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Almost Complex and Hypercomplex Norden Structures Induced by Natural Riemann Extensions †

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† In memory of Professor Simeon Zamkovoy (1977–2020).

Abstract: The Riemann extension, introduced by E. K. Patterson and A. G. Walker, is a semi-Riemannian metric with a neutral signature on the cotangent bundle T^*M of a smooth manifold M, induced by a symmetric linear connection ∇ on M. In this paper we deal with a natural Riemann extension $\overline{g}$, which is a generalization (due to M. Sekizawa and O. Kowalski) of the Riemann extension. We construct an almost complex structure $\overline{J}$ on the cotangent bundle T^*M of an almost complex manifold (M, J, ∇) with a symmetric linear connection ∇ such that $(T^*M, \overline{J}, \overline{g})$ is an almost complex manifold, where the natural Riemann extension $\overline{g}$ is a Norden metric. We obtain necessary and sufficient conditions for $(T^*M, \overline{J}, \overline{g})$ to belong to the main classes of the Ganchev–Borisov classification of the almost complex manifolds with Norden metric. We also examine the cases when the base manifold is an almost complex manifold with Norden metric or it is a complex manifold (M, J, ∇') endowed with an almost complex connection ∇' ($\nabla' J = 0$). We investigate the harmonicity with respect to $\overline{g}$ of the almost complex structure $\overline{J}$, according to the type of the base manifold. Moreover, we define an almost hypercomplex structure $(\overline{J}_1, \overline{J}_2, \overline{J}_3)$ on the cotangent bundle T^*M^{4n} of an almost hypercomplex manifold $(M^{4n}, J_1, J_2, J_3, \nabla)$ with a symmetric linear connection ∇. The natural Riemann extension $\overline{g}$ is a Hermitian metric with respect to $\overline{J}_1$ and a Norden metric with respect to $\overline{J}_2$ and $\overline{J}_3$.

Keywords: natural Riemann extension; almost complex manifolds with Norden metric; almost hypercomplex manifolds with Hermitian and Norden metrics; harmonicity

MSC: 53C15

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1. Introduction

Almost complex Norden structures were introduced in the literature by A. P. Norden [1]. On an almost complex manifold with Norden metric (N, J, g), the almost complex structure J acts as an anti-isometry with respect to the semi-Riemannian metric g, called Norden metric, in each tangent fibre. The metric g is necessarily of neutral signature. Almost complex manifolds with Norden metric were studied in Ref. [2], where they were called generalized B-manifolds. A classification of the considered manifolds with respect to the covariant derivative of the almost complex structure was given by G. Ganchev and A. Borisov in [3]. Beside Riemannian and Lorentzian geometry, a special role is played by manifolds with a metric of neutral signature, among which almost complex manifolds with Norden metric constitute a particular class. These manifolds are investigated by many authors and several examples are given in the literature (e.g., [4–9] and the references therein). Several papers constructed almost complex Norden structures on the total space of the tangent bundle (see [4]); however, such structures on the total space of the cotangent bundle are not so rich. We mention here Ref. [10] as a paper concerning almost complex Norden structures on the cotangent bundle, but we note that the metric of our paper is different, as we work with natural Riemann extensions (which generalize the Riemann extension).

Let (M, ∇) be an n-dimensional manifold endowed with a symmetric linear connection ∇. Patterson and Walker defined in Ref.[11] a semi-Riemannian metric on the cotangent bundle T^*M of (M, ∇), called Riemann extension. This metric is of neutral signature (n, n) and it was generalized by M. Sekizawa and O. Kowalski in Ref. [12,13] to a natural Riemann extension $\overline{g}$, which has the same signature. Recently, the metric $\overline{g}$ has been studied by many authors. For instance, the first author and Kowalski characterized in Ref. [14] some harmonic functions on $(T^*M, \overline{g})$. In Ref. [15], the first author and Eken defined a canonical almost para-complex structure on $(T^*M, \overline{g})$ and investigated its harmonicity with respect to $\overline{g}$. In Ref. [16], the authors constructed a family of hypersurfaces of $(T^*M, \overline{g})$, which are Einstein manifolds with positive scalar curvature.

Our goal in the present work is to construct and study almost complex and hypercomplex Norden structures on the total space of the cotangent bundle, endowed with a natural Riemann extension.

The paper consists of five sections. In Section 2 we recall some notions and results about the cotangent bundle of a manifold and the lifting of objects from the base manifold to its cotangent bundle. In Section 3 we provide some basic information about almost complex manifolds with Norden metric and we obtain some auxiliary results for later use. In Section 4 we consider the cotangent bundle of a 2n-dimensional almost complex manifold (M, J, ∇) with an almost complex structure J and a symmetric linear connection ∇. Motivated by the fact that the natural Riemann extension $\overline{g}$ on T^*M is of a neutral signature, we define an almost complex structure $\overline{J}$ on T^*M, which is an anti-isometry with respect to $\overline{g}$. Thus, the natural Riemann extension $\overline{g}$ is a Norden metric and $(T^*M, \overline{J}, \overline{g})$ is an almost complex manifold with a Norden metric. We give necessary and sufficient conditions for $(T^*M, \overline{J}, \overline{g})$ to belong to the following classes of the Ganchev–Borisov classification in Ref. [3]: $\mathcal{W}_0$ (Kähler–Norden manifolds), $\mathcal{W}_2$ (special complex manifolds with Norden metric), $\mathcal{W}_3$ (quasi-Kähler manifolds with Norden metric). We prove that $(T^*M, \overline{J}, \overline{g})$ is never contained in class $\mathcal{W}_1$. In the case when the base manifold is an almost complex manifold with Norden metric (N, J, g) we also find necessary and sufficient conditions for $(T^*N, \overline{J}, \overline{g})$ to be a manifold from the classes $\mathcal{W}_0$, $\mathcal{W}_2$ and we show that $(T^*N, \overline{J}, \overline{g})$ is never contained in classes $\mathcal{W}_1$ and $\mathcal{W}_3$. At the end of this section, we consider the special case when the base manifold (M, J, ∇') is a complex manifold, endowed with an almost complex connection ∇', i.e., $\nabla' J = 0$. Moreover, we investigate the harmonicity of the almost complex structure $\overline{J}$ with respect to $\overline{g}$ in the three cases above for the base manifold. In the last Section 5 we define an almost hypercomplex structure $\overline{H} = (\overline{J}_1, \overline{J}_2, \overline{J}_3)$ on the cotangent bundle T^*M^{4n} of an almost hypercomplex manifold $(M^{4n}, J_1, J_2, J_3, \nabla)$ with a symmetric linear connection ∇. The hypercomplex manifold $(T^*M^{4n}, \overline{H})$ endowed with the natural Riemann extension $\overline{g}$ turns out to be an almost hypercomplex manifold with Hermitian–Norden metrics.

2. Preliminaries

To fix notations, the cotangent bundle T^*M of a connected smooth n-dimensional manifold M $(n \geq 2)$ consists of all pairs (x, ω), where $x \in M$ and $\omega \in T^*_x M$. Any local chart $(U; x^1, \dots, x^n)$ on M induces a local chart $(p^{-1}(U); x^1, \dots, x^n, x^{1*}, \dots, x^{n*})$ on T^*M, where $p : T^*M \longrightarrow M$, $p(x, \omega) = x$, is the natural projection of T^*M to M. For any $i = 1, \dots, n$ the function $x^i \circ p$ on $p^{-1}(U)$ is identified with the function x^i on U and $x^{i*} = \omega_i = \omega\left(\left(\frac{\partial}{\partial x^i}\right)_x\right)$ at any point $(x, \omega) \in p^{-1}(U)$. We put $\partial_i = \frac{\partial}{\partial x^i}$ and $\partial_{i*} = \frac{\partial}{\partial \omega^i}$ $(i = 1, \dots, n)$.

The vectors $\{(\partial_1)_{(x,\omega)}, \dots, (\partial_n)_{(x,\omega)}, (\partial_{1*})_{(x,\omega)}, \dots, (\partial_{n*})_{(x,\omega)}\}$ form a basis of the tangent space $(T^*M)_{(x,\omega)}$ at each point $(x, \omega)$of any local chart in T^*M. The Liouville type vector field W is a globally defined vector field on T^*M that is expressed in local coordinates by

$$W = \sum_{i=1}^{n} \omega_i \partial_{i*}.$$

Everywhere here we will denote by $\mathcal{F}(M)$, $\chi(M)$ and $\Omega^1(M)$ the set of all smooth real functions, vector fields, and differential 1-forms on M, respectively.

