mathematics

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Matrix Summability of Walsh–Fourier Series

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Abstract: The presented paper discusses the matrix summability of the Walsh–Fourier series. In particular, we discuss the convergence of matrix transforms in L_1 space and in C_W space in terms of modulus of continuity and matrix transform variation. Moreover, we show the sharpness of our result. We also discuss some properties of the maximal operator $t^*(f)$ of the matrix transform of the Walsh–Fourier series. As a consequence, we obtain the sufficient condition so that the matrix transforms $t_n(f)$ of the Walsh–Fourier series are convergent almost everywhere to the function f. The problems listed above are related to the corresponding Lebesgue constant of the matrix transformations. The paper sets out two-sides estimates for Lebesgue constants. The proven theorems can be used in the case of a variety of summability methods. Specifically, the proven theorems are used in the case of Cesàro means with varying parameters.

Keywords: Walsh system; matrix transforms; Cesaro mean; logarithmic means; martingale transform; weak type inequality; convergence in norm; almost everywhere convergence and divergence

MSC: 42C10

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Citation: Goginava, U.; Nagy, K. Matrix Summability of Walsh–Fourier Series. *Mathematics* **2022**, *10*, 2458. https://doi.org/10.3390/math10142458

Academic Editor: Ninoslav Truhar

Received: 13 June 2022
Accepted: 12 July 2022
Published: 14 July 2022

1. Introduction

The issues of summability of Fourier series have been studied by many authors. In particular, different methods of summabilities are known in the literature. The summability methods are concerned with matrix transformations of partial sums of Walsh–Fourier series. It is well known that the partial sums of Walsh–Fourier series are not convergent in the norm both in the classes of continuous functions and in classes of integrable functions [1] (Chapter 4). It is also known that there is an integral function whose Walsh–Fourier series is divergent at all points [1,2].

An example of matrix transformation is the Fejér or arithmetic mean. In this case, there is a matrix transformation where the elements $(t_{k,n} = 1/n, 1 \leq k \leq n)$ of each row of the corresponding triangular matrix are constants. As a result of such a transformation, we obtain a new sequence that can be convergent in the space C_W and L_1, and is also convergent almost everywhere for all integrable functions [1,2].

Another example of matrix summability is summability by the Riesz's logarithmic method $(t_{k,n} = \frac{1}{k \log n})$. The new sequence has "good" properties (convergence in the space C_W and L_1 as well as convergence almost everywhere for all integrable functions).

From the above, we can assume that if the matrix transformations whose first n element of the nth row represents a non-increasing sequence, then the new sequence obtained as a result of such a transformation is characterized by "good" properties (see estimation (29), Theorem 5 and Corollary 4).

Examples of matrix transformations whose first n element of the nth row represents an increasing sequence are:

- $(C,\alpha), \alpha > 0$ summability ($t_{k,n} = A_{n-k}^{\alpha-1}/A_n^{\alpha}, 0 \le k \le n$), where

$$A_n^{\alpha} := \frac{(1+\alpha)\dots(n+\alpha)}{n!};$$

- Nörlund logarithmic summability ($t_{k,n} = \frac{1}{(n-k)\log n}, 0 \le k < n$);
- Cesàro means with varying parameters ($t_{k,n} = A_{n-k}^{\alpha_n-1}/A_n^{\alpha_n}, 0 \le k \le n, \alpha_n \to 0$ as $n \to \infty$).

In the case for (C,α) summability ($\alpha > 0$), it is known that the new sequence obtained by matrix transformation ($t_{k,n} = A_{n-k}^{\alpha-1}/A_n^{\alpha}, 0 \le k \le n$) has "good" properties [1–3]. On the other hand, if ($t_{k,n} = \frac{1}{(n-k)\log n}, 0 \le k < n$) or ($t_{k,n} = A_{n-k}^{\alpha_n-1}/A_n^{\alpha_n}, 0 \le k \le n, \alpha_n \to 0$ as $n \to \infty$), then the new sequences are not characterized by "good" properties [4,5].

Therefore, the sequences obtained by matrix transformations can have "good" or "bad" properties. The article sets out the necessary and sufficient conditions for the sequence obtained as a result of the matrix transformation to be convergence in the space C_W and L_1 (see Theorem 3, Corollarys 2 and 3, Theorem 4).

Sufficient conditions have been established for the sequence obtained as a result of the matrix transformation to be almost everywhere convergent (see Theorem 6).

Note that the behavior of the sequences obtained as a result of the matrix transformation depends on two-sided estimations of the integral norm (Lebesgue's constant) of the corresponding kernel of the matrix transformation (see Theorem 1).

The theorems can be used for various methods of summability. At the end of the article, the theorems are used in the case of Cesàro means with varying parameters; this new result improves the theorem of Gát and Abu Joudeh [6].

2. Definitions

Let $\mathbb{P}$ denote the set of positive integers, $\mathbb{N} := \mathbb{P} \cup \{0\}$. By a dyadic interval in $\mathbb{I} := [0,1)$, we mean one of the form $I(l,k) := \left[\frac{l}{2^k}, \frac{l+1}{2^k}\right)$ for some $k \in \mathbb{N}, 0 \le l < 2^k$. Given $k \in \mathbb{N}$ and $x \in \mathbb{I}$, let $I_k(x)$ denote the dyadic interval of length 2^{-k} which contains the point x. We use also the notation $I_n := I_n(0) (n \in \mathbb{N}), \overline{I}_k(x) := \mathbb{I} \backslash I_k(x)$. Let

$$x = \sum_{n=0}^{\infty} x_n 2^{-(n+1)}$$

be the dyadic expansion of $x \in \mathbb{I}$, where $x_n = 0$ or 1, and if x is a dyadic rational number, we choose the expansion which terminates in $0'$s. We also use the following notation

$$I_k(x) = I_k(x_0, x_1, \dots, x_{k-1}).$$

For any given $n \in \mathbb{N}$, it is possible to write n uniquely as

$$n = \sum_{k=0}^{\infty} \varepsilon_k(n) 2^k,$$

where $\varepsilon_k(n) = 0$ or 1 for $k \in \mathbb{N}$. This expression will be called the binary expansion of n and the numbers $\varepsilon_k(n)$ will be called the binary coefficients of n. Let us denote for $1 \le n \in \mathbb{N}$, $|n| := \max\{j \in \mathbb{N} : \varepsilon_j(n) \neq 0\}$, that is $2^{|n|} \le n < 2^{|n|+1}$.

Let us set the definition of the nth ($n \in \mathbb{N}$) Walsh–Paley function at point $x \in \mathbb{I}$ as:

$$w_n(x) = (-1)^{\sum_{j=0}^{\infty} \varepsilon_j(n) x_j}.$$

Let us denote by $\dotplus$ the logical addition on $\mathbb{I}$. That is, for any $x, y \in \mathbb{I}$ and $k, n \in \mathbb{N}$

$$x \dotplus y := \sum_{n=0}^{\infty} |x_n - y_n| 2^{-(n+1)}.$$

Let us define the binary operator $\oplus : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by

$$k \oplus n = \sum_{i=0}^{\infty} |\varepsilon_i(k) - \varepsilon_i(n)| 2^i. \tag{1}$$

It is well known (see [1], p. 5) that

$$w_{m \oplus n}(x) = w_m(x) w_n(x), x \in \mathbb{I} \quad (n, m \in \mathbb{N}). \tag{2}$$

The Walsh–Dirichlet kernel is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x), D_n^* := w_n D_n.$$

Recall that [1,2]

$$D_{2^n}(x) = 2^n \chi_{I_n}(x), \tag{3}$$

where χ_E is the characteristic function of the set E,

$$D_n = w_n \sum_{k=0}^{\infty} \varepsilon_k(n) r_k D_{2^k}, \tag{4}$$

$$D_{2^n+m} = D_{2^n} + w_{2^n} D_m \quad (m < 2^n). \tag{5}$$

The partial sums of Walsh–Fourier series of a function $f \in L_1(\mathbb{I})$ are defined as follows: $S_0(f) = 0$ and

$$S_n(f; x) := \sum_{k=0}^{n-1} \widehat{f}(k) w_k(x) \quad (n \in \mathbb{N}),$$

where $\widehat{f}(k) = \int_{\mathbb{I}} f w_k$.

3. Triangular Matrix Transforms

Let $T := (t_{k,n})$ be an infinite triangular matrix satisfying the following conditions:

(a) $t_{k,n} \geq 0, k, n \in \mathbb{N}$;
(b) $t_{k,n} = 0, k > n$;
(c) $\sum_{k=1}^{n} t_{k,n} = 1$.

We define the nth triangular matrix transform of the Walsh–Fourier series by

$$t_n(f; x) := \sum_{k=1}^{n} t_{k,n} S_k(f; x) \quad (n \in \mathbb{P}). \tag{6}$$

The triangular matrix transform kernels are defined by

$$F_n(t) := \sum_{k=1}^{n} t_{k,n} D_k(t).$$

We have

$$t_n(f, x) = (f * F_n)(x) = \int_{\mathbb{I}} f(x \dotplus t) F_n(t) d(t).$$

Let us define the following matrices

$$T := \begin{bmatrix} t_1(f;x) \\ \vdots \\ t_n(f,x) \\ \vdots \end{bmatrix}, S := \begin{bmatrix} S_1(f;x) \\ \vdots \\ S_n(f;x) \\ \vdots \end{bmatrix},$$

$$m(T) := \begin{bmatrix} t_{11} & 0 & 0 & \cdots & 0 & 0 & \cdots \\ t_{12} & t_{22} & 0 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \\ t_{1n} & t_{2n} & t_{3n} & \cdots & t_{nn} & 0 & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \cdots & \end{bmatrix}.$$

Then, equality (6) can be written as follows

$$T = m(T) \times S.$$

The Fejér means and kernels are denoted by

$$\sigma = m(\sigma) \times S,$$

where

$$m(\sigma) := \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \cdots & \end{bmatrix}, \sigma := \begin{bmatrix} \sigma_1(f;x) \\ \vdots \\ \sigma_n(f;x) \\ \vdots \end{bmatrix}.$$

It is easily seen that

$$\sigma_n(f,x) := \frac{1}{n}\sum_{k=1}^{n} S_k(f,x), \quad K_n(t) := \frac{1}{n}\sum_{k=1}^{n} D_k(t),$$

$$\sigma_n(f,x) = (f * K_n)(x) = \int_{\mathbb{I}} f(x \dotplus t)K_n(t)d(t).$$

It is well known that L_1 norms of Fejér kernels are uniformly bounded, that is

$$\|K_n\|_1 \leq c \quad \text{for all } n \in \mathbb{N}. \tag{7}$$

Yano [7] estimated the value of c, and he gave $c = 2$. Recently, in paper [8], it was shown that the exact value of c is $\frac{17}{15}$.

