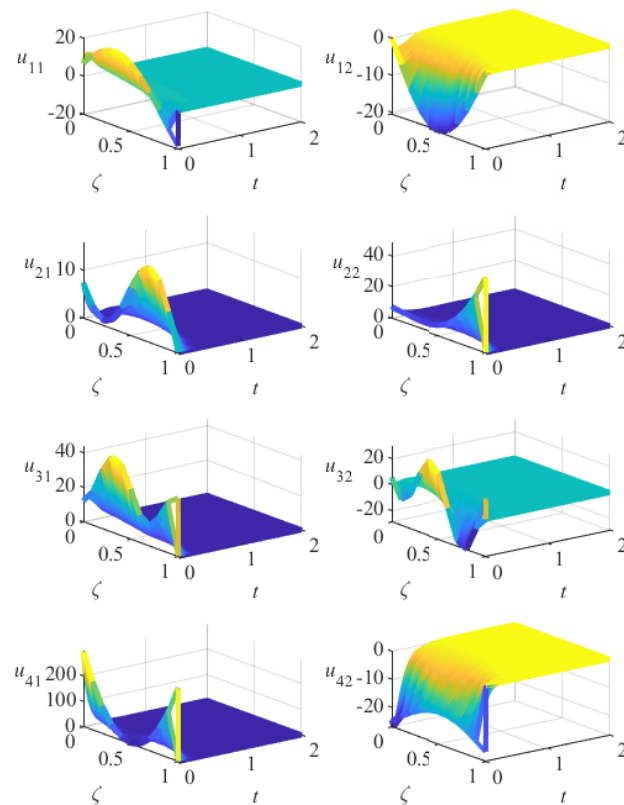**Figure 2.** $e(\zeta, t)$ of HPDECSTN (1) with control.

Figure 3. The control input of HPDECSTN (1).

Example 2. *Consider multi-weighted HPDECSTN (20) with random initial conditions and the same parameters as those of Example 1, except:*

$$c_1 = 0.8, c_2 = 0.3, c_3 = 0.4, c_4 = 0.5 \tag{26}$$

The weights take

$$G_1 = \begin{bmatrix} -5 & 1 & 2 & 2 \\ -1 & 4 & 3 & 0 \\ 1 & 1 & -3 & 1 \\ -3 & -2 & -3 & 8 \end{bmatrix}, G_2 = \begin{bmatrix} 6 & -1 & -2 & -3 \\ -2 & 4 & -3 & 1 \\ 1 & 2 & -3 & 0 \\ -1 & -3 & -3 & 7 \end{bmatrix}, \tag{27}$$

$$G_3 = \begin{bmatrix} -2 & 4 & 1 & -3 \\ 2 & -1 & 3 & -4 \\ -2 & -1 & -2 & 5 \\ 6 & 2 & -3 & -5 \end{bmatrix}, G_4 = \begin{bmatrix} -5 & 1 & 2 & 2 \\ 1 & 3 & -2 & -2 \\ -7 & -2 & 3 & 6 \\ -3 & 1 & -3 & 5 \end{bmatrix}. \tag{28}$$