Now, we recall the constructions of the vertical and complete lifts for which we refer to [17,18].

The vertical lift f^V on T^*M of a function $f \in \mathcal{F}(M)$ is a function on T^*M defined by $f^V = f \circ p$. The vertical lift X^V on T^*M of a vector field $X \in \chi(M)$ is a function on T^*M (called evaluation function) defined by

$$X^V(x,\omega) = \omega(X_x) \text{ or equivalently } X^V(x,\omega) = \omega_i X^i(x), \text{ where } X = X^i\partial_i.$$

In the following proposition it is shown that a vector field $U \in \chi(T^*M)$ is determined by its action on all evaluation functions.

Proposition 1 ([18]). *Let U_1 and U_2 be vector fields on T^*M. If $U_1(Z^V) = U_2(Z^V)$ holds for all $Z \in \chi(M)$, then $U_1 = U_2$.*

The vertical lift α^V on T^*M of a differential 1-form $\alpha \in \Omega^1(M)$ is a tangent vector field to T^*M, which is defined by

$$\alpha^V(Z^V) = (\alpha(Z))^V, \quad Z \in \chi(M).$$

In local coordinates we have

$$\alpha^V = \sum_{i=1}^n \alpha_i \partial_{i*},$$

where $\alpha = \sum_{i=1}^n \alpha_i dx^i$. Hence we obtain $\alpha^V(f^V) = 0$ for all $f \in \mathcal{F}(M)$.

The complete lift X^C on T^*M of a vector field $X \in \chi(M)$ is a tangent vector field to T^*M, which is defined by

$$X^C(Z^V) = [X,Z]^V, \quad Z \in \chi(M).$$

In local coordinates X^C is written as

$$X^C_{(x,\omega)} = \sum_{i=1}^n X^i(x)(\partial_i)_{(x,\omega)} - \sum_{h,i=1}^n \omega_h(\partial_i X^h)(x)(\partial_{i*})_{(x,\omega)},$$

where $X = X^i\partial_i$. Thus we have $X^C(f^V) = (Xf)^V$ for all $f \in \mathcal{F}(M)$.

We note that the vector fields of the form $\alpha^V + X^C$ generate the tangent space $T_{(x,\omega)}T^*M$ at any point $(x,\omega) \in T^*M$.

Let (M,∇) be an n-dimensional manifold endowed with a symmetric linear connection ∇ (i.e., ∇ is torsion-free). In Ref. [13] Sekizawa constructed a semi-Riemannian metric $\overline{g}$ at each point (x,ω) of the cotangent bundle T^*M of M by:

$$\begin{aligned}
&\overline{g}_{(x,\omega)}(X^C,Y^C) = -a\omega(\nabla_{X_x}Y + \nabla_{Y_x}X) + b\omega(X_x)\omega(Y_x),\\
&\overline{g}_{(x,\omega)}(X^C,\alpha^V) = a\alpha_x(X_x),\\
&\overline{g}_{(x,\omega)}(\alpha^V,\beta^V) = 0
\end{aligned} \tag{1}$$

for all vector fields X,Y and all differential 1-forms α,β on M, where a,b are arbitrary constants. We may assume $a > 0$ without loss of generality. The metric $\overline{g}$ defined by (1) and named in Refs. [12,13] as *a natural Riemann extension*, is a semi-Riemannian metric of neutral signature (n,n). When $b \neq 0$, $\overline{g}$ is called a proper natural Riemann extension. In the case when $b = 0$ and $a = 1$, we obtain the notion of the *(classical) Riemann extension* defined by Patterson and Walker (see Ref.[11,19]). Hence, the natural Riemann extension generalizes the (classical) Riemann extension. If $b = 0$ and $a \neq 1$, then $\overline{g}$ is the *(classical) Riemann* *extension, up to a homothety*. From now on, if $\overline{g}$ is the (classical) Riemann extension or the (classical) Riemann extension up to a homothety, we will call $\overline{g}$ briefly a Riemann extension.

The following conventions and formulas will be used later on.

The contracted vector field $C(T) \in \chi(T^*M)$ of a $(1,1)$-tensor field T on a manifold M is defined at any point $(x,\omega) \in T^*M$ by its value on any evaluation function as follows:

$$C(T)(Z^V)_{(x,\omega)} = (TZ)^V_{(x,\omega)} = \omega((TZ)_x), \quad Z \in \chi(M). \tag{2}$$

For a 1-form α on M we denote by $i_\alpha(T)$ the 1-form on M, defined by

$$(i_\alpha(T))(Z) = \alpha(TZ), \quad Z \in \chi(M). \tag{3}$$

By using (3) we obtain

$$(i_\alpha(T))^V(Z)^V_{(x,\omega)} = (\alpha(T))^V(Z)^V_{(x,\omega)} = \alpha((TZ)_x), \quad Z \in \chi(M). \tag{4}$$

Now, the equalities (2), (4), and Proposition 1 imply that at each point $(x,\omega) \in T^*M$, the following equality holds

$$C(T)_{(x,\omega)} = (\omega_x(T))^V. \tag{5}$$

Also, at each point $(x,\omega) \in T^*M$, we have

$$W_{(x,\omega)} = (\omega_x)^V. \tag{6}$$

From (1), (5) and (6) we get

$$\begin{aligned} &\overline{g}_{(x,\omega)}(X^C, C(T)) = a\omega_x((TX)_x), \quad \overline{g}_{(x,\omega)}(W, \alpha^V) = 0, \\ &\overline{g}_{(x,\omega)}(W,W) = \overline{g}_{(x,\omega)}(W, C(T)) = \overline{g}_{(x,\omega)}(C(T_1), C(T_2)) = 0, \end{aligned} \tag{7}$$

where T_1 and T_2 are arbitrary $(1,1)$-tensor fields on M.

In Ref. [12], the following formulas for the Levi–Civita connection $\overline{\nabla}$ of the natural Riemann extension $\overline{g}$ are given:

$$\begin{aligned} (\overline{\nabla}_{X^C} Y^C)_{(x,\omega)} = {} & (\nabla_X Y)^C_{(x,\omega)} + C((\nabla X)(\nabla Y) + (\nabla Y)(\nabla X))_{(x,\omega)} \\ & + C(R(.,X)Y + R(.,Y)X)_{(x,\omega)} \\ & - \frac{b}{2a}\Big\{\omega(Y)X^C + \omega(X)Y^C + 2\omega(Y)C(\nabla X) + 2\omega(X)C(\nabla Y) \\ & + \omega(\nabla_X Y + \nabla_Y X)W\Big\}_{(x,\omega)} + \frac{b^2}{a^2}\omega(X)\omega(Y)W_{(x,\omega)}, \\ (\overline{\nabla}_{X^C}\beta^V)_{(x,\omega)} = {} & (\nabla_X \beta)^V_{(x,\omega)} + \frac{b}{2a}\Big\{\omega(X)\beta^V + \beta(X)W\Big\}_{(x,\omega)}, \\ (\overline{\nabla}_{\alpha^V} Y^C)_{(x,\omega)} = {} & -(i_\alpha(\nabla Y))^V_{(x,\omega)} + \frac{b}{2a}\Big\{\omega(Y)\alpha^V + \alpha(Y)W\Big\}_{(x,\omega)}, \\ (\overline{\nabla}_{\alpha^V}\beta^V)_{(x,\omega)} = {} & 0, \qquad (\overline{\nabla}_{X^C} W)_{(x,\omega)} = -C(\nabla X)_{(x,\omega)} + \frac{b}{a}\omega(X)W_{(x,\omega)}, \\ (\overline{\nabla}_{\alpha^V} W)_{(x,\omega)} = {} & \alpha^V_{(x,\omega)}, \qquad (\overline{\nabla}_W W)_{(x,\omega)} = W_{(x,\omega)}, \end{aligned} \tag{8}$$

where X^C, Y^C, and α^V, β^V are the complete lifts of the vector fields $X, Y \in \chi(M)$ and the vertical lifts of the differential 1-forms α, β on M, respectively. Here $C(\nabla X)$ is the contracted $(1,1)$-tensor field ∇X on M, defined by $(\nabla X)(Z) = \nabla_Z X$, $Z \in \chi(M)$ and R

is the curvature tensor of ∇. By $C(R(.,X)Y)$ is denoted the contracted $(1,2)$-tensor field $R(.,X)Y$ on M given by $(R(.,X)Y)(Z) = R(Z,X)Y)$, $X,Y,Z \in \chi(M)$.