4. Auxiliary Results

This section will mention the definitions and notations from the book [1] (Chapter 3).

For each $n \in \mathbb{N}$, let $\mathcal{A}_n$ represent the σ-algebra generated by the collection of dyadic intervals $\{I(k,n) : k = 0,1,\ldots,2^n - 1\}$. Thus, every element of $\mathcal{A}_n$ is a finite union of intervals of the form $[k2^{-n},(k+1)2^{-n})$ or an empty set.

Let $L(\mathcal{A}_n)$ represent the collection of $\mathcal{A}_n$-measurable functions on $\mathbb{I}$. By the Paley Lemma [1] (Chapter 1, p. 12), $L(\mathcal{A}_n)$ coincides with the collection of Walsh polynomials of order less than 2^n.

A sequence of functions $(f_n : n \in \mathbb{N})$ is called a dyadic martingale if each f_n belongs to $L(\mathcal{A}_n)$ and

$$\int_E f_{n+1} = \int_E f_n (E \in \mathcal{A}_n, n \in \mathbb{N}).$$

Let $\mathbf{A}$ denote the collection of sequences $\boldsymbol{\beta} := \{\beta_n : n \in \mathbb{N}\}$ which satisfy $\beta_n \in L(\mathcal{A}_n)$ for $n \in \mathbb{N}$ and

$$\|\boldsymbol{\beta}\| := \sup_{n\in\mathbb{N}} \|\beta_n\|_\infty < \infty.$$

For a given $\beta \in \mathbf{A}$ and $f \in L_1(\mathbb{I})$, the martingale transform of f is defined by

$$\mathbf{T}(\boldsymbol{\beta})(f) := \sum_{n=0}^{\infty} \beta_n \Delta_n f,$$

where $\Delta_n f := S_{2^{n+1}}(f) - S_{2^n}(f)$ for $n \in \mathbb{N}$. The maximal martingale transform is defined by

$$\mathbf{T}^*(\boldsymbol{\beta})(f) := \sup_{N\in\mathbb{N}} \left| \sum_{n=0}^{N} \beta_n \Delta_n f \right|.$$

The next Lemma plays an important role in our paper and methods [1] [page 97].

Lemma 1 (Schipp, Simon, Wade and Pál [1])**.** *Let $f \in L_1(\mathbb{I}), y > 0$, and $\boldsymbol{\beta} \in \mathbf{A}$. Then, the operator $T^*(\boldsymbol{\beta})$ is of weak type (1,1). That is, there exists an absolute constant C such that*

$$y|\{x \in \mathbb{I} : T^*(\boldsymbol{\beta})(f) > y\}| \leq C\|\boldsymbol{\beta}\|\|f\|_1.$$

5. Kernel Representation and L_1-Norm of the Matrix Transform Kernels

First, we start with a useful decomposition of the kernel function $F_n^* := w_n F_n$. We use the next notation in the proof.

$$T_{n,(k)} := \sum_{l=1}^{k} t_{l,n}, \quad T_n^{(k)} := \sum_{l=k}^{n} t_{l,n}$$

and

$$n^{(s)} := \sum_{j=s}^{\infty} \varepsilon_j(n) 2^j, \quad n_{(s)} := \sum_{j=0}^{s} \varepsilon_j(n) 2^j.$$

We note that $\sum_{l=1}^{n} t_{l,n} = T_{n,(n)} = T_n^{(1)}$.

Lemma 2. *Let $0 < n \in \mathbb{N}$. Then, the next decomposition of the matrix transform kernel holds:*

$$\begin{aligned} F_n^* &= \sum_{s=0}^{|n|} \varepsilon_s(n) T_n^{(n^{(s)})} (D_{2^{s+1}} - D_{2^s}) \\ &\quad + \sum_{s=0}^{|n|} \varepsilon_s(n) w_{n_{(s)}} \sum_{k=1}^{2^s-1} t_{k+n^{(s+1)},n} D_k. \end{aligned}$$

Proof of Lemma 2. For any positive integer n, we write that

$$\begin{aligned} F_n &= \sum_{k=1}^{n} t_{k,n} D_k = -\sum_{k=1}^{n-1} T_{n,(k)} w_k + D_n T_{n,(n)} \\ &= -\sum_{k=1}^{n-1} T_{n,(k)} w_k + \left(\sum_{k=1}^{n-1} w_k\right) T_{n,(n)} + T_{n,(n)} \\ &= \sum_{k=1}^{n-1} \Big(T_{n,(n)} - T_{n,(k)}\Big) w_k + T_{n,(n)} \\ &= \sum_{k=1}^{n-1} T_n^{(k+1)} w_k + T_{n,(n)} \\ &= \sum_{k=0}^{n-1} T_n^{(k+1)} w_k. \end{aligned}$$

Then, from (2), we have that

$$\begin{aligned} F_n &= \sum_{s=0}^{|n|} \varepsilon_s(n) \sum_{k=n^{(s+1)}}^{n^{(s)}-1} T_n^{(k+1)} w_k \\ &= \sum_{s=0}^{|n|} \varepsilon_s(n) \sum_{k=0}^{2^s-1} T_n^{\left(k+1+n^{(s+1)}\right)} w_{k+n^{(s+1)}} \\ &= \sum_{s=0}^{|n|} \varepsilon_s(n) w_{n^{(s+1)}} \sum_{k=0}^{2^s-1} T_n^{\left(k+1+n^{(s+1)}\right)} w_k \\ &= \sum_{s=0}^{|n|} \varepsilon_s(n) w_{n^{(s+1)}} \sum_{k=1}^{2^s-1} \left(T_n^{\left(k+n^{(s+1)}\right)} - T_n^{\left(k+1+n^{(s+1)}\right)}\right) D_k \\ &\quad + \sum_{s=0}^{|n|} \varepsilon_s(n) w_{n^{(s+1)}} T_n^{\left(n^{(s)}\right)} D_{2^s}. \end{aligned}$$

For $x \in I_s$, we have

$$w_{n_{(s)}}(x) = w_{2^s}(x) \tag{8}$$

Hence,

$$\begin{aligned} w_n F_n &= \sum_{s=0}^{|n|} \varepsilon_s(n) T_n^{\left(n^{(s)}\right)} w_{2^s} D_{2^s} \\ &\quad + \sum_{s=0}^{|n|} \varepsilon_s(n) w_{n_{(s)}} \sum_{k=1}^{2^s-1} \left(T_n^{\left(k+n^{(s+1)}\right)} - T_n^{\left(k+1+n^{(s+1)}\right)}\right) D_k \\ &=: F_{n,1}^* + F_{n,2}^*. \end{aligned} \tag{9}$$

This completes the proof of Lemma 2. □

We introduce the notation

$$t_n^*(f) := f * F_n^*, \quad t_{n,1}^* := f * F_{n,1}^*, \quad t_{n,2}^* := f * F_{n,2}^*.$$

Before we discuss the L_1-norm of the kernels F_n, we prove the following lemma.

Lemma 3. *Let* $(\alpha_j : j \in \mathbb{N})$ *be a non-decreasing (in sign* $\alpha_j \uparrow$*) bounded sequence of positive real numbers* $\boldsymbol{\alpha}(n) := (\alpha_j(n) := \alpha_j \varepsilon_j(n) : j \in \mathbb{N})$*. Let the kernel of martingale transform* $T(\boldsymbol{\alpha}(n))f = f * M(\boldsymbol{\alpha}(n))$ *be defined by*

$$M(\boldsymbol{\alpha}(n)) := \sum_{j=1}^{\infty} \varepsilon_j(n)\alpha_j\big(D_{2^{j+1}} - D_{2^j}\big).$$

Then

$$\|M(\boldsymbol{\alpha}(n))\|_1 \sim \sum_{k=1}^{|n|} |\varepsilon_k(n) - \varepsilon_{k+1}(n)|\alpha_k. \tag{10}$$

Proof of Lemma 3. We write that

$$\begin{aligned} M(\boldsymbol{\alpha}(n)) &= \sum_{j=1}^{|n|-1} (\varepsilon_j(n)\alpha_j - \varepsilon_{j+1}(n)\alpha_{j+1})D_{2^{j+1}} \\ &\quad + \varepsilon_{|n|}(n)\alpha_{|n|}D_{2^{|n|+1}} - \varepsilon_1(n)\alpha_1 D_{2^1}. \end{aligned}$$

This and equality (3) yield that

$$\|M(\boldsymbol{\alpha}(n))\|_1 \leq 2\|\boldsymbol{\alpha}\| + \sum_{j=1}^{|n|-1} |\varepsilon_j(n)\alpha_j - \varepsilon_{j+1}(n)\alpha_{j+1}| \tag{11}$$

$$\leq \quad 2\|\boldsymbol{\alpha}\| + \sum_{j=1}^{|n|-1} |\varepsilon_j(n) - \varepsilon_{j+1}(n)|\alpha_j + \sum_{j=1}^{|n|-1} \varepsilon_{j+1}(n)|\alpha_j - \alpha_{j+1}|.$$