Figure 4 shows that HPDECSTN (20) cannot reach synchronization without control. With Theorem 4, solving (23) using Matlab, the feedback gains $d_i = 26.21$ are obtained. Figure 5 shows that HPDECSTN (20) reaches exponential synchronization under controller (2) with $d_i = 26.21$. The controller (2) with the feedback gains $d_i = 26.21$ is shown in Figure 6.
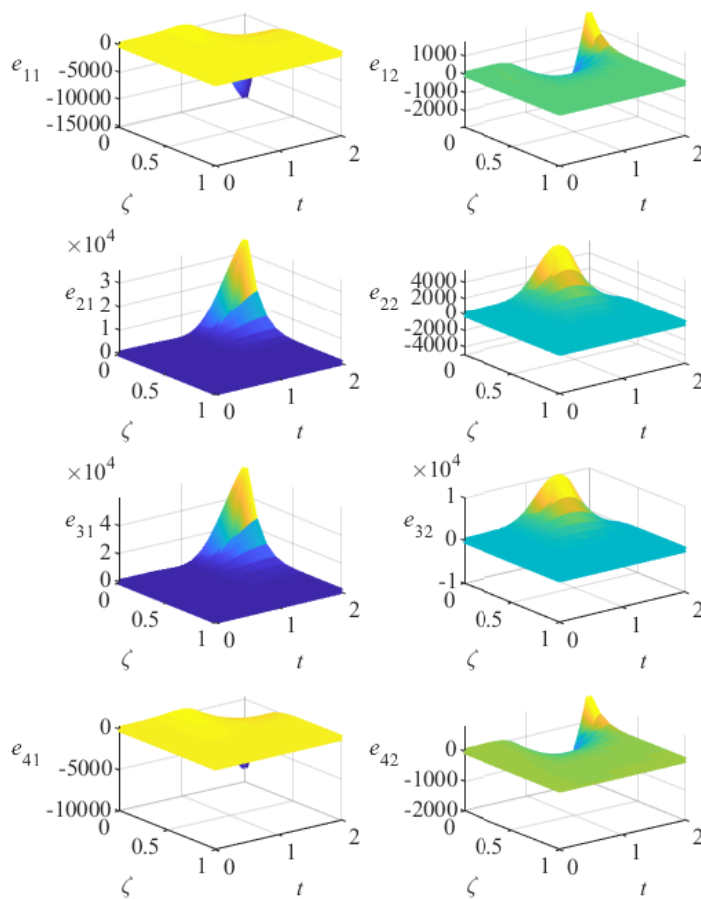

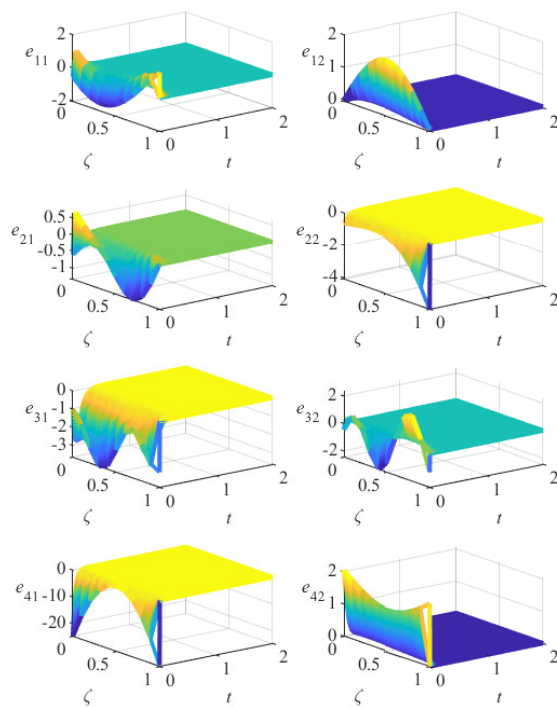**Figure 4.** $e(\zeta, t)$ of HPDECSTN (1) without control.

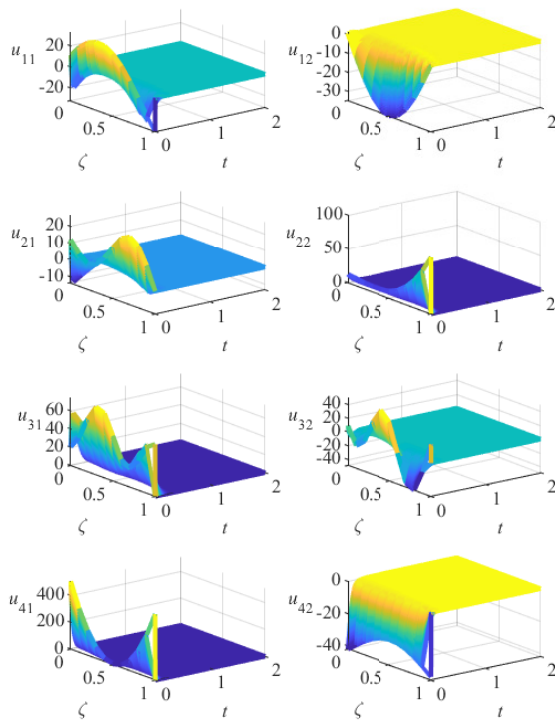**Figure 5.** $e(\zeta, t)$ of HPDECSTN (1) with control.

