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Some Remarks on the Divisibility of the Class Numbers of Imaginary Quadratic Fields

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Abstract: For a given integer n, we provide some families of imaginary quadratic number fields of the form $\mathbb{Q}(\sqrt{4q^2 - p^n})$, whose ideal class group has a subgroup isomorphic to $\mathbb{Z}/n\mathbb{Z}$.

Keywords: class number; imaginary quadratic fields; divisibility of class number

MSC: 11R29; 11R11

1. Introduction

The class number of a number field is by definition the order of the ideal class group of its ring of integers. Thus, a number field has class number one if and only if its ring of integers is a principal ideal domain. In this sense, the ideal class group measures how far R is from being a principal ideal domain, and hence from satisfying unique prime factorization. The divisibility properties of class numbers are very important to know the structure of ideal class groups of number fields. Numerous results about the divisibility of the class numbers of quadratic fields have been introduced by many authors ([1–15]). By their works, it was shown that there exist infinitely many imaginary quadratic number fields whose ideal class numbers are multiples of n. They proved that there exist infinitely many imaginary quadratic number fields such that the ideal class group has a cyclic subgroup of order n. Most of such families are of the type $\mathbb{Q}(\sqrt{x^2 - t^n})$ or of the type $\mathbb{Q}(\sqrt{x^2 - 4t^n})$, where x and t are positive integers with some restrictions. (For the case of $\mathbb{Q}(\sqrt{x^2 - t^n})$, see [1,2,6,7,9,11–13,15] and for the case of $\mathbb{Q}(\sqrt{x^2 - 4t^n})$ see [3–5,8,10,14]).

Recently, K. Chakraborty, A. Hoque, Y. Kishi and P.P. Pandey considered the family $K_{p,q} = \mathbb{Q}(\sqrt{q^2 - p^n})$ when p and q were distinct odd prime numbers and $n \geq 3$ was an odd integer (see Theorem 1.2 of [2]). However, they just dealt with the case when n was an odd integer. We want to deal with the case when n is an even integer. In this article, we treat the family $K_{p,2q} = \mathbb{Q}(\sqrt{4q^2 - p^n})$ when p and q are distinct odd prime numbers.

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2. Preliminaries

In this section, we review some previous results which we will use.

2.1. Being a pth Power

Proposition 1. *(Proposition 2.2 in [2]). Let $d \equiv 5 \pmod 8$ be an integer and ℓ be a prime. For odd integers a, b, we have*

$$\left(\frac{a + b\sqrt{d}}{2}\right)^{\ell} \in \mathbb{Z}[d] \text{ if and only if } \ell = 3.$$

Definition 1. *If L/K is a Galois extension and α is in L, then the trace of α is the sum of all the Galois conjugates of α, i.e.,*

$$Tr(\alpha) = \sum_{\sigma \in \mathrm{Gal}(L/K)} \sigma(\alpha),$$

where Gal(L/K) *denotes the Galois group of* L/K.

Lemma 1. *(Lemma 4 in [10]). Let K be a quadratic number field and O_K be its ring of algebraic integers. If $\alpha \in O_K$, then α is a square in O_K if and only if there exists $A \in \mathbb{Z}$ such that $N(\alpha) = A^2$ and such that $Tr(\alpha) + 2A$ is a square in $\mathbb{Z}$. If K is imaginary, we may assume that $A \geq 0$.*

2.2. Result of Y. Bugeaud and T. N. Shorey

In this section, we review a result of Y. Bugeaud and T.N. Shorey (see [16]). Let F_n be the nth Fibonacci sequence and L_n be the nth Lucas sequence. Let us define the sets $\mathcal{F}$ and $\mathcal{G} \subset \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ by

$$\mathcal{F} := \{(F_{h_1-2\epsilon}, L_{h_1+\epsilon}, F_{h_1}) | h_1 \in \mathbb{N} \text{ s.t. } h_1 \geq 2 \text{ and } \epsilon \in \{\pm 1\}\}$$

and

$$\mathcal{G} := \{(1, 4p_1^{h_2} - 1, p_1) | p_1 \text{ is a prime number and } h_2 \in \mathbb{N}\}.$$