Since $\boldsymbol{\alpha} := (\alpha_n : n \in \mathbb{N})$ is non-decreasing, we can write

$$\sum_{j=2}^{|n|-1} \varepsilon_{j+1}(n)|\alpha_j - \alpha_{j+1}| \leq \sum_{j=1}^{|n|-1} |\alpha_j - \alpha_{j+1}| = \alpha_{|n|} - \alpha_1 \leq \|\mathbf{ff}\|. \tag{12}$$

This yields

$$\|M_n(\boldsymbol{\alpha})\|_1 \leq 3\|\boldsymbol{\alpha}\| + \sum_{j=2}^{|n|-1} |\varepsilon_j(n) - \varepsilon_{j+1}(n)|\alpha_j. \tag{13}$$

Now, we show the lower estimate for $\|M_n(\boldsymbol{\alpha})\|_1$. We use the construction in the book ([1], p. 35). Let us choose the strictly monotone increasing sequences a_i and b_i $(i = 1, \ldots, s)$ such that

$$0 < a_1 \leq b_1 < a_2 \leq b_2 < \ldots < a_s \leq b_s < a_{s+1} = \infty.$$

It is easy to see that

$$b_j + 1 < a_{j+1}$$

holds. We define the nature number $n = \sum_{j=0}^{\infty} \varepsilon_j(n)2^j$ by

$$\varepsilon_j(n) := \begin{cases} 1, & \text{if } a_i \leq j \leq b_i \text{ for an } i \in \{1, \ldots, s\}, \\ 0, & \text{if } b_i < j < a_{i+1} \text{ for an } i \in \{1, \ldots, s\} \text{ or } j < a_1. \end{cases} \tag{14}$$

Let us set the sets

$$A_k := \left(\frac{1}{2^{a_k+1}}, \frac{1}{2^{a_k}}\right), \quad B_k := \left(\frac{1}{2^{b_k+2}}, \frac{1}{2^{b_k+1}}\right), \quad k = 1, \ldots, s.$$

For $x \in A_k$, we have that

$$\begin{aligned}|M(\boldsymbol{\alpha}(n))(x)| &= \left|\sum_{j=1}^{|n|} \varepsilon_j(n)\alpha_j(D_{2^{j+1}}(x) - D_{2^j}(x))\right| \\ &= \left|\sum_{i=1}^{k-1}\sum_{j=a_i}^{b_i} \alpha_j(D_{2^{j+1}}(x) - D_{2^j}(x)) + \sum_{j=a_k}^{b_k} \alpha_j(D_{2^{j+1}}(x) - D_{2^j}(x))\right| \\ &= \left|\sum_{i=1}^{k-1}\sum_{j=a_i}^{b_i} \alpha_j 2^j - \alpha_{a_k}2^{a_k}\right|.\end{aligned}$$

The construction of the sequences $\{a_k\}$ and $\{b_k\}$ yields

$$\begin{aligned}\sum_{i=1}^{k-1}\sum_{j=a_i}^{b_i} \alpha_j 2^j &\leq \alpha_{b_{k-1}} \sum_{i=1}^{k-1}(2^{b_i+1} - 2^{a_i}) \\ &\leq \alpha_{b_{k-1}} \sum_{i=1}^{k-1}(2^{b_i+1} - 2^{b_{i-1}+1}) \\ &\leq \alpha_{b_{k-1}} 2^{b_{k-1}+1} \leq \alpha_{a_k} 2^{b_{k-1}+1}\end{aligned}$$

and

$$|M_n(\boldsymbol{\alpha})(x)| \geq \alpha_{a_k}2^{a_k} - \alpha_{a_k}2^{b_{k-1}+1} \geq \alpha_{a_k}2^{a_k-1}.$$

That is, we obtain that

$$\int_{A_k} |M(\boldsymbol{\alpha}(n))(x)|dx \geq \alpha_{a_k}2^{a_k-1}2^{-a_k-1} \geq \frac{\alpha_{a_k-1}}{4}. \tag{15}$$

Now, we set $x \in B_k$.

$$\begin{aligned}|M(\boldsymbol{\alpha}(n))(x)| &= \left|\sum_{i=1}^{k}\sum_{j=a_i}^{b_i} \alpha_j(D_{2^{j+1}}(x) - D_{2^j}(x))\right| \\ &= \sum_{i=1}^{k}\sum_{j=a_i}^{b_i} \alpha_j 2^j \geq \alpha_{b_k}2^{b_k}\end{aligned}$$

and

$$\int_{B_k} |M(\boldsymbol{\alpha}(n))(x)|dx \geq \alpha_{b_k}2^{b_k}2^{-b_k-2} = \frac{\alpha_{b_k}}{4}. \tag{16}$$

The sets A_k and B_k are pairwise disjoint intervals ($k = 1, \ldots, s$), and we have

$$\begin{aligned}\|M(\boldsymbol{\alpha}(n))\|_1 &\geq \sum_{k=1}^{s}\left(\int_{A_k} |M(\boldsymbol{\alpha}(n))(x)|dx + \int_{B_k} |M(\boldsymbol{\alpha}(n))(x)|dx\right) \\ &\geq \frac{1}{4}\sum_{k=1}^{s}(\alpha_{a_k-1} + \alpha_{b_k})\end{aligned}$$

(see inequalities (15) and (16) as well). Taking into account that

$$|\varepsilon_j(n) - \varepsilon_{j+1}(n)| = \begin{cases} 1, & \text{if } j = a_k - 1 \text{ or } j = b_k \text{ for a } k \in \{1, \ldots, s\}, \\ 0, & \text{otherwise,} \end{cases}$$

we conclude that

$$\|M(\boldsymbol{\alpha}(n))\|_1 \geq \frac{1}{4}\sum_{j=1}^{|n|} |\varepsilon_j(n) - \varepsilon_{j+1}(n)|\alpha_j. \tag{17}$$

Summarizing our results in inequalities (13) and (17), we complete the proof. □

Theorem 1. *(a) If the sequence* $\{t_{k,n} : 1 \le k \le n\}$ *is monotone non-increasing (in sign* $t_{k,n} \downarrow$*) for any fixed* n*, then there exists a positive constant* c *such that*

$$\|F_n\|_1 \le c \tag{18}$$

holds for all $n \in \mathbb{P}$*.*

(b) If the sequence $\{t_{k,n} : 1 \le k \le n\}$ *is monotone non-decreasing (in sign* $t_{k,n} \uparrow$*) for any fixed* n*, then*

$$\|F_n\|_1 \sim \sum_{s=1}^{|n|} |\varepsilon_s(n) - \varepsilon_{s+1}(n)| T_n^{(n^{(s)})}. \tag{19}$$

Proof of Theorem 1. First, let the sequence $\{t_{k,n} : 1 \le k \le n\}$ be monotone non-increasing (in sign $t_{k,n} \downarrow$). For the kernel F_n, we apply Abel's transformation

$$F_n = \sum_{k=1}^{n-1} (t_{k,n} - t_{k+1,n}) k K_k + t_{n,n} n K_n. \tag{20}$$

Inequality (7) implies that

$$\begin{aligned} \|F_n\|_1 &\le \sum_{k=1}^{n-1} |t_{k,n} - t_{k+1,n}| k \|K_k\|_1 + t_{n,n} n \|K_n\|_1 \\ &\le c\left(\sum_{k=1}^{n-1} (t_{k,n} - t_{k+1,n}) k + t_{n,n} n\right) \\ &\le c \sum_{k=1}^{n} t_{k,n} \le c. \end{aligned} \tag{21}$$

Second, let the sequence $\{t_{k,n} : 1 \le k \le n\}$ be monotone non-decreasing (in sign $t_{k,n} \uparrow$). Theorem 2 yields that

$$\|F_n\|_1 = \|F_n^*\|_1 \le \|F_{n,1}^*\|_1 + \|F_{n,2}^*\|_1.$$

Applying Lemma 3 with setting $\alpha_s := T_n^{(n^{(s)})}$, we obtain

$$\|F_{n,1}^*\|_1 \sim \sum_{s=1}^{|n|} |\varepsilon_s(n) - \varepsilon_{s+1}(n)| T_n^{(n^{(s)})}.$$

At last, we discuss the norm $\|F_{n,2}^*\|_1$. In case $\varepsilon_s(n) = 1$, we write that

$$\begin{aligned} I_s := & \sum_{k=1}^{2^s-1} t_{k+n^{(s+1)},n} D_k = \sum_{k=1}^{2^s-1} t_{k+n^{(s)}-2^s,n} D_k \\ = & \sum_{l=1}^{2^s-1} t_{n^{(s)}-l,n} D_{2^s-l} \quad (s = 0, \dots, |n|-1). \end{aligned} \tag{22}$$

For $s = |n|$, we have that

$$I_{|n|} := \sum_{k=1}^{2^{|n|}-1} t_{k,n} D_k = \sum_{l=1}^{2^{|n|}-1} t_{2^{|n|}-l,n} D_{2^{|n|}-l}.$$

It is known that

$$D_{2^k-j} = D_{2^k} - w_{2^k-1} D_j \quad \text{for } j = 1, \dots, 2^k - 1. \tag{23}$$

Applying equality (23) and Abel's transformation, we obtain

$$\begin{aligned} I_s &= D_{2^s}\sum_{l=1}^{2^s-1} t_{n^{(s)}-l,n} - w_{2^s-1}\sum_{l=1}^{2^s-1} t_{n^{(s)}-l,n} D_l \\ &= D_{2^s}\sum_{l=1}^{2^s-1} t_{n^{(s)}-l,n} \\ &\quad - w_{2^s-1}\left(\sum_{l=1}^{2^s-2}(t_{n^{(s)}-l,n} - t_{n^{(s)}-l-1,n})lK_l + t_{n^{(s)}-2^s+1,n}(2^s-1)K_{2^s-1}\right). \end{aligned} \tag{24}$$

Analogously, we transform the expression $I_{|n|}$. Inequality (7) yields

$$\|I_s\|_1 \le c\sum_{l=1}^{2^s-1} t_{n^{(s)}-l,n} \quad (s = 0, \ldots, |n|)$$

and

$$\|I_{|n|}\|_1 \le \sum_{l=1}^{2^{|n|}-1} t_{2^{|n|}-l,n}.$$

Thus,

$$\|F^*_{n,2}\|_1 = \left\|\sum_{s=0}^{|n|} \varepsilon_s(n) w_{n_{(s)}} I_s\right\|_1 \le \sum_{s=0}^{|n|} \|I_s\|_1 \le c\sum_{k=1}^{n} t_{k,n} \le c. \tag{25}$$

Theorem 1 is proved. □

6. Convergence In Measure of Matrix Transform of Walsh–Fourier Series

Theorem 2. *Let $\{t_{k,n} : 1 \le k \le n\}$ be a monotone non-decreasing (or monotone non-increasing) sequence for any fixed n. Then, there exists a positive constant c such that*

$$y|\{x \in \mathbb{I} : |t_n(f)| > y\}| \le c\|f\|_1$$

holds for all $f \in L^1(\mathbb{I})$ and $y > 0$.

