*mathematics*

MDPI

*Article*

# Hyers–Ulam Stability of Order $k$ for Euler Equation and Euler–Poisson Equation in the Calculus of Variations

**Daniela Marian [1,*], Sorina Anamaria Ciplea [2] and Nicolaie Lungu [1]**

[1] Department of Mathematics, Technical University of Cluj-Napoca, 28 Memorandumului Street, 400114 Cluj-Napoca, Romania; nlungu@math.utcluj.ro

[2] Department of Management and Technology, Technical University of Cluj-Napoca, 28 Memorandumului Street, 400114 Cluj-Napoca, Romania; sorina.ciplea@ccm.utcluj.ro

* Correspondence: daniela.marian@math.utcluj.ro

**Abstract:** In this paper, we define and study Hyers–Ulam stability of order 1 for Euler's equation and Hyers–Ulam stability of order $m-1$ for the Euler–Poisson equation in the calculus of variations in two special cases, when these equations have the form $y''(x) = f(x)$ and $y^{(m)}(x) = f(x)$, respectively. We prove some estimations for $|J[y(x)] - J[y_0(x)]|$, where $y$ is an approximate solution and $y_0$ is an exact solution of the corresponding Euler and Euler-Poisson equations, respectively. We also give two examples.

**Keywords:** Euler equation; Euler–Poisson equation; calculus of variations; Hyers–Ulam stability

**MSC:** 49K15; 34K20

check for updates

**Citation:** Marian, D.; Ciplea, S.A.; Lungu, N. Hyers–Ulam Stability of Order *k* for Euler Equation and Euler–Poisson Equation in the Calculus of Variations. *Mathematics* **2022**, *10*, 2556. https://doi.org/10.3390/math10152556

Academic Editor: Ana-Maria Acu

Received: 27 June 2022
Accepted: 21 July 2022
Published: 22 July 2022



## 1. Introduction

Hyers–Ulam stability has been the subject of many papers. Ulam stability was proposed by Ulam in [1] in 1940. The first result in this direction was given in 1941 by Hyers [2]. The first authors which started the study of Hyers–Ulam stability of differential equations was Obloza [3] and Alsina and Ger [4]. After that, many types of differential equations were studied. First-order linear differential equations and linear differential equations of higher order were investigated, for example, in [5–13]. Integral equations in [14–24] and partial differential equations in [25–30] have also been studied. The books [31,32] can be consulted for more details. The Hyers–Ulam stability of fractional differential equations and of fractional integral equations has recently begun to be studied (see [33–42]).

In what follows, we define and study Hyers–Ulam stability of order 1 for Euler's equation and Hyers–Ulam stability of order $m-1$ for the Euler–Poisson equation in the calculus of variations. Section 2 is dedicated to the study of Euler's equation, and Section 3 is dedicated to the Euler–Poisson equation. We also establish an estimation for $|J[y(x)] - J[y_0(x)]|$, where $y$ is an approximate solution and $y_0$ is an exact solution for the considered equations.

This paper is a continuation of the paper [43]. In [43], Hyers–Ulam stability of the Euler equation in two special cases was studied when $F = F(x, y')$ and when $F = F(y, y')$. An estimation for $|J[y(x)] - J[y_0(x)]|$ is also given in [43] for the case $F = F(x, y')$. This was the first time, in [43], that the problem of Ulam stability of extremals for functionals represented in integral form was studied. In [43], a direct method and the Laplace transform method were used. Here, we will use Taylor's formula.

## 2. Hyers–Ulam Stability of Euler's Equation

We consider a class of functions $A \subseteq C^2(I, \mathbb{R}), I \subseteq \mathbb{R}$ an open interval. Let $[a, b] \subset I,\ a < b,\ a \geq 0$. Let $y \in A,\ y = y(x)$ be an element in $A$. Let the function $F : M \longrightarrow \mathbb{R}$, $M = I \times \mathbb{R}^2,\ F \in C^2(M)$. We consider the functional

$$J[y(x)] = \int_a^b F(x, y, y')dx, \; J : A \to \mathbb{R}, \; y \in A, \; J[y(x)] \in \mathbb{R}, \tag{1}$$

and the conditions

$$y(a) = y_a, y'(a) = y'_a, \tag{2}$$

where $y_a, \; y'_a \in \mathbb{R}$ are given.

