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Article

A Novel Projection Method for Cauchy-Type Systems of Singular Integro-Differential Equations

Saeed Althubiti [1,†] and Abdelaziz Mennouni [2,*,†]

1 Department of Mathematics and Statistics, College of Science, Taif University, P.O. Box 11099, Taif 21944, Saudi Arabia; sathubiti@tu.edu.sa
2 Department of Mathematics, LTM, University of Batna 2, Mostefa Ben Boulaïd, Fesdis, Batna 05078, Algeria
* Correspondence: a.mennouni@univ-batna2.dz
† These authors contributed equally to this work.

Abstract: This article introduces a new projection method via shifted Legendre polynomials and an efficient procedure for solving a system of integro-differential equations of the Cauchy type. The proposed computational process solves two systems of linear equations. We demonstrate the existence of the solution to the approximate problem and conduct an error analysis. Numerical tests provide theoretical results.

Keywords: Cauchy type; projection approach; integro-differential system; shifted Legendre polynomials

MSC: 45E05; 45F15; 45F05; 47G20

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1. Introduction

Numerous scientific domains, including hydrodynamics, biology, electromagnetism, and elasticity rely heavily on operator equations theory. Integro-differential and integral equations of Cauchy type are a subset of these equations.

Much research has recently been conducted on using ordinary differential, integro-differential, and integral equations in mathematical physics (see [1–9]). An essential class of these problems is singular integro-differential equations with boundary conditions (cf. [10–16]).

The authors of [17] investigated a category of integro-differential equations with variable-order. They suggested a new approach based on fifth-order shifted Chebyshev polynomials. In [18], the superconvergence error estimate for integro-differential equations of semilinear parabolic type without time-step constraints is generated by spatial discretization with a bilinear element and temporal discretization with a modeled backward Euler formula.

Projection approximation methods are fundamental to approximation theory and have a variety of uses, including the solution of integro-differential and integral equations. The authors of [19] focused on several finite rank approximations with bounded, limited rank projections. For solving second-kind singular Fredholm integral equations with weak singularities, the authors employed a projection approximation in [20]. A projection approximation for solving integro-differential problems of Cauchy type using first-order airfoil polynomials is investigated in [21].

Recently, the authors of [22] demonstrated how to solve fuzzy integro-differential equations with weak singularities via airfoil polynomials. In [23], Mennouni introduced a novel projection approach based on Legendre polynomials for examining integro-differential equations with a Cauchy type on $L^2([-1,1],\mathbb{C})$.

Several results for solving compact operator equations using the Galerkin and Kulkarni methods have been established over the last two decades. These two methods are used in [24] to approach the solution of the following bounded equation:

$$u - Bu = f.$$

The author considered an approximate equation of the form:

$$u_n^G - B_n u_n^G = \Pi_n f.$$

To begin, the author regarded the approximate operator B_n as a Galerkin one:

$$B_n^G := \Pi_n B \Pi_n,$$

for some sequence of bounded projections $(\Pi_n)_{n\geq 1}$.

Secondly, he employed Kulkarni's approximation as follows:

$$B_n^K := \Pi_n B + B\Pi_n - \Pi_n B \Pi_n.$$

This work presents a new projection method for solving the following system of Cauchy integro-differential equations on $L^2([0,1],\mathbb{C})$

$$\begin{cases} v'(s) + \oint_0^1 \frac{u(\eta)}{\eta - s} d\eta = f(s), & 0 \leq s \leq 1, \\ u'(s) + \oint_0^1 \frac{v(\eta)}{\eta - s} d\eta = g(s), & 0 \leq s \leq 1, \\ u(0) = 0, \ \ v(0) = 0. \end{cases}$$

We will turn the problem into a system of two separate equations that looks like this:

$$\begin{cases} \mathcal{A}\mathcal{U} + \mathcal{T}\mathcal{U} = F, \\ \mathcal{A}\mathcal{V} - \mathcal{T}\mathcal{V} = G. \end{cases}$$

Unlike previously, we use the shifted Legendre polynomial and introduce an approximation system of the form:

$$\begin{cases} \mathcal{U}_n + \mathcal{A}^{-1}\mathcal{P}_n^S \mathcal{T}\mathcal{U}_n = \mathcal{A}^{-1}\mathcal{P}_n^S \mathcal{T} F, \\ \mathcal{V}_n - \mathcal{A}^{-1}\mathcal{P}_n^S \mathcal{T}\mathcal{V}_n = \mathcal{A}^{-1}\mathcal{P}_n^S \mathcal{T} G. \end{cases}$$

The results from the introduced computational technique were used to solve two systems of linear equations. We show that the approximation equation has a solution, and we conduct an error analysis. Numerical examples illustrate the theories.

