mathematics

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3-Hom–Lie Yang–Baxter Equation and 3-Hom–Lie Bialgebras

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Abstract: In this paper, we first introduce the notion of a 3-Hom–Lie bialgebra and give an equivalent description of the 3-Hom–Lie bialgebras, the matched pairs and the Manin triples of 3-Hom–Lie algebras. In addition, we define $\mathcal{O}$-operators of 3-Hom–Lie algebras and construct solutions of the 3-Hom–Lie Yang–Baxter equation in terms of $\mathcal{O}$-operators and 3-Hom–pre-Lie algebras. Finally, we show that a 3-Hom–Lie algebra has a phase space if and only if it is sub-adjacent to a 3-Hom–pre-Lie algebra.

Keywords: 3-Hom–Lie algebra; Manin triple; matched pair; symplectic structure; representation

MSC: 17A40; 17B38

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Citation: Guo, S.; Wang, S.; Zhang, X. 3-Hom–Lie Yang–Baxter Equation and 3-Hom–Lie Bialgebras. *Mathematics* **2022**, *10*, 2485. https://doi.org/10.3390/math10142485

Academic Editor: Shuanhong Wang

Received: 10 June 2022
Accepted: 14 July 2022
Published: 17 July 2022

1. Introduction

Hom–algebras were first introduced in the Lie algebra setting [1] with motivation from physics though the origin can be traced back in earlier literature such as [2], where the Jacobi identity was twisted by an endomorphism, namely $[\alpha(x),[y,z]]+[\alpha(y),[z,x]]+[\alpha(z),[x,y]]=0$. In [3], Yau extended the notion of Lie bialgebras to Hom–Lie bialgebras and studied the classical Hom–Yang–Baxter equation using the twisted map, namely

$$CH(r)=[r^{12},r^{13}]+[r^{12},r^{23}]+[r^{13},r^{23}]=0.$$

In [4], Sheng and Bai defined a new kind of Hom–Lie bialgebra which was equivalent to Manin triples of Hom–Lie algebras and constructed solutions of the classical Hom–Yang–Baxter equation in terms of $\mathcal{O}$-operators. Later, in [5], Tao, Bai and Guo introduced the notion of a Hom–Lie bialgebra with emphasis on its compatibility with a Manin triple of Hom–Lie algebras associated to a nondegenerate symmetric bilinear form satisfying a new invariance condition.

3-Lie algebras were special types of n-Lie algebras and played an important role in string theory [6,7]. In [8], Sheng and Tang proved that a 3-Lie algebra has a phase space if and only if it is sub-adjacent to a 3-pre-Lie algebra. In [9], Ataguema, Makhlouf and Silvestrov extended the notion of 3-Lie algebras to 3-Hom–Lie algebras and presented constructions from 3-Lie algebras. Because of close relation to discrete and conformal vector fields, 3-Lie algebras and 3-Hom–Lie algebras were widely studied in the following aspects. In [10], Liu, Chen and Ma described the representations and module-extensions of 3-Hom–Lie algebras. In [11], Abdaoui, Mabrouk, Makhlouf and Massoud introduced and studied 3-Hom–Lie bialgebras, which are a ternary version of Hom–Lie bialgebras introduced by Yau. In [12], Ben Hassine, Chtioui and Mabrouk introduced the notion of 3-Hom–L-dendriform algebras which is the dendriform version of 3-Hom–Lie-algebras and studied their properties, the authors introduced the classical Yang–Baxter equation and Manin triples for 3-Lie algebras in [13,14]. Recently, we introduced the notion of 3-Hom–Lie-Rinehart algebras and systematically described a cohomology complex by considering coefficient modules in [15]. Motivated by the work of [4,8], it is natural and meaningful to study 3-Hom–Lie bialgebras and the phase space on 3-Hom–Lie algebras. This becomes our first motivation for writing the present paper.

The classical Yang–Baxter equation was investigated by Sklyanin [16] in the context of quantum inverse scattering method, which has a close connection with many branches of mathematical physics and pure mathematics. In [3], Yau extended the notion of classical Yang–Baxter equation to classical Hom–Yang–Baxter equation and presented some solutions using the twisting method. In [17], Wang, Wu and Cheng studied the 3-Lie classical Hom–Yang–Baxter equation on coboundary local cocycle 3-Hom–Lie bialgebras. Recently, the classical Hom–Yang–Baxter equation in Hom–Lie algebras has been studied widely in terms of Hom–$\mathcal{O}$-operators [18] and quasitriangular structures [3]. Motivated by the recent work on the classical Hom–Yang–Baxter equation, in this paper, we will study 3-Lie classical Hom–Yang–Baxter equation in terms of $\mathcal{O}$-operators. This becomes another motivation for writing the present paper.

In this paper, we continue the study of 3-Hom–Lie algebras and give a new description of 3-Hom–Lie bialgebras. It needs to be emphasized that there are results on 3-Hom–Lie algebras in this paper which are not "parallel" to the case of Hom–Lie algebras given in [4]. Because of the complexity of 3-Hom–Lie algebras, we need some technique to complete this paper. Now given a 3-Hom–Lie bialgebra (L, L^*), $L \oplus L^*$ is a 3-Hom–Lie algebra such that $(L \oplus L^*, L, L^*)$ is a Manin triple of 3-Hom–Lie algebras. We also study the 3-Lie classical Hom–Yang–Baxter equation in detail, and construct a solution in the semidirect 3-Hom–Lie algebra by introducing a notion of an $\mathcal{O}$-operator for a 3-Hom–Lie algebra. Finally, we describe symplectic structures and phase spaces of 3-Hom–Lie algebras from 3-Hom–pre-Lie algebra structures.

This paper is organized as follows. In Section 2, we recall some concepts and results, and introduce the notions of the matched pairs of 3-Hom–Lie algebras, the 3-Hom–Lie bialgebras and the Manin triples of 3-Hom–Lie algebras. In Section 3, we introduce the notion of the $\mathcal{O}$-operator and construct solutions of the 3-Lie classical Hom–Yang–Baxter equation in terms of $\mathcal{O}$-operators and 3-Hom–pre-Lie algebras. In Section 4, we introduce the notion of the phase space of a 3-Hom–Lie algebra and show that a 3-Hom–Lie algebra has a phase space if and only if it is sub-adjacent to a 3-Hom–pre-Lie algebra.

2. 3-Hom–Lie Bialgebras

In this section, we will recall some basic notions and facts about 3-Hom–Lie algebras and present some examples. Then we give an equivalent description of the 3-Hom–Lie bialgebras, the matched pairs and the Manin triples of 3-Hom–Lie algebras.

Definition 1 ([19])**.** *A 3-Hom–Lie algebra is a triple $(L, [\cdot,\cdot,\cdot], \alpha)$ consisting of a vector space L, a 3-ary skew-symmetric operation $[\cdot,\cdot,\cdot] : \wedge^3 L \to L$ and an algebra morphism $\alpha : L \to L$ satisfying the following 3-Hom–Jacobi identity*

$$\begin{aligned}[\alpha(x),\alpha(y),[u,v,w]] &= [[x,y,u],\alpha(v),\alpha(w)] + [\alpha(u),[x,y,v],\alpha(w)] \\ &\quad +[\alpha(u),\alpha(v),[x,y,w]],\end{aligned}$$

for any $x, y, u, v, w \in L$.

A 3-Hom–Lie algebra is called regular if α is an algebra automorphism.

Example 1. *Let $(L, [\cdot,\cdot,\cdot])$ be a 3-Lie algebra and $\alpha : L \to L$ an algebra morphism, then the algebra $(L, [\cdot,\cdot,\cdot]_\alpha, \alpha)$ is a 3-Hom–Lie algebra, where $[\cdot,\cdot,\cdot]_\alpha$ is defined by*

$$[x_1, x_2, x_3]_\alpha = [\alpha(x_1), \alpha(x_2), \alpha(x_3)].$$

Example 2. *Let $(L, [\cdot,\cdot,\cdot], \alpha)$ be a 3-Hom–Lie algebra and $\beta : L \to L$ an algebra morphism such that $\alpha\beta = \beta\alpha$, then $(L, [\cdot,\cdot,\cdot]_{\alpha\beta} = [\cdot,\cdot,\cdot] \circ (\beta \otimes \beta \otimes \alpha), \alpha \circ \beta)$ is a 3-Hom–Lie algebra.*

Example 3. *Let $(L, [\cdot,\cdot,\cdot], \alpha)$ be a 3-Hom–Lie algebra over a filed F and t an indeterminate, define $\overline{L} : \{\sum(x \otimes t + y \otimes t^n) \subset L \otimes (F[t]/t^{n+1}) | x, y \in L)\}$, $\overline{\alpha}(\overline{L}) = \{\sum(\alpha(x) \otimes t + \alpha(y) \otimes t^n) : x, y \in L\}$. Then $(\overline{L}, \overline{\alpha})$ is a 3-Hom–Lie algebra with the operation $[x_1 \otimes t^{i_1}, x_2 \otimes t^{i_2}, x_3 \otimes t^{i_3}] = [x_1, x_2, x_3] \otimes t^{\sum i_j}$ for all $x_1, x_2, x_3 \in L$ and $i_1, i_2, i_3 \in \{1, 2, 3\}$.*

Definition 2 ([10]). *A representation of a 3-Hom–Lie algebra $(L, [\cdot,\cdot,\cdot], \alpha)$ on the vector space V with respect to $A \in gl(V)$ is a bilinear map $\rho : L \wedge L \to gl(V)$, such that for any $x, y, z, u, v \in L$, the following equalities are satisfied:*

$$\begin{aligned}
&\rho(\alpha(u), \alpha(v)) \circ A = A \circ \rho(u, v), \\
&\rho([x, y, z], \alpha(u)) \circ A = \rho(\alpha(y), \alpha(z))\rho(x, u) + \rho(\alpha(z), \alpha(x))\rho(y, u) \\
&\qquad\qquad\qquad\qquad\qquad\qquad + \rho(\alpha(x), \alpha(y))\rho(z, u), \\
&\rho(\alpha(x), \alpha(y))\rho(z, u) = \rho(\alpha(z), \alpha(u))\rho(x, y) + \rho([x, y, z], \alpha(u)) \circ A \\
&\qquad\qquad\qquad\qquad\qquad\qquad + \rho(\alpha(z), [x, y, u]) \circ A.
\end{aligned}$$

Then (V, A, ρ) is called a representation of L.