3. Almost Complex Manifolds with Norden Metric

Definition 1. *Let (N,J) be an almost complex $2n$-dimensional manifold (whose almost complex structure J is a (1,1)-tensor field satisfying $J^2 = -\mathrm{Id}$). If, moreover, the almost complex manifold (N,J) carries a semi-Riemannian metric g with respect to which J is an anti-isometry, i.e.,*

$$g(JX,JY) = -g(X,Y), \quad X,Y \in \chi(N),$$

then (J,g) is called an almost complex Norden structure and (N,J,g) is an almost complex manifold with Norden metric.

The tensor $\widetilde{g}$ given by

$$\widetilde{g}(X,Y) = g(X,JY), \quad X,Y \in \chi(N)$$

is a Norden metric, which is called *an associated metric* of g. Both metrics g and $\widetilde{g}$ are necessarily of neutral signature, which means (n,n). Let F be a tensor field of type $(0,3)$ on an almost complex manifold with Norden metric, defined by

$$F(X,Y,Z) = g((\nabla_X J)Y,Z), \tag{9}$$

where ∇ is the Levi–Civita connection of g. The tensor field F has the following properties:

$$F(X,Y,Z) = F(X,Z,Y), \quad F(X,JY,JZ) = F(X,Y,Z),\ X,Y,Z \in \chi(N). \tag{10}$$

The Lee form θ associated with F is defined by

$$\theta(Z) = g^{ij}F(f_i,f_j,Z), \tag{11}$$

where $\{f_1,\ldots,f_{2n}\}$ is a local basis on N and g^{ij} are the components of the inverse matrix of the matrix (g_{ij}).

From (10) and (11), by direct computation, we obtain

Proposition 2. *Let (N,J,g) be an almost complex manifold with Norden metric. Then $\theta(Z) = \mathrm{trace}(\nabla J)Z$, where $(\nabla J)Z$ is the linear map $(\nabla J)Z : X \longrightarrow (\nabla_X J)Z$ and ∇ is the Levi–Civita connection of g.*

The Nijenhuis tensor $\mathcal{N}$ of an almost complex manifold with Norden metric (N,J,g) is expressed in terms of the Levi–Civita connection ∇ of g and the almost complex structure J as follows:

$$\mathcal{N}(X,Y) = (\nabla_X J)JY - (\nabla_Y J)JX + (\nabla_{JX} J)Y - (\nabla_{JY} J)X.$$

Ref. [3] introduced an associated with $\mathcal{N}$ tensor $\widetilde{\mathcal{N}}$ given by

$$\widetilde{\mathcal{N}}(X,Y) = (\nabla_X J)JY + (\nabla_Y J)JX + (\nabla_{JX} J)Y + (\nabla_{JY} J)X.$$

A classification of the almost complex manifolds with Norden metric was given in Ref. [3]. Here we recall the characteristic conditions of the eight classes of this classification:

- Kähler manifolds with Norden metric (also called Kähler–Norden manifolds)

$$\mathcal{W}_0 : F(X,Y,Z) = 0 \text{ or } \ \text{equivalently } \ (\nabla_X J)Y = 0.$$

- Conformally Kähler manifolds with Norden metric

$$\mathcal{W}_1 : F(X,Y,Z) = \frac{1}{2n}\{g(X,Y)\theta(Z) + g(X,Z)\theta(Y)$$
$$+g(X,JY)\theta(JZ) + g(X,JZ)\theta(JY)\}.$$

- Special complex manifolds with Norden metric

$$\mathcal{W}_2 : F(X,Y,JZ) + F(Y,Z,JX) + F(Z,X,JY) = 0, \quad \theta = 0$$
$$\text{or equivalently } \mathcal{N} = 0, \ \theta = 0.$$

- Quasi-Kähler manifolds with Norden metric

$$\mathcal{W}_3 : F(X,Y,Z) + F(Y,Z,X) + F(Z,X,Y) = 0,$$
$$\text{or equivalently } \widetilde{\mathcal{N}} = 0.$$

- Complex manifolds with Norden metric

$$\mathcal{W}_1 \oplus \mathcal{W}_2 : F(X,Y,JZ) + F(Y,Z,JX) + F(Z,X,JY) = 0,$$
$$\text{or equivalently } \mathcal{N} = 0.$$

- Semi-Kähler manifolds with Norden metric

$$\mathcal{W}_2 \oplus \mathcal{W}_3 : \theta = 0.$$

- $\mathcal{W}_1 \oplus \mathcal{W}_3 : \underset{(X,Y,Z)}{\mathfrak{S}} F(X,Y,Z) = \frac{1}{n}\{g(X,Y)\theta(Z) + g(Z,X)\theta(Y) + g(Y,Z)\theta(X)$

$$+g(X,JY)\theta(JZ) + g(Z,JX)\theta(JY) + g(Y,JZ)\theta(JX)\},$$

  where $\underset{(X,Y,Z)}{\mathfrak{S}}$ denotes the cyclic sum over X,Y,Z.
- $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$: The whole class of almost complex manifolds with Norden metric.

An almost complex manifold with Norden metric (N,J,g) belonging to the class $\mathcal{W}_i$ will be briefly called a $\mathcal{W}_i$-manifold, $i \in \{0,1,2,3\}$.

The special class $\mathcal{W}_0$ of the Kähler–Norden manifolds belongs to any other class. On a Kähler–Norden manifold the curvature tensor field R of ∇ defined by $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$, has the so called Kähler property

$$R(X,Y,JZ,JU) = -R(X,Y,Z,U), \quad X,Y,Z,U \in \chi(N).$$

Lemma 1. *If (N,J,g) is an almost complex manifold with Norden metric such that either $F(X,Y,Z) = F(Y,X,Z)$ or $F(X,Y,Z) = -F(Y,X,Z)$, then F vanishes identically.*

Proof. Let $F(X,Y,Z) = F(Y,X,Z)$. By using the properties (10) of F we obtain

$$F(JX,JY,Z) = F(JY,JX,Z) = -F(JY,X,JZ) = -F(X,JY,JZ) = -F(X,Y,Z)$$

and

$$F(JX,JY,Z) = F(JX,Z,JY) = F(Z,JX,JY) = F(Z,X,Y) = F(X,Y,Z).$$

Hence, $F \equiv 0$. Analogously, one can prove that $F(X,Y,Z) = -F(Y,X,Z)$ implies $F \equiv 0$. □

For later use, we recall the following.

Definition 2 ([20]). *On a (semi-) Riemannian manifold (N, h), a (1,1)-tensor field T is called harmonic if T viewed as an endomorphism field $T : (TN, h^C) \longrightarrow (TN, h^C)$ is a harmonic map, where h^C denotes the complete lift (see [17]) of the (semi-) Riemannian metric h.*

We recall the following characterization result:

Proposition 3 ([20]). *Let (N, h) be a (semi-) Riemannian manifold and let ∇ be the Levi–Civita connection of h. Then, any (1,1)-tensor field T on (N, h) is harmonic if and only if $\delta T = 0$, where*

$$\delta T = \mathrm{trace}_h(\nabla T) = \mathrm{trace}_h\{(X, Y) \longrightarrow (\nabla_X T)Y\}.$$

By using (11) and Proposition 3 we obtain the following equivalence:

Lemma 2. *Let (N, J, g) be an almost complex manifold with Norden metric. Then the following assertions are equivalent:*

(i) J is harmonic;
(ii) $\theta = 0$;
(iii) M belongs to the one of the classes $\mathcal{W}_0$, $\mathcal{W}_2$, $\mathcal{W}_3$, $\mathcal{W}_2 \oplus \mathcal{W}_3$.

