

Article

Fuzzy Evaluation of Process Quality with Process Yield Index

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Abstract: With the rapid development and evolution of the Internet-of-Things (IoT) and big-data analysis technologies, faster and more accurate production data analysis and process capability evaluation models will bring industries closer to the goal of smart manufacturing. Small sample sizes are also common, due to destructive testing, the high costs of detection, and insufficient technological capacity, and these undermine the reliability of the statistical method. Many studies have pointed out that a confidence-interval-based fuzzy decision model can incorporate accumulated data and expert experiences to increase testing accuracy for small samples. Therefore, this study came up with a confidence-interval-based fuzzy decision model based on a process yield index. The index not only reflects process capability but also has a one-to-one mathematical relation with the process yield so that it is convenient to apply in practice. The proposed model not only diminishes the probability of misjudgment resulting from sampling error but also improves the accuracy of testing under the situation of small sample sizes, thereby contributing to the development of smart manufacturing.

Keywords: process yield index; process capability; confidence interval; fuzzy evaluation and decision model; mathematical programming method

MSC: 62C05; 62C86



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1. Introduction

Central Taiwan is the home to a machine-tool industry cluster that is facing keen global competition. Nowadays, numerous manufacturers emphasize their core competencies by making components that they excel at making to improve their competitive advantages. They then outsource non-core manufacturing to suppliers [1–4]. The machine-tool supply chain is comprised of critical suppliers, machine-tool manufacturers (which sell their machine tools via online platforms to countries all over the world for finishing), and their final customers. Taiwan's electronics industry also represents a comprehensive ecological chain of the electronics industry within the worldwide supply chain of information and communication technology. It boasts, therefore, an established foothold in the global electronics industry. According to the Taiwanese IC industry, in 2021Q1, the global semiconductor market sales value was US \$123.1 billion, and Taiwan's overall IC industry output value reached US \$29.565 billion (accounting for 24% of global output). In addition, many researchers noted that Taiwan's electronics industry plays a major part in the production of consumer electronics [5–8].

Process capability indexes are commonly applied in the evaluation and analysis of process quality in the above-mentioned industries. These evaluations are usually conducted under statistical control mechanisms. In other words, products are sampled only when the production process is stable [9,10]. The advantages of process capability indexes are large sample sizes and comprehensive data. These characteristics offer high levels of accuracy. However, these indexes are time-consuming to apply in practice, so they cannot meet corporate demands for rapid responses.

With the rapid development and evolution of the Internet-of-Things (IoT) and big-data analysis technologies, faster and more accurate production data analysis and process capability evaluation models will bring industries closer to the goal of smart manufacturing. At the same time, it is important to develop analysis methods that are appropriate for small sample sizes. Destructive testing is sometimes necessary to meet the rapid response needs of enterprises or to acquire production data, thereby increasing costs. Technology capabilities may also be insufficient. The resulting small sample sizes will result in excessive interval lengths, reducing the effectiveness of interval estimation.

The process yields index and the process yield have a one-to-one mathematical relationship [11,12]. Huang et al. [13] applied this index to evaluate the production capability of a backlight module with multiple process characteristics. Wisnowski, Simpson, and Montgomery [14] proposed an asymptotic distribution for an estimator of this index. This asymptotic distribution can assist statistical inferences of the process yield [15]. As the process yield index concurrently reflects process capability and yield [16,17], this study uses this index for the evaluation of process quality.

As noted above, this model offers rapid-response low-cost detection technology for the evaluation of small samples. In addition, many studies have pointed out that a confidence-interval-based fuzzy decision model can incorporate accumulated data and expert experiences to increase testing accuracy for small samples [18–20]. Therefore, this study puts forward a confidence-interval-based fuzzy decision model based on a process yield index. The index not only reflects process capability but also has a one-to-one mathematical relation with process yield so that it is convenient to apply in practice. The proposed model not only declines the risk of misjudgment incurred by sampling errors but also improves the accuracy of testing in the case of small sample sizes, thereby contributing to the ongoing development of smart manufacturing.

The remainder of this paper is arranged below. Section 2 derives the confidence region of process mean and standard. Section 3 uses the mathematical programming method to find the confidence interval of the process yield index. Section 4 elaborates on the proposed fuzzy decision model according to the confidence interval of this index. Section 5 provides an example to present the efficacy and applicability of the proposed method. Lastly, conclusions are made in the final section.