Lemma 1 ([10]). *Let (V, A, ρ) be a representation of a 3-Hom–Lie algebra $(L, [\cdot,\cdot,\cdot], \alpha)$. Then there is a 3-Hom–Lie algebra structure on the direct sum of vector spaces $L \oplus V$, defined by*

$$\begin{aligned}
[x_1 + v_1, x_2 + v_2, x_3 + v_3] &= [x_1, x_2, x_3] + \rho(x_1, x_2)v_3 + \rho(x_2, x_3)v_1 + \rho(x_3, x_1)v_2, \\
(\alpha \oplus A)(x_1 + v_1) &= \alpha(x_1) + Av_1,
\end{aligned}$$

for any $x_1, x_2, x_3 \in L$ and $v_1, v_2, v_3 \in V$.

Example 4. *Let $(L, [\cdot,\cdot,\cdot], \alpha)$ be a 3-Hom–Lie algebra and $ad(x, y)z = [x, y, z]$, for all $x, y, z \in L$. Then, (L, α, ad) is called a regular representation of L.*

Definition 3. *Let $(L, [\cdot,\cdot,\cdot], \alpha)$ and $(L', [\cdot,\cdot,\cdot]', \alpha')$ be two 3-Hom–Lie algebras. A morphism from $(L, [\cdot,\cdot,\cdot], \alpha)$ to $(L', [\cdot,\cdot,\cdot]', \alpha')$ is a 3-Lie algebra morphism $f : L \to L'$ satisfying $f \circ \alpha = \alpha' \circ f$.*

Proposition 1. *If $f : (L, [\cdot,\cdot,\cdot], \alpha) \longrightarrow (L', [\cdot,\cdot,\cdot]', \alpha')$ is a 3-Hom–Lie algebras morphism, then (L', ρ, α') becomes a representation of L via f, that is, for all $(x, y, z) \in L^2 \times L'$, $\rho(x, y)z = [f(x), f(y), z]'$.*

Proof. First, for any $x, y \in L, z \in L'$ we have

$$\begin{aligned}
\rho(\alpha(x), \alpha(y))\alpha'(z) &= [f(\alpha(x)), f(\alpha(y)), \alpha'(z)]' \\
&= [\alpha'(f(x)), \alpha'(f(y)), \alpha'(z)]' \\
&= \alpha'[f(x), f(y), z]' \\
&= \alpha'\rho(x, y)z.
\end{aligned}$$

Next, for all $x, y, z, u \in L, z \in L'$ we have

$$\begin{aligned}
&\rho([x,y,z],\alpha(u))\circ\alpha' - \rho(\alpha(y),\alpha(z))\rho(x,u) - \rho(\alpha(z),\alpha(x))\rho(y,x) - \rho(\alpha(x),\alpha(y))\rho(z,u)\\
&\quad = [f([x,y,z],f(\alpha(x)),\alpha'(v)]' - [f(\alpha(y)),f(\alpha(z)),\rho(x,u)v]'\\
&\qquad - [f(\alpha(z)),f(\alpha(x)),\rho(y,u)v]' - [f(\alpha(x)),f(\alpha(y)),\rho(z,u)v]'\\
&\quad = [[f(x),f(y),f(z)]'\alpha' f(u)),\alpha'(v)]' - [\alpha' f(y)),\alpha'(f(z)),[f(x),f(u),v]']'\\
&\qquad - [\alpha' f(z)),\alpha'(f(x)),[f(y),f(u),v]']' - [\alpha' f(x)),\alpha'(f(y)),[f(z),f(u),v]']'\\
&\quad = 0 (by 3 - Hom Jacobi identity),\\
&\rho(\alpha(x),\alpha(y))\rho(z,u) - \rho(\alpha(z),\alpha(u))\rho(x,y) - \rho([x,y,z],\alpha(u))\alpha'(v) - \rho(\alpha(z),[x,y,u])\alpha'(v)\\
&\quad = [f(\alpha(x),f(\alpha(y),\rho(z,u)v]' - [f(\alpha(z),f(\alpha(u),\rho(x,y)v]'\\
&\qquad - [f([x,y,z],f(\alpha(u)),\alpha'(v)]' - [f(\alpha(z)),f([x,y,u]),\alpha'(v)]'\\
&\quad = [\alpha'(f(x)),\alpha'(f(y)),[f(z),f(u),v]']' - [\alpha'(f(u)),[f(x),f(y),v]']'\\
&\qquad - [[f(x),f(y),f(z)]',\alpha'(f(u)),\alpha'(v)]' - [\alpha'(f(z)),[f(x),f(y),f(u)]',\alpha'(v)]'\\
&\quad = 0 (by 3 - Hom Jacobi identity).
\end{aligned}$$

This finishes the proof. □

Proposition 2. *Let $(L, [\cdot,\cdot,\cdot], \alpha)$ and $(L', [\cdot,\cdot,\cdot]', \alpha')$ be two 3-Hom–Lie algebras. Suppose that there are two skew-symmetric linear maps $\rho : L \otimes L \to gl(L')$ and $\mu : L' \otimes L' \to gl(L)$ which are representations of L and L' respectively, satisfying the following equations:*

$$\mu(\alpha'(a_4),\alpha'(a_5))[x_1,x_2,x_3] - [\mu(a_4,a_5)x_1,\alpha(x_2),\alpha(x_3)] \\ -[\alpha(x_1),\mu(a_4,a_5)x_2,\alpha(x_3)] - [\alpha(x_1),\alpha(x_2),\mu(a_4,a_5)x_3] = 0, \quad (1)$$

$$\mu(\rho(x_1,x_4)a_5,\alpha'(a_3))\alpha(x_2) - \mu(\rho(x_2,x_4)a_5,\alpha'(a_3))\alpha(x_1) \\ -\mu(\rho(x_1,x_2)a_3,\alpha'(a_3))\alpha(x_4) + [\alpha(x_1),\alpha(x_2),\mu(a_3,a_5)x_4] = 0, \quad (2)$$

$$[\mu(a_2,a_3)x_1,\alpha(x_4),\alpha(x_5)] - \mu(\alpha'(a_2),\alpha'(a_3))[x_1,x_4,x_5] \\ -\mu(\rho(x_4,x_5)a_2,\alpha'(a_3))\alpha(x_1) - \mu(\alpha'(a_2),\rho(x_4,x_5)a_3)\alpha(x_1) = 0, \quad (3)$$

$$\rho(\alpha(x_4),\alpha(x_5))[a_1,a_2,a_3]' - [\rho(x_4,x_5)a_1,\alpha'(a_2),\alpha'(a_3)]' \\ -[\alpha'(a_1),\rho(x_4,x_5)a_2,\alpha'(a_3)]' - [\alpha'(a_1),\alpha'(a_2),\rho(x_4,x_5)a_3]' = 0, \quad (4)$$

$$\rho(\mu(a_1,a_4)x_5,\alpha(x_3))\alpha'(a_2) - \rho(\mu(a_2,a_4)x_5,\alpha(x_3))\alpha'(a_1) \\ -\rho(\mu(a_1,a_2)x_3,\alpha(x_5))\alpha'(a_4) + [\alpha'(a_1),\alpha'(a_2),\rho(x_3,x_5)a_4]' = 0, \quad (5)$$

$$[\rho(x_2,x_3)a_1,\alpha'(a_4),\alpha'(a_5)] - \rho(\alpha(x_2),\alpha(x_3))[a_1,a_4,a_5]' \\ -\rho(\mu(a_4,a_5)x_2,\alpha(x_3))\alpha'(a_1) - \rho(\alpha(x_2),\mu(a_4,a_5)x_3)\alpha'(a_1) = 0, \quad (6)$$

for any $x_i \in L$ and $a_i \in L', 1 \le i \le 5$. Then, there is a 3-Hom–Lie algebra structure on $L \oplus L'$ defined by

$$\begin{aligned}
&(\alpha \oplus \alpha')(x_1 + a_1) = \alpha(x_1) + \alpha'(a_1),\\
&[x_1 + a_1, x_2 + a_2, x_3 + a_3]_{L \oplus L'} = [x_1,x_2,x_3] + \rho(x_1,x_2)a_3 + \rho(x_3,x_1)a_2 + \rho(x_2,x_3)a_1\\
&\qquad + [a_1,a_2,a_3]' + \mu(a_1,a_2)x_3 + \mu(a_3,a_1)x_2 + \mu(a_2,a_3)x_1.
\end{aligned}$$

Moreover, $(L, L', [\cdot,\cdot,\cdot], [\cdot,\cdot,\cdot]', \rho, \mu, \alpha, \alpha')$ satisfying the above conditions is called a matched pair of 3-Hom–Lie algebras.

Proof. Straightforward. □

Definition 4. *Let $(L, [\cdot,\cdot,\cdot], \alpha)$ be a 3-Hom–Lie algebra. A bilinear form $\langle\cdot,\cdot\rangle$ on L is called invariant if it satisfies*

$$\langle [x,y,z], \alpha(u)\rangle + \langle [x,y,u], \alpha(z)\rangle = 0, \forall x,y,z,u \in L.$$

A 3-Hom–Lie algebra L is called pseudo-metric if there is a non-degenerate symmetric invariant bilinear form on L.