Proof. $(i) \Longleftrightarrow (ii)$ Let $\{e_1, \ldots, e_n, Je_1, \ldots, Je_n\}$ be a local orthonormal basis on N, such that $g(e_i, e_i) = -g(Je_i, Je_i) = 1$ $(i = 1, \ldots, n)$. J is harmonic if and only if

$$\delta J = \mathrm{trace}_g \nabla J = \sum_{i=1}^{n} \{(\nabla_{e_i} J)e_i - (\nabla_{Je_i} J)Je_i\} = 0.$$

Since g is non-degenerate, we have for any $Z \in \chi(N)$

$$\delta J = 0 \Longleftrightarrow g\left(\sum_{i=1}^{n} \{(\nabla_{e_i} J)e_i - (\nabla_{Je_i} J)Je_i\}, Z\right) = 0$$

$$\Longleftrightarrow \sum_{i=1}^{n} \{F(e_i, e_i, Z) - F(Je_i, Je_i, Z)\} = 0 \Longleftrightarrow \theta = 0.$$

$(ii) \Longleftrightarrow (iii)$ We establish the equivalence of (ii) and (iii) by using the classification of the almost complex manifolds with Norden metric given above. Let us remark that the defining condition of the class $\mathcal{W}_3$ implies the vanishing of the Lee form θ for this class. □

Remark 1. *Further, we assume that (M, J, ∇) is a $2n$-dimensional almost complex manifold with an almost complex structure J and a symmetric linear connection ∇. If (N, J, g) is an almost complex manifold with a Norden metric we denote the Levi–Civita connection of g also by ∇. It is clear that all the formulas and statements given when ∇ is an arbitrary symmetric linear connection are also valid when ∇ is the Levi–Civita connection, but the converse is not true.*

4. Cotangent Bundles with Natural Riemann Extensions as Almost Complex Manifolds with Norden Metric

On the cotangent bundle T^*M of an almost complex manifold (M, J, ∇) endowed with a natural Riemann extension $\bar{g}$, we define the endomorphism

$$\bar{J} : T(T^*M) \longrightarrow T(T^*M) \quad \text{by}$$

$$\begin{aligned} \bar{J}X^C &= (JX)^C - ((\nabla X) \circ J)^V + (\nabla JX)^V + \frac{b}{2a} X^V J^V - \frac{b}{2a} (JX)^V W, \\ \bar{J}\alpha^V &= (\alpha(J))^V, \end{aligned} \tag{12}$$

where $X, Y \in \chi(M)$ and $\alpha \in \Omega^1(M)$. One can check by a straightforward computation that $\bar{J}$ is an almost complex structure on T^*M. Moreover, taking into account (1) and (12), we establish that the natural Riemann extension $\overline{g}$ is a Norden metric with respect to $\overline{J}$. Thus, we state the following:

Theorem 1. *Let the total space of the cotangent bundle T^*M of a $2n$-dimensional almost complex manifold (M, J, ∇) be endowed with the natural Riemann extension $\overline{g}$, defined by (1), and the endomorphism $\overline{J}$, defined by (12). Then $(T^*M, \overline{J}, \overline{g})$ is an almost complex manifold with Norden metric.*

Further, we define the tensor field $\overline{F}$ on $(T^*M, \overline{J}, \overline{g})$ given by

$$\overline{F}(\overline{X}, \overline{Y}, \overline{Z}) = \overline{g}((\overline{\nabla}_{\overline{X}}\overline{J})\overline{Y}, \overline{Z}), \tag{13}$$

where $\overline{X}, \overline{Y}, \overline{Z} \in \chi(T^*M)$. By using (1), (7), (8) and (12) we obtain

$$\begin{aligned}
&\overline{F}_{(x,\omega)}(X^C, Y^C, Z^C) = -\frac{b^2}{4a}\{\omega(X)\omega(Y)\omega(JZ) - 2\omega(JX)\omega(Y)\omega(Z) \\
&+\omega(X)\omega(JY)\omega(Z)\} + \frac{b}{2}\{\omega(JY)\omega(\nabla_X Z) + \omega(JZ)\omega(\nabla_X Y) \\
&-\omega(Z)\omega(\nabla_{JX}Y) - \omega(Y)\omega(\nabla_{JX}Z) + \omega(Y)\omega((\nabla_X J)Z) \\
&+\omega(Z)\omega((\nabla_X J)Y)\} - a\Big\{\omega(\nabla_{(\nabla_X J)Z}Y) + \omega(\nabla_{(\nabla_X J)Y}Z)\Big\} \\
&+a\{\omega(R_x(Z, JY)X) - \omega(R_x(JZ, Y)X)\},
\end{aligned} \tag{14}$$

$$\begin{aligned}
&\overline{F}_{(x,\omega)}(X^C, \alpha^V, Z^C) = \overline{F}_{(x,\omega)}(X^C, Z^C, \alpha^V) \\
&= a\alpha((\nabla_X J)Z) + \frac{b}{2}\{\omega(Z)\alpha(JX) - \omega(JZ)\alpha(X)\},
\end{aligned} \tag{15}$$

$$\begin{aligned}
&\overline{F}_{(x,\omega)}(\alpha^V, \beta^V, Z^C) = \overline{F}_{(x,\omega)}(\alpha^V, Z^C, \beta^V) = 0, \\
&\overline{F}_{(x,\omega)}(\alpha^V, Y^C, Z^C) = \overline{F}_{(x,\omega)}(X^C, \beta^V, \gamma^V) = \overline{F}_{(x,\omega)}(\alpha^V, \beta^V, \gamma^V) = 0.
\end{aligned} \tag{16}$$

Let (x, ω), $\omega \neq 0$, be an arbitrary fixed point of T^*M and let $\{f_1, \ldots, f_{2n}\}$ be a local frame around x in M such that $(\nabla_{f_i} f_j)_x = 0$, $i, j = 1, \ldots 2n$. We denote by $\{\alpha_1 = \omega, \alpha_2, \ldots, \alpha_{2n}\}$ the local coframe around x in M, which is dual to $\{f_1, \ldots, f_{2n}\}$, i.e., $\alpha_i(f_j) = \delta_{ij}$, $i, j = 1, \ldots 2n$. We consider the following orthonormal basis $\{E_i, E_{i*}\}$ $(i = 1, \ldots 2n)$ with respect to $\overline{g}$ in $T_{(x,\omega)}(T^*M)$, constructed in [14]:

$$\begin{aligned}
&E_1 = f_1^C + \frac{1-b}{2a}\alpha_1^V; \quad E_{1*} = f_1^C - \frac{1+b}{2a}\alpha_1^V; \\
&E_k = \frac{1}{\sqrt{2a}}(f_k^C + \alpha_k^V); \quad E_{k*} = \frac{1}{\sqrt{2a}}(f_k^C - \alpha_k^V),\ k = 2, \ldots, 2n; \\
&\overline{g}(E_i, E_i) = -\overline{g}(E_{i*}, E_{i*}) = 1,\ i = 1, \ldots, 2n.
\end{aligned} \tag{17}$$

Proposition 4. *Let (M, J, ∇) and $(T^*M, \overline{J}, \overline{g})$ be as in Theorem 1. Let $\overline{F}$ be defined by (13) and $\overline{\theta}$ be its associated Lee form. Then we have*

$$\overline{\theta}_{(x,\omega)}(\alpha^V) = 0, \qquad \overline{\theta}_{(x,\omega)}(Z^C) = \mathrm{trace}(\nabla J)Z - \frac{bn}{a}\omega(JZ). \tag{18}$$

Moreover, if (N, J, g) is an almost complex manifold with Norden metric, then

$$\overline{\theta}_{(x,\omega)}(\alpha^V) = 0, \qquad \overline{\theta}_{(x,\omega)}(Z^C) = \theta(Z) - \frac{bn}{a}\omega(JZ), \tag{19}$$

where $\theta(Z)$ is the Lee form associated with the tensor field F on (N, J, g), given by (9).