We consider the following problem in the calculus of variations (see [44]): find the extremum of this functional. The necessary condition of extremum (see [44]) is given by Euler's equation

$$F'_y(x, y, y') - \frac{d}{dx}\left[F'_{y'}(x, y, y')\right] = 0, \; y \in C^2[a, b]. \tag{3}$$

Equation (3) can be represented (by derivation) in the form:

$$\frac{\partial^2 F}{\partial y'^2} \cdot y'' + \frac{\partial^2 F}{\partial y' \partial y} \cdot y' + \frac{\partial^2 F}{\partial y' \partial x} - \frac{\partial F}{\partial y} = 0, \; y \in C^2[a, b]. \tag{4}$$

The solutions of Equation (3) or (4) are called extremals.

We will study Hyers–Ulam stability of Equation (3) (or (4)).

Let $\varepsilon > 0$ and $a, b \in (0, \infty)$.

We consider the following inequalities:

$$\left| F'_y(x, y, y') - \frac{d}{dx}\left[F'_{y'}(x, y, y')\right] \right| \leq \varepsilon, \; y \in C^2[a, b], \tag{5}$$

or

$$\left| \frac{\partial^2 F}{\partial y'^2} \cdot y'' + \frac{\partial^2 F}{\partial y' \partial y} \cdot y' + \frac{\partial^2 F}{\partial y' \partial x} - \frac{\partial F}{\partial y} \right| \leq \varepsilon, \; y \in C^2[a, b]. \tag{6}$$

**Definition 1.** *Equation (3) (or (4)) is called Hyers–Ulam stable if there is a real number $c > 0$ such that for any solution $y(x)$ of the inequality (5) (or (6)), there is a solution $y_0(x)$ of the Equation (3) (or (4)) such that*

$$|y(x) - y_0(x)| \leq c \cdot \varepsilon, \; \forall x \in [a, b].$$

In the following definition, we give a new notion of stability, named stability of order 1.

**Definition 2.** *Equation (3) (or (4)) is called Hyers–Ulam stable of order 1 if there are real numbers $c_1 > 0, c_2 > 0$ so that for any solution $y(x)$ of the inequality (5) (or (6)), there is a solution $y_0(x)$ of Equation (3) (or (4)) such that*

$$|y(x) - y_0(x)| \leq c_1 \cdot \varepsilon, \; \forall x \in [a, b],$$

*and*

$$\left|y'(x) - y'_0(x)\right| \leq c_2 \cdot \varepsilon, \; \forall x \in [a, b].$$

*$y$ is called an approximate solution and $y_0$ is called an exact solution for Equation (3) (or (4)).*

In the following, we will study the case where Euler's equation is

$$y''(x) = f(x), f \in C[a, b], \; x \in [a, b]. \tag{7}$$

**Remark 1.** *If $y = y(x)$ is a solution of (7), $x \in [a, b]$, then (see [45]),*

$$y(x) = y(a) + \frac{x - a}{1!} y'(a) + \int_a^x \frac{x - s}{1!} f(s)ds. \tag{8}$$

Let $\varepsilon > 0$. We consider the inequality

$$\left|y''(x) - f(x)\right| \le \varepsilon,\ y \in C^2[a,b]. \tag{9}$$

**Remark 2.** *A function $y = y(x)$ is a solution of (9) if and only if there exists a function $g \in C[a,b]$ such that*

*(1)* $|g(x)| \le \varepsilon,\ \forall x \in [a,b]$,
*(2)* $y''(x) - f(x) = g(x),\ \forall x \in [a,b]$.

**Remark 3.** *If $y = y(x)$ is a solution of (9), using Remark 2 and Remark 1, we have*

$$y(x) = y(a) + \frac{x-a}{1!}y'(a) + \int_a^x \frac{x-s}{1!}(f(s)+g(s))ds. \tag{10}$$

**Theorem 1.** *(i) For each solution $y = y(x)$ of (9), there exists a unique solution $y_0 = y_0(x)$ of (7) such that*

$$\begin{cases} y_0(a) = y(a) \\ y_0'(a) = y'(a). \end{cases} \tag{11}$$

*(ii) The Equation (7) is Hyers–Ulam stable of order 1. If $y$ is a solution of (9) and $y_0$ is a solution of (7) satisfying conditions (11), then*

$$|y(x) - y_0(x)| \le \varepsilon\frac{(b-a)^2}{2}, \forall x \in [a,b] \tag{12}$$

*and*

$$\left|y'(x) - y_0'(x)\right| \le \varepsilon(b-a). \tag{13}$$

*(iii) If there exists $l_1, l_2 : [a,b] \longrightarrow [0,\infty)$ continuous, such that*

$$\begin{aligned}&\left|F(x,y_1(x),y_1'(x)) - F(x,y_2(x),y_2'(x))\right| \\ &\le l_1(x)\cdot|y_1(x)-y_2(x)| + l_2(x)\left|y_1'(x)-y_2'(x)\right|, \forall x \in [a,b],\ \ y_1,y_2 \in A,\end{aligned}$$