2. Confidence Region of Process Mean and Standard

According to Boyles [21], the following process yield index features a one-to-one mathematical relation with the process yield:

$$S_{PK} = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{USL - \mu}{\sigma}\right) + \frac{1}{2}\Phi\left(\frac{\mu - LSL}{\sigma}\right)\right\} \tag{1}$$

where $\Phi(\cdot)$ is the cumulative distribution function of standard normal distribution; USL and LSL are the upper and lower specification limits, respectively. The one-to-one mathematical relation between index S_{PK} and the process yield is expressed as follows:

$$Yield\% = 2\Phi(3S_{PK}) - 1 \tag{2}$$

For example, process yield equals 99.73% ($2\Phi(3) - 1$) when $S_{PK} = 1$.

If we let (X_1, X_2, \dots, X_n) be a random sample of X , then the maximum likelihood estimator (MLE) of process mean μ and process standard deviation S are, respectively, as follows:

$$\text{process sample mean } \bar{X} = \frac{1}{n}\sum_{i=1}^n X_i \tag{3}$$

$$\text{process sample standard deviation } S = \sqrt{\frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2} \tag{4}$$

Furthermore, let $Z = \sqrt{n}(\bar{X} - \mu) / \sigma$ and $K = nS^2 / \sigma^2$. On the assumption of normality, Z and K are distributed as $N(0, 1)$ and χ^2_{n-1} , respectively. Thus, $P\{-Z_{\alpha'/2} \leq Z \leq Z_{\alpha'/2}\} = \sqrt{1 - \alpha}$ and $P\{\chi^2_{\alpha'/2;n-1} \leq K \leq \chi^2_{1-(\alpha'/2);n-1}\} = \sqrt{1 - \alpha}$, where $Z_{\alpha'/2}$ is the upper $\alpha'/2$ quintile of $N(0, 1)$, $\chi^2_{\alpha'/2;n-1}$ is the upper $\alpha'/2$ quintile of χ^2_{n-1} , $\alpha' = 1 - \sqrt{1 - \alpha}$, and α represents the confidence level. \bar{X} and S^2 are distinct from each other, and so are Z and K . Furthermore, we can obtain the equation from their relations below:

$$\begin{aligned}
 1 - \alpha &= P\left\{-Z_{\alpha'/2} \leq Z \leq Z_{\alpha'/2}, \chi^2_{\alpha'/2;n-1} \leq K \leq \chi^2_{1-(\alpha'/2);n-1}\right\} \\
 &= P\left\{-Z_{\alpha'/2} \leq \frac{\sqrt{n}(\bar{X}-\mu)}{\sigma} \leq Z_{\alpha'/2}, \chi^2_{\alpha'/2;n-1} \leq \frac{nS^2}{\sigma^2} \leq \chi^2_{1-(\alpha'/2);n-1}\right\}
 \end{aligned}
 \tag{5}$$

Equivalently,

$$1 - \alpha = P\left\{\bar{X} - \frac{Z_{\alpha'/2}\sigma}{\sqrt{n}} \leq \mu \leq \bar{X} + \frac{Z_{\alpha'/2}\sigma}{\sqrt{n}}, \sqrt{\frac{nS^2}{\chi^2_{1-(\alpha'/2);n-1}}} \leq \sigma \leq \sqrt{\frac{nS^2}{\chi^2_{\alpha'/2;n-1}}}\right\}
 \tag{6}$$

If we let (x_1, x_2, \dots, x_n) represent the observed value of (X_1, X_2, \dots, X_n) , then \bar{x} and s are the observed values of \bar{X} and S , respectively, as displayed below:

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i
 \tag{7}$$

$$s = \sqrt{\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2}
 \tag{8}$$

Then, the confidence region of (μ, σ) can be defined in the following equation:

$$CR(\mu, \sigma) = P\{\bar{x} - e \leq \mu \leq \bar{x} + e, \sigma_L \leq \sigma \leq \sigma_U\}
 \tag{9}$$

where $\sigma_L = \sqrt{ns^2 / \chi^2_{0.5+\sqrt{1-\alpha}/2;n-1}}$, and $\sigma_U = \sqrt{ns^2 / \chi^2_{0.5-\sqrt{1-\alpha}/2;n-1}}$. Obviously, process yield index S_{PK} is a function of (μ, σ) . We use process yield index $S_{PK}(\mu, \sigma)$ as the objective function and use confidence region $CR(\mu, \sigma)$ as a feasible solution area. For any $\sigma \leq \sigma_U$, $S_{PK}(\mu, \sigma) \geq S_{PK}(\mu, \sigma_U)$. Thus, the mathematical programming model for the lower confidence limit is illustrated as follows:

$$\begin{cases}
 LS_{PK} = \min S_{PK}(\mu, \sigma_U) \\
 \text{subject to} \\
 \bar{x} - e_U \leq \mu \leq \bar{x} + e_U
 \end{cases}
 \tag{10}$$

where LS_{PK} is the lower confidence limit of index S_{PK} and

$$e_U = \frac{Z_{0.5-\sqrt{1-\alpha}/2} s}{\sqrt{\chi^2_{0.5-\sqrt{1-\alpha}/2;n-1}}}
 \tag{11}$$