Proof of Theorem 2. First, let the sequence $\{t_{k,n} : 1 \le k \le n\}$ be monotone non-increasing (in sign $t_{k,n} \downarrow$). Since, by Theorem 1, we write that

$$\|t_n(f)\|_1 = \|f * F_n\|_1 \le \|f\|_1\|F_n\|_1 \le c\|f\|_1. \tag{26}$$

(for more details, see [1,2]). We immediately learn that the operator t_n is of weak type (1,1).

Second, let the sequence $\{t_{k,n} : 1 \le k \le n\}$ be monotone non-decreasing (in sign $t_{k,n} \uparrow$). Lemma 2 yields that

$$t^*_n(f) = f * F^*_n = f * F^*_{n,1} + f * F^*_{n,2}.$$

Since $t^*_{n,1}(f) = f * F^*_{n,1}$ is a martingale transform with coefficients $\varepsilon_s(n)T_n^{(n^{(s)})}$, we apply Lemma 1. This lemma gives immediately that the operator $t^*_{n,1}$ is of weak type $(1,1)$. That is, there exists a positive constant c such that

$$y\left|\{x \in \mathbb{I} : |t^*_{n,1}(f)| > y\}\right| \le c\|f\|_1 \quad (y > 0) \tag{27}$$

holds for all $f \in L_1(\mathbb{I})$.

For the operator $t^*_{n,2}$, we apply inequality (25) and write that

$$\|t^*_{n,2}(f)\|_1 = \|f * F_{n,2}\|_1 \le \|f\|_1\|F^*_{n,2}\|_1 \le c\|f\|_1. \tag{28}$$

(for more details, see [1,2]). That is, the operator $t^*_{n,2}$ is of weak type (1,1).

Inequalities (26)–(28) complete the proof of Theorem 2. □

Theorem 2 implies that the following is valid.

Corollary 1. *Let* $\{t_{k,n} : 1 \le k \le n\}$ *be a monotone non-decreasing (or monotone non-increasing) sequence for any fixed* n*. Then, for all* $f \in L_1(\mathbb{I})$*,* $t_n(f) \to f$ *in measure as* $n \to \infty$*.*

Remark 1. *In the case that the sequence* $\{t_{k,n} : 1 \le k \le n\}$ *is not increasing for any fixed* n*, below, more is proved. In particular, the weak type inequality for the maximal operator* $t^*(f)$ *is proved (see Theorem 5).*

7. Convergence in L_1-Norm and C_W-Norm

Let $C_W(\mathbb{I})$ represent the collection of functions f which are continuous at every dyadic irrational, continuous from the right on $\mathbb{I}$, and have a finite limit from the left on $\mathbb{I}$, all this in the usual topology.

Set $\|f\|_{C_W} := \sup_{x\in\mathbb{I}} |f(x)|$. Let us denote by $L_p(\mathbb{I})$ the usual Lebesgue spaces on $\mathbb{I}$ with the corresponding norm $\|.\|_p$ $(1 \le p < \infty)$. Let $X := X(\mathbb{I})$ be either $L_1(\mathbb{I})$ or $C_W(\mathbb{I})$ with the corresponding norm denoted by $\|.\|_X$. The modulus of continuity, when $X = C_W(\mathbb{I})$, and the integrated modulus of continuity, while $X = L_1(\mathbb{I})$ are defined by

$$\omega\left(\frac{1}{2^n}, f\right)_X := \sup_{h\in I_n} \|f(.\dot{+}h) - f(.)\|_X.$$

In this section, we discuss the convergence of matrix transforms in L_1 space and in C_W in terms of modulus of continuity and matrix transform variation. Moreover, in Theorem 4, we show the sharpness of our result.

For non-negative integer n, the variation of n is defined by

$$V(n) := \sum_{k=0}^{\infty} |\varepsilon_k(n) - \varepsilon_{k+1}(n)| + \varepsilon_0(n)$$

(see [1], p. 34). Motivated by this definition for the monotone non-decreasing sequence $\{t_{k,n} : 1 \le k \le n\}$ (in sign $t_{k,n} \uparrow$), we introduce the matrix transform variation of n by

$$V(n, \{t_{k,n}\}) := \sum_{k=1}^{|n|} |\varepsilon_k(n) - \varepsilon_{k+1}(n)| T_n^{(n^{(k)})}.$$

For the convenience of the reader, the main theorems of this section will be formulated first, and the proofs will be given below.

Theorem 3. *Let* $f \in X(\mathbb{I})$ *and* $\{t_{k,n} : 1 \le k \le n\}$ *be a sequence of non-negative numbers.*

(a) If the sequence $\{t_{k,n} : 1 \le k \le n\}$ *is monotone non-increasing (in sign* $t_{k,n} \downarrow$*), then*

$$\begin{aligned} \|t_n(f) - f\|_X \le\ & c_1\omega\left(\frac{1}{2^{|n|}}, f\right)_X + c_2\omega\left(\frac{1}{2^{|n|-1}}, f\right)_X \\ & + c_3 \sum_{r=0}^{|n|-2} 2^r t_{2^r,n}\omega\left(\frac{1}{2^r}, f\right)_X. \end{aligned} \tag{29}$$

(b) If the sequence $\{t_{k,n} : 1 \le k \le n\}$ *is monotone non-decreasing (in sign* $t_{k,n} \uparrow$*), then*

$$\begin{aligned} \|t_n(f) - f\|_X \le\ & c_1 V(n, \{t_{k,n}\})\omega\left(\frac{1}{2^{|n|}}, f\right)_X \\ & + c_2\omega\left(\frac{1}{2^{|n|-1}}, f\right)_X \\ & + c_3 \sum_{r=0}^{|n|-2} 2^r t_{2^{r+1}-1,n}\omega\left(\frac{1}{2^r}, f\right)_X. \end{aligned} \tag{30}$$

Proof of Theorem 3. We carry out the proof of Theorem 3 for space $X = L_1(\mathbb{I})$. The proof for $X = C_W$ is similar and even simpler. Keeping in mind that $\sum_{k=1}^{n} t_{k,n} = 1$, we write that

$$\begin{aligned} t_n(f,x) - f(x) &= \int_{\mathbb{I}} (f(x \dotplus t) - f(x)) \sum_{k=2^{|n|}}^{n} t_{k,n} D_k(t) dt \\ &\quad + \int_{\mathbb{I}} (f(x \dotplus t) - f(x)) \sum_{k=2^{|n|-1}}^{2^{|n|}-1} t_{k,n} D_k(t) dt \\ &\quad + \int_{\mathbb{I}} (f(x \dotplus t) - f(x)) \sum_{k=1}^{2^{|n|-1}-1} t_{k,n} D_k(t) dt \\ &=: I_1 + I_2 + I_3. \end{aligned} \tag{31}$$

First, we discuss the expression I_1. We write that

$$\begin{aligned} I_1 &= \int_{\mathbb{I}} (f(x \dotplus t) - S_{2^{|n|}}(f, x \dotplus t)) \sum_{k=2^{|n|}}^{n} t_{k,n} D_k(t) dt \\ &\quad + \int_{\mathbb{I}} (S_{2^{|n|}}(f, x \dotplus t) - S_{2^{|n|}}(f,x)) \sum_{k=2^{|n|}}^{n} t_{k,n} D_k(t) dt \\ &\quad + \int_{\mathbb{I}} (S_{2^{|n|}}(f,x) - f(x)) \sum_{k=2^{|n|}}^{n} t_{k,n} D_k(t) dt \\ &=: I_{1,1} + I_{1,2} + I_{1,3}. \end{aligned} \tag{32}$$

It is easily seen that $I_{1,2} = 0$. Applying generalized Minkowski's inequality, we have

$$\|I_{1,1}\|_X \leq \omega\left(\frac{1}{2^{|n|}}, f\right)_X \int_{\mathbb{I}} \left| \sum_{k=2^{|n|}}^{n} t_{k,n} D_k(t) \right| dt. \tag{33}$$

For sequence $t_{k,n} \uparrow$, we learn immediately that

$$\|I_{1,1}\|_X \leq \omega\left(\frac{1}{2^{|n|}}, f\right)_X (1 + V(n, \{t_{k,n}\})).$$

Analogously, we can prove that

$$\|I_{1,3}\|_X \leq \omega\left(\frac{1}{2^{|n|}}, f\right)_X (1 + V(n, \{t_{k,n}\})).$$

That is, we have that

$$\|I_1\|_X \leq \omega\left(\frac{1}{2^{|n|}}, f\right)_X (1 + V(n, \{t_{k,n}\})). \tag{34}$$

For sequence $t_{k,n} \downarrow$ we apply the equality (5), and we obtain

$$\begin{aligned} \sum_{k=2^{|n|}}^{n} t_{k,n} D_k &= \sum_{l=0}^{n-2^{|n|}} t_{2^{|n|}+l,n} D_{2^{|n|}+l} \\ &= \sum_{l=0}^{n-2^{|n|}} t_{2^{|n|}+l,n} D_{2^{|n|}} + w_{2^{|n|}} \sum_{l=1}^{n-2^{|n|}} t_{2^{|n|}+l,n} D_l. \end{aligned}$$