Definition 5. *A Manin triple of 3-Hom–Lie algebras consists of a pseudo-metric 3-Hom–Lie algebra* $(L, [\cdot,\cdot,\cdot], \langle\cdot,\cdot\rangle, \alpha)$ *and 3-Hom–Lie algebras* L_1 *and* L_2 *such that*

(1) L_1, L_2 *are isotropic 3-Hom–Lie subalgebras of L;*

(2) $L = L_1 \oplus L_2$ *as the direct sum of vector spaces;*

(3) For all $x_1, y_1 \in L_1$ *and* $x_2, y_2 \in L_2$*, we have* $pr_1[x_1, y_1, x_2] = 0$ *and* $pr_2[x_2, y_2, x_1] = 0$*, where* pr_1 *and* pr_2 *denote the projections from* $L_1 \oplus L_2$ *to* L_1, L_2*, respectively.*

Given a representation (V, A, ρ), define $\rho^* : L \wedge L \to gl(V^*)$ by

$$\langle \rho^*(x,y)(f), v\rangle = -\langle f, \rho(x,y)(v)\rangle, \forall x, y \in L, f \in V^*, v \in V.$$

As observed in [4], (V^*, A^*, ρ^*) is not a representation of L on V^* with respect to A^* in general. It is easy to obtain the following result by Proposition 2.

Proposition 3. *Let* (V, A, ρ) *be a representation of a 3-Hom–Lie algebra* $(L, [\cdot,\cdot,\cdot], \alpha)$*. Then* (V^*, A^*, ρ^*) *is a representation of the 3-Hom–Lie algebra* $(L, [\cdot,\cdot,\cdot], \alpha)$ *if the following conditions hold:*

$$\begin{aligned}
(i) \qquad & A \circ \rho(\alpha(u), \alpha(v)) = \rho(u,v) \circ A, \\
(ii) \qquad & A \circ \rho([x,y,z], \alpha(u)) = \rho(y,z)\rho(\alpha(x), \alpha(u)) + \rho(z,x)\rho(\alpha(y), \alpha(u)) \\
& \qquad + \rho(x,y)\rho(\alpha(z), \alpha(u)), \\
(iii) \qquad & \rho(x,y)\rho(\alpha(z), \alpha(u)) = \rho(z,u)\rho(\alpha(x), \alpha(y)) + A \circ \rho([x,y,z], \alpha(u)) \\
& \qquad + A \circ \rho(\alpha(z), [x,y,u]),
\end{aligned}$$

for all $x, y, z, u, v \in L$.

A representation (V, A, ρ) is called admissible if (V^*, A^*, ρ^*) is also a representation, i.e., conditions (i),(ii) and (iii) in Proposition 3 are satisfied. When we focus on the adjoint representation, we have the following corollary:

Corollary 1. *Let* $(L, [\cdot,\cdot,\cdot], \alpha)$ *be a 3-Hom–Lie algebra. The adjoint representation* (L, α, ad) *is admissible if the following three equations hold:*

$$[(id - \alpha^2)(u), (id - \alpha^2)(v), \alpha(w)] = 0, \tag{7}$$

$$\begin{aligned}
[[\alpha(x), \alpha(y), \alpha(z)], \alpha^2(u), \alpha(w)] = {} & [y, z, [\alpha(x), \alpha(u), w]] + [z, x, [\alpha(y), \alpha(u), w]] \\
& + [x, y, [\alpha(z), \alpha(u, w]],
\end{aligned} \tag{8}$$

$$\begin{aligned}
[x, y, [\alpha(z), \alpha(u), w]] = {} & [z, u, [\alpha(x), \alpha(y)]] + [[\alpha(x), \alpha(y), \alpha(z)], \alpha^2(u), \alpha(w)] \\
& + [\alpha^2(z), [\alpha(x), \alpha(y), \alpha(u)], \alpha(w)],
\end{aligned} \tag{9}$$

for all $x, y, z, u, v, w \in L$.

Definition 6. *A 3-Hom–Lie algebra* $(L, [\cdot,\cdot,\cdot], \alpha)$ *is called admissible if its adjoint representation is admissible, i.e., Equations (7)–(9) are satisfied.*

In the following, we concentrate on the case that L' is L^* , the dual space of L, and $\alpha' = \alpha^*$, $\rho = ad^*$, $\mu = a\partial^*$, where $a\partial^*$ is the dual map of $a\partial$.

Let $(L, [\cdot,\cdot,\cdot], \alpha)$ be an admissible 3-Hom–Lie algebra. Then, we have a natural nondegenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$ on $L \oplus L^*$ given by

$$\langle x + \xi, y + \eta\rangle = \langle x, \eta\rangle + \langle y, \xi\rangle, \forall x, y \in L, \xi, \eta \in L^*. \tag{10}$$

There is also a twist map $\alpha \oplus \alpha^*$ and a bracket operation $[\cdot,\cdot,\cdot]_{L\oplus L^*}$ on $L \oplus L^*$ given by

$$\begin{aligned}
&(\alpha \oplus \alpha^*)(x+\xi) = \alpha(x) + \alpha^*(\xi), \\
&[x+\xi, y+\eta, z+\gamma]_{L\oplus L^*} = [x,y,z] + ad^*_{x,y}\gamma + ad^*_{y,z}\xi + ad^*_{z,x}\eta \\
&\qquad\qquad + a\partial^*_{\xi,\eta}z + a\partial^*_{\eta,\gamma}x + a\partial^*_{\gamma,\xi}y + [\xi,\eta,\gamma]^*.
\end{aligned} \tag{11}$$

Note that the bracket operation $[\cdot,\cdot,\cdot]_{L\oplus L^*}$ is naturally invariant with respect to the symmetric bilinear form $\langle\cdot,\cdot\rangle$ and satisfies the condition (10). Assume that $(L \oplus L^*, [\cdot,\cdot,\cdot]_{L\oplus L^*}, \alpha \oplus \alpha^*)$ is a 3-Hom–Lie algebra, then obviously L and L^* are isotropic subalgebras. Consequently, $((L \oplus L^*, \langle\cdot,\cdot\rangle, \alpha \oplus \alpha^*), L, L^*)$ is a Manin triple, which is called the standard Manin triple of 3-Hom–Lie algebras.

Next we will show a close relation between the matched pair and the Manin triple of admissible 3-Hom–Lie algebras.

Lemma 2. *Let $(L, [\cdot,\cdot,\cdot], \alpha)$ and $(L^*, [\cdot,\cdot,\cdot]^*, \alpha^*)$ be two admissible 3-Hom–Lie algebras. If Equations (1)–(3) hold. Then, $(L, L^*, ad^*, a\partial^*, \alpha, \alpha^*)$ is a matched pair.*

Proof. For any $x_1, x_2, x_4 \in L$ and $a_3, a_5, a_6 \in L^*$, we have

$$\begin{aligned}
& \langle -a\partial^*_{ad^*_{x_1,x_2}a_3,a_5}\alpha(x_4) + a\partial^*_{ad^*_{x_1,x_4}a_5,a_3}\alpha(x_2) - a\partial^*_{ad^*_{x_2,x_4}a_5,a_3}\alpha(x_1) \\
& +[\alpha(x_1), \alpha(x_2), a\partial^*_{a_3,a_5}x_4], a_6\rangle \\
= \;& \langle [ad^*_{x_1,x_2}a_3, a_5, a_6]^*, \alpha(x_4) > - < [ad^*_{x_1,x_4}a_5, a_3, a_6]^*, \alpha(x_2)\rangle \\
& +\langle [ad^*_{x_2,x_4}a_5, a_3, a_6]^* - ad^*_{\alpha(x_2), a\partial^*_{a_3,a_5}x_4}a_6, \alpha(x_1)\rangle \\
= \;& \langle x_1, ad^*_{x_2, a\partial^*_{a_5,a_6}x_4}\alpha^*(a_3) + ad^*_{x_4, a\partial^*_{a_3,a_6}x_2}\alpha^*(a_5) \\
& +ad^*_{x_2,x_4}a_5, \alpha^*(a_3), \alpha^*(a_6)]^* - ad^*_{x_2, a\partial^*_{a_3,a_5}x_4}\alpha^*(a_6)\rangle,
\end{aligned}$$

which implies the equivalence between Equations (2) and (5). The proofs of Equation (1) $\Longleftrightarrow$ Equation (4), Equation (3) $\Longleftrightarrow$ Equation (6) are similar. □

Proposition 4. *Let $(L, [\cdot,\cdot,\cdot], \alpha)$ and $(L^*, [\cdot,\cdot,\cdot]^*, \alpha^*)$ be two admissible 3-Hom–Lie algebras. Then $(L \oplus L^*, \langle\cdot,\cdot\rangle, \alpha \oplus \alpha^*, L, L^*)$ under the nondegenerate symmetric bilinear form (10) and the bracket operation (11) is a standard Manin triple if and only if $(L, L^*, ad^*, a\partial^*, \alpha, \alpha^*)$ is a matched pair.*

Proof. Straightforward from Lemma 2. □

Theorem 1. *Let $(L, [\cdot,\cdot,\cdot], \alpha)$ and $(L^*, [\cdot,\cdot,\cdot]^*, \alpha^*)$ be two admissible 3-Hom–Lie algebras, $\Delta : L \to L \otimes L \otimes L$ a linear map. Suppose that $\Delta^* : L^* \otimes L^* \otimes L^* \to L^*$ defines a 3-Hom–Lie algebra structure $[\cdot,\cdot,\cdot]^*$ on L^*. Then, $(L, L^*, ad^*, a\partial^*, \alpha, \alpha^*)$ is a matched pair if and only if the following equations are satisfied:*

$$\Delta([x,y,z]) = (\alpha \otimes \alpha \otimes ad_{y,z})\Delta(x) + (\alpha \otimes \alpha \otimes ad_{z,x})\Delta(y) + (\alpha \otimes \alpha \otimes ad_{x,y})\Delta(z) \tag{12}$$

$$\Delta([x,y,z]) = (\alpha \otimes \alpha \otimes ad_{y,z})\Delta(x) + (\alpha \otimes ad_{y,z} \otimes \alpha)\Delta(x) + (ad_{y,z} \otimes \alpha \otimes \alpha)\Delta(x) \tag{13}$$

$$(ad_{x,y} \otimes \alpha \otimes \alpha + \alpha \otimes \alpha \otimes ad_{x,y})\Delta(z) = (\alpha \otimes ad_{z,x} \otimes \alpha)\Delta(y) + (\alpha \otimes ad_{y,z} \otimes \alpha)\Delta(x) \tag{14}$$

for any $x, y, z \in L$.