Similarly, for any $\sigma \geq \sigma_L$, $S_{PK}(\mu, \sigma) \leq S_{PK}(\mu, \sigma_L)$, and the mathematical programming model for the upper confidence limit is as follows:

$$\begin{cases}
 US_{PK} = \max S_{PK}(\mu, \sigma_L) \\
 \text{subject to} \\
 \bar{x} - e_L \leq \mu \leq \bar{x} + e_L
 \end{cases}
 \tag{12}$$

where US_{PK} is the upper confidence limit of index S_{PK} and

$$e_L = \frac{Z_{0.5-\sqrt{1-\alpha}/2} s}{\sqrt{\chi^2_{0.5+\sqrt{1-\alpha}/2; n-1}}} \tag{13}$$

3. Results Confidence Interval of S_{PK}

Mathematical programming can be applied to figure out the confidence interval of index S_{PK} . First, we find the lower and upper confidence limits by means of the following three cases:

Case 1 $\bar{x} - e_U \leq T \leq \bar{x} + e_U$

For this case, we can conclude that $\mu = T$. Based on Equations (10) and (12), the lower and upper confidence limits are then displayed as follows:

$$LS_{PK} = \frac{1}{3} \Phi^{-1} \left\{ \Phi \left(\frac{d \sqrt{\chi^2_{0.5-\sqrt{1-\alpha}/2; n-1}}}{s Z_{0.5-\sqrt{1-\alpha}/2}} \right) \right\} \tag{14}$$

$$US_{PK} = \frac{1}{3} \Phi^{-1} \left\{ \Phi \left(\frac{d \sqrt{\chi^2_{0.5+\sqrt{1-\alpha}/2; n-1}}}{s Z_{0.5-\sqrt{1-\alpha}/2}} \right) \right\} \tag{15}$$

Case 2 $T < \bar{x} - e_U$

In this case, for any $\mu \leq \bar{x} + e_U$, $S_{PK}(\mu, \sigma_U) \geq S_{PK}(\bar{x} + e_U, \sigma_U)$. Based on Equation (10), the lower confidence limit is as follows:

$$LS_{PK} = \frac{1}{3} \Phi^{-1} \left\{ \frac{1}{2} \Phi \left(\frac{USL - (\bar{x} + e_U)}{\sigma_U} \right) + \frac{1}{2} \Phi \left(\frac{(\bar{x} + e_U) - LSL}{\sigma_U} \right) \right\} \tag{16}$$

Similarly, for any $\bar{x} - e_L \leq \mu \leq \bar{x} + e_L$ and $\mu \geq \bar{x} - e_L$, $S_{PK}(\mu, \sigma_L) \leq S_{PK}(\bar{x} - e_L, \sigma_L)$. Based on Equation (12), the upper confidence limit is then as follows:

$$US_{PK} = \frac{1}{3} \Phi^{-1} \left\{ \frac{1}{2} \Phi \left(\frac{USL - (\bar{x} - e_L)}{\sigma_L} \right) + \frac{1}{2} \Phi \left(\frac{(\bar{x} - e_L) - LSL}{\sigma_L} \right) \right\} \tag{17}$$

Case 3 $\bar{x} + e_U < T$

For any $\mu \geq \bar{x} - e_U$, $S_{PK}(\mu, \sigma_U) \geq S_{PK}(\bar{x} - e_U, \sigma_U)$. Based on Equation (10), the lower confidence limit is then as follows:

$$LS_{PK} = \frac{1}{3} \Phi^{-1} \left\{ \frac{1}{2} \Phi \left(\frac{USL - (\bar{x} - e_U)}{\sigma_U} \right) + \frac{1}{2} \Phi \left(\frac{(\bar{x} - e_U) - LSL}{\sigma_U} \right) \right\} \tag{18}$$

Similarly, for any $\mu \leq \bar{x} + e_L$, $S_{PK}(\mu, \sigma_U) \geq S_{PK}(\bar{x} - e_U, \sigma_U)$. Based on Equation (12), the upper confidence limit is as follows:

$$US_{PK} = \frac{1}{3} \Phi^{-1} \left\{ \frac{1}{2} \Phi \left(\frac{USL - (\bar{x} + e_L)}{\sigma_L} \right) + \frac{1}{2} \Phi \left(\frac{(\bar{x} + e_L) - LSL}{\sigma_L} \right) \right\} \tag{19}$$