Other sections of this study are described as follows: The system of logarithmic integro-differential equations is covered in the next section. Section 3 discusses some key aspects of shifted Legendre polynomials and the development of the method. Section 6 improves the convergence of the approximate solution and estimates the error analysis. Section 5 explores some numerical tests.

2. Cauchy-Type Systems of Singular Integro-Differential Equations

Let $\mathcal{H} := L^2([0,1],\mathbb{C})$ be the space of complex-valued Lebesgue square integrable functions on $[0,1]$. The goal of this paper is to introduce a new projection method that uses shifted Legendre polynomials to solve Cauchy-type systems of singular integro-differential equations in $\mathcal{H}$.

Consider the following Cauchy-type system of singular integro-differential equations:

$$\begin{cases} v'(s) + \oint_0^1 \frac{u(\eta)}{\eta - s} d\eta = f(s), & 0 \leq s \leq 1, \\ u'(s) + \oint_0^1 \frac{v(\eta)}{\eta - s} d\eta = g(s), & 0 \leq s \leq 1, \\ u(0) = 0, \ \ v(0) = 0. \end{cases} \tag{1}$$

Both integrals denote the main value of Cauchy:

$$\oint_0^1 \frac{v(\eta)}{\eta - s} d\eta = \lim_{\varepsilon\to 0}\left(\int_0^{s-\varepsilon} \frac{v(\eta)}{\eta - s} d\eta + \int_{s+\varepsilon}^1 \frac{v(\eta)}{\eta - s} d\eta\right),$$
$$\oint_0^1 \frac{u(\eta)}{\eta - s} d\eta = \lim_{\varepsilon\to 0}\left(\int_0^{s-\varepsilon} \frac{u(\eta)}{\eta - s} d\eta + \int_{s+\varepsilon}^1 \frac{u(\eta)}{\eta - s} d\eta\right).$$

We follow [25] in letting

$$\mathcal{V} := u - v, \quad F := g + f;$$

$$\mathcal{U} := u + v, \quad G := g - f.$$

Lemma 1. *Problem (1) can be expressed in the form:*

$$\begin{cases} \mathcal{U}'(s) + \oint_0^1 \frac{\mathcal{U}(\eta)}{\eta - s} d\eta = F(s), & 0 \le s \le 1, \\ \mathcal{V}'(s) - \oint_0^1 \frac{\mathcal{V}(\eta)}{\eta - s} d\eta = G(s), & 0 \le s \le 1, \\ \mathcal{U}(0) = 0, \ \mathcal{V}(0) = 0. \end{cases} \tag{2}$$

Proof. We note that

$$u = \frac{\mathcal{U} + \mathcal{V}}{2}, \quad f = \frac{F - G}{2};$$
$$v = \frac{\mathcal{U} - \mathcal{V}}{2}, \quad g = \frac{F + G}{2}.$$

Substituting these into (1), we obtain

$$(\mathcal{U} - \mathcal{V})'(s) + \oint_0^1 \frac{(\mathcal{U} + \mathcal{V})(\eta)}{\eta - s} d\eta = (F - G)(s), \tag{3}$$
$$(\mathcal{U} + \mathcal{V})'(s) + \oint_0^1 \frac{(\mathcal{U} - \mathcal{V})(\eta)}{\eta - s} d\eta = (F + G)(s). \tag{4}$$

By adding Equations (3) and (4) together, and by subtracting (3) from (4), we obtain (2). □

System (2) can be rewritten in operator form as follows:

$$\begin{cases} \mathcal{A}\mathcal{U} + \mathcal{T}\mathcal{U} = F, \\ \mathcal{A}\mathcal{V} - \mathcal{T}\mathcal{V} = G, \end{cases}$$

where $\mathcal{T}$ is the Cauchy integral operator

$$\mathcal{T}\varphi(s) := \oint_0^1 \frac{\varphi(\eta)}{\eta - s} d\eta, \quad 0 \le s \le 1,$$

and $\mathcal{A}$ is the differential operator

$$\mathcal{A}\varphi(s) := \varphi'(s), \quad 0 \le s \le 1,$$

with domain

$$\mathcal{D} := \{\varphi \in \mathcal{H} : \ \varphi' \in \mathcal{H}, \quad \varphi(0) = 0\}.$$

Following [23], the operator $\mathcal{T}$ is bounded from $\mathcal{H}$ into itself.