Proof. Let $\{e_1, e_2, ..., e_n\}$ be a basis of L and $\{e_1^*, e_2^*, ..., e_n^*\}$ the dual basis. Suppose

$$[e_i, e_j, e_k] = \sum_{l=1}^{n} c^l_{ijk} e_l, \; [e_i^*, e_j^*, e_k^*]^* = \sum_{l=1}^{n} d^l_{ijk} e_l^*.$$

Let

$$\alpha(e_i) = \sum_s f_s e_s, \alpha(e_j) = \sum_n g_n e_n, \alpha(e_k) = \sum_n h_m e_m,$$
$$\alpha^*(e_\xi^*) = \sum_s f_s^* e_s^*, \alpha^*(e_\eta^*) = \sum_n g_n^* e_n^*, \alpha^*(e_k^*) = \sum_m h_m^* e_m^*.$$

Then we have

$$ad^*_{e_i,e_j}e_k^* = -\sum_{l=1}^{n} c_{ijk}^l e_l^*, a\partial^*_{e_i^*,e_j^*}e_k = -\sum_{l=1}^{n} d_{ijk}^l e_l, \Delta(e_k) = \sum_{i,j,l=1}^{n} d_{ijl}^k e_i \otimes e_j \otimes e_k.$$

By Equation (1), we have

$$a\partial^*_{\alpha^*(e_\xi^*),\alpha^*(e_\eta^*)}[e_i, e_j, e_k] - [a\partial^*_{e_\xi^*,e_\eta^*}e_i, \alpha(e_j), \alpha(e_k)]$$
$$-[\alpha(e_i), a\partial^*_{e_\xi^*,e_\eta^*}e_j, \alpha(e_k)] - [\alpha(e_i), \alpha(e_j), a\partial^*_{e_\xi^*,e_\eta^*}e_k] = 0.$$

It follows that

$$\sum_{l=1}^{n}(-f_s^* g_n^* d_{snm}^l c_{ijk}^l + g_n h_m d_{\xi\eta l}^i c_{lnm}^m + f_s h_m d_{\xi\eta l}^j c_{slm}^m + f_s g_n d_{\xi\eta l}^k c_{snl}^m) = 0,$$

as the coefficient of e_m. On the other hand, the left hand side of the above equation is also the coefficient of $e_\xi \otimes e_\eta \otimes e_m$ in Equation (12). Thus, we deduce that Equation (1) is equivalent to Equation (12). The proofs of the other case are similar. □

Definition 7. *Let $(L, [\cdot,\cdot,\cdot], \alpha)$ and $(L^*, [\cdot,\cdot,\cdot]^*, \alpha^*)$ be two admissible 3-Hom–Lie algebras, $\Delta : L \to L \otimes L \otimes L$ be a linear map. Suppose that $\Delta^* : L^* \otimes L^* \otimes L^* \to L^*$ defines a 3-Hom–Lie algebra structure $[\cdot,\cdot,\cdot]^*$ on L^*. If Δ satisfies Equations (12)–(14), then we call (L, L^*, α, Δ) a double construction 3-Hom–Lie bialgebra.*

Example 5. *Consider the 4-dimensional 3-Hom–Lie algebra $(L, [\cdot,\cdot,\cdot], \alpha)$ with respect to a basis $\{e_1, e_2, e_3, e_4\}$ given by*

$$[e_2, e_3, e_4] = e_1, \alpha(e_1) = -e_1, \alpha(e_2) = e_2, \alpha(e_3) = e_3, \alpha(e_4) = e_4.$$

Define the skew-symmetric linear map $\Delta : L \to L \otimes L \otimes L$ satisfying Equation (12) is given as follows

$$\Delta(e_1) = 0, \Delta(e_2) = e_1 \wedge e_2 \wedge e_3 + e_1 \wedge e_2 \wedge e_4 + e_1 \wedge e_3 \wedge e_4,$$
$$\Delta(e_3) = e_1 \wedge e_2 \wedge e_3 - e_1 \wedge e_3 \wedge e_4 + e_1 \wedge e_2 \wedge e_4,$$
$$\Delta(e_4) = -e_1 \wedge e_2 \wedge e_4 + e_1 \wedge e_3 \wedge e_4 + e_1 \wedge e_2 \wedge e_3,$$

then (L, Δ) is a double construction 3-Hom–Lie bialgebra.

Combining Lemma 2, Proposition 6, Theorem 1 and Definition 7, we have

Theorem 2. *Let $(L, [\cdot,\cdot,\cdot], \alpha)$ and $(L^*, [\cdot,\cdot,\cdot]^*, \alpha^*)$ be two admissible 3-Hom–Lie algebras, $\Delta : L \to L \otimes L \otimes L$ be a linear map. Suppose that $\Delta^* : L^* \otimes L^* \otimes L^* \to L^*$ defines a 3-Hom–Lie algebra structure $[\cdot,\cdot,\cdot]^*$ on L^*. Then, the following statements are equivalent:*
(1) (L, L^*, α, Δ) is a double construction 3-Hom–Lie bialgebra.
(2) $(L \oplus L^*, \langle\cdot,\cdot\rangle, \alpha \oplus \alpha^*)$ is a standard Manin triple of admissible 3-Hom–Lie algebras.
(3) $(L, L^*, ad^*, a\partial^*, \alpha, \alpha^*)$ is a matched pair of admissible 3-Hom–Lie algebras.

Example 6. *Consider the 4-dimensional 3-Hom–Lie algebra* $(L, [\cdot,\cdot,\cdot], \alpha)$ *in Example 5 and* $\{e_1^*, e_2^*, e_3^*, e_4^*\}$ *is the dual basis. On the vector space* $L \oplus L^*$ *define a bilinear form* $\langle \cdot,\cdot \rangle$ *by Equation (10), the non-zero product of 3-Hom–Lie algebra structure on* $L \oplus L^*$ *is given by*

$$\begin{aligned}
&[e_2, e_3, e_4] = e_1, \alpha(e_1) = -e_1, \alpha(e_2) = e_2, \alpha(e_3) = e_3, \alpha(e_4) = e_4,\\
&[e_1^*, e_2^*, e_3^*]^* = e_2^* + e_3^* + e_4^*, [e_1^*, e_2^*, e_4^*]^* = e_2^* + e_3^* - e_4^*,\\
&[e_1^*, e_3^*, e_4^*]^* = e_2^* - e_3^* - e_4^*, [e_1, e_2, e_1^*] = -e_3^*, [e_1, e_3, e_1^*] = e_2^*,\\
&[e_2, e_3, e_1^*] = -e_1^*, [e_2, e_1^*, e_2^*] = -e_3 - e_4, [e_2, e_2^*, e_3^*] = -e_1,\\
&[e_2, e_1^*, e_3^*] = e_2 - e_4, [e_2, e_2^*, e_4^*] = -e_1, [e_2, e_1^*, e_4^*] = e_2 + e_3,\\
&[e_2, e_3^*, e_4^*] = -e_1, [e_3, e_1^*, e_2^*] = -e_3 - e_4, [e_3, e_2^*, e_3^*] = -e_1,\\
&[e_3, e_2^*, e_4^*] = -e_1, [e_3, e_1^*, e_4^*] = e_2 - e_3, [e_3, e_3^*, e_4^*] = e_1,\\
&[e_3, e_3^*, e_4^*] = e_1, [e_4, e_1^*, e_2^*] = -e_3 + e_4, [e_4, e_2^*, e_3^*] = -e_1,\\
&[e_4, e_1^*, e_3^*] = e_2 - e_4, [e_3, e_1^*, e_3^*] = e_2 + e_4.
\end{aligned}$$

They correspond to the double construction 3-Hom–Lie bialgebra (L, Δ) *given in Example 5.*

3. $\mathcal{O}$-Operators and 3-Hom–pre-Lie Algebras

In this section, we mainly study the $\mathcal{O}$-operator of a 3-Hom–Lie algebra and present a class of solutions of 3-Hom–Lie Yang–Baxter equations.

Definition 8. *Let* $(L, [\cdot,\cdot,\cdot], \alpha)$ *be a 3-Hom–Lie algebra and* (V, A, ρ) *a representation. A linear operator* $T : V \to L$ *is called an* $\mathcal{O}$*-operator associated to* (V, A, ρ) *if* T *satisfies: for any* $u, v, w \in L$*,*

$$\alpha \circ T = T \circ A, \tag{15}$$

$$[Tu, Tv, Tw] = T(\rho(Tu, Tv)w + \rho(Tv, Tw)u + \rho(Tw, Tu)v). \tag{16}$$

Example 7. *Let* $(L, [\cdot,\cdot,\cdot], \alpha)$ *be a 3-Hom–Lie algebra. An* $\mathcal{O}$*-operator of* L *associated to the adjoint representation* (L, ad, α) *is nothing but the Rota-Baxter operator of weight zero introduced in [17].*

