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Representing Functions in H^2 on the Kepler Manifold via WPOAFD Based on the Rational Approximation of Holomorphic Functions

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Abstract: The central problem of this study is to represent any holomorphic and square integrable function on the Kepler manifold in the series form based on Fourier analysis. Because these function spaces are reproducing kernel Hilbert spaces (RKHS), three different domains on the Kepler manifold are considered and the weak pre-orthogonal adaptive Fourier decomposition (POAFD) is proposed on the domains. First, the weak maximal selection principle is shown to select the coefficient of the series. Furthermore, we prove the convergence theorem to show the accuracy of our method. This study is the extension of work by Wu et al. on POAFD in Bergman space.

Keywords: algebraic varieties; Kepler manifold; reproducing kernel Hilbert space

MSC: 14M12; 30E10; 32A50; 46E22

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1. Introduction

Algebraic varieties which describe solutions of a system of polynomial equations are extremely important because they are the fundamental objectives in the field of algebraic geometry [1]. For example, the Jacobian variety [2] and moduli varieties [3] are two classical cases of algebraic varieties. In addition, the term "algebraic manifold" defines special kinds of algebraic varieties which are smooth manifolds of dimension n themselves except for some singular points. The Kepler manifold arises from the Jordan–Kepler variety, which is a class of algebraic variety on the framework of Jordan theoretic terms. In addition, the classical Kepler manifold is a submanifold of $\mathbb{C}^{n+1}$ and is a peculiar example of the Jordan–Kepler variety. In general, there are many measures one must be equipped with to understand the Kepler manifold.

To determine the measure associated with a manifold is of great importance. Meanwhile, there are also many measures to be selected according to different problems in the field of mathematics, physics, and engineering. There is the Riemann measure and other measures relative to Kähler potentials focused on the Kepler manifold in the previous research [4] which are K-invariant and have nice polar decompositions. In addition, under these measures, one can focus on the function space consisting of holomorphic (analytic) and square integrable functions i.e., the weighted Bergman space on the given manifold. It is well-known that, if a Hilbert space has a reproducing kernel, the function space will be an RKHS. Furthermore, if one decides a set of n reproducing kernels associated with n distinct points on the given Kepler manifold, the set is dense in the weighted Bergman space, and every holomorphic function on the Kepler manifold can be represented in the form of combinations of kernels in this set. A new decomposition of holomorphic and square integrable functions have been developed on the Kepler manifold replacing Hua–Schmid–Kostant decomposition [5] and Peter–Weyl expansion [6] previously studied.

The main problem is to explore a novel decomposition of any function, which is on the Kepler manifold and is square integrable i.e., $f \in H^2$. The motivation of this study is to extend the POAFD and weak pre-orthogonal adaptive Fourier decomposition (WPOAFD) proposed by Qian [7] initially in Hardy space on the unit disc, which is applicable in signal processing. For Bergman space, Qu et al. [8] studied functions on the unit disc and unit ball; Wu et al. [9] generalized it to the symmetry bounded domain with the kernel proposed by Hua [10]. In numerical analysis, Song has proposed the WPOAFD method for the Helmholtz equation [11]. The RKHS method is applied to fractional partial differential equations [12] and shows the potential to perform well compared with the finite difference method, the finite element method, and the finite volume method. There is some other work, please see [13–15] for details.

In this study, we generalize the WPOAFD method to the weighted Bergman space on the Kepler manifold associated with the smooth measure, measures with Kähler potentials, and the rotation measure. In addition, for the Kepler ball, the POAFD is also studied. In addition, the convergence of this method is shown. The decomposition allows an infinite series sequentially determined by the orthonormal sequence by the so-called weak maximal selection principle.

The organization of this paper is as follows: in the first section, we sketch the procedure of this study and show the main results. After that, some preliminaries are reviewed including basic definitions, K-invariant measures and the corresponding reproducing kernels. The weak maximal selection principle is proved in Section 4 and the convergence is shown in Section 5. In addition, applications are given in Section 6, and conclusions are drawn in Section 7.

2. Brief Procedure of This Study

In this section, a brief motivation and procedure of our main results is introduced, and rigorous proof will be given in the subsequent section.