It is well known that the operator $\mathcal{A}$ is invertible, and its inverse is the following Voletrra integral operator

$$(\mathcal{A}^{-1}y)(s) = \int_0^s y(t)dt.$$

Moreover, $\mathcal{A}^{-1}\colon \mathcal{H} \to \mathcal{D}$ is compact.

3. Shifted Legendre Polynomials and Development of the Method

The well-known Legendre polynomials represented on $[-1, 1]$ are described by the following recurrence

$$\begin{aligned} L_0(\sigma) =& 1, \quad L_1(\sigma) = \sigma, \\ L_{j+1}(\sigma) =& \frac{2j+1}{j+1}\sigma L_j(\sigma) - \frac{j}{j+1}L_{j-1}(\sigma), \quad j = 1, 2, 3, \ldots \end{aligned}$$

The first Legendre polynomials are given below.

$$\begin{aligned} L_0(\sigma) &= 1; \\ L_1(\sigma) &= \sigma; \\ L_2(\sigma) &= \frac{1}{2}\Big(3\sigma^2 - 1\Big); \\ L_3(\sigma) &= \frac{1}{2}\Big(5\sigma^3 - 3\sigma\Big); \\ L_4(\sigma) &= \frac{1}{8}\Big(35\sigma^4 - 30\sigma^2 + 3\Big); \\ L_5(\sigma) &= \frac{1}{8}\Big(63\sigma^5 - 70\sigma^3 + 15\sigma\Big). \end{aligned}$$

Let the shifted Legendre polynomials $L_i(2\sigma - 1)$ be denoted by $L_i^S(\sigma)$, $\sigma \in [0, 1]$. We recall that

$$\int_0^1 L_n^S(\sigma) L_m^S(\sigma) d\sigma = \begin{cases} \dfrac{1}{2n+1} & n = m, \\ 0 & n \neq m. \end{cases} \tag{5}$$

Let

$$\Lambda_p := \sqrt{2p+1} L_p^S, \quad p = 0, 2, 3, \ldots$$

denote the corresponding normalized sequence. Let $(\mathcal{P}_n^S)_{n \geq 0}$ be the chain of bounded finite rank orthogonal projections described by

$$\mathcal{P}_n^S \psi := \sum_{j=0}^{n-1} \langle \psi, \Lambda_j \rangle \Lambda_j, \quad \text{where} \quad \langle \psi, \Lambda_j \rangle := \int_0^1 \psi(\sigma) \Lambda_j(\sigma) d\sigma.$$

Denote by $\|.\|$ the corresponding norm on $\mathcal{H}$. Thus,

$$\lim_{n \to \infty} \|\mathcal{P}_n^S \vartheta - \vartheta\| = 0, \quad \text{for all} \quad \vartheta \in \mathcal{H}.$$

Let $\mathcal{H}_n$ represent the space covered by the first n-shifted Legendre polynomials. It is obvious that $\mathcal{A}^{-1}(\mathcal{H}_n) = \mathcal{H}_{n+1}$. The approximate solution $(\mathcal{U}_n, \mathcal{V}_n)$ solves the following system:

$$\begin{cases} \mathcal{U}_n + \mathcal{A}^{-1}\mathcal{P}_n^S \mathcal{T} \mathcal{U}_n = \mathcal{A}^{-1}\mathcal{P}_n^S \mathcal{T} F, \\ \mathcal{V}_n - \mathcal{A}^{-1}\mathcal{P}_n^S \mathcal{T} \mathcal{V}_n = \mathcal{A}^{-1}\mathcal{P}_n^S \mathcal{T} G. \end{cases}$$