For $\lambda \in \{1, \sqrt{2}, 2\}$, we define the set $\mathcal{H}_\lambda \subset \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ by

$$\mathcal{H}_\lambda := \left\{ (D_1, D_2, p) \;\middle|\; \begin{array}{l} D_1, D_2 \text{ and } p \text{ are mutually coprime positive integers with} \\ p \text{ an odd prime and there exist positive integers } r, s \text{ such} \\ \text{that } D_1 s^2 + D_2 = \lambda^2 p^r \text{ and } 3D_1 s^2 - D_2 = \pm\lambda^2 \end{array} \right\}$$

Theorem 1. *(Theorem 1 in [16]). Let D_1, D_2 and p be mutually coprime positive integers with p a prime number. Let $\lambda \in \{1, \sqrt{2}, 2\}$ be such that $\lambda = 2$ if $p = 2$. We assume that D_2 is odd if $\lambda \in \{\sqrt{2}, 2\}$. Then, the number of positive integer solutions (x, y) of the equation*

$$D_1 x^2 + D_2 = \lambda^2 p^y \tag{1}$$

is at most one except for

$$(\lambda, D_1, D_2, p) \in \mathcal{E} := \left\{ \begin{array}{l} (2,13,3,2), (\sqrt{2},7,11,3), (1,2,1,3), (2,7,1,2), \\ (\sqrt{2},1,1,5), (\sqrt{2},1,1,13), (2,1,3,7). \end{array} \right\}$$

or

$$(D_1, D_2, p) \in \mathcal{F} \cup \mathcal{G} \cup \mathcal{H}_\lambda.$$

We recall the result of J.H.E Cohn [17] about the appearance of squares in the Lucas sequence.

Theorem 2. *Let L_n be the nth Lucas sequence. Then, the only perfect square appearing in the Lucas sequences are $L_1 = 1$ and $L_3 = 4$.*

3. Main Result

In this section, we will describe the main result. Here is the crucial theorem.

Theorem 3. *Suppose that $n \geq 3$ is an integer and q is an odd prime number such that $(q, n) = 1$ and $q \not\equiv \pm 1 \pmod{\ell}$ for all odd prime number $\ell \neq 3$ dividing n. Let p be an odd prime number with $4q^2 < p^n$ and $(q, p) = 1$. Let d be the square-free part of $4q^2 - p^n$, i.e., $4q^2 - p^n = m^2 d$ for some positive integer m. Assume that $2q \not\equiv \pm 1 \pmod{|d|}$. Moreover, we assume $q \not\equiv 2 \pmod 3$ when $3|n$. Then, we have the following:*

(i) Assume that n is an even integer or $p \equiv 1 \pmod 4$. Then, the class number of $K_{p,2q} = \mathbb{Q}(\sqrt{d})$ is divisible by n.

(ii) Assume that n is an odd integer and $p \equiv 3 \pmod 4$. Moreover, we assume $p^{n/3} \neq (4q+1)/3$ when $3|n$. Then, the class number of $K_{p,2q} = \mathbb{Q}(\sqrt{d})$ is divisible by n.

Remark 1. *By Dirichlet's theorem on arithmetic progressions, we know that there exist infinitely many q such that $q \not\equiv \pm 1 \pmod{\ell}$ for all odd prime number $\ell \neq 3$ dividing n.*

Theorem 4. *Let n, q be as in Theorem 3. For each q, the class number of $K_{p,2q}$ is divisible by n for all but finitely many p's. Furthermore, for each q there are infinitely many fields $K_{p,2q}$.*

4. Proof of Main Theorem

4.1. Crucial Proposition

Lemma 2. *Let p, d and m be as in Theorem 3 (i) or (ii). Let ℓ be an odd prime such that*

$$\alpha = 2q + m\sqrt{d} = (a + b\sqrt{d})^{\ell}$$

for some integer a and b. Then, $a|2q$ if and only if $-a|2q$.