Before the presentation of the procedure, it is necessary to define

$$H^2(\mathring{Z}_r, \mathbf{d}\rho) = \left\{ f \text{ is holomorphic on } \mathring{Z}_r \middle| \int_{\mathring{Z}} f^2(z)\mathbf{d}\rho < \infty \right\},$$

where

$$\mathring{Z}_r := \{z \in Z | R(z) = r\}.$$

(Further definitions in Equations (6)–(9) will be reviewed in Section 3.2.)

Engliš et al. [4] constructed the reproducing kernel $k_w(z) = K(z, w)$ for $H^2(\mathring{Z}_r, \mathbf{d}\rho)$ for $z, w \in \mathring{Z}_r$

$$K(z,w) = \sum_{m \in Z^r_+} \frac{H^m(z,w)}{\int_\Omega N_m \mathbf{d}\widetilde{\rho}}. \tag{1}$$

where N_m ($|m| := m_1 + \cdots + m_r$) is the highest weight polynomial

$$N_m = N_1^{m_1-m_2} N_2^{m_2-m_3} \cdots N_r^{m_r};$$
$$H_m(z,w) = (\frac{d}{r})_m E^m(z,w).$$

$E^m(z,w)$ is the Fischer–Fock reproducing kernel for the Peter–Weyl space; $(\cdot)_m$ is the generalized Pochhammer symbol (please see [4] (Section 4)).

The reproducing property $f(w) = \langle f, K(\cdot, w)\rangle$ for any $f \in H^2(\mathring{Z}_r, \mathbf{d}\rho)$ is satisfied. Therefore, one can say the Hilbert space $H^2(\mathring{Z}_r, \mathbf{d}\rho)$ is an RKHS with reproducing kernel $k_w(z)$ in Equation (1).

It is worth mentioning that $\{k_{ai}\}_{a_i \in \mathring{Z}_r}$ will span a dense space of $H^2(\mathring{Z}_r, \mathbf{d}\rho)$. It means any function in $H^2(\mathring{Z}_r, \mathbf{d}\rho)$ will be represented by the linear combination of k_{a_i} with $a_i \in H^2(\mathring{Z}_r, \mathbf{d}\rho)$.

In this paper, we consider a rational approximation of function $f \in H^2(\mathring{Z}_r, \mathbf{d}\rho)$ based on WPOAFD proposed by Qian [7,16], which decomposes a function (or signal) in an RKHS into infinite terms associated with orthonormal reproducing kernels. WPOAFD is a highly efficient method to give the approximation of any function on RKHS in the domain without a boundary vanishing property. In addition, it is a method not related to the form of an inner product of a Hilbert space.

In general, one common inner product of Bergman space is

$$\langle f, g \rangle = \int_\Omega f\overline{g}\mathbf{d}\rho \tag{2}$$

with respect to measure ρ, and we use inner product Equation (2) without loss of generality.

To be specific, for any distinct points $a_1, a_2, \ldots, a_n$ in $\mathring{Z}_r$, the set $\{k_{a_1} \cdots k_{a_n}\}$ is linear independent in $H^2(\mathring{Z}_r, \mathbf{d}\rho)$; then, one can apply the Gram–Schmidt orthonormalization method to obtain a set of orthonormal sequences $\{\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n\}$ of $H^2(\mathring{Z}_r)$ such that $\mathcal{B}_1 = \frac{k_{a_1}}{||k_{a_1}||}$ and $\mathcal{B}_m = \frac{k_{a_m} - \sum_{i=1}^{m-1}\langle k_{a_m}, \mathcal{B}_i\rangle \mathcal{B}_i}{||k_{a_m} - \sum_{i=1}^{m-1}\langle k_{a_m}, \mathcal{B}_i\rangle \mathcal{B}_i||}$, for $m = 2, 3, \ldots, n$.

Then, one can consider $f \in H^2(\mathring{Z}_r, \mathbf{d}\rho)$. Let h_n be the image of the orthonormal projection of f onto the span$\{\mathcal{B}_1, \ldots, \mathcal{B}_n\} =$ span$\{k_{a_i} | i = 1, \ldots, n\}$; then, it follows:

$$g_n = \sum_{i=1}^{n} \langle f, \mathcal{B}_i \rangle \mathcal{B}_i, \tag{3}$$

and

$$||g_n||^2 = \sum_{i=1}^{n} |\langle f, \mathcal{B}_i \rangle|^2.$$

Furthermore, there is still one problem that remains, which is how to select an optimal sequence $\{a_1, \ldots, a_n\}$ of the points in $\mathring{Z}_r$ such that $|\langle f, \mathcal{B}_i \rangle|$ is as large as possible (like greedy algorithms).