We note that $\mathcal{U}_n, \mathcal{V}_n \in \mathcal{D} \cap \mathcal{H}_{n+1}$. Thus, the system

$$\begin{cases} \mathcal{U} + \mathcal{A}^{-1}\mathcal{T}\mathcal{U} = \mathcal{A}^{-1}F, \\ \mathcal{V} - \mathcal{A}^{-1}\mathcal{T}\mathcal{V} = \mathcal{A}^{-1}G \end{cases}$$

is approximated by

$$\begin{cases} \mathcal{U}_n + \mathcal{A}^{-1}\mathcal{P}_n^S \mathcal{T} \mathcal{U}_n = \mathcal{A}^{-1}\mathcal{P}_n^S F, \\ \mathcal{V}_n - \mathcal{A}^{-1}\mathcal{P}_n^S \mathcal{T} \mathcal{V}_n = \mathcal{A}^{-1}\mathcal{P}_n^S G. \end{cases}$$

We assume that -1 and 1 are not eigenvalues of $\mathcal{A}^{-1}\mathcal{T}$. Thus, both operators $I + \mathcal{A}^{-1}\mathcal{T}$ and $I - \mathcal{A}^{-1}\mathcal{T}$ are invertible.

We recall that $\mathcal{A}^{-1}$ is compact, and

$$\lim_{n\to\infty} \|\mathcal{A}^{-1}\mathcal{P}_n^S \mathcal{T} x - \mathcal{A}^{-1}\mathcal{T} x\| = 0, \text{ for all } x \in \mathcal{H}.$$

This implies that

$$\lim_{n\to\infty} \|\left(\mathcal{A}^{-1}\mathcal{P}_n^S \mathcal{T} - \mathcal{A}^{-1}\mathcal{T}\right)\mathcal{A}^{-1}\mathcal{T}\| = 0, \quad \lim_{n\to\infty} \|\left(\mathcal{A}^{-1}\mathcal{P}_n^S \mathcal{T} - \mathcal{A}^{-1}\mathcal{T}\right)\mathcal{A}^{-1}\mathcal{P}_n^S\mathcal{T}\| = 0.$$

Writing

$$\begin{cases} \mathcal{U}_n = \sum_{j=0}^{n} x_{n,j}\Lambda_j, \\ \mathcal{V}_n = \sum_{j=0}^{n} y_{n,j}\Lambda_j. \end{cases}$$

We obtain the $2n+2$ unknowns $x_n(j)$ and $y_n(j)$ by solving the two separate linear systems

$$\begin{cases} \sum_{j=0}^{n} x_n(j)\left[\Lambda_j' + \mathcal{P}_n^S T \Lambda_j\right] = \mathcal{P}_n^S \mathcal{A}^{-1} F, \text{ with } \sum_{j=0}^{n} x_n(j)\Lambda_j(0) = 0, \\ \sum_{j=0}^{n} y_n(j)\left[\Lambda_j' - \mathcal{P}_n^S T \Lambda_j\right] = \mathcal{P}_n^S \mathcal{A}^{-1} G, \text{ with } \sum_{j=0}^{n} y_n(j)\Lambda_j(0) = 0. \end{cases}$$

As a result, two separate linear systems are produced:

$$\begin{cases} A_n x_n = b_n, \\ \widehat{A}_n y_n = \widehat{b}_n, \end{cases}$$

where, for $i = 0 \cdots n-1$ and $j = 0 \cdots n$,

$$A_n(i,j) := \sqrt{(2i+1)(2j+1)}\left[\int_0^1 L_j^{S'}(s) L_i(s) ds + \int_0^1 \left(\oint_0^1 \frac{L_j^S(\sigma)}{\sigma - s} d\sigma\right) L_i^S(s) ds\right],$$

$$\widehat{A}_n(i,j) := \sqrt{(2i+1)(2j+1)}\left[\int_0^1 L_j^{S'}(s) L_i^S(s) ds - \int_0^1 \left(\oint_0^1 \frac{L_j^S(\sigma)}{\sigma - s} d\sigma\right) L_i^S(s) ds\right],$$

$$A_n(n,j) := \Lambda_j(0), \quad \widehat{A}_n(n,j) := \Lambda_j(0),$$

$$b_n(i) := \sqrt{2i+1}\int_0^1 F(s) L_i^S(s) ds, \quad \widehat{b}_n(i) := \sqrt{2i+1}\int_0^1 G(s) L_i^S(s) ds,$$

$$b_n(n) := 0, \quad \widehat{b}_n(n) := 0.$$

4. Convergence Analysis

We will now show how the current method converges. To that end, consider $H^s(0,1)$ to be the classical Sobolev space for some $s > 0$, and $\|.\|_s$ to be its norm.