Proof. Suppose that

$$\alpha = 2q + m\sqrt{d} = (a + b\sqrt{d})^{\ell}.$$

If we compare the real parts, we know that

$$2q = a^{\ell} + \sum_{i=1}^{(\ell-1)/2} \binom{\ell}{2i} a^{\ell-2i} b^{2i} d^{i}.$$

This implies that $a|2q$. Since $a|2q$, we also know that $-a|2q$. Similarly, $-a|2q$ implies that $a|2q$. □

Proposition 2. *Let n, q, p, d and m be as in Theorem 3 (i) or (ii). Then, the element $\alpha = 2q + m\sqrt{d}$ is not an ℓth power of an element in the ring of integers of $K_{p,2q}$ for any odd prime divisor ℓ of n. In addition, α and $-\alpha$ are not a square in $\mathcal{O}_{K_{p,2q}}$.*

Proof. (i) Assume that n is an even integer or $p \equiv 1 \pmod 4$. Moreover, we assume $p^{n/3} \neq (q + 16)/3$ when $3|n$. Since n is an even integer or $p \equiv 1 \pmod 4$, we know that $d \equiv 3 \pmod 4$. Let ℓ be an odd prime divisor of n. If $\alpha = 2q + m\sqrt{d}$ is an ℓth power, then

$$\alpha = 2q + m\sqrt{d} = (a + b\sqrt{d})^{\ell}$$

for some integer a and b. If we compare the real parts, we know that

$$2q = a^{\ell} + \sum_{i=1}^{(\ell-1)/2} \binom{\ell}{2i} a^{\ell-2i} b^{2i} d^{i}.$$

This implies that $a|2q$. By Lemma 2, we can assume that $a = 2q$, $a = q$, $a = 2$ or $a = 1$.

Case (i-A1): $a = 2$, $\ell \neq 3$

Comparing the real parts, we have

$$2q = (\pm 2)^{\ell} + \sum_{i=1}^{(\ell-1)/2} \binom{\ell}{2i} (\pm 2)^{\ell-2i} b^{2i} d^{i} \equiv \pm 2 \pmod{\ell}.$$

From these, we have $q \equiv \pm 1 \pmod{\ell}$, which violates our assumption.

Case (i-A2): $a = 2$, $\ell = 3$

Suppose that

$$\alpha = 2q + m\sqrt{d} = (2 + b\sqrt{d})^{3}.$$

Comparing the real parts, we have

$$2q = 8 + 6b^2 d. \tag{2}$$

Since $d < 0$, we have $q = 4 + 3b^2 d < 0$. This is impossible.

Case (i-B1) : $a = q$, $\ell \neq 3$
Comparing the real parts, we have

$$2q = (\pm q)^\ell + \sum_{i=1}^{(\ell-1)/2} \binom{\ell}{2i} (\pm q)^{\ell-2i} b^{2i} d^i \equiv \pm q \pmod{\ell}.$$

Thus, we get $3q \equiv 0 \pmod{\ell}$ or $q \equiv 0 \pmod{\ell}$, which contradicts the assumption "$(q, n) = 1$" and "$\ell \neq 3$".

Case (i-B2) : $a = q$, $\ell = 3$
Suppose that

$$\alpha = 2q + m\sqrt{d} = (q + b\sqrt{d})^3.$$

Comparing the real parts, we have

$$2q = q^3 + 3qb^2 d. \tag{3}$$

By (3), we have $2 = q^2 + 3b^2 d$, and hence $2 \equiv q^2 \pmod{3}$. This is impossible.

Case (i-C) : $a = 2q$
We have $2q + m\sqrt{d} = (2q + b\sqrt{d})^\ell$. Taking the norm on both sides, we obtain

$$p^n = (4q^2 - b^2 d)^\ell.$$

If we write $D_1 = -d > 0$, we have

$$D_1 b^2 + 4q^2 = p^{n/\ell}.$$

We also obtain

$$D_1 m^2 + 4q^2 = p^n.$$

Then, we easily know that $(|b|, n/\ell)$ and (m, n) are distinct solutions of (1) for $D_1 = -d > 0$, $D_2 = 4q^2$, $\lambda = 1$. The next thing we have to do is to show that $(1, D_1, D_2, p) \notin \mathcal{E}$ and $(D_1, D_2, p) \notin \mathcal{F} \cup \mathcal{G} \cup \mathcal{H}_\lambda$. Clearly, $(1, D_1, D_2, p) \notin \mathcal{E}$ and $(D_1, D_2, p) \notin \mathcal{G}$. By Theorem 2, we know that $(D_1, D_2, p) \notin \mathcal{F}$. Finally suppose that $(D_1, D_2, p) \in \mathcal{H}_\lambda$. Then, there exist positive integers r, s such that