Therefore, interval $[LS_{PK}, US_{PK}]$ is the $100(1 - \alpha)\%$ confidence interval of index S_{PK} with $\alpha' = 1 - \sqrt{1 - \alpha}$, where

$$LS_{PK} = \begin{cases} \frac{1}{3}\Phi^{-1}\left\{\Phi\left(\frac{d}{s}\frac{\sqrt{\chi_{0.5-\sqrt{1-\alpha}/2;n-1}^2}}{Z_{0.5-\sqrt{1-\alpha}/2}}\right)\right\}, & \bar{x} - e_U \leq T \leq \bar{x} + e_U \\ \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{USL-(\bar{x}+e_U)}{\sigma_U}\right) + \frac{1}{2}\Phi\left(\frac{(\bar{x}+e_U)-LSL}{\sigma_U}\right)\right\}, & T < \bar{x} - e_U \\ \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{USL-(\bar{x}-e_L)}{\sigma_L}\right) + \frac{1}{2}\Phi\left(\frac{(\bar{x}-e_L)-LSL}{\sigma_L}\right)\right\}, & \bar{x} + e_U < T \end{cases} \quad (20)$$

and

$$US_{PK} = \begin{cases} \frac{1}{3}\Phi^{-1}\left\{\Phi\left(\frac{d}{s}\frac{\sqrt{\chi_{0.5+\sqrt{1-\alpha}/2;n-1}^2}}{Z_{0.5-\sqrt{1-\alpha}/2}}\right)\right\}, & \bar{x} - e_U \leq T \leq \bar{x} + e_U \\ \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{USL-(\bar{x}-e_L)}{\sigma_L}\right) + \frac{1}{2}\Phi\left(\frac{(\bar{x}-e_L)-LSL}{\sigma_L}\right)\right\}, & T < \bar{x} - e_U \\ \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{USL-(\bar{x}+e_U)}{\sigma_U}\right) + \frac{1}{2}\Phi\left(\frac{(\bar{x}+e_U)-LSL}{\sigma_U}\right)\right\}, & \bar{x} + e_U < T \end{cases} \quad (21)$$

4. Fuzzy Decision Model

Fuzzy decision models based on confidence intervals have been effectively applied to evaluations of process capability [22–24]. If the customer requests a process yield index of $c(S_{PK}=c)$, then $H_0:S_{PK}=c$ and $H_1:S_{PK} \neq c$ can be applied to hypothesis testing. As described by Chen [25], the α -cuts of triangular fuzzy number S_{PK} is expressed as follows:

$$S_{PK}[\alpha] = \begin{cases} [S_{PK1}(\alpha), S_{PK2}(\alpha)], & \text{for } 0.01 \leq \alpha \leq 1 \\ [S_{PK1}(0.01), S_{PK2}(0.01)], & \text{for } 0 \leq \alpha \leq 0.01 \end{cases} \quad (22)$$

$S_{PK1}(\alpha)$ and $S_{PK2}(\alpha)$ in Equation (22) differ according to the following cases:

Case 1 $\bar{x} - e_U \leq T \leq \bar{x} + e_U$

If $\bar{x} - e_U \leq T \leq \bar{x} + e_U$, then

$$S_{PK1}(\alpha) = \frac{1}{3}\Phi^{-1}\left\{\Phi\left(\frac{d}{s}\frac{\sqrt{\chi_{0.5-\sqrt{1-\alpha}/2;n-1}^2}}{Z_{0.5-\sqrt{1-\alpha}/2}}\right)\right\} \quad (23)$$

$$S_{PK2}(\alpha) = \frac{1}{3}\Phi^{-1}\left\{\Phi\left(\frac{d}{s}\frac{\sqrt{\chi_{0.5+\sqrt{1-\alpha}/2;n-1}^2}}{Z_{0.5-\sqrt{1-\alpha}/2}}\right)\right\} \quad (24)$$

Case 2 $T < \bar{x} - e_U$

If $T < \bar{x} - e_U$, then

$$S_{PK1}(\alpha) = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{USL-\bar{x}}{s}\sqrt{\frac{\chi_{0.5-\sqrt{1-\alpha}/2;n-1}^2}{n} - \frac{Z_{0.5-\sqrt{1-\alpha}/2}}{\sqrt{n}}}\right) + \frac{1}{2}\Phi\left(\frac{\bar{x}-LSL}{s}\sqrt{\frac{\chi_{0.5-\sqrt{1-\alpha}/2;n-1}^2}{n} + \frac{Z_{0.5-\sqrt{1-\alpha}/2}}{\sqrt{n}}}\right)\right\} \quad (25)$$

$$S_{PK2}(\alpha) = \frac{1}{3}\Phi^{-1} \left\{ \frac{1}{2}\Phi \left(\frac{USL-\bar{x}}{s} \sqrt{\frac{\chi_{0.5+\sqrt{1-\alpha}/2;n-1}^2}{n}} - \frac{Z_{0.5-\sqrt{1-\alpha}/2}}{\sqrt{n}} \right) + \frac{1}{2}\Phi \left(\frac{\bar{x}-LSL}{s} \sqrt{\frac{\chi_{0.5+\sqrt{1-\alpha}/2;n-1}^2}{n}} + \frac{Z_{0.5-\sqrt{1-\alpha}/2}}{\sqrt{n}} \right) \right\} \tag{26}$$