Remark that
$$(I + \mathcal{A}^{-1}\mathcal{T})(H^s([0,1],\mathbb{C})) = H^s([0,1],\mathbb{C}).$$

Also,
$$(I - \mathcal{A}^{-1}\mathcal{T})(H^s([0,1],\mathbb{C})) = H^s([0,1],\mathbb{C}).$$

We recall that there exists $c > 0$ such that
$$\left\|(I - \mathcal{P}_n^S)\psi\right\| \le cn^{-s}\|\psi\|_s, \quad \text{for all } \ \psi \in H^s([0,1],\mathbb{C}).$$

Because $\mathcal{A}^{-1}\mathcal{T}$ is compact, according to [23], the operators $(I + \mathcal{A}^{-1}\mathcal{P}_n^S\mathcal{T})^{-1}$ and $(I - \mathcal{A}^{-1}\mathcal{P}_n^S\mathcal{T})^{-1}$ exist for n large enough and are uniformly bounded with respect to n.

Theorem 1. *Assume that $f, g \in H^s([0,1],\mathbb{C})$ for some $s > 0$. Then, there exist $\alpha, \beta > 0$ such that*
$$\|\mathcal{U}_n - \mathcal{U}\| \le \alpha[n^{1-s}\|\mathcal{T}(u+v)\|_{s-1} + n^{-s}\|(g+f)\|_s],$$
and
$$\|\mathcal{V}_n - \mathcal{V}\| \le \beta[n^{1-s}\|\mathcal{T}(u-v)\|_{s-1} + n^{-s}\|(g-f)\|_s].$$

Proof. In fact,
$$\begin{cases} \mathcal{U}_n + \mathcal{A}^{-1}\mathcal{P}_n^S\mathcal{T}\mathcal{U}_n = \mathcal{A}^{-1}\mathcal{P}_n^S F, \\ \mathcal{V}_n - \mathcal{A}^{-1}\mathcal{P}_n^S\mathcal{T}\mathcal{V}_n = \mathcal{A}^{-1}\mathcal{P}_n^S G. \end{cases}$$

As in [23], we have
$$\begin{aligned} \mathcal{U}_n - \mathcal{U} = &\left[\left(I + \mathcal{A}^{-1}\mathcal{P}_n^S\mathcal{T}\right)^{-1}\mathcal{A}^{-1}\mathcal{P}_n^S F - \left(I + \mathcal{A}^{-1}\mathcal{T}\right)^{-1}\mathcal{A}^{-1}F\right] \\ &+ \left(I + \mathcal{A}^{-1}\mathcal{P}_n^S\mathcal{T}\right)^{-1}\mathcal{A}^{-1}F - \left(I + \mathcal{A}^{-1}\mathcal{P}_n^S\mathcal{T}\right)^{-1}\mathcal{A}^{-1}F \\ &= \left(I + \mathcal{A}^{-1}\mathcal{P}_n^S\mathcal{T}\right)^{-1}\left[\left(\mathcal{A}^{-1}\mathcal{T} - \mathcal{A}^{-1}\mathcal{P}_n^S\mathcal{T}\right)\mathcal{U} + \mathcal{A}^{-1}\mathcal{P}_n^S F - \mathcal{A}^{-1}F\right]. \end{aligned}$$

Further,
$$\begin{aligned} \mathcal{V}_n - \mathcal{V} = &\left[\left(I - \mathcal{A}^{-1}\mathcal{P}_n^S\mathcal{T}\right)^{-1}\mathcal{A}^{-1}\mathcal{P}_n^S G - \left(I - \mathcal{A}^{-1}\mathcal{T}\right)^{-1}\mathcal{A}^{-1}G\right] \\ &+ \left(I - \mathcal{A}^{-1}\mathcal{P}_n^S\mathcal{T}\right)^{-1}\mathcal{A}^{-1}G - \left(I - \mathcal{A}^{-1}\mathcal{P}_n^S\mathcal{T}\right)^{-1}\mathcal{A}^{-1}G \\ &= \left(I - \mathcal{A}^{-1}\mathcal{P}_n^S\mathcal{T}\right)^{-1}\left[\left(\mathcal{A}^{-1}\mathcal{P}_n^S\mathcal{T} - \mathcal{A}^{-1}T\right)\mathcal{V} + \mathcal{A}^{-1}\mathcal{P}_n^S G - \mathcal{A}^{-1}G\right]. \end{aligned}$$