$$3D_1 s^2 - 4q^2 = \pm 1 \tag{4}$$

and

$$D_1 s^2 + 4q^2 = p^r. \tag{5}$$

By (4), we have $q \neq 3$, and hence we have $3D_1 s^2 - 4q^2 = -1$. From this together with (5), we obtain

$$16q^2 = 3p^r + 1,$$

that is,

$$(4q - 1)(4q + 1) = 3p^r.$$

This implies that $4q - 1 = 1$ or $4q - 1 = 3$. It contradicts the fact that q is an odd prime number. Hence, $(D_1.D_2, p) \notin \mathcal{H}_1$. By Theorem 1, the equation

$$-dx^2 + 4q^2 = p^y$$

has at most one integer solutions (x, y). Thus, $a \neq 2q$

Case (i-D) : $a = 1$
Comparing the real parts, we have

$$2q = (1)^\ell + \sum_{i=1}^{(\ell-1)/2} \binom{\ell}{2i} (1)^{\ell-2i} b^{2i} d^i \equiv 1 \pmod{|d|}.$$

It contradicts our assumption "$2q \equiv 1 \pmod{|d|}$".

(ii) Assume that n is an odd integer and $p \equiv 3 \pmod 4$. Then, we know that $d \equiv 1 \pmod 4$. Moreover, we assume $p^{n/3} \neq (4q+1)/3$ when $3|n$. Let ℓ be an odd prime divisor of n. If $\alpha = 2q + m\sqrt{d}$ is an ℓth power, then

$$\alpha = 2q + m\sqrt{d} = \left(\frac{a + b\sqrt{d}}{2}\right)^\ell, \; a \equiv b \pmod 2.$$

for some integer a and b. In case both a and b are even, then we can proceed as in the above and obtain a contradiction. Thus, we can assume that both a and b are odd. If we take the norm on both sides we obtain

$$4p^{n/\ell} = a^2 - b^2 d. \tag{6}$$

Since a and b are odd integers and $p \neq 2$, we can get $d \equiv 5 \pmod 8$. By Proposition 1, we know that $\ell = 3$. Thus, we have

$$\alpha = 2q + m\sqrt{d} = \left(\frac{a + b\sqrt{d}}{2}\right)^3.$$

Comparing the real parts, we have

$$16q = a(a^2 + 3b^2 d). \tag{7}$$

Since a is an odd integer, we have $a = 1$ or $a = q$.

Case (ii-A) : $a = 1$
By (7) and $d < 0$, we have $16q = 1 + 3b^2 d < 0$. This is not possible.

Case (ii-B) : $a = q$
By (6) and (7), we have

$$4p^{n/3} = q^2 - b^2 d \text{ and } 16 = q^2 + 3b^2 d.$$

From these, we have $3p^{n/3} = q^2 - 4 = (q-2)(q+2)$. This implies that $q - 2 = 3$ or $q + 2 = 3$. Since q is a prime, we have $q - 2 = 3$ and $p^{n/3} = q + 2 = 7$. These violate our assumption $p^{n/3} \neq (4q+1)/3$.
□

4.2. *Proof of Theorem 3*

Next, we prove Theorem 3.