To answer this problem, we study the following result Equation (4) called weak maximal selection principle on the domain $\mathbb{B}$ and $\mathring{Z}_r$, respectively. In addition, the case of $\mathbb{B}$ is a special case of the case of $\mathring{Z}_r$. The difference is the reproducing kernel and the sequence $\{a_1, \ldots, a_{m-1}\}$. In addition, an obvious fact is that the Kepler ball is a bounded domain but not necessarily symmetrical:

$$|\langle f, \mathcal{B}_m \rangle| \geq \rho_m \sup\left\{ |\langle f, \mathcal{B}_m^b \rangle| \;\middle|\; b \in \mathring{Z}_r \setminus \{a_1, \ldots, a_{m-1}\} \right\}, \tag{4}$$

where $\mathcal{B}_m = \frac{k_{a_m} - \sum_{i=1}^{m-1}\langle k_{a_m}, \mathcal{B}_i\rangle \mathcal{B}_i}{||k_{a_m} - \sum_{i=1}^{m-1}\langle k_{a_m}, \mathcal{B}_i\rangle \mathcal{B}_i||}$ and $\mathcal{B}_m^b = \frac{k_b - \sum_{i=1}^{m-1}\langle k_b, \mathcal{B}_i\rangle \mathcal{B}_i}{||k_b - \sum_{i=1}^{m-1}\langle k_b, \mathcal{B}_i\rangle \mathcal{B}_i||}$.

By the above-mentioned weak maximal selection principle, we have the convergence both in $\mathbb{B}$ and $H^2(\mathring{Z}_r, \mathbf{d}\rho)$ as follows:

$$f = \sum_{i \geq 1} \langle f, \mathcal{B}_i \rangle \mathcal{B}_i. \tag{5}$$

Equations (4) and (5) are the main results of this study, and the strict expression can be seen in Theorem 1 and Theorem 2.

3. Preliminaries

3.1. RKHS

In this part, the definition of RKHS is reviewed according to [17].

Definition 1 (RKHS). *Let H be a Hilbert space of complex-valued functions defined on a non-empty set X with an inner product $\langle \cdot, \cdot \rangle$. H is called a reproducing kernel Hilbert space on X, if, for any point $x \in X$, the evaluation functional $L_x : H \to C$ defined by $L_x(f) = f(x)$ is continuous on H.*

Definition 2 (Reproducing kernel)**.** *Let H be a RKHS on X and $x, y \in X$. The function $K : X \times X \to C$ satisfying*

$$\langle f, K(\cdot, x)\rangle = f(x)$$

is called the reproducing kernel for H.

Remark 1. *The reproducing kernel of X in Definition 2 is also defined by*

$$K(x, y) = \langle K(\cdot, y), K(\cdot, x)\rangle = k_y(x)$$

for $x, y \in X$.

Remark 2. *There is an example that all the weighted Bergman spaces corresponding to $H^2_\beta, \beta < 0$ are reproducing kernel Hilbert space [8].*

Remark 3. *A Hilbert space is a reproducing kernel Hilbert space if and only if the point-evaluating linear functional is a bounded functional.*

3.2. Research Objective

Let $\mathring{Z}$ be the open dense subset of all elements of maximal rank r. It is mentioned that, under the condition, Z is of tube type with Jordan determinant N. Then, it follows:

$$\mathring{Z} = \{z \in Z | N(z) \neq 0\}. \tag{6}$$

The rank of supporting tripotent of $z \in Z, 1 \leq l \leq r$ is denoted by $R(z)$. In addition, we define $\mathring{Z}_l$ as follows:

$$\mathring{Z}_l = \{z \in Z | R(z) = l\}. \tag{7}$$

Actually, $\mathring{Z}_l$ is a complex manifold called the Kepler manifold [4] with respect to Z in Equation (6).