Applying Abel's transform and inequalities (7) and (33), we learn that

$$\|I_{1,1}\|_X \leq c\omega\left(\frac{1}{2^{|n|}}, f\right)_X.$$

Analogously, we can prove that

$$\|I_{1,3}\|_X \le c\omega\left(\frac{1}{2^{|n|}}, f\right)_X.$$

That is, we have that

$$\|I_1\|_X \le c\omega\left(\frac{1}{2^{|n|}}, f\right)_X. \tag{35}$$

The estimation of the I_2 is analogous to the estimation of the I_1, and we have

$$\|I_2\|_X \le c\omega\left(\frac{1}{2^{|n|-1}}, f\right)_X \int_{\mathbb{I}}\left|\sum_{k=2^{|n|-1}}^{2^{|n|}-1} t_{k,n} D_k(t)\right| dt.$$

Now, we discuss the integral $I := \int_{\mathbb{I}}\left|\sum_{k=2^{|n|-1}}^{2^{|n|}-1} t_{k,n} D_k(t)\right| dt$. We apply equality (23), Abel's transformation and inequality (7). We have that

$$\begin{aligned}
I &\le \int_{\mathbb{I}}\left|\sum_{k=1}^{2^{|n|-1}} t_{2^{|n|}-k,n} D_{2^{|n|}-k}(t)\right| dt \\
&\le \sum_{k=1}^{2^{|n|-1}} t_{2^{|n|}-k,n} + \int_{\mathbb{I}}\left|\sum_{k=1}^{2^{|n|-1}} t_{2^{|n|}-k,n} D_k(t)\right| dt \\
&\le c + \int_{\mathbb{I}}\left|\sum_{k=1}^{2^{|n|-1}-1} (t_{2^{|n|}-k,n} - t_{2^{|n|}-k-1,n}) k K_k(t)\right| dt \\
&\quad + \int_{\mathbb{I}} t_{2^{|n|-1},n} 2^{|n|-1} \left|K_{2^{|n|-1}}(t)\right| dt \\
&\le c + c\left(\sum_{k=1}^{2^{|n|-1}-1} \left|t_{2^{|n|}-k,n} - t_{2^{|n|}-k-1,n}\right| k + t_{2^{|n|-1},n} 2^{|n|-1}.\right)
\end{aligned}$$

For sequence $t_{k,n} \uparrow$, we learn that

$$I \le c + c\sum_{l=1}^{2^{|n|-1}} t_{2^{|n|}-l} \le c.$$

For sequence $t_{k,n} \downarrow$, we write

$$I \le c + ct_{2^{|n|-1},n} 2^{|n|-1} \le c + c\sum_{k=0}^{2^{|n|-1}-1} t_{2^{|n|-1}-k,n} \le c.$$

That is, we have that

$$\|I_2\|_X \le c\omega\left(\frac{1}{2^{|n|-1}}, f\right)_X \tag{36}$$

in both cases (a) and (b).

At last, we discuss the expression I_3.

$$\begin{aligned} I_3 &= \sum_{r=0}^{|n|-2} \sum_{j=2^r}^{2^{r+1}-1} t_{j,n} \int_{\mathbb{I}} (f(x \dotplus t) - f(x)) D_j(t) dt \\ &= \sum_{r=0}^{|n|-2} \int_{\mathbb{I}} (f(x \dotplus t) - S_{2^r}(f, x \dotplus t)) \sum_{j=2^r}^{2^{r+1}-1} t_{j,n} D_j(t) dt \\ &\quad + \sum_{r=0}^{|n|-2} \int_{\mathbb{I}} (S_{2^r}(f, x \dotplus t) - S_{2^r}(f, x)) \sum_{j=2^r}^{2^{r+1}-1} t_{j,n} D_j(t) dt \\ &\quad + \sum_{r=0}^{|n|-2} \int_{\mathbb{I}} (S_{2^r}(f, x) - f(x)) \sum_{j=2^r}^{2^{r+1}-1} t_{j,n} D_j(t) dt \\ &=: I_{3,1} + I_{3,2} + I_{3,3}. \end{aligned}$$

It can be proved that $I_{3,2} = 0$. By generalized Minkowski's inequality, we have that

$$\|I_{3,i}\|_1 \le \sum_{r=0}^{|n|-2} \omega\left(\frac{1}{2^r}, f\right)_X \int_{\mathbb{I}} \left| \sum_{j=2^r}^{2^{r+1}-1} t_{j,n} D_j(t) \right| dt \quad (i = 1, 3).$$

Equality (5) and Abel's transformation yield that

$$\begin{aligned} &\sum_{j=2^r}^{2^{r+1}-1} t_{j,n} D_j = \sum_{j=0}^{2^r-1} t_{2^r+j,n} D_{2^r} + w_{2^r} \sum_{j=1}^{2^r-1} t_{2^r+j,n} D_j \\ = &\sum_{j=0}^{2^r-1} t_{2^r+j,n} D_{2^r} \\ + &\ w_{2^r} \left(\sum_{j=1}^{2^r-2} (t_{2^r+j,n} - t_{2^r+j+1,n}) j K_j + t_{2^{r+1}-1,n} (2^{r+1}-1) K_{2^{r+1}-1} \right). \end{aligned}$$

Inequality (7) gives

$$\|\sum_{j=2^r}^{2^{r+1}-1} t_{j,n} D_j\|_1 \le \sum_{j=0}^{2^r-1} t_{2^r+j,n} + c\left(\sum_{j=1}^{2^r-2} |t_{2^r+j,n} - t_{2^r+j+1,n}| j + t_{2^{r+1}-1,n} (2^{r+1}-1) \right).$$

For sequence $t_{k,n} \downarrow$, we write

$$\|\sum_{j=2^r}^{2^{r+1}-1} t_{j,n} D_j\|_1 \le c \sum_{j=0}^{2^r-1} t_{2^r+j,n} \le c 2^r t_{2^r,n}.$$

For sequence $t_{k,n} \uparrow$, we have

$$\|\sum_{j=2^r}^{2^{r+1}-1} t_{j,n} D_j\|_1 \le c 2^r t_{2^{r+1}-1,n}.$$

That is, for a monotone non-increasing sequence (in sign $t_{k,n} \downarrow$), we have

$$\|I_{3,i}\|_X \le c \sum_{r=0}^{|n|-2} 2^r t_{2^r,n} \omega\left(\frac{1}{2^r}, f\right)_X \quad (i = 1, 3) \tag{37}$$

and for a monotone non-decreasing sequence (in sign $t_{k,n} \uparrow$),

$$\|I_{3,i}\|_X \leq c \sum_{r=0}^{|n|-2} 2^r t_{2^{r+1}-1,n} \omega\left(\frac{1}{2^r}, f\right)_X \quad (i = 1, 3). \tag{38}$$

For a monotone non-increasing sequence (in sign $t_{k,n} \downarrow$), we proved that

$$\|I_3\|_X \leq c \sum_{r=0}^{|n|-2} 2^r t_{2^r,n} \omega\left(\frac{1}{2^r}, f\right)_X. \tag{39}$$

For a monotone non-decreasing sequence (in sign $t_{k,n} \uparrow$), we reached that

$$\|I_3\|_X \leq c \sum_{r=0}^{|n|-2} 2^r t_{2^{r+1}-1,n} \omega\left(\frac{1}{2^r}, f\right)_X. \tag{40}$$

Combining (31), (34)–(36), (39) and (40), we complete the proof. □

Corollary 2. *Let $f \in X(\mathbb{I})$ and $\{m_n : n \in \mathbb{P}\}$ be a strictly monotone increasing sequence. Let $\{t_{l,m_n} : 1 \leq l \leq m_n\}$ be a monotone non-decreasing sequence of non-negative numbers (in sign $t_{l,m_n} \uparrow$). Let the condition*

$$\omega\left(\frac{1}{2^{|m_n|}}, f\right)_X = o\left(\frac{1}{V(m_n, \{t_{l,m_n}\})}\right) \tag{41}$$

be satisfied. Then, the subsequence $t_{m_n}(f)$ converges in the norm of the space $X(\mathbb{I})$.

Corollary 3. *Let $f \in X(\mathbb{I})$ and $\{t_{l,m_n} : 1 \leq l \leq m_n\}$ be a monotone non-decreasing sequence of non-negative numbers (in sign $t_{l,m_n} \uparrow$). Let the sequence $\{m_n : n \in \mathbb{P}\}$ be such that the next condition holds*

$$\sup_n V(m_n, \{t_{l,m_n}\}) < \infty.$$

Then, the subsequence $t_{m_n}(f)$ converges in the norm of the space $X(\mathbb{I})$.

The next theorem proofs the sharpness of condition (41).