Proof. By using (11), (13), and the orthonormal basis (17) in $T_{(x,\omega)}(T^*M)$, we obtain

$$\overline{\theta}_{(x,\omega)}(\overline{Z}) = \overline{F}_{(x,\omega)}(E_1, E_1, \overline{Z}) + \sum_{k=2}^{2n} \overline{F}_{(x,\omega)}(E_k, E_k, \overline{Z})$$

$$-\overline{F}_{(x,\omega)}(E_{1*}, E_{1*}, \overline{Z}) - \sum_{k=2}^{2n} \overline{F}_{(x,\omega)}(E_{k*}, E_{k*}, \overline{Z})$$

$$= \frac{1}{a}\Big\{\overline{F}_{(x,\omega)}(f_1^C, \alpha_1^V, \overline{Z}) + \overline{F}_{(x,\omega)}(\alpha_1^V, f_1^C, \overline{Z})$$

$$+ \sum_{k=2}^{2n} \overline{F}_{(x,\omega)}(f_k^C, \alpha_k^V, \overline{Z}) + \sum_{k=2}^{2n} \overline{F}_{(x,\omega)}(\alpha_k^V, f_k^C, \overline{Z})\Big\}, \quad \overline{Z} \in \chi(T^*M).$$

In the latter equality we substitute $\overline{Z}$ with α^V and Z^C. Taking into account (16) we get $\overline{\theta}_{(x,\omega)}(\alpha^V) = 0$ and $\overline{\theta}_{(x,\omega)}(Z^C) = \frac{1}{a}\sum_{k=1}^{2n} \overline{F}_{(x,\omega)}(f_k^C, \alpha_k^V, Z^C)$, respectively. Now, using (15) and $\alpha_1 = \omega$, for $\overline{\theta}_{(x,\omega)}(Z^C)$ we have

$$\overline{\theta}_{(x,\omega)}(Z^C) = \frac{1}{a}\sum_{k=1}^{2n}\Big\{a\alpha_k((\nabla_{f_k}J)Z) + \frac{b}{2}[\alpha_1(Z)\alpha_k(Jf_k) - \alpha_1(JZ)]\Big\}$$

$$= \sum_{k=1}^{2n} \alpha_k((\nabla_{f_k}J)Z) + \frac{b}{2a}\alpha_1(Z)\sum_{k=1}^{2n}\alpha_k(Jf_k) - \frac{b}{2a}\sum_{k=1}^{2n}\alpha_1(JZ)$$

$$= \mathrm{trace}(\nabla J)Z + \frac{b}{2a}\alpha_1(Z)\mathrm{trace}(J) - \frac{bn}{a}\alpha_1(JZ).$$

Since $\mathrm{trace}(J) = 0$, the equality (18) holds.

By using Proposition 2 and (18), we obtain (19). □

One can easily prove the following:

Lemma 3. *Let (M, J, ∇) be an almost complex manifold.*

(i) The following conditions are equivalent:

$$R(JX, Y)Z = R(X, JY)Z, \quad X, Y, Z \in \chi(M), \tag{20}$$

$$R(JX, X)Z = 0. \tag{21}$$

(ii) If (N, J, g) is a Kähler–Norden manifold, then (20) and (21) are both equivalent to the Kähler property of R.

Theorem 2. *Let (M, J, ∇) be an almost complex manifold. Then $(T^*M, \overline{J}, \overline{g})$ is a Kähler–Norden manifold if and only if at each point $(x, \omega) \in T^*M$ the conditions*

$$(\nabla_X J)Z = -\frac{b}{2a}\{\omega(Z)JX - \omega(JZ)X\} \tag{22}$$

and (20) are fulfilled.

Proof. The manifold $(T^*M, \overline{J}, \overline{g})$ is Kähler–Norden if and only if $\overline{F}_{(x,\omega)}(\overline{X}, \overline{Y}, \overline{Z}) = 0$ at each point $(x, \omega) \in T^*M$ and for all $\overline{X}, \overline{Y}, \overline{Z} \in \chi(T^*M)$. By using (14)–(16) we conclude that $\overline{F}_{(x,\omega)}(\overline{X}, \overline{Y}, \overline{Z}) = 0$ is equivalent to $\overline{F}_{(x,\omega)}(X^C, Y^C, Z^C) = 0$ and $\overline{F}_{(x,\omega)}(X^C, \alpha^V, Z^C) = \overline{F}_{(x,\omega)}(X^C, Z^C, \alpha^V) = 0$. The latter equality is equivalent to the condition (22). Substituting (22) in (14) we obtain that $\overline{F}_{(x,\omega)}(X^C, Y^C, Z^C) = 0$ if and only if (20) holds. □

Corollary 1. *Let (M, J, ∇) be an almost complex manifold, and let $\overline{g}$ be a Riemann extension on T^*M. Then $(T^*M, \overline{J}, \overline{g})$ is a Kähler–Norden manifold if and only if J is parallel with respect to ∇ and (20) is satisfied.*

Corollary 2. *Let (M, J, ∇) be an almost complex manifold, such that J is parallel with respect to ∇. Then $(T^*M, \overline{J}, \overline{g})$ is a Kähler–Norden manifold if and only if (20) is satisfied and $\overline{g}$ is a Riemann extension.*

Theorem 3. *Let (N, J, g) be an almost complex manifold with Norden metric. Then $(T^*N, \overline{J}, \overline{g})$ is a Kähler–Norden manifold if and only if $\overline{g}$ is a Riemann extension and (N, J, g) is a Kähler–Norden manifold.*

Proof. "$\Longrightarrow$" Let $(T^*N, \overline{J}, \overline{g})$ be a Kähler–Norden manifold. From the condition $\overline{F}_{(x,\omega)}(X^C, \alpha^V, Z^C) = 0$ and (15) it follows that (22) is fulfilled. By using (22), we have

$$F(X, Y, Z) = -\frac{b}{2a}\{\omega(Y)g(JX, Z) - \omega(JY)g(X, Z)\}.$$

Now, we find $\theta(Z) = 0$. Substituting $\overline{\theta} = \theta = 0$ in (19) we obtain $b = 0$, which implies $F(X, Y, Z) = 0$, i.e., (N, J, g) is a Kähler–Norden manifold.

"$\Longleftarrow$" Conversely, if (N, J, g) is a Kähler–Norden manifold and $b = 0$, then (15) and (14) become $\overline{F}_{(x,\omega)}(X^C, \alpha^V, Z^C) = 0$ and

$$\overline{F}_{(x,\omega)}(X^C, Y^C, Z^C) = a\{\omega(R_x(Z, JY)X) - \omega(R_x(JZ, Y)X)\},$$

respectively. Since R has the Kähler property, we get $\overline{F}_{(x,\omega)}(X^C, Y^C, Z^C) = 0$, which completes the proof. $\square$

Several examples of Kähler–Norden manifolds are given in Ref. [4–6,8] and other papers. Theorem 3 allows us to construct many new examples of Kähler–Norden manifolds as the total spaces of the cotangent bundles of some Kähler–Norden manifolds. Here we give another example of a Kähler–Norden manifold, whose cotangent bundle is also a Kähler–Norden manifold.

Example 1. *Let $N = S^1 \times \ldots \times S^1$ be the $2n$-dimensional torus and let $\{X_1, \ldots, X_{2n}\}$ be a global frame of vector fields, each of them tangent respectively to each cycle. With respect to this frame, let J be the almost complex structure and let g be the Norden metric given respectively by*

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} A & B \\ B & -A \end{pmatrix},$$

where I_n denotes the identity matrix of order n, A, B are symmetric real matrices of order n, with A non-singular. In particular, g can be taken as $g = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}$. In this case, (N, J, g) is a Kähler–Norden manifold. From Theorem 3 it follows that $(T^*N, \overline{J}, \overline{g})$ is a Kähler–Norden manifold, provided $b = 0$.

Theorem 4. *Let (M, J, ∇) and $(T^*M, \overline{J}, \overline{g})$ be as in Theorem 1. Then the manifold $(T^*M, \overline{J}, \overline{g})$ is never contained in class $\mathcal{W}_1$.*

Proof. Let us assume that (M, J, ∇) is an almost complex manifold and $(T^*M, \overline{J}, \overline{g})$ is a $\mathcal{W}_1$-manifold. Then for the non-zero components of $\overline{F}$ we have

$$\begin{aligned}\overline{F}_{(x,\omega)}(X^C, \alpha^V, Z^C) = \frac{1}{4n}\Big\{&\overline{g}_{(x,\omega)}(X^C, \alpha^V)\overline{\theta}_{(x,\omega)}(Z^C)\\ &+\overline{g}_{(x,\omega)}(X^C, Z^C)\overline{\theta}_{(x,\omega)}(\alpha^V) + \overline{g}_{(x,\omega)}(X^C, \overline{J}\alpha^V)\overline{\theta}_{(x,\omega)}(\overline{J}Z^C)\\ &+\overline{g}_{(x,\omega)}(X^C, \overline{J}Z^C)\overline{\theta}_{(x,\omega)}(\overline{J}\alpha^V)\Big\},\end{aligned} \tag{23}$$

$$\begin{aligned}\overline{F}_{(x,\omega)}(X^C, Y^C, Z^C) = \frac{1}{4n}\Big\{&\overline{g}_{(x,\omega)}(X^C, Y^C)\overline{\theta}_{(x,\omega)}(Z^C)\\ &+\overline{g}_{(x,\omega)}(X^C, Z^C)\overline{\theta}_{(x,\omega)}(Y^C) + \overline{g}_{(x,\omega)}(X^C, \overline{J}Y^C)\overline{\theta}_{(x,\omega)}(\overline{J}Z^C)\\ &+\overline{g}_{(x,\omega)}(X^C, \overline{J}Z^C)\overline{\theta}_{(x,\omega)}(\overline{J}Y^C)\Big\}.\end{aligned} \tag{24}$$