Proof of Theorem 3. Let n, q, p, d and m be as in Theorem 3 (i) or (ii). Set $\alpha = 2q + m\sqrt{d}$. We can easily check that α and $\bar{\alpha}$ are coprime and $N(\alpha) = \alpha\bar{\alpha} = p^n$. This implies that $(\alpha) = \mathfrak{a}^n$ for some integral ideal $\mathfrak{a}$ of $K_{p,2q}$. It suffices to show that the order of $[\mathfrak{a}]$ in the ideal class group of $K_{p,2q}$ is n. If this is not the case, we have $(\alpha) = (\beta)^\ell$ for some integer β in $\mathcal{O}_{K_{p,2q}}$ and some prime divisor ℓ of n. Since $K_{p,2q}$ is an imaginary quadratic field, the only units of $\mathcal{O}_{K_{p,2q}}$ are ± 1. Thus, we have $\alpha = \pm\beta^{\ell}$. If ℓ is an odd prime, we have $\alpha = \gamma^{\ell}$ where $\gamma = \pm\beta$. This contradicts Proposition 2. Next, let us consider the case of $\ell = 2$. Then, we have $\alpha = \pm\beta^2$. It means that α or $-\alpha$ is a square in $\mathcal{O}_{K_{p,2q}}$, which contradicts Proposition 2. Hence, the order of $[\mathfrak{a}]$ in the ideal class group of $K_{p,2q}$ is n. □

4.3. Proof of Theorem 4

We are now in a position to prove the main theorem

Proof. Let n and q be as in Theorem 3. For any positive integer D, the curve

$$DX^2 + 4q^2 = Y^n \tag{8}$$

is an irreducible algebraic curve of genus > 0 (see [18]). By Siegel's theorem (see [19]), there are only finitely many integral points (X, Y) on the curve (8). Thus, for each $d < 0$, there are at most finitely many primes p such that

$$-dx^2 + 4q^2 = p^n.$$

It means that there are infinitely many fields $K_{p,2q}$ for the fixed prime q. In addition, we have $|d| > 2q + 1$ for sufficiently large p, so $2q \not\equiv \pm 1 \pmod{|d|}$. Further, if p is large enough, then $p^{n/3} \neq (q+16)/3$ and $p^{n/3} \neq (4q+1)/3$. Hence, the class number of $K_{p,2q}$ is divisible by n for a sufficiently large p. □

5. Numerical Examples

In this section, we give several examples. All computations in this section are based on the Magma program. For example, Table 1 is the list of imaginary quadratic fields $K_{p,2q}$ corresponding to $n = 3$ and $p \leq 19$. In the below Tables 2–8, we use * in the column for class number to indicate the failure of condition "$p^{n/3} \neq (q+16)/3$" or "$p^{n/3} \neq (4q+1)/3$". Furthermore, the appearance of ** in the column for a class number indicates the failure of condition "$2q \not\equiv \pm 1 \pmod{|d|}$". Finally, the appearance of *** in the column for a class number indicates the failure of condition "$q \not\equiv \pm 1 \pmod{\ell}$" for an odd prime divisor $\ell \neq 3$ of n.

Table 1. Numerical examples for $n = 3$.

p	q	$4q^2 - p^3$	d	$h(d)$	p	q	$4q^2 - p^3$	d	$h(d)$
7	5	−243	−3	1 *	11	5	−1231	−1231	27
11	7	−1135	−1135	18	11	13	−655	−655	12
11	17	−175	−7	1 *	13	5	−2097	−233	12
13	7	−2001	−2001	48	13	11	−1713	−1713	36
13	17	−1041	−1041	36	13	19	−753	−753	12
17	5	−4813	−4813	30	17	7	−4717	−4717	24
17	11	−4429	−4429	60	17	13	−4237	−4237	24
17	19	−3469	−3469	30	17	23	−2797	−2797	18
17	29	−1549	−1549	18	17	31	−1069	−1069	30
19	5	−6759	−751	15	19	7	−6663	−6663	60
19	11	−6375	−255	12	19	13	−6183	−687	12
19	17	−5703	−5703	54	19	23	−4743	−527	18
19	29	−3495	−3495	36	19	31	−3015	−335	18
19	37	−1383	−1383	18	19	41	−135	−15	2 *

Table 2. Numerical examples for $n = 4$.