Case 3 $\bar{x} + e_U < T$

If $\bar{x} + e_U < T$, then

$$S_{PK1}(\alpha) = \frac{1}{3}\Phi^{-1} \left\{ \frac{1}{2}\Phi \left(\frac{USL-\bar{x}}{s} \sqrt{\frac{\chi_{0.5-\sqrt{1-\alpha}/2;n-1}^2}{n}} + \frac{Z_{0.5-\sqrt{1-\alpha}/2}}{\sqrt{n}} \right) + \frac{1}{2}\Phi \left(\frac{\bar{x}-LSL}{s} \sqrt{\frac{\chi_{0.5-\sqrt{1-\alpha}/2;n-1}^2}{n}} - \frac{Z_{0.5-\sqrt{1-\alpha}/2}}{\sqrt{n}} \right) \right\} \tag{27}$$

$$S_{PK2}(\alpha) = \frac{1}{3}\Phi^{-1} \left\{ \frac{1}{2}\Phi \left(\frac{USL-\bar{x}}{s} \sqrt{\frac{\chi_{0.5+\sqrt{1-\alpha}/2;n-1}^2}{n}} - \frac{Z_{0.5-\sqrt{1-\alpha}/2}}{\sqrt{n}} \right) + \frac{1}{2}\Phi \left(\frac{\bar{x}-LSL}{s} \sqrt{\frac{\chi_{0.5+\sqrt{1-\alpha}/2;n-1}^2}{n}} + \frac{Z_{0.5-\sqrt{1-\alpha}/2}}{\sqrt{n}} \right) \right\} \tag{28}$$

Therefore, triangular fuzzy number $S_{PK} = \Delta(S_L, S_M, S_R)$, where $S_L = S_{PK1}(0.01)$, $S_M = S_{PK1}(1) = S_{PK2}(1)$, and $S_R = S_{PK2}(0.01)$. Therefore, the membership function of fuzzy number S_{PK} is

$$\eta(x) = \begin{cases} 0 & \text{if } x < S_L \\ \alpha_1 & \text{if } S_L \leq x < S_M \\ 1 & \text{if } x = S_M \\ \alpha_2 & \text{if } S_M < x \leq S_R \\ 0 & \text{if } x > S_R \end{cases} \tag{29}$$

where α_h is determined by $S_{PKh}(\alpha_h) = x$, $h = 1, 2$. The proposed model is built on the statistical testing rules as follows:

- (1) If $c < LS_{PK}$, t , then decline H_0 and infer that $S_{PK} < c$.
- (2) If $US_{PK} < c$, then decline H_0 and infer that $S_{PK} > c$.
- (3) If $LS_{PK} \leq c \leq US_{PK}$, then do not decline H_0 and infer that $S_{PK} = c$.

The proposed model further considers two cases: $c \leq S_M$ and $c > S_M$.

Case 1 $c \leq S_M$

Based on Chen [25] and Chen et al. [26], let set A_T represent the area in the graph of $\eta(x)$ and set A_R^- represent the area in the graph of $\eta(x)$ to the left of vertical line $x = c$. Figure 1 illustrates membership functions of $\eta(x)$ with vertical line $x = c$ and $c \leq S_M$.

Thus,

$$A_T = \{(x, \alpha) | S_{PK1}(\alpha) \leq x \leq S_{PK2}(\alpha), 0 \leq \alpha \leq 1\} \tag{30}$$

and

$$A_R^- = \{(x, \alpha) | S_{PK1}(\alpha) \leq x \leq c, 0 \leq \alpha \leq \alpha_1\}$$