Definition 9. *A 3-Hom–pre-Lie algebra is a triple* $(L, \{\cdot,\cdot,\cdot\}, \alpha)$ *consisting of a vector space* L*, with a trilinear map* $\{\cdot,\cdot,\cdot\} : L \otimes L \otimes L \to L$ *and an algebra morphism* $\alpha : L \to L$ *satisfying*

$$\{x, y, z\} = -\{y, x, z\}, \tag{17}$$

$$\{\alpha(x), \alpha(y), \{z, u, v\}\} = \{[x, y, z]_C, \alpha(u), \alpha(v)\} + \{\alpha(z), [x, y, u]_C, \alpha(v)\} + \{\alpha(z), \alpha(u), [x, y, v]_C\}, \tag{18}$$

$$\{[x, y, z], \alpha(u), \alpha(v)\} = \{\alpha(x), \alpha(y), [z, u, v]_C\} + \{\alpha(y), \alpha(z), [x, u, v]_C\} + \{\alpha(z), \alpha(x), [y, u, v]_C\}, \tag{19}$$

for any $x, y, z, u, v \in L$*.*

Proposition 5. *Let* $(L, \{\cdot,\cdot,\cdot\}, \alpha)$ *be a 3-Hom–pre-Lie algebra. Then, the induced 3-commutator*

$$[x, y, z]_C = \{x, y, z\} + \{y, z, x\} + \{z, x, y\}, \tag{20}$$

defines a 3-Hom–Lie algebra $(L^c, \{\cdot,\cdot,\cdot\}_C, \alpha)$*.*

Proof. It is easy to check that $[\cdot,\cdot,\cdot]_C$ is skew-symmetric. For any $x_1, x_2, x_3, x_4, x_5 \in L$, we have

$$\begin{aligned}
& [\alpha(x_1),\alpha(x_2),[x_3,x_4,x_5]_C]_C-[[x_1,x_2,x_3]_C,\alpha(x_4),\alpha(x_5)]_C-[\alpha(x_3),[x_1,x_2,x_4]_C,\alpha(x_5)]_C \\
& -[\alpha(x_3),\alpha(x_4),[x_1,x_2,x_5]_C,]_C \\
= \; & \{\alpha(x_1),\alpha(x_2),\{x_3,x_4,x_5\}\}+\{\alpha(x_1),\alpha(x_2),\{x_4,x_5,x_3\}\}+\{\alpha(x_1),\alpha(x_2),\{x_5,x_3,x_4\}\} \\
& +\{\alpha(x_2),[x_3,x_4,x_5]_C,\alpha(x_1)\}+\{[x_3,x_4,x_5],\alpha(x_1),\alpha(x_2)\} \\
& -\{\alpha(x_4),\alpha(x_5),\{x_1,x_2,x_3\}\}-\{\alpha(x_4),\alpha(x_5),\{x_2,x_3,x_1\}\}-\{\alpha(x_4),\alpha(x_5),\{x_3,x_1,x_2\}\} \\
& -\{[x_1,x_2,x_3],\alpha(x_4),\alpha(x_5)\}-\{\alpha(x_5),[x_1,x_2,x_3]_C,\alpha(x_4)\} \\
& -\{\alpha(x_5),\alpha(x_3),\{x_1,x_2,x_4\}\}-\{\alpha(x_5),\alpha(x_3),\{x_2,x_4,x_1\}\}-\{\alpha(x_5),\alpha(x_3),\{x_2,x_4,x_1\}\} \\
& -\{[x_1,x_2,x_4],\alpha(x_5),\alpha(x_3)\}-\{\alpha(x_3),[x_1,x_2,x_4]_C,\alpha(x_5)\} \\
& -\{\alpha(x_3),\alpha(x_4),\{x_1,x_2,x_5\}\}-\{\alpha(x_3),\alpha(x_4),\{x_2,x_5,x_1\}\}-\{\alpha(x_3),\alpha(x_4),\{x_5,x_1,x_2\}\} \\
& -\{[x_1,x_2,x_5]_C,\alpha(x_3),\alpha(x_4)\}-\{\alpha(x_4),[x_1,x_2,x_5]_C,\alpha(x_3)\} \\
= \; & 0.
\end{aligned}$$

Thus the proof is finished. □

Definition 10. *Let $(L,\{\cdot,\cdot,\cdot\},\alpha)$ be a 3-Hom–pre-Lie algebra. The 3-Hom–Lie algebra $(L^c,[\cdot,\cdot,\cdot]_C,\alpha)$ is called the sub-adjacent 3-Hom–Lie algebra of $(L,\{\cdot,\cdot,\cdot\},\alpha)$ and $(L,\{\cdot,\cdot,\cdot\},\alpha)$ is called a compatible 3-Hom–pre-Lie algebra of the 3-Hom–Lie algebra $(L^c,[\cdot,\cdot,\cdot]_C,\alpha)$.*

Definition 11. *Let $(L,\{\cdot,\cdot,\cdot\},\alpha)$ and $(L',\{\cdot,\cdot,\cdot\}',\alpha')$ be two 3-Hom–pre-Lie algebras. A morphism from $(L,\{\cdot,\cdot,\cdot\},\alpha)$ to $(L',\{\cdot,\cdot,\cdot\}',\alpha')$ is a 3-pre-Lie algebra morphism $f:L\to L'$ satisfying $f\circ\alpha=\alpha'\circ f$.*

Theorem 3. *Let $\mathcal{L}=(L,\{\cdot,\cdot,\cdot\},\alpha)$ be a* 3*-Hom–pre-Lie algebra and $\alpha':\mathcal{L}\to\mathcal{L}$ be a* 3*-pre-Lie algebras morphism such that α and α' commute. Define*

$$\{\cdot,\cdot,\cdot\}_{\alpha'}:L\times L\to L,\{x,y,z\}_{\alpha'}=\alpha'(\{x,y,z\}),\forall x,y,z\in L.$$

Then $\mathcal{L}'_\alpha=(L'_\alpha=L,\{x,y,z\}_{\alpha'},\alpha')$ is a 3*-Hom–pre-Lie algebra, called α'-twist or Yau twist of $\mathcal{L}$. Moreover, assume that $\mathcal{L}'=(L',\{\cdot,\cdot,\cdot\}',\beta)$ is another* 3*-Hom–pre-Lie algebra, and $\beta':\mathcal{L}'\to\mathcal{L}'$ is a* 3*-Hom–pre-pre-Lie algebras morphism such that α and α' commute. Let $f:\mathcal{L}\to\mathcal{L}'$ be a* 3*-Hom–pre-Lie algebras morphism satisfying $f\circ\alpha'=\beta'\circ f$. Then, $f:\mathcal{L}_{\alpha'}\to\mathcal{L}'_{\beta'}$ is a* 3*-Hom–pre-Lie algebras morphism.*

Proof. Let $x, y, z \in L$,

$$\begin{aligned}
\{x,y,z\}_{\alpha'} &= \alpha'(\{x,y,z\}) \\
&= \{\alpha'(x), \alpha'(y), \alpha'(z)\} \\
&= -\{\alpha'(y), \alpha'(x), \alpha'(z)\} \\
&= -\alpha'(\{y,x,z\}) \\
&= -\{y,x,z\}_{\alpha'}, \\
\{\alpha\alpha'(x), \alpha\alpha'(y), \{z,u,v\}_{\alpha'}\}_{\alpha'} &= \{\alpha\alpha'^2(x), \alpha\alpha'^2(y), \alpha'\{\alpha'(z), \alpha'(u), \alpha'(v)\}\} \\
&= \{\alpha\alpha'^2(x), \alpha\alpha'^2(y), \{\alpha'^2(z), \alpha'^2(u), \alpha'^2(v)\}\} \\
&= \{[\alpha'^2(x), \alpha'^2(y), \alpha'^2(z)]_C, \alpha\alpha'^2(u), \alpha\alpha'^2(v)\} \\
&\quad + \{\alpha\alpha'^2(z), [\alpha'^2(x), \alpha'^2(y), \alpha'^2(u)], \alpha\alpha'^2(v)\} \\
&\quad + \{\alpha\alpha'^2(z), \alpha\alpha'^2(u), [\alpha'^2(x), \alpha'^2(y), \alpha'^2(v)]\} \\
&= \{([x,y,z]_{\alpha'})_C, \alpha\alpha'(u), \alpha\alpha'(v)\}_{\alpha'} \\
&\quad + \{\alpha\alpha'(z), ([x,y,u]_{\alpha'})_C, \alpha\alpha'(v)\}_{\alpha'} \\
&\quad + \{\alpha\alpha'(z), \alpha\alpha'(u)\}_{\alpha'}, ([x,y,v]_{\alpha'})_C.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\{[x,y,z], \alpha\alpha'(u), \alpha\alpha'(v)\} &= \{\alpha\alpha'(x), \alpha\alpha'(y), [z,u,v]\} \\
&\quad + \{\alpha\alpha'(y), \alpha\alpha'(z), [x,u,v]_C\} \\
&\quad + \{\alpha\alpha'(z), \alpha\alpha'(x), [y,u,v]_C\}.
\end{aligned}$$

For the second assertion, we have

$$\begin{aligned}
f(\{x,y,z\}_{\alpha'}) &= f(\{\alpha'(x), \alpha'(y), \alpha'(z)\}) \\
&= \{f(\alpha'(x)), f(\alpha'(y)), f(\alpha'(z))\}') \\
&= \{\beta'(f(x)), \beta'(f(y)), \beta'(f(z))\}').
\end{aligned}$$

□

Corollary 2. *If $\mathcal{A} = (A, \{\cdot,\cdot,\cdot\}, \alpha)$ is a 3-Hom–pre-Lie algebra, for any $n \in \mathbb{N}^*$, the following results hold:*

1. *The nth derived 3-Hom–pre-Lie algebra of type 1 of $\mathcal{A}$ is defined by $\mathcal{A}_1^n = (A, \{\cdot,\cdot,\cdot\}^{(n)} = \alpha^n \circ \{\cdot,\cdot,\cdot\}, \alpha^{n+1})$.*
2. *The nth derived 3-Hom–pre-Lie algebra of type 2 of A is defined by $\mathcal{A}_2^n = (A, \{\cdot,\cdot,\cdot\}^{(2^n-1)} = \alpha^{2^n-1} \circ \{\cdot,\cdot,\cdot\}, \alpha^{2^n})$.*

Proof. Apply Theorem 3 with $\alpha' = \alpha^n$ and $\alpha' = \alpha^{2^n-1}$ respectively. □

Define the left multiplication $\mathcal{L} : \wedge^2 L \to gl(L)$ by $\mathcal{L}(x,y)z = \{x,y,z\}$ for all $x, y, z \in L$. Then $(L, \mathcal{L}, \alpha)$ is a representation of the 3-Hom–Lie algebra L. Similarly, we define the right multiplication $\mathcal{R} : \wedge^2 L \to gl(L)$ by $\mathcal{R}(x,y)z = \{z,x,y\}$. If there is an admissible 3-Hom–pre-Lie algebra structure on its dual space L^*, we denote the left multiplication and right multiplication by $\mathcal{L}^*$ and $\mathcal{R}^*$ respectively.