p	q	$4q^2 - p^4$	d	$h(d)$	p	q	$4q^2 - p^4$	d	$h(d)$
5	3	−589	−589	16	5	7	−429	−429	16
5	11	−141	−141	8	7	3	−2365	−2365	32
7	5	−2301	−2301	48	7	11	−1917	−213	8
7	13	−1725	−69	8	7	17	−1245	−1245	32
7	19	−957	−957	16	7	23	−285	−285	16
11	3	−14,605	14, 605	80	11	5	−14,541	−14,541	64
11	7	−14,445	1605	16	11	13	−13,965	−285	16
11	17	−13,485	−13,485	128	11	19	−13,197	−13,197	48
11	23	−12,525	−501	16	11	29	−11,277	−11,277	32
11	31	−10,797	−10,797	64	11	37	−9165	−9165	64
11	41	−7917	−7917	32	11	43	−7245	−805	16
11	47	−5805	−645	16	11	53	−3405	−3405	48
11	59	−717	−717	16					

Table 3. Numerical examples for $n = 5$.

p	q	$4q^2 - p^5$	d	$h(d)$	p	q	$4q^2 - p^5$	d	$h(d)$
3	7	−47	−47	5	5	3	−3089	−3089	40
5	7	−2929	−2929	40	5	11	−2641	−2641	20
5	13	−2449	−2449	40	5	17	−1969	−1969	20
5	19	−1681	−1	1 **	5	23	−1009	−1009	20
7	3	−16,771	−16,771	40	7	5	−16,707	−16,707	20
7	11	−16,323	−16,323	30	7	13	−16,131	−16,131	40
7	17	−15,651	−1739	20	7	19	−15,363	−1707	10
7	23	−14,691	−14,691	40	7	29	−13,443	−13,443	30
7	31	−12,963	−12,963	20	7	37	−11,331	−1259	15
7	41	−10,083	−10,083	20	7	43	−9411	−9411	30
7	47	−7971	−7971	30	7	53	−5571	−619	5
7	59	−2883	−3	1 **	7	61	−1923	−1923	10

Table 4. Numerical examples for $n = 6$.

p	q	$4q^2 - p^6$	d	$h(d)$	p	q	$4q^2 - p^6$	d	$h(d)$
3	5	−629	−629	36	3	7	−533	−533	12
3	11	−245	−5	2 *	3	13	−53	−53	6
5	3	−15,589	−15,589	72	5	7	−15,429	−15,429	96
5	11	−15,141	−309	12	5	13	−14,949	−1661	48
5	17	−14,469	−14,469	96	5	19	−14,181	−14,181	96
5	23	−13,509	−1501	24	5	29	−12,261	−12,261	72
5	31	−11,781	−1309	24	5	37	−10,149	−10,149	120
5	41	−8901	−989	36	5	43	−8229	−8229	48
5	47	−6789	−6789	72	5	53	−4389	−4389	48
5	59	−1701	−21	4 *	5	61	−741	−741	24
7	3	−117,613	−117,613	168	7	5	−117,549	−13,061	156
7	11	−117,165	−117,165	240	7	13	−116,973	−12,997	60
7	17	−116,493	−116,493	192	7	19	−116,205	−116,205	192
7	23	−115,533	−12,837	72	7	29	−114,285	−114,285	240
7	31	−113,805	−1405	24	7	37	−112,173	−112,173	240
7	41	−110,925	−493	12	7	43	−110,253	−110,253	288

Table 4. *Cont.*

p	q	$4q^2-p^6$	d	$h(d)$	p	q	$4q^2-p^6$	d	$h(d)$
7	47	−108,813	−108,813	240	7	53	−106,413	−106,413	216
7	59	−103,725	−461	30	7	61	−102,765	−102,765	192
7	67	−99,693	−11,077	48	7	71	−97,485	−97,485	192
7	73	−96,333	−96,333	192	7	79	−92,685	−92,685	288
7	83	−90,093	−90,093	192	7	89	−85,965	−85,965	240
7	97	−80,013	−80,013	192	7	101	−76,845	−76,845	192
7	103	−75,213	−8357	72	7	107	−71,853	−71,853	144
7	109	−70,125	−2805	48	7	113	−66,573	−7397	72
7	127	−53,133	−53,133	120	7	131	−49,005	−5	2 *
7	137	−42,573	−42,573	120	7	139	−40,365	−4485	48
7	149	−28,845	−3205	24	7	151	−26,445	−26,445	96
7	157	−19,053	−2117	36	7	163	−11,373	−11,373	72
7	167	−6093	−677	30					

Table 5. Numerical examples for $n = 7$.