where $S_{PK1}(\alpha_1) = c$. We let

$$d_T = S_R - S_L \text{ and } d_R^- = c - S_L \tag{31}$$

Then,

$$\frac{d_R^-}{d_T} = \frac{c - S_L}{S_R - S_L} \tag{32}$$

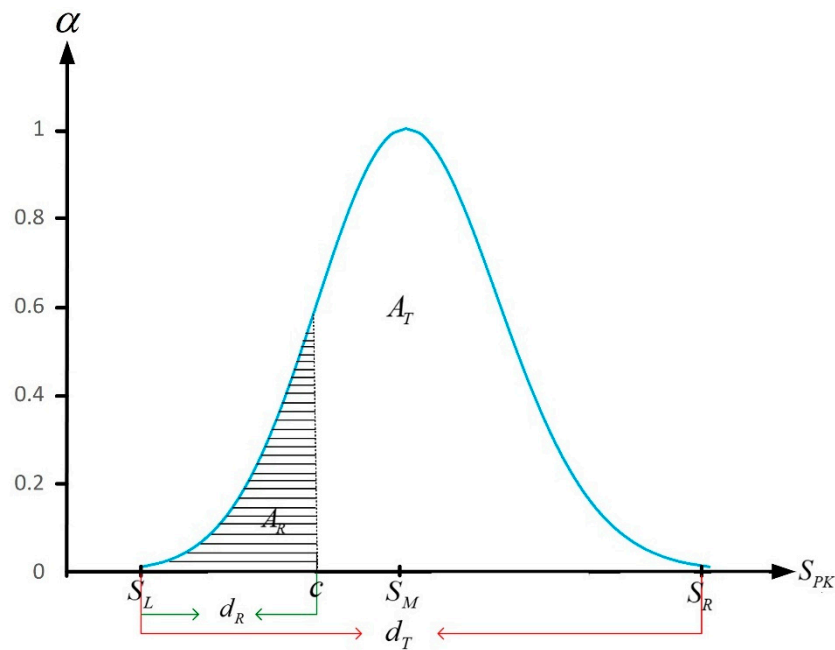


Figure 1. Membership functions of $\eta(x)$ with $x = c$ and $c \leq S_M$.

Case 2 $c > S_M$

Similar to case 1, we let set A_R^+ represent the area in the graph of $\eta(x)$ to the right of vertical line $x = c$. Figure 2 exhibits membership functions of $\eta(x)$ with vertical line $x = c$ and $c > S_M$.

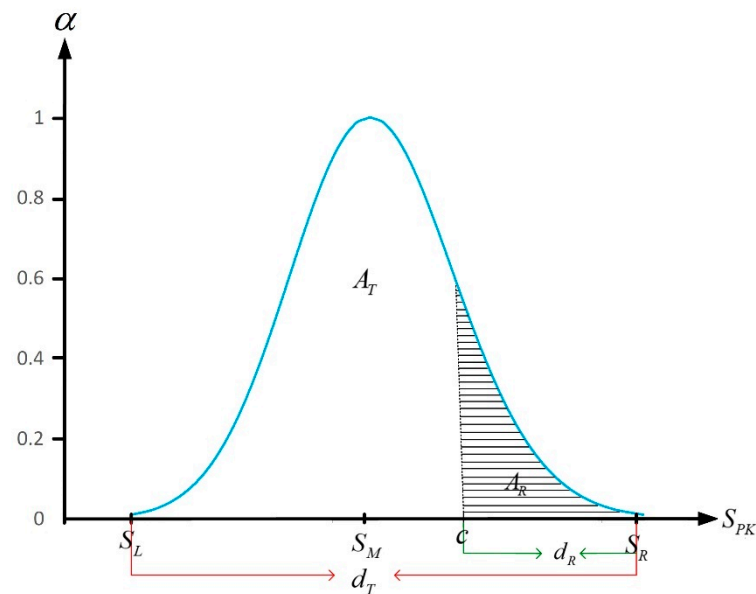


Figure 2. Membership functions of $\eta(x)$ with $x = c$ and $c > S_M$.

Thus,

$$A_R^+ = \{ (x, \alpha) | c \leq x \leq S_{PK2}(\alpha), 0 \leq \alpha \leq a_2 \} \tag{33}$$

where $S_{PK1}(a_2) = c$. We let $d_R^+ = S_R - c$. Then,

$$\frac{d_R^+}{d_T} = \frac{S_R - c}{S_R - S_L} \tag{34}$$

According to Yu et al. [23], let $0 < \phi \leq 0.5$. The value of ϕ can be determined based on accumulated production data or expert experience. Decision rules are as follows:

- (1) If $c \leq S_M$ and $d_R^- / d_T < \phi$, then decline H_0 and infer that $S_{PK} > c$. That is, the process requires improvement.
- (2) If $c \leq S_M$ and $\phi \leq d_R^- / d_T \leq 0.5$, then do not decline H_0 and infer that $S_{PK} = c$. That is, maintain the process at current quality levels.
- (3) If $c > S_M$ and $\phi \leq d_R^+ / d_T \leq 0.5$, then do not decline H_0 and infer that $S_{PK} = c$. That is, maintain the process at current quality levels.
- (4) If $c > S_M$ and $d_R^+ / d_T < \phi$, then decline H_0 and infer that $S_{PK} < c$. That is, consider reducing quality levels to reduce costs.