Figure 6. The control input of HPDECSTN (1).

6. Conclusions

This paper has dealt with the leader-following synchronization control of two classes of semi-linear HPDECSTNs: one HPDECSTN with a single weight, and the other with multi-weights. To drive HPDECSTNs to synchronization, one new distributed controller was constructed. Dealing with HPDECSTNs with a single weight, sufficient conditions for synchronization and exponential synchronization of CSTNHPDE were presented by providing gain ranges. Furthermore, the proposed distributed controller was used to address

CSTNHPDE with multi-weights and gain ranges, which were obtained for synchronization and exponential synchronization, respectively. Two examples illustrated the effectiveness of the developed theoretical results. In future work, the event-triggered control and pinning control of HPDECSTNs will be studied.

References

1. Wang, D.; Che, W.W.; Yu, H.; Li, J.Y. Adaptive pinning synchronization of complex networks with negative weights and its application in traffic road network. *Int. J. Control. Autom. Syst.* **2018**, *16*, 782–790. [CrossRef]
2. Rodriguez, M.; Fathy, H.K. Distributed Kuramoto self-synchronization of vehicle speed trajectories in traffic networks. *IEEE Trans. Intell. Transp. Syst.* **2021**, *23*, 6786–6796 . [CrossRef]
3. Xu, Y.; Sun, J.; Wang, G.; Wu, Z.G. Dynamic triggering mechanisms for distributed adaptive synchronization control and its application to circuit systems. *IEEE Trans. Circuits Syst. I Regul. Pap.* **2021**, *68*, 2246–2256. [CrossRef]
4. Sheng, S.; Zhang, X.; Lu, G. Finite-time outer-synchronization for complex networks with Markov jump topology via hybrid control and its application to image encryption. *J. Frankl. Inst.* **2018**, *355*, 6493–6519. [CrossRef]
5. Moon, S.; Baik, J.J.; Seo, J.M. Chaos synchronization in generalized Lorenz systems and an application to image encryption. *Commun. Nonlinear Sci. Numer. Simul.* **2021**, *96*, 105708. [CrossRef]
6. Tan, X.; Xiang, C.; Cao, J.; Xu, W.; Wen, G.; Rutkowski, L. Synchronization of neural networks via periodic self-triggered impulsive control and its application in image encryption. *IEEE Trans. Cybern.* **2021**. [CrossRef]
7. Chowdhury, D.D. Synchronization for smart grid infrastructure. In *Next Gen Network Synchronization*; Springer: Berlin/Heidelberg, Germany, 2021; pp. 181–207.
8. Behinfaraz, R.; Ghaemi, S.; Khanmohammadi, S.; Badamchizadeh, M.A. Time-varying parameters identification and synchronization of switching complex networks using the adaptive fuzzy-impulsive control with an application to secure communication. *Asian J. Control* **2022**, *24*, 377–387. [CrossRef]
9. Yang, T.; Chua, L.O. Impulsive control and synchronization of nonlinear dynamical systems and application to secure communication. *Int. J. Bifurc. Chaos* **1997**, *7*, 645–664. [CrossRef]
10. Tan, X.; Cao, M.; Cao, J.; Xu, W.; Luo, Y. Event-triggered synchronization of multiagent systems with partial input saturation. *IEEE Trans. Control Netw. Syst.* **2021**, *8*, 1406–1416. [CrossRef]
11. Zhu, L.; Zhao, H.; Wang, H. Partial differential equation modeling of rumor propagation in complex networks with higher order of organization. *Chaos Interdiscip. J. Nonlinear Sci.* **2019**, *29*, 053106. [CrossRef]
12. Li, J.F.; Jahanshahi, H.; Kacar, S.; Chu, Y.M.; Gómez-Aguilar, J.; Alotaibi, N.D.; Alharbi, K.H. On the variable-order fractional memristor oscillator: Data security applications and synchronization using a type-2 fuzzy disturbance observer-based robust control. *Chaos Solitons Fractals* **2021**, *145*, 110681. [CrossRef]
13. Rosenblum, M.G.; Pikovsky, A.S.; Kurths, J. Synchronization approach to analysis of biological systems. In *The Random and Fluctuating World: Celebrating Two Decades of Fluctuation and Noise Letters*; World Scientific: Singapore, 2022; pp. 335–344.
14. Yang, S.; Hu, C.; Yu, J.; Jiang, H. Finite-time cluster synchronization in complex-variable networks with fractional-order and nonlinear coupling. *Neural Netw.* **2021**, *135*, 212–224. [CrossRef] [PubMed]
15. Ji, X.; Lu, J.; Jiang, B.; Zhong, J. Network synchronization under distributed delayed impulsive control: Average delayed impulsive weight approach. *Nonlinear Anal. Hybrid Syst.* **2022**, *44*, 101148. [CrossRef]
16. Chai, L.; Liu, J.; Chen, G.; Zhao, X. Dynamics and synchronization of a complex-valued star network. *Sci. China Technol. Sci.* **2021**, *64*, 2729–2743. [CrossRef]
17. Wang, Z.; Jin, X.; Pan, L.; Feng, Y.; Cao, J. Quasi-synchronization of delayed stochastic multiplex networks via impulsive pinning control. *IEEE Trans. Syst. Man Cybern. Syst.* **2021**. [CrossRef]
18. Ding, X.; Cao, J.; Alsaadi, F.E. Pinning synchronization of fractional-order complex networks with adaptive coupling weights. *Int. J. Adapt. Control Signal Process.* **2019**, *33*, 1478–1490. [CrossRef]