*then*

$$|J[y(x)] - J[y_0(x)]| \le \varepsilon\cdot\frac{(b-a)^2}{2}\int_a^b l_1(x)dx + \varepsilon(b-a)\int_a^b l_2(x)dx, \tag{14}$$

*where $y$ is a solution of (9) and $y_0$ is a solution of (7), both satisfying the conditions of (2).*

**Proof.** (i) This results from Cauchy–Picard's theorem of existence and uniqueness (see [46]).

(ii) Let $y = y(x)$ be a solution of (9). Let $y_0 = y_0(x)$ be the unique solution of (7) which verifies the corresponding Cauchy conditions of (11). We have

$$\begin{aligned}&|y(x) - y_0(x)| = \\ &\left|y(a) + \frac{x-a}{1!}y'(a) + \int_a^x \frac{x-s}{1!}(f(s)+g(s))ds - y(a) - \frac{x-a}{1!}y'(a) - \int_a^x \frac{x-s}{1!}f(s)ds\right|,\end{aligned}$$

hence

$$|y(x) - y_0(x)| = \left|\int_a^x \frac{x-s}{1!}g(s)ds\right| \le \int_a^x \left|\frac{x-s}{1!}g(s)\right|ds,$$

so

$$|y(x) - y_0(x)| \le \varepsilon\frac{(b-a)^2}{2}, \forall x \in [a,b].$$

We also have

$$\left|y'(x) - y_0'(x)\right| \le \int_a^x g(s)ds \le \varepsilon(b-a).$$

Thus, Equation (7) is Hyers–Ulam stable of order 1.

(iii) If $y$ is a solution of (9) and $y_0$ is a solution of (7), both satisfying conditions (2), then

$$\begin{aligned} |J[y(x)] - J[y_0(x)]| &\leq \int_a^b [l_1(x) \cdot |y(x) - y_0(x)| + l_2(x)|y'(x) - y_0'(x)|]dx \\ &\overset{(12),(13)}{\leq} \varepsilon \cdot \frac{(b-a)^2}{2} \int_a^b l_1(x)dx + \varepsilon(b-a) \int_a^b l_2(x)dx. \end{aligned} \quad (15)$$

□

**Example 1.** *We consider* $J : A \longrightarrow R,\ A \subseteq C^2(I),\ [1,2] \subset I,$

$$J[y(x)] = \int_1^2 \left(y'^2 - 2xy\right)dx, \quad (16)$$

*and the conditions*

$$y(1) = 0,\ y'(1) = -\frac{1}{3}. \quad (17)$$

*The Euler equation becomes*

$$y'' + x = 0. \quad (18)$$

*Let* $\varepsilon > 0$. *We consider the inequality*

$$|y'' + x| \leq \varepsilon. \quad (19)$$

*We remark that*

$$y_0(x) = -\frac{x^3}{6} + \frac{x}{6} \quad (20)$$

*is a solution of Equation (18), satisfying (17).*

*If $y$ is a solution of (19) and $y_0$ is a solution of (18), both satisfying (17), then applying Theorem 1, we get*

$$|y(x) - y_0(x)| \leq \varepsilon\frac{(x-1)^2}{2} \leq \frac{\varepsilon}{2},\ \forall x \in [1,2], \quad (21)$$

*and*

$$|y'(x) - y_0'(x)| \leq \varepsilon(x-1) \leq \varepsilon,\ \forall x \in [1,2], \quad (22)$$

*hence, Equation (18) is Hyers–Ulam stable of order* 1.

*Moreover,*

$$\begin{aligned} |J[y(x)] - J[y_0(x)]| &\leq \int_1^2 \left|y'^2(x) - 2xy(x) - y_0'^2(x) + 2xy_0(x)\right| dx \\ &\leq \int_1^2 |y'(x) - y_0'(x)||y'(x) + y_0'(x)| + 2x|y(x) - y_0(x)|\, dx \\ &\leq \int_1^2 \left[\varepsilon(x-1)\left(\varepsilon(x-1) + x^2 + \frac{1}{3}\right) + \varepsilon x(x-1)^2\right]dx = \frac{\varepsilon(2\varepsilon+13)}{4}. \end{aligned}$$

## 3. Hyers–Ulam Stability of the Euler–Poisson Equation

Now, we consider functionals dependent on higher derivatives.