5. Practical Application

The output of Taiwanese machine tools comes out top in the globe, while the export volume ranks fifth. Central Taiwan is home to an industry cluster for machine tool and machinery industries. In addition to the production of various professional machine tools, this cluster brings together industries focused on the processing and maintenance of various important components [26–28]. To demonstrate the proposed approach, we consider an axis, in which a double-linked chain is placed in a U-shaped groove. The distance tolerance of the groove pitch for this component is 4 ± 0.05 (as indicated by D in Figure 3). When the gap is too large, the sprocket is likely to break. When the gap is too small, the chain cannot be replaced. If the customer requires that process yield index $S_{PK} 1.1$, then null hypothesis $H_0: S_{PK} = 1.1$, alternative hypothesis $H_1: S_{PK} \neq 1.1$.

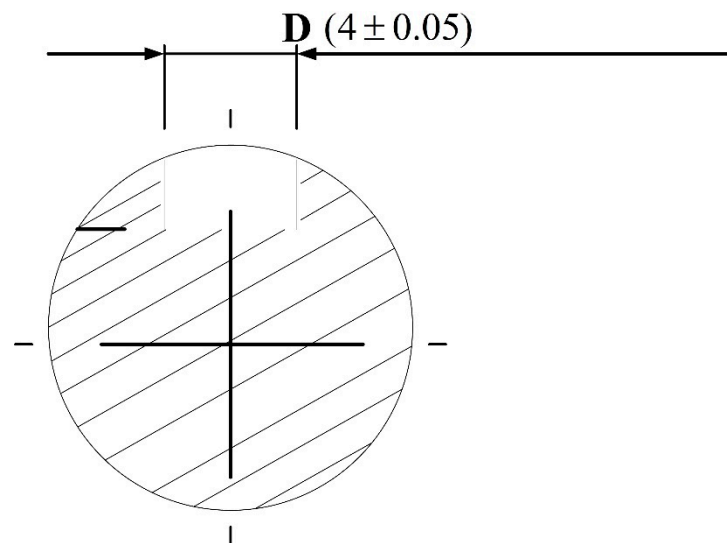


Figure 3. Distance tolerance of groove pitch gap.

The upper specification limit is $USL = 4.05$, and the lower specification limit is $LSL = 3.95$. The observed value of a randomly-selected sample with size $n = 36$ is $(x_1, x_2, \dots, x_{36})$. Therefore,

$$\bar{x} = \frac{1}{36} \sum_{i=1}^{36} x_i = 4.012$$

$$s = \sqrt{\frac{1}{36} \sum_{i=1}^{36} (x_i - 4.012)^2} = 0.016$$

The observed value of the estimator for process yield index S_{PK} can be shown as follows:

$$\hat{S}_{PK} = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{4.05 - 4.012}{0.016}\right) + \frac{1}{2}\Phi\left(\frac{4.012 - 3.95}{0.016}\right)\right\} = 0.873$$

Based on Equation (18), the value of e_U with $\alpha = 0.01$ is

$$e_U = \frac{Z_{0.0025} \times s}{\sqrt{\chi_{0.0025,35}^2}} = \frac{2.807 \times 0.016}{\sqrt{16.032}} = 0.011$$

Therefore,

$$[\bar{x} - e_U, \bar{x} + e_U] = [4.012 - 0.011, 4.012 + 0.011] = [4.001, 4.023]$$

Obviously, if target value $T = 4 < \bar{x} - e_U = 4.005$, then $S_L = LS_{PK} = S_{PK1}(0.01)$, $S_M = S_{PK1}(1)$, and $S_R = US_{PK} = S_{PK2}(0.01)$ as follows:

$$\begin{aligned} S_L &= \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{4.05 - 4.012}{0.016}\sqrt{\frac{\chi_{0.0025,35}^2}{36}} - \frac{Z_{0.0025}}{\sqrt{36}}\right) + \right. \\ &\quad \left. \frac{1}{2}\Phi\left(\frac{4.012 - 3.95}{0.016}\sqrt{\frac{\chi_{0.0025,35}^2}{36}} + \frac{Z_{0.0025}}{\sqrt{36}}\right)\right\} \\ &= \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(2.375\sqrt{\frac{16.032}{36}} - \frac{2.807}{6}\right) + \frac{1}{2}\Phi\left(3.875\sqrt{\frac{16.032}{36}} + \frac{2.807}{6}\right)\right\} \\ &= \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi(1.117) + \frac{1}{2}\Phi(3.054)\right\} \\ &= 0.500 \end{aligned}$$