p	q	$4q^2-p^7$	d	$h(d)$	p	q	$4q^2-p^7$	d	$h(d)$
3	5	−2087	−2087	35	3	11	−1703	−1703	28
3	13	−1511	−1511	49	3	17	−1031	−1031	35
3	19	−743	−743	21	3	23	−71	−71	7
5	3	−78,089	−78,089	280	5	11	−77,641	−77,641	112
5	13	−77,449	−77,449	112	5	17	−76,969	−76,969	196
5	19	−76,681	−76,681	140	5	23	−76,009	−76,009	224
5	29	−74,761	−74,761	140	5	31	−74,281	−74,281	140
5	37	−72,649	−72,649	168	5	41	−71,401	−71,401	140
5	43	−70,729	−70,729	140	5	47	−69,289	−69,289	196
5	53	−66,889	−66,889	112	5	59	−64,201	−64,201	112
5	61	−63,241	−63,241	196	5	67	−60,169	−60,169	112
5	71	−57,961	−57,961	112	5	73	−56,809	−56,809	112
5	79	−53,161	−53,161	168	5	83	−50,569	−50,569	168
5	89	−46,441	−46,441	140	5	97	−40,489	−40,489	140
5	101	−37,321	−37,321	84	5	103	−35,689	−35,689	112
5	107	−32,329	−32,329	140	5	109	−30,601	−30,601	112
5	113	−27,049	−27,049	84	5	127	−13,609	−13,609	56
5	131	−9481	−9481	84	5	137	−3049	−3049	28
5	139	−841	−1	1 **					

Table 6. Numerical examples for $n = 8$.

p	q	$4q^2-p^8$	d	$h(d)$	p	q	$4q^2-p^8$	d	$h(d)$
3	5	−6461	−6461	96	3	7	−6365	−6365	64
3	11	−6077	−6077	48	3	13	−5885	−5885	96
3	17	−5405	−5405	64	3	19	−5117	−5117	64
3	23	−4445	−4445	64	3	29	−3197	−3197	64
3	31	−2717	−2717	32	3	37	−1085	−1085	32

Table 7. Numerical examples for $n = 9$.

p	q	$4q^2 - p^9$	d	$h(d)$	p	q	$4q^2 - p^9$	d	$h(d)$
3	5	−19,583	−19,583	99	3	7	−19,487	−19,487	144
3	11	−19,199	−19,199	162	3	13	−19,007	−19,007	108
3	17	−18,527	−18,527	108	3	19	−18,239	−18,239	144
3	23	−17,567	−17,567	90	3	29	−16,319	−16,319	153
3	31	−15,839	−15,839	180	3	37	−14,207	−14,207	81
3	41	−12,959	−12,959	99	3	43	−12,287	−12,287	90
3	47	−10,847	−10,847	63	3	53	−8447	−8447	99
3	59	−5759	−5759	108	3	61	−4799	−4799	63
3	67	−1727	−1727	36					

Table 8. Numerical examples for $n = 10$.

p	q	$4q^2 - p^{10}$	d	$h(d)$	p	q	$4q^2 - p^{10}$	d	$h(d)$
3	7	−58,853	−58,853	180	3	11	−58,565	−58,565	240
3	13	−58,373	−58,373	240	3	17	−57,893	−57,893	280
3	23	−56,933	−197	10	3	29	−55,685	−55,685	160
3	31	−55,205	−55,205	240	3	37	−53,573	−317	10
3	41	−52,325	−2093	40	3	43	−51,653	−51,653	160
3	47	−50,213	−50,213	120	3	53	−47,813	−47,813	260
3	59	−45,125	−5	2 ***	3	61	−44,165	−365	20
3	67	−41,093	−41,093	240	3	71	−38,885	−38,885	160
3	73	−37,733	−37,733	160	3	79	−34,085	−34,085	200
3	83	−31,493	−31,493	120	3	89	−27,365	−27,365	120
3	97	−21,413	−437	20	3	101	−18,245	−18,245	160
3	103	−16,613	−16,613	100	3	107	−13,253	−13,253	80
3	109	−11,525	−461	30	3	113	−7973	−7973	80

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