Let $n \in \mathbb{N}, n \geq 2$, $A \subseteq C^{2n}(I,\mathbb{R})$, $I \subseteq \mathbb{R}$ be an open interval. Let $[a,b] \subset I,\ a < b$. Let $y \in A,\ y = y(x)$ be an element in $A$. Let the function $F : M \longrightarrow \mathbb{R}$, $M = I \times \mathbb{R}^{n+1}$ . We suppose that $F$ is $n+2$ times differentiable with respect to all arguments.

Let

$$J[y(x)] = \int_a^b F\left(x, y(x), y'(x), \ldots, y^{(n)}(x)\right)dx, \quad (23)$$

and the conditions

$$y(a) = y_a, y'(a) = y_a', \ldots, y^{(2n-1)}(a) = y_a^{(2n-1)}, \quad (24)$$

where $y_a, y'_a, \dots, y_a^{(2n-1)} \in \mathbb{R}$ are given.

The extremals of the functional (23), given conditions (24), are the integral curves of the Euler–Poisson equation (see [44]):

$$F'_y - \frac{d}{dx}[F'_{y'}] + \frac{d^2}{dx^2}[F'_{y''}] - \dots + (-1)^n \frac{d^n}{dx^n}[F'_{y^{(n)}}] = 0. \tag{25}$$

Let $\varepsilon > 0$. We consider the inequality

$$\left| F'_y - \frac{d}{dx}[F'_{y'}] + \frac{d^2}{dx^2}[F'_{y''}] - \dots + (-1)^n \frac{d^n}{dx^n}[F'_{y^{(n)}}] \right| \le \varepsilon. \tag{26}$$

We give a new notion of stability, named stability of order $k, k \ge 1, k \in \mathbb{N}$.

**Definition 3.** *Equation (25) is called Hyers–Ulam stable of order k if there are real numbers $C_1 > 0, C_2 > 0, \cdots, C_{k+1} > 0$ such that for any solution $y(x)$ of the inequality (26) there is a solution $y_0(x)$ of the Equation (25) such that*

$$|y(x) - y_0(x)| \le C_1 \cdot \varepsilon,\ \forall x \in [a,b],$$

*and*

$$\left|y'(x) - y'_0(x)\right| \le C_2 \cdot \varepsilon,\ \forall x \in [a,b],$$

...

$$\left|y^{(k)}(x) - y_0^{(k)}(x)\right| \le C_{k+1} \cdot \varepsilon,\ \forall x \in [a,b].$$

*$y$ is called an approximate solution and $y_0$ is called an exact solution for Equation (25).*

In the following, we will study the case where the Euler–Poisson equation is

$$y^{(m)}(x) = f(x),\ m = 2n,\ f \in C[a,b],\ x \in [a,b]. \tag{27}$$

**Remark 4.** *If $y = y(x)$ is a solution of (27), then (see [45])*

$$y(x) = y(a) + \frac{x-a}{1!}y'(a) + \cdots + \frac{(x-a)^{m-1}}{(m-1)!}y^{(m-1)}(a) + \int_a^x \frac{(x-s)^{m-1}}{(m-1)!}f(s)ds. \tag{28}$$

*Let $\varepsilon > 0$. We also consider the inequality*

$$\left|y^{(m)}(x) - f(x)\right| \le \varepsilon, \forall x \in [a,b],\ y \in C^m[a,b]. \tag{29}$$

**Remark 5.** *A function $y = y(x)$ is a solution of (29) if and only if there exists a function $g \in C[a,b]$ such that*

*(1)* $|g(x)| \le \varepsilon,\ \forall x \in [a,b]$,
*(2)* $y^{(m)}(x) - f(x) = g(x),\ \forall x \in [a,b]$.

**Remark 6.** *If $y = y(x)$ is a solution of (29), using Remark 5 and Remark 4, we have*

$$y(x) = y(a) + \tfrac{x-a}{1!}y'(a) + \cdots + \tfrac{(x-a)^{m-1}}{(m-1)!}y^{(m-1)}(a) + \int_a^x \tfrac{(x-s)^{m-1}}{(m-1)!}(f(s) + g(s))ds. \tag{30}$$

**Theorem 2.** *(i) For each solution $y = y(x)$ of (29), there exists a unique solution $y_0 = y_0(x)$ of (27) such that*

$$\begin{cases} y_0(a) = y(a) \\ y_0'(a) = y'(a) \\ \cdots \\ y_0^{(m-1)}(a) = y^{(m-1)}(a). \end{cases} \tag{31}$$