$$\begin{aligned} S_M &= \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{4.05 - 4.012}{0.016}\sqrt{\frac{\chi_{0.5,35}^2}{36}}\right) + \frac{1}{2}\Phi\left(\frac{4.012 - 3.95}{0.016}\sqrt{\frac{\chi_{0.5,35}^2}{36}}\right)\right\} \\ &= \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(2.375\sqrt{\frac{34.336}{36}}\right) + \frac{1}{2}\Phi\left(3.875\sqrt{\frac{34.336}{36}}\right)\right\} \\ &= \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi(2.319) + \frac{1}{2}\Phi(3.784)\right\} \\ &= 0.856 \end{aligned}$$

$$\begin{aligned} S_R &= \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{4.05 - 4.012}{0.016}\sqrt{\frac{\chi_{0.9975,35}^2}{36}} + \frac{Z_{0.0025}}{\sqrt{36}}\right) + \right. \\ &\quad \left. \frac{1}{2}\Phi\left(\frac{4.012 - 3.95}{0.016}\sqrt{\frac{\chi_{0.9975,35}^2}{36}} - \frac{Z_{0.0025}}{\sqrt{36}}\right)\right\} \\ &= \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(2.375\sqrt{\frac{63.076}{36}} + \frac{2.807}{6}\right) + \frac{1}{2}\Phi\left(3.875\sqrt{\frac{63.076}{36}} - \frac{2.807}{6}\right)\right\} \\ &= \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi(3.612) + \frac{1}{2}\Phi(4.661)\right\} \\ &= 1.262 \end{aligned}$$

According to $(S_L, S_M, S_R) = (0.500, 0.856, 1.262)$ in the Figure 4. Since $1.1 > S_M$

$$\frac{d_R^+}{d_T} = \frac{S_R - c}{S_R - S_L} = \frac{1.262 - 1.1}{1.262 - 0.500} = 0.134$$

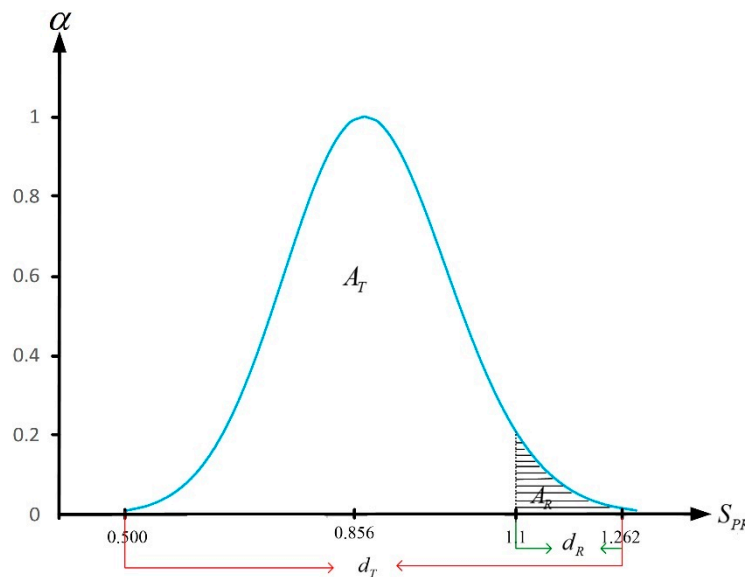


Figure 4. Membership functions of $\eta(x)$ with $x = 1.1$.

Engineers analyze relevant production data and experience to set $\phi = 0.15$. The proposed model then suggests that since $1.1 > S_M$ and $d_R^+/d_T = 0.134 < \phi$, the user should reject H_0 and conclude that $S_{PK} < 1.33$. This means the firm can consider reducing quality levels to reduce costs. In fact, when $\hat{S}_{PK} = 0.856$, the value of US_{PK} with $\alpha = 0.01$ is 1.262. When the result of the statistical inference is $S_{PK} = 1.1$, it is obvious that the proposed model offers a more reasonable solution than the conventional statistical inference.

6. Conclusions

This study uses a process yield index S_{PK} to develop a fuzzy decision model based on confidence intervals to evaluate process quality. Process yield index S_{PK} is superior to other process capability indexes in its ability to reflect process capability and yield simultaneously.

We derived the $100(1 - \alpha)\%$ confidence region of (μ, σ) based on the independence of sample average \bar{X} and sample variance S^2 under normal manufacturing conditions. Taking the confidence region as the feasible region and process yield index S_{PK} as the objective function, we derived the $100(1 - \alpha)\%$ confidence interval of index S_{PK} to develop a fuzzy decision model on the basis of the confidence interval. By further incorporating expert input and accumulated data, this approach lowers the risk of misjudgment incurred by sampling errors associated with small sample sizes. Moreover, we demonstrated the efficacy of the proposed approach by taking an example of a machined product.

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