19. Feng, Y.; Wang, Y.; Wang, J.W.; Li, H.X. Backstepping-based distributed abnormality localization for linear parabolic distributed parameter systems. *Automatica* **2022**, *135*, 109930. [CrossRef]
20. Wang, J.W.; Liu, Y.Q.; Sun, C.Y. Pointwise exponential stabilization of a linear parabolic PDE system using non-collocated pointwise observation. *Automatica* **2018**, *93*, 197–210. [CrossRef]
21. Wang, J.W.; Wang, J.M. Mixed H_2/H_∞ sampled-data output feedback control design for a semi-linear parabolic PDE in the sense of spatial L_∞ norm. *Automatica* **2019**, *103*, 282–293. [CrossRef]
22. Ferrari-Trecate, G.; Buffa, A.; Gati, M. Analysis of coordination in multi-agent systems through partial difference equations. *IEEE Trans. Autom. Control* **2006**, *51*, 1058–1063. [CrossRef]
23. Wu, K.; Chen, B.S. Synchronization of partial differential systems via diffusion coupling. *IEEE Trans. Circuits Syst. I Regul. Pap.* **2012**, *59*, 2655–2668. [CrossRef]
24. Huang, C.; Zhang, X.; Lam, H.K.; Tsai, S.H. Synchronization analysis for nonlinear complex networks with reaction-diffusion terms using fuzzy-model-based approach. *IEEE Trans. Fuzzy Syst.* **2020**, *29*, 1350–1362. [CrossRef]
25. Luo, Y.; Yao, Y.; Cheng, Z.; Xiao, X.; Liu, H. Event-triggered control for coupled reaction–diffusion complex network systems with finite-time synchronization. *Phys. A Stat. Mech. Its Appl.* **2021**, *562*, 125219. [CrossRef]
26. Yang, Y.; Hu, C.; Yu, J.; Jiang, H.; Wen, S. Synchronization of fractional-order spatiotemporal complex networks with boundary communication. *Neurocomputing* **2021**, *450*, 197–207. [CrossRef]
27. Zheng, B.; Hu, C.; Yu, J.; Jiang, H. Synchronization analysis for delayed spatio-temporal neural networks with fractional-order. *Neurocomputing* **2021**, *441*, 226–236. [CrossRef]
28. Shen, H.; Wang, X.; Wang, J.; Cao, J.; Rutkowski, L. Robust composite H_∞ synchronization of Markov jump reaction-diffusion neural networks via a disturbance observer-based method. *IEEE Trans. Cybern.* **2021**. [CrossRef]
29. Kabalan, A.; Ferrante, F.; Casadei, G.; Cristofaro, A.; Prieur, C. Leader-follower synchronization of a network of boundary-controlled parabolic equations with in-domain coupling. *IEEE Control Syst. Lett.* **2021**, *6*, 2006–2011. [CrossRef]
30. Schneider, K.A.; Gallardo, J.M.; Escalante, C. Efficient GPU implementation of multidimensional incomplete Riemann solvers for hyperbolic nonconservative systems: Applications to shallow water systems with topography and dry areas. *J. Sci. Comput.* **2022**, *92*, 1–42. [CrossRef]
31. Kitsos, C.; Besancon, G.; Prieur, C. High-gain observer design for a class of quasi-linear integro-differential hyperbolic systems: Application to an epidemic model. *IEEE Trans. Autom. Control* **2021**, *67*, 292–303. [CrossRef]