*(ii)* *The Equation (27) is Hyers–Ulam stable of order $m-1$, and if $y$ is a solution of (29) and $y_0$ is a solution of (27) satisfying conditions (31), then*

$$|y(x) - y_0(x)| \le \varepsilon \frac{(b-a)^m}{m!}, \forall x \in [a,b], \tag{32}$$

$$|y'(x) - y_0'(x)| \le \varepsilon \frac{(b-a)^{m-1}}{(m-1)!}, \forall x \in [a,b], \tag{33}$$

$$|y''(x) - y_0''(x)| \le \varepsilon \frac{(b-a)^{m-2}}{(m-2)!}, \forall x \in [a,b], \tag{34}$$

$\cdots$

$$\left|y^{(m-1)}(x) - y_0^{(m-1)}(x)\right| \le \varepsilon(b-a), \forall x \in [a,b]. \tag{35}$$

*(iii)* *If there exists $l_1, l_2, \cdots l_n : [a,b] \longrightarrow [0,\infty)$ continuous, such that*

$$\left|F\left(x, y_1(x), y_1'(x), \cdots, y_1^{(n)}(x)\right) - F\left(x, y_2(x), y_2'(x), \cdots, y_2^{(n)}(x)\right)\right|$$
$$\le l_1(x)|y_1(x) - y_2(x)| + l_2(x)|y_1'(x) - y_2'(x)| + \cdots + l_n(x)\left|y_1^{(n)}(x) - y_2^{(n)}(x)\right|,$$

*$\forall x \in [a,b]$, $\forall y_1, y_2 \in A$, then*

$$|J[y(x)] - J[y_0(x)]|$$
$$\le \varepsilon \cdot \frac{(b-a)^m}{m!} \int_a^b l_1(x)dx + \varepsilon \cdot \frac{(b-a)^{m-1}}{(m-1)!} \int_a^b l_2(x)dx + \cdots + \varepsilon \cdot \frac{(b-a)^n}{n!} \int_a^b l_n(x), \tag{36}$$

*where $y$ is a solution of (29) and $y_0$ is a solution of (27), both satisfying the conditions of (24).*

**Proof.** (i) This results from Cauchy–Picard's theorem of existence and uniqueness (see [46]).

(ii) Let $y = y(x)$ be a solution of (29). Let $y_0 = y_0(x)$ the unique solution of (27) satisfying the conditions of (31). Using Remark 6, we have

$$|y(x) - y_0(x)| =$$
$$\Bigg| y(a) + \frac{x-a}{1!}y'(a) + \cdots + \int_a^x \frac{(x-s)^{m-1}}{(m-1)!}(f(s) + g(s))ds$$
$$- y(a) - \frac{x-a}{1!}y'(a) - \cdots - \int_a^x \frac{(x-s)^{m-1}}{(m-1)!} f(s)ds \Bigg|,$$

hence

$$|y(x) - y_0(x)| = \left| \int_a^x \frac{(x-s)^{m-1}}{(m-1)!} g(s)ds \right| \le \int_a^x \left| \frac{(x-s)^{m-1}}{(m-1)!} g(s) \right| ds,$$

so

$$|y(x) - y_0(x)| \le \varepsilon \frac{(b-a)^m}{m!}, \forall x \in [a,b]. \tag{37}$$

We also have

$$\left|y'(x)-y_0'(x)\right| = \left|\int_a^x \frac{(x-s)^{m-2}}{(m-2)!}g(s)ds\right| \le \int_a^x \left|\frac{(x-s)^{m-2}}{(m-2)!}g(s)\right|ds \le \varepsilon\frac{(b-a)^{m-1}}{(m-1)!},$$

$$\left|y''(x)-y_0''(x)\right| = \left|\int_a^x \frac{(x-s)^{m-3}}{(m-3)!}g(s)ds\right| \le \int_a^x \left|\frac{(x-s)^{m-3}}{(m-3)!}g(s)\right|ds \le \varepsilon\frac{(b-a)^{m-2}}{(m-2)!},$$

...

$$\left|y^{(m-1)}(x)-y_0^{(m-1)}(x)\right| = \left|\int_a^x g(s)ds\right| \le \int_a^x |g(s)|ds \le \varepsilon(b-a), \forall x\in[a,b],$$

thus the Equation (27) is Hyers–Ulam stable of order $m-1$.