However,
$$\left(\mathcal{A}^{-1}\mathcal{T} - \mathcal{A}^{-1}\mathcal{P}_n^S\mathcal{T}\right)\mathcal{U} = \mathcal{A}^{-1}\left(I - \mathcal{P}_n^S\right)\mathcal{T}\mathcal{U},$$

and
$$\left(\mathcal{A}^{-1}\mathcal{P}_n^S\mathcal{T} - \mathcal{A}^{-1}\mathcal{T}\right)\mathcal{V} = \mathcal{A}^{-1}\left(\mathcal{P}_n^S - I\right)\mathcal{T}\mathcal{V}.$$

Since $f, g \in H^s([0,1],\mathbb{C})$, we get $\mathcal{U}, \mathcal{V} \in H^s([0,1],\mathbb{C})$ and $\mathcal{T}\mathcal{U}, \mathcal{T}\mathcal{V} \in H^{s-1}([0,1],\mathbb{C})$. Moreover, we have
$$u_n - u = \frac{\mathcal{U}_n - \mathcal{U}}{2} + \frac{\mathcal{V}_n - \mathcal{V}}{2}, \quad v_n - v = \frac{\mathcal{U}_n - \mathcal{U}}{2} + \frac{\mathcal{V} - \mathcal{V}_n}{2}.$$

Hence,

$$\begin{aligned}\mathcal{U}_n - \mathcal{U} &= \left(I + \mathcal{A}^{-1}\mathcal{P}_n^S\mathcal{T}\right)^{-1}\left[\mathcal{A}^{-1}\left(I - \mathcal{P}_n^S\right)\mathcal{T}\mathcal{U} + \mathcal{A}^{-1}\left(I - \mathcal{P}_n^S\right)F\right] \\ &= \left(I + \mathcal{A}^{-1}\mathcal{P}_n^S\mathcal{T}\right)^{-1}\mathcal{A}^{-1}\left[\left(I - \mathcal{P}_n^S\right)\mathcal{T}(u+v) + \left(I - \mathcal{P}_n^S\right)(g+f)\right],\end{aligned}$$

and

$$\begin{aligned}\mathcal{V}_n - \mathcal{V} &= \left(I - \mathcal{A}^{-1}\mathcal{P}_n^S\mathcal{T}\right)^{-1}\left[\mathcal{A}^{-1}\left(I - \mathcal{P}_n^S\right)\mathcal{T}\mathcal{V} + \mathcal{A}^{-1}\left(I - \mathcal{P}_n^S\right)G\right] \\ &= \left(I - \mathcal{A}^{-1}\mathcal{P}_n^S\mathcal{T}\right)^{-1}\mathcal{A}^{-1}\left[\left(I - \mathcal{P}_n^S\right)\mathcal{T}(u-v) + \left(I - \mathcal{P}_n^S\right)(g-f)\right].\end{aligned}$$

Thus,

$$\begin{aligned}\|\mathcal{U}_n - \mathcal{U}\| &\le \left\|\left(I + \mathcal{A}^{-1}\mathcal{P}_n^S\mathcal{T}\right)^{-1}\right\|\left\|\mathcal{A}^{-1}\right\|\left[\left\|\left(I - \mathcal{P}_n^S\right)\mathcal{T}(u+v)\right\| + \left\|\left(I - \mathcal{P}_n^S\right)(g+f)\right\|\right] \\ &\le \alpha\left[\left\|\left(I - \mathcal{P}_n^S\right)\mathcal{T}(u+v)\right\| + \left\|\left(I - \mathcal{P}_n^S\right)(g+f)\right\|\right]\end{aligned}$$

and

$$\begin{aligned}\|\mathcal{V}_n - \mathcal{V}\| &= \left\|\left(I - \mathcal{A}^{-1}\mathcal{P}_n^S\mathcal{T}\right)^{-1}\left[\mathcal{A}^{-1}\left(I - \mathcal{P}_n^S\right)\mathcal{T}\mathcal{V} + \mathcal{A}^{-1}\left(I - \mathcal{P}_n^S\right)G\right]\right\| \\ &\le \left\|\left(I - \mathcal{A}^{-1}\mathcal{P}_n^S\mathcal{T}\right)^{-1}\right\|\left\|\mathcal{A}^{-1}\right\|\left[\left\|\left(I - \mathcal{P}_n^S\right)\mathcal{T}(u-v)\right\| + \left\|\left(I - \mathcal{P}_n^S\right)(g-f)\right\|\right].\end{aligned}$$