Theorem 4. *Let the sequences $\{t_{l,n} : 1 \leq l \leq n\}$ be monotone non-decreasing (in sign $t_{l,n} \uparrow$) for all $n \in \mathbb{P}$. Let $\{m_A : A \in \mathbb{N}\}$ be a sequence of natural numbers such that*

$$\sup_A V(m_A, \{t_{l,m_A}\}) = \infty.$$

Then, there exists a sequence $\{p_j : j \in \mathbb{N}\}$ and a function $f \in X(\mathbb{I})$ such that

$$\omega\left(\frac{1}{2^{|m_{p_j}|}}, f\right)_X = O\left(\frac{1}{V(m_{p_j}, \{t_{l,m_{p_j}}\})}\right)$$

and

$$\|t_{m_{p_l}}(f) - f\|_X \not\to 0 \quad \text{as} \quad l \to \infty.$$

Proof of Theorem 4. Let the sequence $\{t_{k,n} : 1 \leq k \leq n\}$ be monotone non-decreasing (in sign $t_{k,n} \uparrow$) for all $n \in \mathbb{P}$. Then, condition

$$\sup_n V(m_n, \{t_{k,m_n}\}) = \infty$$

yields that there exists a sequence $\{p_l : l \in \mathbb{N}\}$ such that the following two conditions hold

$$|m_{p_l}| > |m_{p_{l-1}}| + 2\log(l+1) \tag{42}$$

and

$$V\left(m_{p_l}, \{t_{s,m_{p_l}}\}\right) \geq 2lV\left(m_{p_{l-1}}, \{t_{s,m_{p_{l-1}}}\}\right). \tag{43}$$

First, let us discuss $X(\mathbb{I}) = L_1(\mathbb{I})$. Now, we set

$$g(x) := \sum_{j=1}^{\infty} g_j(x), \quad g_j(x) := \frac{D_{2^{|m_{p_j}|+1}}(x)}{V\left(m_{p_j}, \{t_{s,m_{p_j}}\}\right)}.$$

It is easy to check that $g \in L_1(\mathbb{I})$. Let us calculate $\omega\left(\frac{1}{m_{p_k}}, g\right)_{L_1}$. We set $y \in I_{|m_{p_k}|}$, and we learn that

$$D_{2^{|m_{p_l}|}}(x \dotplus y) - D_{2^{|m_{p_l}|}}(x) = 0 \quad \text{for } l = 1, 2, \ldots, k-1. \tag{44}$$

Inequalities (43) and (44) yield that

$$\begin{aligned} &\int_{\mathbb{I}} |g(x \dotplus y) - g(x)| dx \\ \leq\ & \sum_{j=k}^{\infty} \frac{1}{V\left(m_{p_j}, \{t_{s,m_{p_j}}\}\right)} \int_{\mathbb{I}} \left| D_{2^{|m_{p_j}|+1}}(x \dotplus y) - D_{2^{|m_{p_j}|+1}}(x) \right| dx \\ \leq\ & \frac{c}{V\left(m_{p_k}, \{t_{s,m_{p_k}}\}\right)}. \end{aligned}$$

Consequently, taking the supremum for all $y \in I_{|m_{p_k}|}$, we have that

$$\omega\left(\frac{1}{m_{p_k}}, g\right)_{L_1} = O\left(\frac{1}{V\left(m_{p_k}, \{t_{s,m_{p_k}}\}\right)}\right).$$

We can write

$$\begin{aligned} \left\| t_{m_{p_k}}(g) - g \right\|_1 \quad \geq \quad & \left\| t_{m_{p_k}}\left(\sum_{j=k}^{\infty} g_j\right) \right\|_1 - \sum_{j=k}^{\infty} \|g_j\|_1 \\ & - \left\| t_{m_{p_k}}\left(\sum_{j=1}^{k-1} g_j\right) - \sum_{j=1}^{k-1} g_j \right\|_1. \end{aligned} \tag{45}$$

For $j \geq k$

$$\begin{aligned} t_{m_{p_k}}(g_j) &= g_j * F_{m_{p_k}} = F_{m_{p_k}} * g_j = \frac{1}{V\left(m_{p_j}, \{t_{s,m_{p_j}}\}\right)} S_{2^{|m_{p_j}|+1}}\left(F_{m_{p_k}}\right) \\ &= \frac{1}{V\left(m_{p_j}, \{t_{s,m_{p_j}}\}\right)} F_{m_{p_k}}. \end{aligned}$$

From inequality (19), we have that

$$\left\| t_{m_{p_k}}\left(\sum_{j=k}^{\infty} g_j\right) \right\|_1 = \sum_{j=k}^{\infty} \frac{\left\| F_{m_{p_k}} \right\|_1}{V\left(m_{p_j}, \{t_{s,m_{p_j}}\}\right)} \geq \frac{\left\| F_{m_{p_k}} \right\|_1}{V\left(m_{p_k}, \{t_{s,m_{p_k}}\}\right)} \geq 1 > 0. \tag{46}$$

Equality (3) and condition (43) yield that

$$\sum_{j=k}^{\infty}\|g_j\|_1 \leq \sum_{j=k}^{\infty}\frac{1}{V\left(m_{p_j},\{t_{s,m_{p_j}}\}\right)} \leq \frac{2}{V\left(m_{p_k},\{t_{s,m_{p_k}}\}\right)}. \tag{47}$$

By Theorem 3 and (44), we obtain the following inequality $(j<k)$

$$\begin{aligned}\left\|t_{m_{p_k}}(g_j)-g_j\right\|_1 &\leq c\sum_{r=0}^{|m_{p_k}|-2}2^r t_{2^{r+1}-1,m_{p_k}}\,\omega\left(\frac{1}{2^r},g_j\right)_{L_1}\\ &\leq c\sum_{r=0}^{|m_{p_j}|-1}2^r t_{2^{r+1}-1,m_{p_k}}\,\omega\left(\frac{1}{2^r},g_j\right)_{L_1}\\ &\leq c\sum_{r=0}^{|m_{p_j}|-1}2^r t_{2^{r+1}-1,m_{p_k}}\\ &\leq c t_{2^{|m_{p_k}|-1},m_{p_k}}\sum_{r=0}^{|m_{p_j}|-1}2^r.\end{aligned}$$

Since the sequence $\{t_{s,m_{p_k}}\}$ is non-decreasing, we write

$$2^{|m_{p_k}|-1}t_{2^{|m_{p_k}|-1},m_{p_k}} \leq \sum_{s=2^{|m_{p_k}|-1}}^{2^{|m_{p_k}|}-1}t_{s,m_{p_k}} \leq \sum_{s=1}^{m_{p_k}}t_{s,m_{p_k}}=1$$

and

$$t_{2^{|m_{p_k}|-1},m_{p_k}} \leq \frac{1}{2^{|m_{p_k}|-1}}. \tag{48}$$

By inequality (42), we obtain

$$\left\|t_{m_{p_k}}(g_j)-g_j\right\|_1 \leq \frac{c}{2^{|m_{p_k}|-1}}\sum_{r=0}^{|m_{p_j}|-1}2^r \leq \frac{c}{k^2}$$

and

$$\sum_{j=1}^{k-1}\left\|t_{m_{p_k}}(g_j)-g_j\right\|_1 \leq \frac{c}{k}. \tag{49}$$

Combining (45)–(49), we have that

$$\overline{\lim_{k\to\infty}}\left\|t_{m_{p_k}}(g)-g\right\|_1>0.$$

Second, we discuss the case $X(\mathbb{I})=C_W(\mathbb{I})$. Let the condition (42) and (43) hold as well. We define the function h by

$$h(x):=\sum_{j=1}^{\infty}\frac{h_j(x)}{V\left(m_{p_j},\{t_{l,m_{p_j}}\}\right)},$$

where

$$h_j(x):=\operatorname{sgn}\left(F_{m_{p_j}}\right).$$

It is easily seen that $h\in C_w(\mathbb{I})$. Now, we calculate the modulus of continuity in C_W. Let $y\in I_{|m_{p_k}|}$, then for $j=1,2,\dots,k-1$, we obtain

$$h_j(x\dot{+}y)-h_j(x)=0.$$

Applying condition (43), we obtain

$$\begin{aligned} |h(x \dotplus y) - h(x)| &\leq 2\sum_{j=k}^{\infty} \frac{1}{V\left(m_{p_j}, \{t_{l,m_{p_j}}\}\right)} \\ &= O\left(\frac{1}{V\left(m_{p_k}, \{t_{l,m_{p_k}}\}\right)}\right). \end{aligned}$$

That is,

$$\omega\left(\frac{1}{m_{p_k}}, h\right)_{C_W} = O\left(\frac{1}{V\left(m_{p_k}, \{t_{l,m_{p_k}}\}\right)}\right).$$

It is easily seen that

$$\begin{aligned} \left|t_{m_{p_k}}(h,0) - h(0)\right| &\geq \frac{\left|t_{m_{p_k}}(h_k,0)\right|}{V\left(m_{p_k}, \{t_{l,m_{p_k}}\}\right)} - \sum_{j=k}^{\infty} \frac{|h_j(0)|}{V\left(m_{p_j}, \{t_{l,m_{p_j}}\}\right)} \\ &- \sum_{j=k+1}^{\infty} \frac{\left|t_{m_{p_k}}(h_j,0)\right|}{V\left(m_{p_j}, \{t_{l,m_{p_j}}\}\right)} - \sum_{j=1}^{k-1} \frac{\left|t_{m_{p_k}}(h_j,0) - h_j(0)\right|}{V\left(m_{p_j}, \{t_{l,m_{p_j}}\}\right)} \\ =: &\quad Q_1 - Q_2 - Q_3 - Q_4. \end{aligned} \tag{50}$$

Theorem 1, conditions (42) and (43) yield that

$$Q_1 \geq \frac{\left\|F_{m_{p_k}}\right\|_1}{V\left(m_{p_k}, \{t_{l,m_{p_k}}\}\right)} \geq c > 0, \tag{51}$$

$$Q_2 \leq \frac{c}{V\left(m_{p_k}, \{t_{l,m_{p_k}}\}\right)}, \tag{52}$$

$$Q_3 \leq \sum_{j=k+1}^{\infty} \frac{\left\|F_{m_{p_k}}\right\|_1}{V\left(m_{p_j}, \{t_{l,m_{p_j}}\}\right)} \leq \frac{cV\left(m_{p_k}, \{t_{l,m_{p_k}}\}\right)}{V\left(m_{p_{k+1}}, \{t_{l,m_{p_{k+1}}}\}\right)} \leq \frac{c}{k}. \tag{53}$$

We apply Theorem 3, inequality (48), conditions (42) and (43); we have that

$$\begin{aligned} Q_4 &\leq c\sum_{j=1}^{k-1} \sum_{r=0}^{|m_{p_k}|-2} 2^r t_{2^{r+1}-1,m_{p_k}} \omega\left(\frac{1}{2^r}, h_j\right)_{C_W} \\ &\leq c\sum_{j=1}^{k-1} \sum_{r=0}^{|m_{p_j}|-1} 2^r t_{2^{r+1}-1,m_{p_k}} \omega\left(\frac{1}{2^r}, h_j\right)_{C_W} \\ &\leq c\sum_{j=1}^{k-1} t_{2^{|m_{p_k}|-1},m_{p_k}} \sum_{r=0}^{|m_{p_j}|-1} 2^r \\ &\leq c\sum_{j=1}^{k-1} \frac{1}{2^{|m_{p_k}|-1}} \sum_{r=0}^{|m_{p_j}|-1} 2^r \\ &\leq \frac{ck2^{|m_{p_{k-1}}|}}{2^{|m_{p_k}|}} \leq \frac{c}{k}. \end{aligned} \tag{54}$$

Combining (50)–(54), we complete the proof of Theorem 4. □

8. Almost Everywhere Convergence of Matrix Transforms of Walsh–Fourier Series

Let us set $E_n(f;x) = S_{2^n}(f;x)$. The maximal function is defined by

$$E^*(f;x) = \sup_{n\in\mathbb{N}} |E_n(f;x)|.$$

It is known that ([1], p. 81) there exists a positive constant c such that

$$y|\{x \in \mathbb{I} : E^*(f;x) > y\}| \leq c\|f\|_1 \tag{55}$$

holds for all $f \in L_1(\mathbb{I})$ and $y > 0$.