32. Rein, M.; Mohring, J.; Damm, T.; Klar, A. Model order reduction of hyperbolic systems focusing on district heating networks. *J. Frankl. Inst.* **2021**, *358*, 7674–7697. [CrossRef]
33. Dostál, J.; Havlena, V. Mixed mesh finite volume method for 1D hyperbolic systems with application to plug-flow heat exchangers. *Mathematics* **2021**, *9*, 2609. [CrossRef]
34. Aksikas, I. Optimal control and duality-based observer design for a hyperbolic PDEs system with application to fixed-bed reactor. *Int. J. Syst. Sci.* **2021**, *52*, 2493–2504. [CrossRef]
35. Chueshov, I. Invariant manifolds and nonlinear master-slave synchronization in coupled systems. *Appl. Anal.* **2007**, *86*, 269–286. [CrossRef]
36. Li, D.; Rao, B. *Boundary Synchronization for Hyperbolic Systems*; Springer: Berlin/Heidelberg, Germany, 2019.
37. Li, T.; Lu, X. Exact boundary synchronization for a kind of first order hyperbolic system. *ESAIM Control. Optim. Calc. Var.* **2022**, *28*, 34. [CrossRef]
38. Lu, X. Local exact boundary synchronization for a kind of first order quasilinear hyperbolic systems. *Chin. Ann. Math. Ser. B* **2019**, *40*, 79–96. [CrossRef]
39. Wang, J.L.; Zhao, L.H. PD and PI Control for passivity and synchronization of coupled neural networks with multi-weights. *IEEE Trans. Netw. Sci. Eng.* **2021**, *8*, 790–802. [CrossRef]
40. Wang, J.L.; Wu, H.N.; Huang, T.; Ren, S.Y. Pinning Synchronization of CDNs with Multi-weights. In *Analysis and Control of Output Synchronization for Complex Dynamical Networks*; Springer: Berlin/Heidelberg, Germany, 2019; pp. 145–174.
41. Jia, Y.; Wu, H.; Cao, J. Non-fragile robust finite-time synchronization for fractional-order discontinuous complex networks with multi-weights and uncertain couplings under asynchronous switching. *Appl. Math. Comput.* **2020**, *370*, 124929. [CrossRef]
42. Li, X.; Wu, H.; Cao, J. Synchronization in finite time for variable-order fractional complex dynamic networks with multi-weights and discontinuous nodes based on sliding mode control strategy. *Neural Netw.* **2021**, *139*, 335–347. [CrossRef]
43. Sakthivel, R.; Sakthivel, R.; Kwon, O.M.; Selvaraj, P.; Anthoni, S.M. Observer-based robust synchronization of fractional-order multi-weighted complex dynamical networks. *Nonlinear Dyn.* **2019**, *98*, 1231–1246. [CrossRef]
44. Bahuguna, D.; Sakthivel, R.; Chadha, A. Asymptotic stability of fractional impulsive neutral stochastic partial integro-differential equations with infinite delay. *Stoch. Anal. Appl.* **2017**, *35*, 63–88. [CrossRef]
45. Long, H.V.; Thao, H.T.P. Hyers-Ulam stability for nonlocal fractional partial integro-differential equation with uncertainty. *J. Intell. Fuzzy Syst.* **2018**, *34*, 233–244. [CrossRef]