Taking into account that $\overline{\theta}_{(x,\omega)}(\alpha^V) = 0$ and (15), the equality (23) becomes

$$\begin{aligned}&a\alpha((\nabla_X J)Z) + \frac{b}{2}\{\omega(Z)\alpha(JX) - \omega(JZ)\alpha(X)\}\\ &= \frac{a}{4n}\Big\{\overline{\theta}_{(x,\omega)}(\overline{J}Z^C)\alpha(JX) + \overline{\theta}_{(x,\omega)}(Z^C)\alpha(X)\Big\}.\end{aligned}$$

From the latter it follows

$$(\nabla_X J)Z = \left[\frac{b\omega(JZ)}{2a} + \frac{\overline{\theta}_{(x,\omega)}(Z^C)}{4n}\right]X + \left[-\frac{b\omega(Z)}{2a} + \frac{\overline{\theta}_{(x,\omega)}(\overline{J}Z^C)}{4n}\right]JX.$$

Now, we find

$$\operatorname{trace}(\nabla J)Z = 2n\left[\frac{b\omega(JZ)}{2a} + \frac{\overline{\theta}_{(x,\omega)}(Z^C)}{4n}\right] = \frac{bn}{a}\omega(JZ) + \frac{\overline{\theta}_{(x,\omega)}(Z^C)}{2}.$$

Substituting $\operatorname{trace}(\nabla J)Z$ in (18) we obtain $\overline{\theta}_{(x,\omega)}(Z^C) = 0$. Then from (23) and (24) we get $\overline{F}_{(x,\omega)}(X^C, \alpha^V, Z^C) = \overline{F}_{(x,\omega)}(X^C, Y^C, Z^C) = 0$ at each point $(x, \omega) \in T^*M$, which means that $(T^*M, \overline{J}, \overline{g})$ is a Kähler–Norden manifold.

Having in mind Proposition 2 and (19), the conclusion of the theorem is valid also when (N, J, g) is an almost complex manifold with Norden metric. □

Theorem 5. *Let (M, J, ∇) be an almost complex manifold. Then $(T^*M, \overline{J}, \overline{g})$ belongs to the class $\mathcal{W}_3$ if and only if at each point $(x, \omega) \in T^*M$ the following conditions are fulfilled:*

$$(\nabla_X J)Z + (\nabla_Z J)X = \frac{b}{2a}\{\omega(JZ)X - \omega(Z)JX + \omega(JX)Z - \omega(X)JZ\}, \tag{25}$$

$$\begin{aligned}&R(JX, Z)Y + R(JY, X)Z + R(JZ, Y)X\\ &= R(X, JZ)Y + R(Y, JX)Z + R(Z, JY)X,\end{aligned} \tag{26}$$

where $X, Y, Z \in \chi(M)$ and R is the curvature tensor of M.

Proof. "$\Longrightarrow$" Let $(T^*M, \overline{J}, \overline{g}) \in \mathcal{W}_3$. Then $\underset{(\overline{X},\overline{Y},\overline{Z})}{\mathfrak{S}} \overline{F}_{(x,\omega)}(\overline{X}, \overline{Y}, \overline{Z}) = 0$ at each point (x, ω) in T^*M and for arbitrary vector fields $\overline{X} = X^C + \alpha^V$, $\overline{Y} = Y^C + \beta^V$, $\overline{Z} = Z^C + \gamma^V$ on T^*M. Taking into account (16) we find

$$\begin{aligned}
&\underset{(X^C+\alpha^V, Y^C+\beta^V, Z^C+\gamma^V)}{\mathfrak{S}} \overline{F}_{(x,\omega)}(X^C + \alpha^V, Y^C + \beta^V, Z^C + \gamma^V) = 0 \\
&\Longleftrightarrow \underset{(X^C,Y^C,Z^C)}{\mathfrak{S}} \overline{F}_{(x,\omega)}(X^C, Y^C, Z^C) + \overline{F}_{(x,\omega)}(X^C, Z^C, \beta^V) \\
&+\overline{F}_{(x,\omega)}(Z^C, X^C, \beta^V) + \overline{F}_{(x,\omega)}(X^C, Y^C, \gamma^V) \\
&+\overline{F}_{(x,\omega)}(Y^C, X^C, \gamma^V) + \overline{F}_{(x,\omega)}(Y^C, Z^C, \alpha^V) + \overline{F}_{(x,\omega)}(Z^C, Y^C, \alpha^V) = 0.
\end{aligned} \tag{27}$$

Replacing Y^C, α^V and γ^V with 0 in (27) we get

$$\overline{F}_{(x,\omega)}(X^C, Z^C, \beta^V) + \overline{F}_{(x,\omega)}(Z^C, X^C, \beta^V) = 0. \tag{28}$$

By using (15) and (28) we obtain (25). If we take $\alpha^V = \beta^V = \gamma^V = 0$ in (27) we have

$$\underset{(X^C,Y^C,Z^C)}{\mathfrak{S}} \overline{F}_{(x,\omega)}(X^C, Y^C, Z^C) = 0. \tag{29}$$

By direct calculations, from (14) and (25), we obtain that (26) is a consequence from (29).

"$\Longleftarrow$" Conversely, let the conditions (25) and (26) be valid. Then (25) and (15) imply (28). With the help of (14), (25) and (26) we obtain (29). Now, having in mind (27)–(29), we complete the proof. □

Theorem 6. *Let (N, J, g) be an almost complex manifold with Norden metric. Then the manifold $(T^*N, \overline{J}, \overline{g})$ is never contained in class $\mathcal{W}_3$.*

Proof. Let us assume that there exists a $\mathcal{W}_3$-manifold $(T^*N, \overline{J}, \overline{g})$ whose base manifold (N, J, g) is an almost complex manifold with a Norden metric. Then, according to Theorem 5, the condition (25) holds. Hence, for arbitrary $X, Y, Z \in \chi(N)$, we have

$$\begin{aligned}
F(X, Z, Y) + F(Z, X, Y) = \frac{b}{2a}\{&\omega(JZ)g(X, Y) \\
&-\omega(Z)g(X, JY) + \omega(JX)g(Y, Z) - \omega(X)g(Y, JZ)\},
\end{aligned}$$

from where we find $\theta(Y) = \dfrac{bn}{2a}\omega(JY)$. Now, since $\overline{\theta} = 0$ for the class $\mathcal{W}_3$, by using (19) we obtain $b = 0$. Thus $F(X, Z, Y) = -F(Z, X, Y)$. Applying Lemma 1 we get $F \equiv 0$. Because (N, J, g) is a Kähler–Norden manifold and $b = 0$, from Theorem 3, it follows that $(T^*N, \overline{J}, \overline{g})$ is also Kähler–Norden, which is a contradiction. □

We will omit the proofs of the following two theorems because one can prove them in a similar manner as Theorems 5 and 6.

Theorem 7. *Let (M, J, ∇) be an almost complex manifold. Then $(T^*M, \overline{J}, \overline{g})$ belongs to the class $\mathcal{W}_2$ if and only if at each point $(x, \omega) \in T^*M$ the following conditions are fulfilled:*

$$(\nabla_X J)Y - (\nabla_Y J)X = \frac{b}{2a}\{\omega(JY)X - \omega(Y)JX + \omega(X)JY - \omega(JX)Y\}, \tag{30}$$

$$\begin{aligned}
&\omega(R(JX, JZ)Y + R(JY, JX)Z + R(JZ, JY)X) \\
&= \frac{b^2}{4a^2}\{\omega(JX)\omega(JY)\omega(Z) + \omega(JY)\omega(JZ)\omega(X) + \omega(JZ)\omega(JX)\omega(Y)\},
\end{aligned} \tag{31}$$

$$\mathrm{trace}(\nabla J)Z = \frac{bn}{a}\omega(JZ), \tag{32}$$

where $X, Y, Z \in \chi(M)$ and R is the curvature tensor of M.