Proposition 6. *Let $(L, [\cdot,\cdot,\cdot], \alpha)$ be a 3-Hom–Lie algebra and (V, A, ρ) a representation. Suppose that the linear map $T : V \to L$ is an $\mathcal{O}$-operator associated to (V, A, ρ). Then, there exists a 3-Hom–pre-Lie algebra structure on V given by*

$$\{u,v,w\} = \rho(Tu, Tv)w, \forall u, v, w \in V.$$

Proof. For any $u, v, w \in V$, we have

$$\{u, v, w\} = \rho(Tu, Tv)w = -\rho(Tv, Tu)w = -\{v, u, w\}.$$

Since $[u, v, w]_C = \{u, v, w\} + \{v, w, u\} + \{w, u, v\}$, we have

$$[u, v, w]_C = \rho(Tu, Tv)w + \rho(Tv, Tw)u + \rho(Tw, Tu)v.$$

Because T is an $\mathcal{O}$-operator, we have

$$T[u, v, w]_C = [Tu, Tv, Tw].$$

For any $v_1, v_2, v_3, v_4, v_5 \in V$, we have

$$\begin{aligned}
& \{\beta(v_1), \beta(v_2), \{v_3, v_4, v_5\}\} = \rho(T \circ A(v_1), T \circ A(v_2))\rho(Tv_3, Tv_4)v_5, \\
& \{[v_1, v_2, v_3], \beta(v_4), \beta(v_5)\} \\
= \; & \rho(T[v_1, v_2, v_3], TA(v_4))A(v_5) = \rho([Tv_1, Tv_2, Tv_3], T \circ A(v_4))A(v_5), \\
& \{\beta(v_3), [v_1, v_2, v_4], \beta(v_5)\} \\
= \; & \rho(T \circ A(v_3), T[v_1, v_2, v_4])A(v_5) = \rho(T \circ A(v_3), [Tv_1, Tv_2, Tv_4])A(v_5), \\
& \{\beta(v_3), \beta(v_4), \{v_1, v_2, v_5\}\} = \rho(T \circ A(v_3), T \circ A(v_4))\rho(Tv_1, Tv_2)v_5.
\end{aligned}$$

Since (V, A, ρ) is a representation, we can check that Equations (18) and (19) hold. This finishes the proof. □

Corollary 3. *Let $T : V \to L$ be an $\mathcal{O}$-operator on a 3-Hom–Lie algebra $(L, [\cdot, \cdot, \cdot], \alpha)$ associated to the representation (V, A, ρ). Then, T is a morphism from the 3-Hom–Lie algebra $(V, [\cdot, \cdot]_C, A)$ to $(A, [\cdot, \cdot], \alpha)$.*

Proof. For all $u, v, w \in V$, we have

$$\begin{aligned}
T([u, v, w]_C) &= T(\{u, v, w\} + \{w, u, v\} + \{v, w, u\}) \\
&= T(\rho(Tu, Tv)w + \rho(Tw, Tu)v + \rho(Tv, Tw)u) \\
&= [Tu, Tv, Tw],
\end{aligned}$$

as desired. □

Example 8. *Let $(A, [\cdot, \cdot, \cdot], \alpha)$ be a 3-Hom–Lie algebra and $R : A \longrightarrow A$ a Rota-Baxter operator. Define a new operation on A by $\{x, y, z\} = [R(x), R(y), z]$. Then, $(A, \{\cdot, \cdot, \cdot\}, \alpha)$ is a 3-Hom–pre-Lie algebra and R is a homomorphism from the sub-adjacent 3-Hom–Lie algebra $(A, [\cdot, \cdot, \cdot]_C, \alpha)$ to $(A, [\cdot, \cdot, \cdot], \alpha)$.*

Proposition 7. *Let $(L, [\cdot, \cdot, \cdot], \alpha)$ be a 3-Hom–Lie algebra. Then there exists a compatible 3-Hom–pre-Lie algebra if and only if there exists an invertible $\mathcal{O}$-operator of L.*

Proof. Let T be an invertible $\mathcal{O}$-operator of L associated to a representation (V, A, ρ). Then there exists a 3-Hom–pre-Lie algebra structure on (V, A, ρ) defined by

$$\{u, v, w\} = \rho(Tu, Tv)(w), \forall u, v, w \in V.$$

Moreover, there is an induced 3-Hom–pre-Lie algebra structure $\{\cdot, \cdot, \cdot\}$ on $L = T(V)$ given by

$$\{x, y, z\} = T\{T^{-1}x, T^{-1}y, T^{-1}z\} = T\rho(x, y)T^{-1}z.$$

Since T is an $\mathcal{O}$-operator, we have

$$\begin{aligned} [x,y,z] &= T\rho(y,z)T^{-1}x + T\rho(z,x)T^{-1}y + T\rho(x,y)T^{-1}z \\ &= \{x,y,z\} + \{y,z,x\} + \{z,x,y\}. \end{aligned}$$

Therefore, $(L, \{\cdot,\cdot,\cdot\}, \alpha)$ is a compatible 3-Hom–pre-Lie algebra.
Conversely, the identity map id is an $\mathcal{O}$-operator of L. □

Definition 12 ([17]). *Let $(L, [\cdot,\cdot,\cdot], \alpha)$ be a 3-Hom–Lie algebra and $r \in L \otimes L$. The equation*

$$[[r,r,r]]^{\alpha} = 0$$

is called the 3-Hom–Lie Yang–Baxter equation.

Let $(L, [\cdot,\cdot,\cdot], \alpha)$ be an admissible 3-Hom–Lie algebra. For any $r \in L \otimes L$, the induced skew-symmetric linear map $r : L^* \to L$ is defined by

$$\langle r(\xi), \eta \rangle = \langle r, \xi \wedge \eta \rangle.$$

We denote the ternary operation $\Delta^* : L^* \otimes L^* \otimes L^* \to L^*$ by $[\cdot,\cdot,\cdot]^*$. According to [17], for any $r = \sum_i x_i \otimes y_i \in L \otimes L$ and $x \in L$, one can define

$$\begin{aligned} \Delta_1(x) &= \sum_{i,j} [x, x_i, x_j] \otimes \alpha(y_j) \otimes \alpha(y_i), \\ \Delta_2(x) &= \sum_{i,j} \alpha(y_i) \otimes [x, x_i, x_j] \otimes \alpha(y_j), \\ \Delta_3(x) &= \sum_{i,j} \alpha(y_j) \otimes \alpha(y_i) \otimes [x, x_i, x_j]. \end{aligned}$$

Proposition 8. *Let $(L, [\cdot,\cdot,\cdot], \alpha)$ be an admissible 3-Hom–Lie algebra and $r \in L \otimes L$ such that $\alpha^{\otimes^2}(r) = r$. Suppose that r is skew-symmetric and $\Delta = \Delta_1 + \Delta_2 + \Delta_3 : L \to L \otimes L \otimes L$. Then*

$$[\xi, \eta, \gamma]^* = ad^*_{r(\xi),r(\eta)}\gamma + ad^*_{r(\eta),r(\gamma)}\xi + ad^*_{r(\gamma),r(\xi)}\eta. \tag{21}$$

Furthermore, we have

$$[r(\xi), r(\eta), r(\gamma)] - r([\xi, \eta, \gamma]^*) = [[r,r,r]](\xi, \eta, \gamma), \tag{22}$$

for any $\xi, \eta, \gamma \in L^*$.

Proof. Let $r = \sum_i x_i \otimes y_i$, then for any $x, y \in L$ and $\xi, \eta, \gamma \in L^*$, we have

$$\begin{aligned} \langle x, ad^*_{r\circ\alpha^*(\xi), r\circ\alpha^*(\eta)}\gamma \rangle &= \langle -[r\circ\alpha^*(\xi), r\circ\alpha^*(\eta), x], \gamma \rangle \\ &= -\langle r, \alpha^*(\eta) \otimes ad^*_{r\circ\alpha^*(\xi),x}\gamma \rangle \\ &= \sum_i \langle y_i, \alpha^*(\eta) \rangle \langle r, \alpha^*(\xi) \otimes ad^*_{x,x_i}\gamma \rangle \\ &= \sum_i \langle y_i, \alpha^*(\eta) \rangle \langle y_j, \alpha^*(\xi) \rangle \langle [x, x_i, x_j], \gamma \rangle \\ &= \sum_{i,j} \langle \alpha(y_i), \eta \rangle \langle \alpha(y_j), \xi \rangle \langle [x, x_i, x_j], \gamma \rangle \\ &= \langle \sum_{i,j} \alpha(y_j) \otimes \alpha(y_i) \otimes [x, x_i, x_j], \xi \otimes \eta \otimes \gamma \rangle \\ &= \langle \Delta_3(x), \xi \otimes \eta \otimes \gamma \rangle. \end{aligned}$$

Similarly, we have

$$\begin{aligned}\langle x, ad^*_{r\circ\alpha^*(\eta),r\circ\alpha^*(\gamma)}(\xi)\rangle &= \langle \Delta_1(x), \xi\otimes\eta\otimes\gamma\rangle, \langle x, ad^*_{r\circ\alpha^*(\gamma),r\circ\alpha^*(\xi)}(\eta)\rangle \\ &= \langle \Delta_1(x), \xi\otimes\eta\otimes\gamma\rangle.\end{aligned}$$

It follows that

$$\begin{aligned}&\langle \Delta(x), \xi\otimes\eta\otimes\gamma\rangle \\ =\ &\langle \Delta_1(x)+\Delta_2(x)+\Delta_3(x), \xi\otimes\eta\otimes\gamma\rangle \\ =\ &\langle x, ad^*_{r\circ\alpha^*(\eta),r\circ\alpha^*(\gamma)}\xi\rangle + \langle x, ad^*_{r\circ\alpha^*(\gamma),r\circ\alpha^*(\xi)}\eta\rangle + \langle x, ad^*_{r\circ\alpha^*(\xi),r\circ\alpha^*(\eta)}\gamma\rangle \\ =\ &\langle x, [\xi,\eta,\gamma]^*\rangle.\end{aligned}$$