(iii) If $y$ is a solution of (29), and $y_0$ is a solution of (27), both satisfying the conditions of (24), then

$$\begin{aligned}&|J[y(x)]-J[y_0(x)]|\\ &\le \int_a^b\Big[l_1(x)|y(x)-y_0(x)|+l_2(x)|y'(x)-y_0'(x)|+\cdots+l_n(x)\Big|y^{(n)}(x)-y_0^{(n)}(x)\Big|\Big]dx\\ &\overset{(ii)}{\le} \varepsilon\cdot\frac{(b-a)^m}{m!}\int_a^b l_1(x)dx+\varepsilon\cdot\frac{(b-a)^{m-1}}{(m-1)!}\int_a^b l_2(x)dx+\cdots+\varepsilon\cdot\frac{(b-a)^n}{n!}\int_a^b l_n(x).\end{aligned}$$

□

**Example 2.** *We consider* $J: A \longrightarrow R,\ A\subseteq C^4(I,\mathbb{R}),\ [0,1]\subset I,$

$$J[y(x)] = \int_0^1 \left(360x^2y - y''^2\right)dx, \tag{38}$$

$$y(0)=0,\ y'(0)=1, y''(0)=-6,\ y'''(0)=9. \tag{39}$$

*The Euler–Poisson equation becomes*

$$y^{IV}(x) - 180x^2 = 0. \tag{40}$$

*Let* $\varepsilon > 0$*. We consider the inequality*

$$\left|y^{IV}(x)-180x^2\right| \le \varepsilon. \tag{41}$$

*We apply Theorem 2; therefore, for each solution* $y = y(x)$ *of* (41) *satisfying* (39)*, there exists a unique solution* $y_0 = y_0(x)$ *of* (40) *satisfying* (39) *such that*

$$|y(x)-y_0(x)| \le \varepsilon\frac{1}{4!},\ \forall x\in[0,1], \tag{42}$$

$$\left|y'(x)-y_0'(x)\right| \le \varepsilon\frac{1}{3!}, \forall x\in[0,1],$$

$$\left|y''(x)-y_0''(x)\right| \le \varepsilon\frac{1}{2!}, \forall x\in[0,1],$$

$$\left|y'''(x)-y_0'''(x)\right| \le \varepsilon\frac{1}{1!}, \forall x\in[0,1];$$

*hence, Equation* (40) *is Hyers–Ulam stable of order 3.*

*Let*

$$y_0(x) = \frac{1}{2}x^6+\frac{3}{2}X^3-3X^2+X, \tag{43}$$

*be the solution of Equation* (40) *satisfying the conditions of* (39)*. We remark that* $-6 \leq y''(x) \leq 18, \forall x \in [0,1]$.

*If* $y$ *is a solution of* (41) *and* $y_0$ *is a solution of* (40)*, both satisfying the conditions of* (39)*, then*

$$|J[y(x)] - J[y_0(x)]| \leq \int_0^1 \left|360x^2 y - y''^2 - 360x^2 y_0 + y_0''^2\right| dx \tag{44}$$

$$\leq \int_0^1 \left(360x^2 |y - y_0| + |y'' - y_0''||y'' + y_0''|\right) dx \tag{45}$$

$$\leq \int_0^1 \left[360x^2 \cdot \frac{\varepsilon}{24} + \frac{\varepsilon}{2}\left(\frac{\varepsilon}{2} + 36\right)\right] dx = \frac{\varepsilon(\varepsilon + 92)}{4}. \tag{46}$$

## 4. Conclusions

In this paper, we have defined and studied Hyers–Ulam stability of order 1 for Euler's equation $y''(x) = f(x)$ and Hyers–Ulam stability of order $m-1$ for the Euler–Poisson equation $y^{(m)}(x) = f(x)$, in the calculus of variations. An example is considered for each case. Some estimations for $|J[y(x)] - J[y_0(x)]|$, where $y$ is a solution of (9) and $y_0$ is a solution of (7), both satisfying the conditions of (2) and where $y$ is a solution of (29) and $y_0$ is a solution of (27), both satisfying the conditions of (24), have been established. This paper is a continuation of the paper [43]. In [43], the Hyers–Ulam stability of Euler's equation in two special cases was studied when $F = F(x, y')$ and when $F = F(y, y')$. The general case will be the subject of future works.

**Author Contributions:** Conceptualisation, D.M. and N.L.; methodology, D.M. and N.L.; investigation, D.M., S.A.C. and N.L.; validation S.A.C.; writing—original draft preparation, D.M.; writing—review and editing, S.A.C. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research received no external funding.

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Data Availability Statement:** Not applicable.