We define the maximal operator t^* of the linear transforms t_n generated by the sequences $\{t_{k,n} : 1 \leq k \leq n\}$

$$t^*(f) := \sup_n |t_n(f)|.$$

In this section, we discuss some properties of the maximal operator $t^*(f)$. As a consequence, we learn that the matrix transforms $t_n(f)$ of the Walsh–Fourier series converge almost everywhere to the function f for all integrable functions. This result is reached with different monotonity conditions.

First, we state the boundedness of the maximal operator of the linear transforms defined by monotone non-increasing sequences.

Theorem 5. *Let $\{t_{k,n} : 1 \leq k \leq n\}$ be monotone non-increasing sequences of non-negative numbers (in sign $t_{k,n} \downarrow$) for all $n \in \mathbb{P}$. Then, the maximal operator t^* is bounded from the Lebesque space L_p to the Lebesque space L_p for all $1 < p \leq \infty$. That is, there exists a positive constant C_p which depends only on p such that*

$$\|t^*(f)\|_p \leq C_p\|f\|_p$$

holds for all $f \in L_p(\mathbb{I})$. Moreover, the maximal operator t^* is of weak type $(1,1)$. That is, there exists a positive constant c such that

$$\sup_{\lambda>0} \lambda|\{t^*(f) > \lambda\}| := \|t^*(f)\|_{weak-L_1(\mathbb{I})} \leq c\|f\|_1$$

holds for all $f \in L_1(\mathbb{I})$, $\lambda > 0$.

Proof of Theorem 5. Since (see (20))

$$f * F_n = \sum_{k=1}^{n-1} (t_{k,n} - t_{k+1,n})k(f * K_k) + t_{n,n}n(f * K_n),$$

$$\sup_{\lambda>0} \lambda \left| \{\sup_k |f * K_k| > \lambda\} \right| \leq c\|f\|_1, f \in L_1(\mathbb{I})$$

$$\left\| \sup_k |f * K_k| \right\|_p \leq c\|f\|_p, p \geq 1, f \in L_p(\mathbb{I})$$

and

$$\begin{aligned} \sup_n |f * F_n| &\leq c \sup_k |f * K_k| \left(\sum_{k=1}^{n-1} (t_{k,n} - t_{k+1,n})k + t_{n,n}n \right) \\ &\leq c \sup_k |f * K_k|, \end{aligned}$$

we complete the proof of Theorem 5. □

By the well-known density argument due to Marcinkiewicz and Zygmund [9], the next corollary holds.

Corollary 4. *Let $\{t_{k,n} : 1 \leq k \leq n\}$ be a monotone non-increasing sequence of non-negative numbers (in sign $t_{k,n} \downarrow$) for all $n \in \mathbb{P}$ and $f \in L_1(\mathbb{I})$. Then*

$$\lim_{n\to\infty} t_n(f;x) = f(x) \text{ for a. e. } x \in \mathbb{I}.$$

Now, we consider the following maximal operator

$$\sup_n |f * |K_n||.$$

We prove that the maximal operator is of weak (1,1) type. That is, there exists a positive constant c such that

$$\sup_{\lambda>0} \lambda \left| \{\sup_n |f * |K_n|| > \lambda\} \right| \leq c\|f\|_1 \tag{56}$$

holds for all $f \in L_1(\mathbb{I})$, $\lambda > 0$. For this, it is enough to prove that the operator $\sup_n |f * |K_n||$ is quasi-local and bounded from the space $L_\infty(I)$ to the space $L_\infty(I)$ (see [1]). The boundedness immediately follows from (7). Now, we prove the quasi-locality. In particular, let $f \in L_1(\mathbb{I})$ such that $\operatorname{supp}(f) \subset I_N(u')$, $\int_{I_N(u')} f = 0$ for some dyadic interval $I_N(u')$. Then, we show that there exists a positive constant c such that the next inequality

$$\int_{\overline{I_N(u')}} \sup_n |f * |K_n|| \leq c\|f\|_1$$

holds. It can be supposed that $u' = 0$. If $n \leq 2^N$, then

$$|f * |K_n|| = 0.$$

Consequently, $n > 2^N$ can be supposed.

It is known that (see Gát [10])

$$\int_{\overline{I}_N} \sup_{n\geq 2^N} |K_n| < \infty, \tag{57}$$

Then, we have

$$\begin{aligned}
& \int_{\overline{I}_N} \sup_{n\geq 2^N} |(f * |K_n|)(x)| dx \\
= & \int_{\overline{I}_N} \sup_{n\geq 2^N} \left| \int_{\mathbb{I}} f(t) |K_n(x \dot{+} t)| dt \right| dx \\
\leq & \int_{\mathbb{I}} |f(t)| \int_{\overline{I}_N} \sup_{n\geq 2^N} |K_n(x \dot{+} t)| dx dt \\
\leq & \; c\|f\|_1.
\end{aligned}$$

Hence, (56) is proved.

From (24), we can write

$$\begin{aligned} F_{n,2} &= w_n \sum_{s=0}^{|n|} \varepsilon_s(n) w_{n_{(s)}} D_{2^s} \sum_{l=1}^{2^s-1} t_{n^{(s)}-l,n} \\ &\quad - w_n \sum_{s=0}^{|n|} \varepsilon_s(n) w_{n_{(s)}} w_{2^s-1} \\ &\quad \times \left(\sum_{l=1}^{2^s-2} (t_{n^{(s)}-l,n} - t_{n^{(s)}-l-1,n}) l K_l + t_{n^{(s)}-2^s+1,n} (2^s-1) K_{2^s-1} \right). \end{aligned}$$

Let us set

$$\begin{aligned} \widetilde{F}_{n,2} \ :&= \sum_{s=0}^{|n|} \varepsilon_s(n) D_{2^s} \sum_{l=1}^{2^s-1} t_{n^{(s)}-l,n} \\ &\quad + \sum_{s=0}^{|n|} \varepsilon_s(n) \sum_{l=1}^{2^s-2} (t_{n^{(s)}-l,n} - t_{n^{(s)}-l-1,n}) l |K_l| \\ &\quad + \sum_{s=0}^{|n|} \varepsilon_s(n) t_{n^{(s)}-2^s+1,n} (2^s-1) |K_{2^s-1}|. \end{aligned}$$

It is easy to see that

$$|F_{n,2}| \le \widetilde{F}_{n,2}.$$

In order to prove Theorem 6, we need the following lemmas.

Lemma 4. *Let $\{t_{k,n} : k = 1, \dots, n\}$ be a monotone non-decreasing sequence of non-negative numbers for every fixed $n \in \mathbb{N}$. The operator $\sup_{n\in\mathbb{N}} \left| f * \widetilde{F}_{n,2} \right|$ is of weak type $(1,1)$. That is, there exists a positive constant c such that*

$$\sup_{\lambda>0} \lambda \left| \left\{ \sup_{n\in\mathbb{N}} \left| f * \widetilde{F}_{n,2} \right| > \lambda \right\} \right| \le c \|f\|_1$$

holds for all $f \in L_1(\mathbb{I})$, $\lambda > 0$.

Proof of Lemma 4. We can write

$$\begin{aligned} &\left| f * \widetilde{F}_{n,2} \right| \\ \le\ & \sum_{s=0}^{|n|} \varepsilon_s(n) \sum_{l=1}^{2^s-1} t_{n^{(s)}-l,n} |f * D_{2^s}| \\ &+ \sum_{s=0}^{|n|} \varepsilon_s(n) \sum_{l=1}^{2^s-2} (t_{n^{(s)}-l,n} - t_{n^{(s)}-l-1,n}) l |f * |K_l|| \\ &+ \sum_{s=0}^{|n|} \varepsilon_s(n) t_{n^{(s)}-2^s+1,n} (2^s-1) |f * |K_{2^s-1}|| \end{aligned} \tag{58}$$