So Equation (21) holds as required. For Equation (22) we take any $\kappa \in L^*$ and compute

$$\begin{aligned}&[[r,r,r]](\xi,\eta,\gamma,\kappa) \\ =\ &\sum_{i,j,k}([x_i,x_j,x_k]\otimes\alpha(y_i)\otimes\alpha(y_j)\otimes\alpha(y_k)(\xi,\eta,\gamma,\kappa) \\ &+\alpha(x_i)\otimes[y_i,x_j,x_k]\otimes\alpha(y_j)\otimes\alpha(y_k)(\xi,\eta,\gamma,\kappa) \\ &\alpha(x_i)\otimes\alpha(x_j)\otimes[y_i,y_j,x_k]\otimes\alpha(y_k)(\xi,\eta,\gamma,\kappa) + \\ &\alpha(x_i)\otimes\alpha(x_j)\otimes\alpha(x_k)\otimes[y_i,y_j,y_k](\xi,\eta,\gamma,\kappa)) \\ =\ &\sum_{i,j,k}\langle\xi,[x_i,x_j,x_k]\rangle\langle\eta,\alpha(y_i)\rangle\langle\gamma,\alpha(y_j)\rangle\langle\kappa,\alpha(y_k)\rangle + \langle\eta,[y_i,x_j,x_k]\rangle\langle\xi,\alpha(x_i)\rangle \\ &\langle\gamma,\alpha(y_j)\rangle\langle\kappa,\alpha(y_k)\rangle\langle\gamma,[y_i,y_j,x_k]\rangle\langle\xi,\alpha(x_i)\rangle\langle\eta,\alpha(x_j)\rangle\langle\kappa,\alpha(y_k)\rangle + \\ &\langle\kappa,[y_i,y_j,y_k]\rangle\langle\xi,\alpha(x_i)\rangle\langle\eta,\alpha(x_j)\rangle\langle\gamma,\alpha(x_k)\rangle \\ =\ &-\langle\xi,[r\circ\alpha^*(\eta),r\circ\alpha^*(\gamma),r\circ\alpha^*(\kappa)\rangle - \langle\eta,[r\circ\alpha^*(\gamma),r\circ\alpha^*(\xi),r\circ\alpha^*(\kappa)\rangle \\ &-\langle\gamma,[r\circ\alpha^*(\xi),r\circ\alpha^*(\eta),r\circ\alpha^*(\kappa)\rangle + \langle\kappa,[r\circ\alpha^*(\xi),r\circ\alpha^*(\eta),r\circ\alpha^*(\gamma)\rangle \\ =\ &\langle[r\circ\alpha^*(\xi),r\circ\alpha^*(\eta),r\circ\alpha^*(\gamma)] - r\circ\alpha^*([\xi,\eta,\gamma]^*),\kappa\rangle.\end{aligned}$$

So Equation (22) holds and this finishes the proof. □

Proposition 9. *Let $(L,[\cdot,\cdot,\cdot],\alpha)$ be a regular 3-Hom–Lie algebra and $r \in L\otimes L$ such that $\alpha^{\otimes^2}(r) = r$. Suppose r is skew-symmetric and nondegenerate. Then, r is a solution of the 3-Hom–Lie Yang–Baxter equation if and only if the nondegenerate skew-symmetric bilinear form B on L defined by $B(x,y) = \langle r^{-1}(x), y\rangle$ satisfies*

$$B(\alpha[x,y,z],w) - B(\alpha[x,y,w],z) + B(\alpha[x,z,w],y) - B(\alpha[y,z,w],x) = 0,$$

for any $x,y,z,w \in L$.

Proof. For any $x,y,z,w \in L$, there exists $\xi,\eta,\gamma,\kappa \in L^*$ such that $r(\xi) = x, r(\eta) = y$, $r(\gamma) = z$, $r(\kappa) = w$. If $[[r,r,r]]^\alpha = 0$, we have

$$\begin{aligned}&B(\alpha[x,y,z],w) \\ =\ &\langle\alpha[r(\xi),r(\eta),r(\gamma)],\kappa\rangle \\ =\ &\langle r\circ\alpha^*(ad^*_{r\circ\alpha^*(\xi),r\circ\alpha^*(\eta)}\gamma + ad^*_{r\circ\alpha^*(\eta),r\circ\alpha^*(\gamma)}\xi + ad^*_{r\circ\alpha^*(\gamma),r\circ\alpha^*(\xi)}\eta),\kappa\rangle \\ =\ &\langle ad^*_{r\circ\alpha^*(\xi),r\circ\alpha^*(\eta)}\gamma + ad^*_{r\circ\alpha^*(\eta),r\circ\alpha^*(\gamma)}\xi + ad^*_{r\circ\alpha^*(\gamma),r\circ\alpha^*(\xi)}\eta, -\alpha\circ r(\kappa)\rangle \\ =\ &\langle\gamma,\alpha[x,y,w]\rangle - \langle-\xi,\alpha[y,z,w]\rangle - \langle-\eta,\alpha[z,x,w]\rangle \\ =\ &B(\alpha[x,y,w],z) - B(\alpha[x,z,w],y) + B(\alpha[y,z,w],x).\end{aligned}$$

Thus the proof is finished. □

4. Symplectic Structures and Phase Spaces of 3-Hom–Lie Algebras

In this section, we introduce the notions of symplectic structures and phase spaces of 3-Hom–Lie algebras, and prove that a 3-Hom–Lie algebra has a phase space if and only if it is sub-adjacent to a 3-Hom–pre-Lie algebra.

Definition 13. *A symplectic structure on a regular 3-Hom–Lie algebra $(L, [\cdot,\cdot,\cdot], \alpha)$ is a nondegenerate skew-symmetric bilinear form $\omega \in L^* \wedge L^*$ satisfying the following equality*

$$\omega([x,y,z], \alpha(w)) - \omega([y,z,w], \alpha(x)) + \omega([z,w,x], \alpha(y)) - \omega([w,x,y], \alpha(z)) = 0, \qquad (23)$$

for any $x, y, z, w \in L$.

Definition 14 ([20])**.** *Let $(L, [\cdot,\cdot,\cdot], \alpha)$ be a 3-Hom–Lie algebra and $B : L \times L \to F$ be a nondegenerate symmetric bilinear form on L. If B satisfies*

$$B([x,y,z], w) + B(z, [x,y,w]) = 0, \forall x, y, z, w \in L. \qquad (24)$$

Then B is called a metric on 3-Hom–Lie algebra $(L, [\cdot,\cdot,\cdot], \alpha)$ and $(L, [\cdot,\cdot,\cdot], \alpha, B)$ is a metric 3-Hom–Lie algebra.

If there exists a metric B and a symplectic structure ω on the 3-Hom–Lie algebra $(L, [\cdot,\cdot,\cdot], \alpha)$, then $(L, [\cdot,\cdot,\cdot], \alpha, B, \omega)$ is called a metric symplectic 3-Hom–Lie algebra.

Let $(L, [\cdot,\cdot,\cdot], \alpha, B)$ be a metric 3-Hom–Lie algebra, we denote

$$Der_B(L) = \{D \in Der(L) | B(Dx, y) + B(x, Dy) = 0, \forall x, y \in L\}.$$

Theorem 4. *Let $(L, [\cdot,\cdot,\cdot], \alpha, B)$ be a metric 3-Hom–Lie algebra. Then, there exists a symplectic structure on L if and only if there exists a skew-symmetric invertible derivation $D \in Der_B(L)$.*

Proof. Suppose that $(L, [\cdot,\cdot,\cdot], \alpha, B)$ is a metric 3-Hom–Lie algebra, then for any $x, y \in L$, define $D : L \to L$ by

$$B(Dx, y) = \omega(\alpha(x), y). \qquad (25)$$

It is clear that D is invertible. Next we will check that D is a skew-symmetric invertible derivation of $(L, [\cdot,\cdot,\cdot], \alpha, B)$. In fact, for any $x, y, z, w \in L$, we have

$$\begin{aligned} & B([Dx,y,z],w) + B([x,Dy,z],w) + B([x,y,Dz],w) + B(D[x,y,z],w) \\ = & -B([y,z,w],Dx) + B([x,z,w],Dy) - B([x,y,w],Dz) + B([x,y,z],Dw) \\ = & \omega([x,y,z],\alpha(w)) - \omega([y,z,w],\alpha(x)) + \omega([z,w,x],\alpha(y)) - \omega([w,x,y],\alpha(z)) = 0, \end{aligned}$$

that is, $D \in Der_B(L)$.

Conversely, assume that $D \in Der_B(L)$ is a skew-symmetric invertible derivation. Define ω by Equation (25), then there exists a symplectic structure on L satisfies Equation (23). □

Example 9. *Let $(L, [\cdot,\cdot,\cdot], \alpha)$ be a 3-Hom–Lie algebra and*

$$F[t] = \{f(t) = \sum_{i=0}^{m} a_i t^i | a_i \in F, m \in N\}$$

be the algebra of polynomials over F. We consider

$$L_n = L \otimes (tF[t]/t^n F[t]),$$

where $tF[t]/t^nF[t]$ is the quotient space of $tF[t]$ module $t^nF[t]$. Then, L_n is a nilpotent 3-Hom–Lie algebra, with a linear map $\alpha' : L_n \to L_n$ and the following multiplication:

$$\alpha'(x \otimes t^{\overline{p}}) = \alpha(x) \otimes t^{\overline{p}}, [x \otimes t^{\overline{p}}, y \otimes t^{\overline{q}}, z \otimes t^{\overline{r}}]' = [x, y, z] \otimes t^{\overline{p+q+r}},$$

for any $x, y, z \in L$ and $p, q, r \in N \setminus \{0\}$. Define an endomorphism D of L_n by

$$D(x \otimes t^{\overline{p}}) = p(x \otimes t^{\overline{p}}), \forall x \in L, p = 1, ..., n-1.$$

Then D is an invertible derivation of the 3-Hom–Lie algebra L_n.