Theorem 8. *Let (N, J, g) be an almost complex manifold with Norden metric with a Lee form θ.*

(i) If $\theta = 0$, then the manifold $(T^*N, \overline{J}, \overline{g})$ is never contained in class $\mathcal{W}_2$.

(ii) If $\theta \neq 0$, then $(T^*N, \overline{J}, \overline{g})$ belongs to the class $\mathcal{W}_2$ if and only if the following conditions are fulfilled:

$$\begin{aligned} &F(X,Y,Z) - F(Y,X,Z) = \\ &\frac{b}{2a}\{\omega(JY)g(X,Z) - \omega(Y)g(X,JZ) + \omega(X)g(Y,JZ) - \omega(JX)g(Y,Z)\}, \end{aligned} \tag{33}$$

$$\begin{aligned} &\omega(R(JX,JZ)Y + R(JY,JX)Z + R(JZ,JY)X) \\ &= \frac{b^2}{4a^2}\{\omega(JX)\omega(JY)\omega(Z) + \omega(JY)\omega(JZ)\omega(X) + \omega(JZ)\omega(JX)\omega(Y)\}, \end{aligned} \tag{34}$$

$$\theta(Z) = \frac{bn}{a}\omega(JZ), \tag{35}$$

where $X, Y, Z \in \chi(N)$ and R is the curvature tensor of N.

By using Lemma 2, Theorems 2, 5, and 7 and the defining condition of the class $\mathcal{W}_2 \oplus \mathcal{W}_3$ we obtain

Theorem 9. *Let (M, J, ∇) and $(T^*M, \overline{J}, \overline{g})$ be as in Theorem 1. Then $\overline{J}$ is harmonic if and only if one of the following groups of conditions are fulfilled: (22) and (20); (25) and (26); (30)–(32).*

Now, taking into account Lemma 2, Theorems 3 and 8, we state

Theorem 10. *Let (N, J, g) be an almost complex manifold with Norden metric. Then $\overline{J}$ is harmonic if and only if either $\overline{g}$ is a Riemann extension and (N, J, g) is a Kähler–Norden manifold or the conditions (33)–(35) are fulfilled and $\overline{g}$ is a proper natural Riemann extension.*

In the rest of this section we will consider the case when the base manifold (M, J, ∇') is complex and ∇' is an almost complex connection on M.

We recall that the linear connection ∇' on an almost complex manifold (M, J) is said to be *almost complex* (see [21]) if the almost complex structure J is parallel with respect to ∇', i.e.,

$$\nabla' J = 0. \tag{36}$$

In Ref. [21] it is also proved that any almost complex manifold M admits an almost complex connection ∇' defined by

$$\nabla'_X Y = \nabla_X Y - \frac{1}{4}\{(\nabla_{JY} J)X + J((\nabla_Y J)X) + 2J((\nabla_X J)Y)\}, \quad X, Y \in \chi(M),$$

where ∇ is an arbitrary symmetric linear connection on M. The curvature tensor R' of an almost complex connection ∇' satisfies the equality

$$R'(X,Y)JZ = JR'(X,Y)Z, \quad X, Y, Z \in \chi(M). \tag{37}$$

From [21], it is known that ∇' is symmetric if and only if the Nijenhuis tensor $\mathcal{N}$ of J vanishes.

Lemma 4. *Let (M, J, ∇') be a complex manifold and ∇' be an almost complex connection on M. For the curvature tensor R' of ∇', the following equalities are valid:*

$$\underset{(X,Y,Z)}{\mathfrak{S}} R'(X,Y)JZ = 0, \tag{38}$$

$$\underset{(X,Y,Z)}{\mathfrak{S}} R'(JX,JY)Z = 0, \tag{39}$$

$$\underset{(X,Y,Z)}{\mathfrak{S}} (R'(JX,Z)Y + R'(X,JZ)Y) = 0. \tag{40}$$

Proof. Since M is a complex manifold, the almost complex connection ∇' is symmetric. Then (38) is an immediate consequence from the first identity of Bianchi and (37). Replacing X, Y and Z in (38) with JX, JY and JZ, respectively, we obtain (39). Finally, from the first identity of Bianchi, we have

$$R'(Y,X)JZ = -R'(X,JZ)Y - R'(JZ,Y)X$$

and two more relations are obtained as a cyclic permutation of X, Y, Z. By adding together the above three equalities and using (38), we get (40). □

With the help of (36), Theorems 5 and 7, (39), and (40), we prove the following

Theorem 11. *Let (M, J, ∇') be a complex manifold and ∇' be an almost complex connection on M. Then we have*

(i) $(T^*M, \overline{J}, \overline{g})$ is a Kähler–Norden manifold if and only if $\overline{g}$ is a Riemann extension and R' satisfies (20).
(ii) $(T^*M, \overline{J}, \overline{g})$ belongs to the class $\mathcal{W}_2$ if and only if $\overline{g}$ is a Riemann extension and R' does not satisfy (20).
(iii) $(T^*M, \overline{J}, \overline{g})$ belongs to the class $\mathcal{W}_3$ if and only if $\overline{g}$ is a Riemann extension and R' satisfies the following equality:

$$R'(JX,Z)Y + R'(JY,X)Z + R'(JZ,Y)X = 0,$$

where R' is the curvature tensor of ∇'.

Theorem 12. *Let (M, J, ∇') be a complex manifold and let ∇' be an almost complex connection on M. Then for the almost complex manifold with Norden metric $(T^*M, \overline{J}, \overline{g})$ the following assertions are equivalent:*

(i) $\overline{J}$ is integrable;
(ii) $\overline{g}$ is a Riemann extension;
(iii) $\overline{J}$ is harmonic.

Proof. $(i) \Longleftrightarrow (ii)$ In Ref. [3] it is shown that the Nijenhuis tensor $\mathcal{N}$ of an almost complex manifold with Norden metric (N, J, g) vanishes identically on N if and only if the condition $F(X,Y,JZ) + F(Y,Z,JX) + F(Z,X,JY) = 0$ holds for any $X, Y, Z \in \chi(N)$.
Let us assume that the almost complex structure $\overline{J}$ is integrable. Then we have

$$\overline{F}_{(x,\omega)}(\overline{X}, \overline{Y}, \overline{JZ}) + \overline{F}_{(x,\omega)}(\overline{Y}, \overline{Z}, \overline{JX}) + \overline{F}_{(x,\omega)}(\overline{Z}, \overline{X}, \overline{JY}) = 0, \tag{41}$$

where $\overline{X}, \overline{Y}, \overline{Z} \in \chi(T^*M)$. Replacing in (41) $\overline{X}$, $\overline{Y}$, and $\overline{Z}$ with X^C, Y^C, and α^V, respectively, we obtain

$$\overline{F}_{(x,\omega)}(X^C, Y^C, \overline{J}\alpha^V) + \overline{F}_{(x,\omega)}(Y^C, \alpha^V, \overline{J}X^C) + \overline{F}_{(x,\omega)}(\alpha^V, X^C, \overline{J}Y^C) = 0. \tag{42}$$

Taking into account (12), (15), (16), and (36), the equality (42) becomes

$$\frac{b}{2}\{\omega(X)\alpha(Y)+\omega(JX)\alpha(JY)-\omega(Y)\alpha(X)-\omega(JY)\alpha(JX)\}=0.$$

The latter equality implies $b=0$, which means that $\overline{g}$ is a Riemann extension. Conversely, let $b=0$. Substituting $b=0$ and (36) in both relations (14) and (15), we get respectively

$$\overline{F}_{(x,\omega)}(X^C,Y^C,Z^C)=a\{\omega(R'_x(Z,JY)X)-\omega(R'_x(JZ,Y)X)\}, \quad (43)$$

$$\overline{F}_{(x,\omega)}(X^C,\alpha^V,Y^C)=\overline{F}_{(x,\omega)}(X^C,Y^C,\alpha^V)=0, \quad (44)$$

where R' is the curvature tensor of ∇'. With the help of (12), (16), (43) and (44), for any $\overline{X},\overline{Y},\overline{Z}\in\chi(T^*M)$, we obtain

$$\overline{F}_{(x,\omega)}(\overline{X},\overline{Y},\overline{JZ})=\overline{F}_{(x,\omega)}(X^C,Y^C,(JZ)^C)$$

$$=a\{\omega(R'_x(JZ,JY)X)+\omega(R'_x(Z,Y)X)\}.$$

Then by using the first identity of Bianchi and (39) one can check that (41) holds. Hence, the Nijenhuis tensor $\overline{N}$ of $(T^*M,\overline{J},\overline{g})$ vanishes identically, i.e., $\overline{J}$ is integrable.