**Acknowledgments:** We would like to thank the editor and reviewers in advance for their helpful comments.

**Conflicts of Interest:** The authors declare no conflict of interest.

## References

1. Ulam, S.M. *A Collection of Mathematical Problems*; Interscience: New York, NY, USA, 1960.
2. Hyers, D.H. On the stability of the linear functional equation. *Proc. Natl. Acad. Sci. USA* **1941**, *27*, 222–224. [CrossRef] [PubMed]
3. Obloza, M. Hyers stability of the linear differential equation. *Rocz. Nauk-Dydakt. Pr. Mat.* **1993**, *13*, 259–270.
4. Alsina, C.; Ger, R. On some inequalities and stability results related to exponential function. *J. Inequal. Appl.* **1998**, *2*, 373–380. [CrossRef]
5. Takahasi, S.E.; Takagi, H.; Miura, T.; Miyajima, S. The Hyers-Ulam stability constant of first order linear differential operators. *J. Math. Anal. Appl.* **2004**, *296*, 403–409. [CrossRef]
6. Jung, S.-M. Hyers-Ulam stability of linear differential equations of first order. *Appl. Math. Lett.* **2004**, *17*, 1135–1140. [CrossRef]
7. Jung, S.-M. Hyers-Ulam stability of linear differential equations of first order, II. *Appl. Math. Lett.* **2006**, *19*, 854–858. [CrossRef]
8. Jung, S.-M. Hyers-Ulam stability of linear differential equations of first order, III. *J. Math. Anal. Appl.* **2005**, *311*, 139–146. [CrossRef]
9. Brzdek, J.; Popa, D.; Xu, B. Hyers-Ulam stability for linear equations of higher orders. *Acta Math. Hung.* **2008**, *120*, 1–8. [CrossRef]
10. Rezaei, H.; Jung, S.-M.; Rassias T. Laplace transform and Hyers-Ulam stability of linear differential equations. *J. Math. Anal. Appl.* **2013**, *403*, 244–251. [CrossRef]
11. Alqifiary, Q.; Jung, S.-M. Laplace transform and generalized Hyers-Ulam stability of linear differential equations. *Electron. J. Differ. Equ.* **2014**, *2014*, 1–11.
12. Cimpean, D.S.; Popa, D. On the stability of the linear differential equation of higher order with constant coefficients. *Appl. Math. Comput.* **2010**, *217*, 4141–4146. [CrossRef]