Let $\widetilde{L}_n = L_n \oplus L_n^*$, where L_n^* is the dual space of L_n. Then, $(\widetilde{L}_n, B)$ ia a metric 3-Hom–Lie algebra with the multiplication

$$[x+f, y+g, z+h] = [x, y, z] + ad^*(y, z)f - ad^*(x, z)g + ad^*(x, y)h,$$
$$B(x+f, y+g) = f(y) + g(x),$$

for any $x, y, z \in L_n$ and $f, g, h \in L_n^*$. And define linear maps $\widehat{D}, \widetilde{\alpha} : \widetilde{L}_n \to \widetilde{L}_n$ by

$$\widehat{D}(x+f) = Dx + D^*f, \widetilde{\alpha}(x+f) = \alpha(x) + f \circ \alpha,$$

where $D^*f = -fD$. Then, $\widehat{D}$ is invertible. Hence $(\widetilde{L}_n, \widetilde{\alpha}, B, \omega)$ is a metric symplectic 3-Hom–Lie algebra, where ω is defined as follows:

$$\omega(\widetilde{\alpha}(x+f), y+g) = B(\widehat{D}(x+f), y+g) = -f(Dy) + g(Dx).$$

Proposition 10. *Let $(L, [\cdot,\cdot,\cdot], \alpha, \omega)$ be a symplectic 3-Hom–Lie algebra. Then, there exists a compatible 3-Hom–pre-Lie algebra structure $\{\cdot,\cdot,\cdot\}$ on L given by*

$$\omega(\{x, y, z\}, \alpha(w)) = -\omega(\alpha(z), [x, y, w]), \forall x, y, z, w \in L. \tag{26}$$

Proof. For any $x, y \in L$, define the map $T : L^* \to L$ by $\langle T^{-1}x, y\rangle = \omega(x, y)$. By Equation (23), we obtain that T is an invertible $\mathcal{O}$-operator associated to the coadjoint representation (L^*, ad^*, α^*), and there exists a compatible 3-Hom–pre-Lie algebra on L given by $\{x, y, z\} = T(ad^*_{x,y}T^{-1}z)$. For any $x, y, z, w \in L$, we have

$$\begin{aligned}\omega(\{x, y, z\}, \alpha(w)) &= \omega(T(ad^*_{x,y}T^{-1}z), \alpha(w)) = \langle ad^*_{x,y}T^{-1}z, \alpha(w)\rangle \\ &= \langle T^{-1}(\alpha(z)), -[x, y, w]\rangle = -\omega(\alpha(z), [x, y, w]),\end{aligned}$$

as desired. The proof is finished. □

Let V be a vector space and V^* its dual space. Then, there is a natural nondegenerate skew-symmetric bilinear form ω on $T^*V = V \oplus V^*$ given by:

$$\omega(x+f, y+g) = \langle f, y\rangle - \langle g, x\rangle, \forall x, y \in V, f, g \in V^*. \tag{27}$$

Definition 15. *Let $(L, [\cdot,\cdot,\cdot], \alpha)$ and $(L^*, [\cdot,\cdot,\cdot]^*, \alpha^*)$ be two admissible 3-Hom–Lie algebras. If there is a 3-Hom–Lie algebra structure $[\cdot,\cdot,\cdot]$ on the direct sum vector space $T^*L = L \oplus L^*$ such that $(L \oplus L^*, [\cdot,\cdot,\cdot], \alpha \oplus \alpha^*, \omega)$ is a symplectic 3-Hom–Lie algebra, where ω given by Equation (27), $(L, [\cdot,\cdot,\cdot], \alpha)$ and $(L^*, [\cdot,\cdot,\cdot]^*, \alpha^*)$ are two 3-Hom–Lie subalgebras of $(L \oplus L^*, [\cdot,\cdot,\cdot], \alpha \oplus \alpha^*)$. Then the symplectic 3-Hom–Lie algebra $(L \oplus L^*, [\cdot,\cdot,\cdot], \alpha \oplus \alpha^*, \omega)$ is called a phase space of the 3-Hom–Lie algebra $(L, [\cdot,\cdot,\cdot], \alpha)$.*

Next, we will study the relation between 3-Hom–pre-Lie algebras and phase spaces of 3-Hom–Lie algebras.

Theorem 5. *A 3-Hom–Lie algebra has a phase space if and only if it is sub-adjacent to a 3-Hom–pre-Lie algebra.*

Proof. $\Leftarrow$ Assume $(L, \{\cdot,\cdot,\cdot\}, \alpha)$ is a 3-Hom–pre-Lie algebra. By Proposition 5, the left multiplication $\mathcal{L}$ is a representation of the sub-adjacent 3-Lie algebra L^C on L, $\mathcal{L}^*$ is a representation of the sub-adjacent 3-Lie algebra L^c on L^*, then we have a 3-Hom–Lie algebra $(L^c \oplus L^*, [\cdot,\cdot,\cdot]_{L^*}, \alpha \oplus \alpha^*)$. For any $x_1, x_2, x_3, x_4 \in L$ and $f_1, f_2, f_3, f_4 \in L^*$, we have

$$\begin{aligned}
& \omega([x_1 + f_1, x_2 + f_2, x_3 + f_3]_{L^*}, \alpha(x_4) + \alpha^*(f_4)) \\
= \; & \omega([x_1, x_2, x_3]_C + \mathcal{L}^*(x_1, x_3) f_3 + \mathcal{L}^*(x_2, x_3) f_1 + \mathcal{L}^*(x_3, x_1) f_2, \alpha(x_4) + \alpha^*(f_4)) \\
= \; & \langle \mathcal{L}^*(x_1, x_3) f_3 + \mathcal{L}^*(x_2, x_3) f_1 + \mathcal{L}^*(x_3, x_1) f_2, \alpha(x_4) \rangle - \langle \alpha^*(f_4), [x_1, x_2, x_3]_C \rangle \\
= \; & -\langle \alpha^*(f_3), \{x_1, x_2, x_3\} \rangle - \langle \alpha^*(f_1), \{x_2, x_3, x_4\} \rangle - \langle \alpha^*(f_2), \{x_3, x_1, x_4\} \rangle \\
& -\langle \alpha^*(f_4), \{x_1, x_2, x_3\} \rangle - \langle \alpha^*(f_4), \{x_2, x_3, x_1\} \rangle - \langle \alpha^*(f_4), \{x_3, x_1, x_2\} \rangle.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
& \omega([x_2 + f_2, x_3 + f_3, x_4 + f_4]_{L^*}, \alpha(x_1) + \alpha^*(f_1)) \\
= \; & -\langle \alpha^*(f_4), \{x_2, x_3, x_1\} \rangle - \langle \alpha^*(f_2), \{x_3, x_4, x_1\} \rangle - \langle \alpha^*(f_3), \{x_4, x_2, x_1\} \rangle \\
& -\langle \alpha^*(f_1), \{x_2, x_3, x_4\} \rangle - \langle \alpha^*(f_1), \{x_3, x_4, x_2\} \rangle - \langle \alpha^*(f_1), \{x_4, x_2, x_3\} \rangle, \\
& \omega([x_3 + f_3, x_4 + f_4, x_1 + f_1,]_{L^*}, \alpha(x_2) + \alpha^*(f_2)) \\
\\
= \; & -\langle \alpha^*(f_1), \{x_3, x_4, x_2\} \rangle - \langle \alpha^*(f_3), \{x_4, x_1, x_2\} \rangle - \langle \alpha^*(f_4), \{x_1, x_3, x_2\} \rangle \\
& -\langle \alpha^*(f_2), \{x_3, x_4, x_1\} \rangle - \langle \alpha^*(f_2), \{x_4, x_1, x_3\} \rangle - \langle \alpha^*(f_2), \{x_1, x_3, x_4\} \rangle, \\
& \omega([x_4 + f_4, x_1 + f_1, x_2 + f_3,]_{L^*}, \alpha(x_3) + \alpha^*(f_3)) \\
= \; & -\langle \alpha^*(f_2), \{x_4, x_1, x_3\} \rangle - \langle \alpha^*(f_4), \{x_1, x_2, x_3\} \rangle - \langle \alpha^*(f_1), \{x_2, x_4, x_3\} \rangle \\
& -\langle \alpha^*(f_3), \{x_4, x_1, x_2\} \rangle - \langle \alpha^*(f_3), \{x_1, x_2, x_4\} \rangle - \langle \alpha^*(f_3), \{x_2, x_4, x_1\} \rangle.
\end{aligned}$$

So ω is a symplectic structure on the semidirect product 3-Hom–Lie algebra $(L^c \oplus L^*, [\cdot,\cdot,\cdot]_{L^*}, \alpha \oplus \alpha^*)$. Thus the symplectic 3-Hom–Lie algebra $(L^c \oplus L^*, [\cdot,\cdot,\cdot]_{L^*}, \alpha \oplus \alpha^*, \omega)$ is a phase space of the sub-adjacent 3-Hom–Lie algebra $(L^c, [\cdot,\cdot,\cdot]_C, \alpha)$.

$\Rightarrow$ Clearly. □

Author Contributions: Conceptualization, S.G., S.W. and X.Z.; methodology, S.G., S.W. and X.Z.; investigation, S.G., S.W. and X.Z.; resources, S.G., S.W. and X.Z.; writing—original draft preparation, S.G., S.W. and X.Z.; writing—review and editing, S.G., S.W. and X.Z.; visualization, S.G., S.W. and X.Z. All authors have read and agreed to the published version of the manuscript.

Funding: The paper is supported by the NSF of China (No. 12161013), the Key University Science Research Project of Anhui Province (No. KJ2020A0711) and the Anhui Provincial Natural Science Foundation (No. 1908085MA03).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Acknowledgments: The authors are very grateful to the anonymous referee for his/her thorough review of this work and his/her comments.

Conflicts of Interest: The authors declare no conflict of interest.

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