Since

$$\sum_{s=0}^{|n|} \varepsilon_s(n) \sum_{l=1}^{2^s-1} t_{n^{(s)}-l,n} = \sum_{s=0}^{|n|} \varepsilon_s(n) \sum_{l=n^{(s+1)}+1}^{n^{(s)}-1} t_{l,n} \le \sum_{k=1}^{n} t_{k,n} \le c < \infty$$

and

$$
\begin{aligned}
& \sum_{s=0}^{|n|} \varepsilon_s(n) \sum_{l=1}^{2^s-2} (t_{n^{(s)}-l,n} - t_{n^{(s)}-l-1,n}) l \\
& + \sum_{s=0}^{|n|} \varepsilon_s(n) t_{n^{(s)}-2^s+1,n} (2^s - 1) \\
= & \sum_{s=0}^{|n|} \varepsilon_s(n) \sum_{l=1}^{2^s-2} \Big[(l t_{n^{(s)}-l,n} - (l+1) t_{n^{(s)}-l-1,n}) + t_{n^{(s)}-l-1,n} \Big] \\
& + \sum_{s=0}^{|n|} \varepsilon_s(n) t_{n^{(s)}-2^s+1,n} (2^s - 1) \\
= & \sum_{s=0}^{|n|} \varepsilon_s(n) \Bigg[(t_{n^{(s)}-1,n} - (2^s-1) t_{n^{(s+1)}+1,n}) + \sum_{l=1}^{2^s-2} t_{n^{(s)}-l-1,n} \Bigg] \\
& + \sum_{s=0}^{|n|} \varepsilon_s(n) t_{n^{(s+1)}+1,n} (2^s - 1) \\
= & \sum_{s=0}^{|n|} \varepsilon_s(n) \sum_{l=0}^{2^s-2} t_{n^{(s)}-l-1,n} \\
\leq & \sum_{s=0}^{|n|} \varepsilon_s(n) \sum_{l=n^{(s)}-1}^{n^{(s+1)}} t_{l,n} \leq \sum_{k=1}^{n} t_{k,n} \leq c < \infty,
\end{aligned}
$$

from (58), we have

$$
\sup_{n\in\mathbb{N}} \left| f * \widetilde{F}_{n,2} \right| \leq c \left(\sup_{k\in\mathbb{N}} |f * |K_k|| + E^*(f) \right)
$$

and consequently, by (56) and (55), we complete the proof of Lemma 4. □

Theorem 6. *Let $\{m_A : A \in \mathbb{P}\}$ be a strictly monotone increasing sequence. Let $\{t_{k,n} : k = 1, \ldots, n\}$ be a monotone non-decreasing sequence of non-negative numbers for every fixed $n \in \mathbb{N}$. If*

$$
\sup_{A\in\mathbb{P}} V(m_A, \{t_{k,m_A}\}) < \infty \tag{59}
$$

holds, then there exists a positive constant c such that

$$
\sup_{\lambda>0} \lambda \left| \{ \sup_A |t_{m_A}(f)| > \lambda \} \right| \leq c \|f\|_1
$$

holds for all $f \in L_1(\mathbb{I})$, $\lambda > 0$.

Proof of Theorem 6. We have (see (9))

$$
\begin{aligned}
t_n(f) &= f * w_n F_n^* \\
&= f * w_n F_{n,1}^* + f * w_n F_{n,2}^*.
\end{aligned} \tag{60}
$$

We obtain

$$\begin{aligned} & \left|f * w_n F_{n,1}^*\right| \\ = & \left|\sum_{k=0}^{\infty} \alpha_k(n)\left(S_{2^{k+1}}(f w_n)-S_{2^k}(f w_n)\right)\right| \\ \leq & \left|\sum_{k=0}^{\infty}\left(\alpha_k(n)-\alpha_{k+1}(n)\right) S_{2^{k+1}}(f w_n)\right|+\left|\alpha_0(n) S_{2^0}(f w_n)\right| \\ \leq & E^*(|f|) \sum_{k=0}^{\infty}\left|\alpha_k(n)-\alpha_{k+1}(n)\right|+\left|\alpha_0(n)\right| \end{aligned}$$

where $\alpha_k(n)=\varepsilon_k(n) T_n^{(n^{(k)})}$. Since

$$\begin{aligned} & \sup_{n \in \mathbb{N}}\left(\sum_{k=0}^{\infty}\left|\alpha_k(n)-\alpha_{k+1}(n)\right|+\left|\alpha_0(n)\right|\right) \\ \leq & \sup_{n \in \mathbb{N}}\left(\sum_{k=0}^{\infty}\left|\varepsilon_k(n)-\varepsilon_{k+1}(n)\right| T_n^{(n^{(k)})}\right)+c<\infty, \end{aligned}$$

we conclude that

$$\sup_{n \in \mathbb{N}}\left|f * w_n F_{n,1}^*\right| \leq c E^*(|f|, x).$$

Consequently, we can write

$$\left\|\sup_{n \in \mathbb{N}}\left|f * w_n F_{n,1}^*\right|\right\|_{\text{weak}-L_1(\mathbb{I})} \leq c\|f\|_1. \tag{61}$$

By Lemma 4, we obtain

$$\left\|\sup_{n \in \mathbb{N}}\left|f * w_n F_{n,2}^*\right|\right\|_{\text{weak}-L_1(\mathbb{I})} \leq c\|f\|_1. \tag{62}$$

We combine (60), (61) and (62) in order to obtain

$$\left\|\sup_{n \in \mathbb{N}}\left|t_n(f)\right|\right\|_{\text{weak}-L_1(\mathbb{I})} \leq c\|f\|_1.$$

Theorem 6 is proved. □

Let us define for positive real numbers K the subset $L_K(\{t_{k,n}\})$ of natural numbers by

$$L_K(\{t_{k,n}\}):=\left\{n \in \mathbb{N}: \sum_{k=1}^{|n|}\left|\varepsilon_k(n)-\varepsilon_{k+1}(n)\right| T^{(n^{(k)})} \leq K\right\}.$$

The next corollary follows from Theorem 6 by the well-known density argument due to Marcinkiewicz and Zygmund [9].

Corollary 5. *Let $\{t_{k,n}: k=1, \ldots, n\}$ be a monotone non-decreasing sequence of non-negative numbers for every fixed $n \in \mathbb{N}$ and $f \in L_1(\mathbb{I})$. Then, $t_n(f; x) \rightarrow f$ almost everywhere provided that $n \rightarrow \infty$ and $n \in L_K(\{t_{k,n}\})$.*

9. Application: Cesàro Means With Varying Parameters of Walsh–Fourier Series

The theorems can be used for various methods of summability. In this section, the application of the theorems proved above to Cesàro means with varying parameters will be presented.

The (C, α_n) means of the Walsh–Fourier series of the function f is given by

$$\sigma_n^{\alpha_n}(f, x) = \frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=1}^{n} A_{n-j}^{\alpha_n - 1} S_j(f, x),$$

where

$$A_n^{\alpha_n} := \frac{(1+\alpha_n)\ldots(n+\alpha_n)}{n!}$$

for any $n \in \mathbb{N}, \alpha_n \neq -1, -2, \ldots$. The (C, α_n) kernel is defined by

$$K_n^{\alpha_n} = \frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=1}^{n} A_{n-j}^{\alpha_n - 1} D_j.$$

We shall need the following Lemma (see [11]).

Lemma 5. *Let $k, n \in \mathbb{N}$. Then*

$$c_1(d) k^{\alpha_n} < A_k^{\alpha_n} < c_2(d) k^{\alpha_n}, 0 < \alpha_n \leq d. \tag{63}$$

The idea of Cesàro means with variable parameters of numerical sequences is due to Kaplan [12], and the introduction of these (C, α_n) means of Fourier series is due to Akhobadze [11].

The almost everywhere convergence of the subsequence of Cesàro means with variable parameters has been studied by the following authors: Abu Joudeh and Gát [6], Gát and Goginava [13,14], Weisz [15].

Let $t_{k,n} = A_{n-k}^{\alpha_n - 1} / A_{n-1}^{\alpha_n}, 0 \leq k \leq n$. Then, from (63), we have

$$T_n^{(n^{(s)})} = \sum_{l=n^{(s)}}^{n} \frac{A_{n-l}^{\alpha_n - 1}}{A_{n-1}^{\alpha_n}} = \sum_{l=0}^{n_{(s-1)}} \frac{A_l^{\alpha_n - 1}}{A_{n-1}^{\alpha_n}} = \frac{A_{n_{(s-1)}}^{\alpha_n}}{A_{n-1}^{\alpha_n}} \leq \frac{c 2^{s\alpha_n}}{2^{|n|\alpha_n}}.$$

Hence, from Corollary 5, we obtain

Theorem 7 (see [14])**.** *Suppose that $\alpha_n \in (0, 1)$. Let $f \in L_1(\mathbb{I})$. Then, $\sigma_n^{\alpha_n}(f) \to f$ almost everywhere provided that $n \to \infty$ and $n \in L_K\left(\left\{ A_{n-k}^{\alpha_n - 1} / A_{n-1}^{\alpha_n} \right\}\right)$.*

Now, we consider the rate of convergence of the Cesàro means with varying parameters of Walsh–Fourier series. Since

$$V(n, \{t_{k,n}\}) \leq \frac{c}{2^{|n|\alpha_n}} \sum_{k=1}^{|n|} |\varepsilon_k(n) - \varepsilon_{k+1}(n)| 2^{k\alpha_n} \leq \frac{c}{\alpha_n}$$

and (see Lemma 5)

$$t_{2^{r+1}-1,n} = \frac{A_{n-2^{r+1}+1}^{\alpha_n - 1}}{A_n^{\alpha_n}} \sim \alpha_n \frac{\left(n - 2^{r+1} + 1\right)^{\alpha_n - 1}}{n^{\alpha_n}} \sim \alpha_n 2^{-|n|}$$

from Theorem 3, we have

Theorem 8. *Let $f \in X(\mathbb{I})$ and $\alpha_n \in (0,1)$. Then,*

$$\|\sigma_n^{\alpha_n}(f) - f\|_X \leq \frac{c_1}{\alpha_n}\omega\left(\frac{1}{2^{|n|}}, f\right)_X + c_2\omega\left(\frac{1}{2^{|n|-1}}, f\right)_X + c_3\alpha_n \sum_{r=0}^{|n|-2} 2^{r-|n|}\omega\left(\frac{1}{2^r}, f\right)_X.$$

Author Contributions: Investigation, U.G. and K.N.; Methodology, U.G.; Writing—original draft, U.G.; Writing—review & editing, K.N. All authors have read and agreed to the published version of the manuscript.

Funding: The authors are very thankful to United Arab Emirates University (UAEU) for the Start-up Grant 12S100.

Conflicts of Interest: The authors declare no conflict of interest.

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