13. Popa, D.; Rasa, I. Hyers-Ulam stability of the linear differential operator with non-constant coefficients. *Appl. Math. Comput.* **2012**, *219*, 1562–1568.
14. Lungu, N.; Rus, I.A. On a functional Volterra-Fredholm integral equation, via Picard operators. *J. Math. Ineq.* **2009**, *41*, 519–527. [CrossRef]
15. Otrocol, D. Ulam stabilities of differential equation with abstract Volterra operator in a Banach space. *Nonlinear Funct. Anal. Appl.* **2010**, *15*, 613–619.
16. Cadariu, L. The generalized Hyers-Ulam stability for a class of the Volterra nonlinear integral equations. *Sci. Bull. Politeh. Univ. Timisoara Trans. Math. Phys.* **2011**, *56*, 30–38.
17. Ngoc L.T.P.; Long, N.T. On nonlinear Volterra-Hammerstein integral equation in two variables. *Acta Math. Sci.* **2013**, *33B*, 484–494. [CrossRef]
18. Ngoc, L.T.P.; Thuyet, T.M.; Long, N.T. A nonlinear Volterra-Hammerstein integral equation in three variables. *Nonlinear Funct. Anal. Appl.* **2014**, *19*, 193–211.
19. Pachpatte, B.G. On Volterra-Fredholm integral equation in two variables. *Demonstr. Math.* **2007**, *40*, 839–852. [CrossRef]
20. Pachpatte, B.G. On Fredholm type integral equation in two variables. *Differ. Equ. Appl.* **2009**, *1*, 27–39. [CrossRef]
21. Pachpatte, B.G. Volterra integral and integro differential equations in two variables. *J. Inequal. Pure Appl. Math.* **2009**, *10*, 108.
22. Ilea, V.; Otrocol, D. Existence and Uniqueness of the Solution for an Integral Equation with Supremum, via w-Distances. *Symmetry* **2020**, *12*, 1554. [CrossRef]
23. Marian, D.; Ciplea, S.A.; Lungu, N. Ulam-Hyers stability of Darboux-Ionescu problem. *Carpatian J. Math.* **2021**, *37*, 211–216. [CrossRef]
24. Marian, D.; Ciplea, S.A.; Lungu, N. On a functional integral equation. *Symmetry.* **2021**, *13*, 1321. [CrossRef]
25. Prastaro, A.; Rassias, T.M. Ulam stability in geometry of PDE's. *Nonlinear Funct. Anal. Appl.* **2003**, *8*, 259–278.
26. Jung, S.-M. Hyers-Ulam stability of linear partial differential equations of first order. *Appl. Math. Lett.* **2009**, *22*, 70–74. [CrossRef]
27. Jung, S.-M.; Lee, K.-S. Hyers-Ulam stability of first order linear partial differential equations with constant coefficients. *Math. Inequal. Appl.* **2007**, *10*, 261–266. [CrossRef]
28. Lungu, N.; Ciplea, S. Ulam-Hyers-Rassias stability of pseudoparabolic partial differential equations. *Carpatian J. Math.* **2015**, *31*, 233–240. [CrossRef]
29. Lungu, N.; Marian, D. Ulam-Hyers-Rassias stability of some quasilinear partial differential equations of first order. *Carpathian J. Math.* **2019**, *35*, 165–170. [CrossRef]
30. Lungu, N.; Popa, D. Hyers-Ulam stability of a first order partial differential equation. *J. Math. Anal. Appl.* **2012**, *385*, 86–91. [CrossRef]
31. Brzdek, J.; Popa, D.; Rasa, I.; Xu, B. *Ulam Stability of Operators*; Elsevier: Amsterdam, The Netherlands, 2018.
32. Tripathy, A.K. *Hyers-Ulam Stability of Ordinary Differential Equations*; Taylor and Francis: Boca Raton, FL, USA, 2021.
33. Wang, J.; Lv, L.; Zhou, Y. Ulam stability and data dependence for fractional differential equations with Caputo derivative. *Electron. J. Qual. Theory Differ. Equ.* **2011**, *63*, 1–10. [CrossRef]
34. Ibrahim, R.W. Generalized Ulam-Hyers stability for fractional differential equations. *Int. J. Math.* **2012**, *23*, 1–99. [CrossRef]
35. Wang, J.; Lin, Z. Ulam's Type Stability of Hadamard Type Fractional Integral Equations. *Filomat* **2014**, *28*, 1323–1331. [CrossRef]
36. Abbas, S.; Benchohra, M.; Lagreg, J.E.; Alsaedi, A.; Zhou, Y. Existence and Ulam stability for fractional differential equations of Hilfer-Hadamard type. *Adv. Differ. Equ.* **2017**, *2017*, 180. [CrossRef]
37. Benchohra, M.; Lazreg, J.E. Existence and Ulam stability for nonlinear implicit fractional differential equations with Hadamard derivative. *Stud. Univ. Babes-Bolyai Math.* **2017**, *62*, 27–38. [CrossRef]
38. Khan, H.; Tunc, C.; Chen, W.; Khan, A. Existence theorems and Hyers-Ulam stability for a class of hybrid fractional differential equations with p-Laplacian operator. *J. Appl. Anal. Comput.* **2018**, *8*, 1211–1226. [CrossRef]
39. Ameen, R.; Jarad, F.; Abdeljawad, T. Ulam stability for delay fractional differential equations with a generalized Caputo derivative. *Filomat* **2018**, *32*, 5265–5274. [CrossRef]
40. Moharramniaa, A.; Eghbali, N.; Rassias, J.M. Mittag-Leffler-Hyers-Ulam stability of Prabhakar fractional integral equation. *Int. J. Nonlinear Anal. Appl.* **2021**, *12*, 25–33.
41. Wang, S. The Ulam stability of fractional differential Equation with the Caputo-Fabrizio derivative. *J. Funct. Spaces* **2022**, *2022*, 7268518. [CrossRef]
42. Brzdęk, J.; Eghbali, N.; Kalvandi, V. On Ulam Stability of a Generalized Delayed Differential Equation of Fractional Order. *Results Math.* **2022**, *77*, 26. [CrossRef]
43. Marian, D.; Ciplea, S.A.; Lungu, N. Hyers-Ulam Stability of Euler's Equation in the Calculus of Variations. *Mathematics* **2021**, *9*, 3320. [CrossRef]
44. Krasnov, M.L.; Makarenko, G.I.; Kiselev, A.I. *Problems and Exercices in the Calculus of Variations*; Mir: Moscow, Russia, 1975.
45. Stewart, J. *Multivariable Calculus: Concepts and Contexts*; Brooks/Cole, Cengage Learning: Toronto, ON, Canada, 2010.
46. Ionescu, D.V. *Ecuatii Diferentiale si Integrale*; Editura Didactica si Pedagogica: Bucuresti, Romania, 1964.