The desired result follows. □

5. Numerical Example

We establish some numerical tests in this section to highlight the theoretical results described above. In these numerical tests, the Maple programming language was used.

Through this example, we consider the integro-differential system (1), which has the exact solution as follows:

$$u(s) = \frac{s+s^2}{2(s^2+1)}, \quad v(s) = \frac{s-s^2}{2(s^2+1)}.$$

In this instance, we obtain

$$\mathcal{U}(s) = \frac{s}{s^2+1}, \quad \mathcal{V}(s) = \frac{s^2}{s^2+1}.$$

The way of connecting errors for this example is shown in Table 1.

Figures 1 and 2 compare the exact and approximate solutions at $n = 10$.

Table 1. Example 1.

n	$\|\mathcal{U} - \mathcal{U}_n\|_2$	$\|\mathcal{V} - \mathcal{V}_n\|_2$
3	3.182×10^{-3}	4.100×10^{-3}
5	4.555×10^{-5}	1.776×10^{-4}
9	1.000×10^{-6}	3.505×10^{-7}
13	4.472×10^{-8}	2.264×10^{-13}
18	4.960×10^{-19}	3.06×10^{-20}

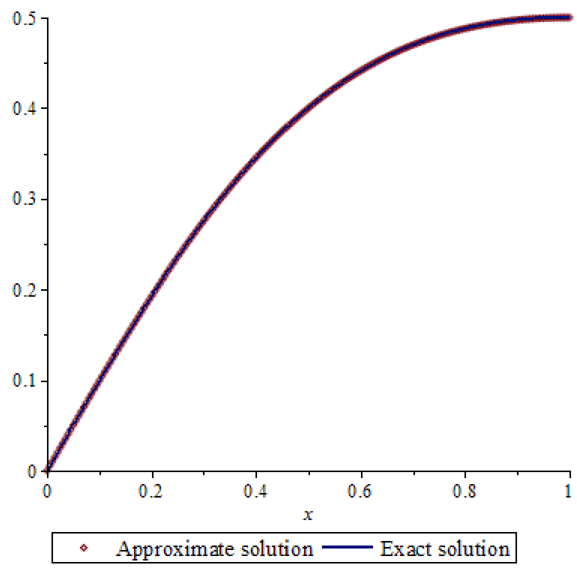

Figure 1. The approach solution $\mathcal{U}_n$ for $n = 10$.

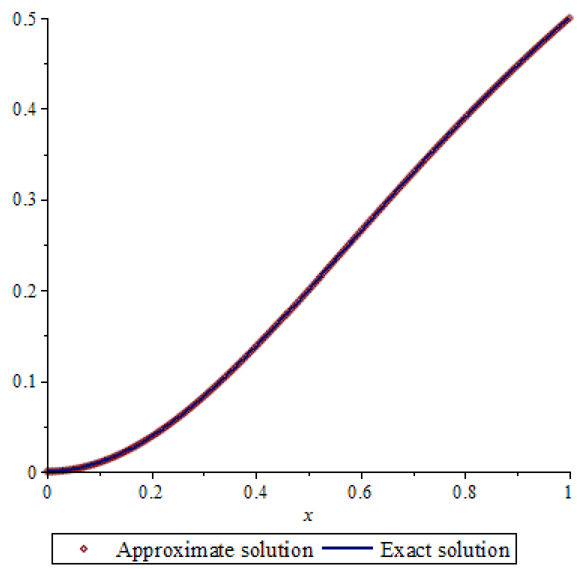

Figure 2. The approach solution $\mathcal{V}_n$ for $n = 10$.