$(ii)\iff(iii)$ According to Lemma 2, $\overline{J}$ is harmonic if and only if $\overline{\theta}=0$. Since $\mathrm{trace}(\nabla J')=0$, from (18) it follows that $\overline{\theta}(\overline{Z})=0$ if and only if $b=0$, which completes the proof. □

5. Cotangent Bundles with Natural Riemann Extensions as Almost Hypercomplex Manifolds with Hermitian-Norden Metrics

An almost hypercomplex structure on a $4n$-dimensional smooth manifold M^{4n} is a triple $H=(J_1,J_2,J_3)$ of almost complex structures having the properties:

$$J_i^2=-\mathrm{Id}\ (i=1,2,3),\quad J_1=J_2\circ J_3=-J_3\circ J_2.$$

A manifold (M^{4n},H), equipped with an almost hypercomplex structure H, is called *an almost hypercomplex manifold* [22]. If J_i $(i=1,2,3)$ are integrable almost complex structures, then (M^{4n},H) is called *a hypercomplex manifold*.

Let g be a pseudo-Riemannian metric on (M^{4n},H), which is Hermitian with respect to J_1 and g is a Norden metric with respect to J_2 and J_3, i.e.,

$$g(J_1X,J_1Y)=-g(J_2X,J_2Y)=-g(J_3X,J_3Y)=g(X,Y),\ \ X,Y\in\chi(M^{4n}). \quad (45)$$

The associated bilinear forms Φ, g_2 and g_3 are determined by

$$\Phi(X,Y)=g(J_1X,Y),\quad g_2(X,Y)=g(J_2X,Y),\quad g_3(X,Y)=g(J_3X,Y). \quad (46)$$

According to (45) and (46), the metric g and the associated bilinear forms g_2 and g_3 are necessarily pseudo-Riemannian metrics of neutral signature $(2n,2n)$ and Φ is the known Kähler 2-form with respect to J_1.

Differentiable manifolds M^{4n} equipped with structures $(H,G)=(J_1,J_2,J_3,g,\Phi,g_2,g_3)$ are studied in Refs. [23–28] under the name *almost hypercomplex pseudo-Hermitian manifolds*, *almost hypercomplex manifolds with Hermitian and anti-Hermitian metrics*, and *almost hypercomplex manifolds with Hermitian–Norden metrics*, respectively. In this paper we refer to (M^{4n},H,G) as *an almost hypercomplex manifold with Hermitian–Norden metrics*.

Let (M^{4n},H) be an almost hypercomplex manifold with an almost hypercomplex structure $H=(J_1,J_2,J_3)$ and a symmetric linear connection ∇. By using (12), we define on the cotangent bundle T^*M^{4n} of (M^{4n},H) the almost complex structures

$$\overline{J}_i:T(T^*M^{4n})\longrightarrow T(T^*M^{4n})\ (i=2,3)$$

$$\bar{J}_i X^C = (J_i X)^C - ((\nabla X)\circ J_i)^V + (\nabla J_i X)^V + \frac{b}{2a} X^V J_i^V - \frac{b}{2a}(J_i X)^V W, \qquad (47)$$
$$\bar{J}_i \alpha^V = (\alpha(J_i))^V,$$

where $X, Y \in \chi(M^{4n})$ and $\alpha \in \Omega^1(M^{4n})$. By standard calculations, taking into account that $J_2 \circ J_3 = -J_3 \circ J_2$, we check that $\bar{J}_2 \circ \bar{J}_3 = -\bar{J}_3 \circ \bar{J}_2$. The latter implies that $\bar{J}_1 = \bar{J}_2 \circ \bar{J}_3$ is an almost complex structure on T^*M^{4n}, given by

$$\bar{J}_1 X^C = (J_1 X)^C + ((\nabla X)\circ J_1)^V + (\nabla J_1 X)^V - \frac{b}{2a} X^V J_1^V - \frac{b}{2a}(J_1 X)^V W, \qquad (48)$$
$$\bar{J}_1 \alpha^V = (\alpha(J_1))^V.$$

Hence, $\overline{H} = (\bar{J}_1, \bar{J}_2, \bar{J}_3)$ is an almost hypercomplex structure on T^*M^{4n} and $(T^*M^{4n}, \overline{H})$ is an almost hypercomplex manifold. According to Theorem 1, the natural Riemann extension $\bar{g}$ on T^*M^{4n} is a Norden metric with respect to $\bar{J}_2$ and $\bar{J}_3$. Then $\bar{g}(\bar{J}_1\overline{X}, \bar{J}_1\overline{Y}) = \bar{g}(\bar{J}_2(\bar{J}_3\overline{X}), \bar{J}_2(\bar{J}_3\overline{Y})) = \bar{g}(\overline{X}, \overline{Y})$, $\overline{X}, \overline{Y}, \overline{Z} \in \chi(T^*M^{4n})$, which means that $\bar{g}$ is a Hermitian metric with respect to $\bar{J}_1$. Let us denote the Kähler 2-form with respect to $\bar{J}_1$ and the Norden metrics with respect to $\bar{J}_i$ with $\overline{\Phi}$ and $\bar{g}_i$ $(i = 2, 3)$, respectively. Then we obtain:

Theorem 13. *Let the total space of the cotangent bundle (T^*M^{4n}) of an almost hypercomplex manifold (M^{4n}, H, ∇) (∇ is a symmetric linear connection) be endowed with the natural Riemann extension $\bar{g}$, defined by (1) and the endomorphisms $\bar{J}_i$ $(i = 1, 2, 3)$, defined by (47), (48). Then $(T^*M^{4n}, \overline{H}, \bar{g}, \overline{\Phi}, \bar{g}_2, \bar{g}_3)$ is an almost hypercomplex manifold with Hermitian–Norden metrics.*

An almost hypercomplex manifold with Hermitian-Norden metrics (M^{4n}, H, G) is called in Ref. [23] a pseudo-hyper-Kähler manifold if $\nabla J_i = 0$ $(i = 1, 2, 3)$ with respect to the Levi–Civita connection of g. It is clear that (M^{4n}, H, G) is pseudo-hyper-Kähler if $F_i(X, Y, Z) = g((\nabla_X J_i)Y, Z) = 0$ $(i = 1, 2, 3)$, i.e., (M^{4n}, H, G) is a Kähler manifold with respect to J_i $(i = 1, 2, 3)$. The relation

$$F_1(X, Y, Z) = F_2(X, J_3 Y, Z) + F_3(X, Y, J_2 Z),$$

obtained in Ref. [23], implies that (M^{4n}, H, G) is pseudo-hyper-Kähler if two of the tensors F_i $(i = 1, 2, 3)$ vanish. Taking into account the latter and Theorem 3, we establish the following:

Theorem 14. *Let (M^{4n}, H, G) be an almost hypercomplex manifold with Hermitian–Norden metrics. Then $(T^*M^{4n}, \overline{H}, \bar{g}, \overline{\Phi}, \bar{g}_2, \bar{g}_3)$ is a pseudo-hyper-Kähler manifold if and only if $\bar{g}$ is a Riemann extension and (M^{4n}, H, G) is a pseudo-hyper-Kähler manifold.*

6. Conclusions

Our framework is the total space of the cotangent bundle, of any manifold endowed with a symmetric linear connection. On this space, Sekizawa-Kowalski constructed a metric of neutral signature, called natural Riemann extension, which generalizes the (classical) Riemann extension, defined by Patterson-Walker. In our paper we construct an almost complex structure which together with the natural Riemann extension becomes an almost complex structure with Norden metric and we classify it according to the classification of almost complex structures with Norden metric obtained by Ganchev-Borisov. Several results provide necessary and sufficient conditions and we also obtain a non-existence result. Then we study the behaviour of such structure for some particular cases of the base manifold, we construct an example and for these particular cases, some harmonic properties are also investigated. At the end we construct an almost hypercomplex structure with a Hermitian-Norden metric on the total space of an almost hypercomplex manifold with a symmetric linear connection. The contribution of our paper is not only to relate some classical structures, but also to create new geometrical structures with interesting properties.

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