Allow us to illustrate our method by providing some examples of approximate solutions. For instance, when $n = 5$, the approximate solutions are as follows:

$$\begin{aligned} x_{5,0} &= 0.271, \quad x_{5,1} = -0.169 \times 10^{-1}, \\ x_{5,2} &= 0.337, \quad x_{5,3} = -0.277, \\ x_{5,4} &= 0.933 \times 10^{-1}, \quad x_{5,5} = -0.125 \times 10^{-1}. \end{aligned}$$

$$\begin{aligned} \mathcal{U}_5(s) &= 2.10^{-30} + 0.995s + 0.804 \times 10^{-1}s^2 \\ &- 1.474s^3 + 1.225s^4 - 0.327s^5. \end{aligned}$$

$$
\begin{aligned}
y_{5,0} &= 0.294, \quad y_{5,1} = -0.106, \\
y_{5,2} &= 0.234, \quad y_{5,3} = -0.300 \times 10^{-1}, \\
y_{5,4} &= -0.288 \times 10^{-1}, \quad y_{5,5} = 0.921 \times 10^{-2}.
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{V}_5(s) &= 10^{-30} - 0.733 \times 10^{-2}s + 1.110s^2 \\
&- 0.465s^3 - 0.378s^4 + 0.240s^5.
\end{aligned}
$$

The reported results show that the stated method is extremely accurate in analyzing this example, as seen in Figures 3 and 4. Observe that the first 18 terms were examined for the performance of the described procedure.

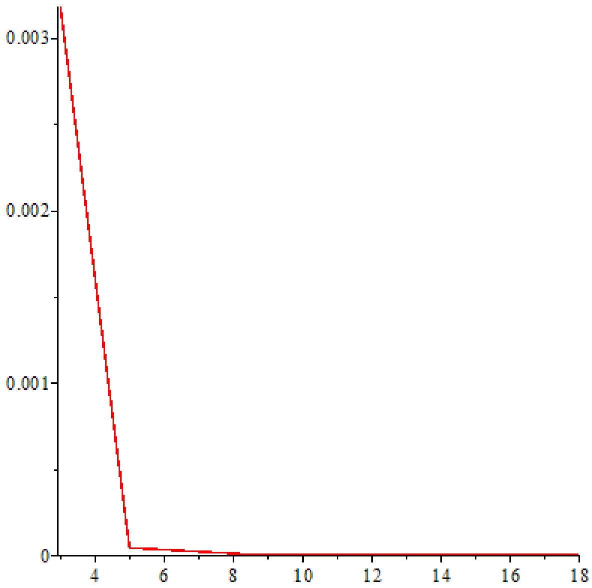

Figure 3. Presentation of $\|\mathcal{U} - \mathcal{U}_n\|_2$ with $n = 18$.

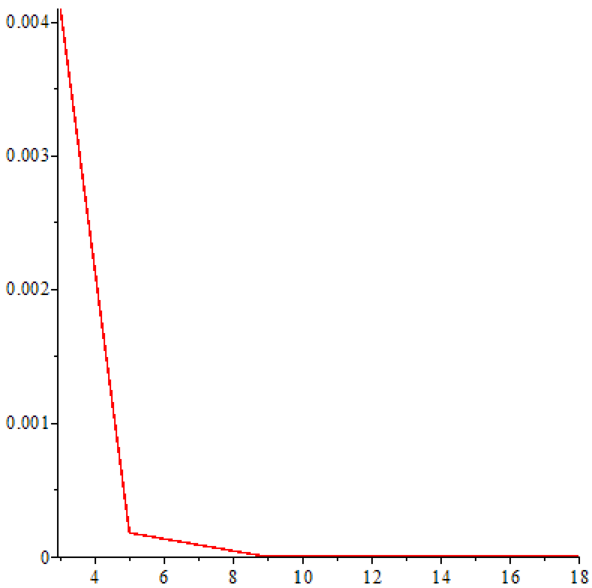

Figure 4. Presentation of $\|\mathcal{V} - \mathcal{V}_n\|_2$ with $n = 18$.

6. Conclusions

Many works have been published on the numerical solution of integro-differential equations using various approximation procedures. The application of projection methods to the Cauchy singular integro-differential system is extended in this paper. A series of orthogonal finite-rank Legendre polynomial projections build the modified projection method. In the Sobolev space, we demonstrated the convergence of the approach solution to the exact one. Numerical experiments demonstrate the utility of our method. Other classes of integro-differential and integral systems can be studied and solved using this method.

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