We consider the Hilbert space $H^2(\mathring{Z}, \mathbf{d}\rho)$ equipped with inner product $\langle\cdot,\cdot\rangle$ in Equation (2) as follows:

$$H^2(\mathring{Z}_r, \mathbf{d}\rho) = \left\{ f \text{ is holomorphic on } \mathring{Z}_r \,\middle|\, \int_{\mathring{Z}} f^2(z)\mathbf{d}\rho < \infty \right\} \tag{8}$$

with respect to a K-invariant measure ρ defined as

$$\int_{\mathring{Z}_r} f(z)\mathbf{d}\rho(z) = \int_\Omega \mathbf{d}\tilde{\rho}(t) \int_K f(k\sqrt{t})\mathbf{d}k, \tag{9}$$

where $\mathbf{d}\tilde{\rho}(t)$ is called the radial part, which is a smooth measure on invariant domain Ω under the group action $L = \mathrm{Aut}(X)$ of the Euclidean Jordan algebra X.

3.3. Kepler Manifold and Kepler Ball

Traditionally, the Kepler manifold is defined as follows, serving as a symplectic manifold associated with the cotangent bundle of unit sphere.

Definition 3 (Classical Kepler manifold)**.** *Denoting $z \cdot w = \sum_j z_j w_j$, the Kepler manifold is $\mathcal{H} = \left\{ z \in \mathbb{C}^{n+1} \middle| z \cdot z = 0, z \neq 0 \right\}$.*

Remark 4. *$\mathcal{H}$ is a complex submanifold of $\mathbb{C}^{n+1}$, and $\mathcal{H} \cup \{0\}$ is the simplest case of Jordan–Kepler varieties.*

Now, the definition of Kepler ball is reviewed.

Definition 4 (Kepler ball)**.** *The Kepler ball is the domain* $\left\{ z \in \mathring{Z}_r \middle| ||z|| < 1 \right\}$.

3.4. K-Invariant Measures on $\mathring{Z}_r$

Equation (9) shows a polar decomposition of a K-invariant measure ρ on $\mathring{Z}_r$. However, there are several forms of K-invariant measures such as Riemann measure which have been studied before.

In this section, we review a measure which comes from Kähler potential.

Consider the pluri-subharmonic function $\phi = \langle z, z \rangle$ on Z_l and associated Kähler form $\omega = \partial\overline{\partial}\phi$. The measure is denoted by $\frac{|\omega^n|}{n!} \exp -\nu\phi$ with polar decomposition [18]

$$\int_{\mathring{Z}_r} \frac{|\omega^n|}{n!} \exp -\nu\phi f(z) = \int_{\Omega} N_c(t)^b \exp -\nu\langle t, c\rangle \mathrm{d}t, \tag{10}$$

where $N_c(t)$ is the rank r Jordan determinant on Ω.

3.5. Rotation Measure

We present the definition of rotation measure.

Definition 5. *A rotation measure* $\mathrm{d}(\rho \otimes \mu)$ *on* $\mathring{Z}_r$ *is defined by*

$$\int_{\mathring{Z}_r} f \mathrm{d}(\rho \otimes \mu) = \int_0^\infty \int_{\partial\mathcal{B}} f(t\zeta)\mathrm{d}\mu(\zeta)\mathrm{d}\rho(t).$$

In addition, the corresponding weighted Bergman space is

$$H^2(h\mathbb{B}, \mathbf{d}(\rho \otimes \mu)) = \left\{ f \text{ is holomorphic on } h\mathbb{B} \middle| \int_{\mathring{Z}} f^2(z)\mathbf{d}(\rho \otimes \mu) < \infty \right\}, \tag{11}$$

where $h = \sup \left\{ |z| \middle| z \in \mathrm{supp}\rho \otimes \mu \right\}$.

In the previous work [19], it is proved that $H^2(h\mathbb{B}, \mathbf{d}(\rho \otimes \mu))$ is an RKHS with reproducing kernel

$$K(z, w) = \sum_{l=0}^{\infty} \frac{(2l + n - 1)(l + n - 2)!(z \cdot \overline{w})^l}{l!(n-1)! \int_0^\infty t^{2l} \mathbf{d}\rho}. \tag{12}$$

3.6. Function Space

First, we consider the holomorphic functions which are square integrable on $\mathring{Z}_r$. The function space $H^2(\mathring{Z}_r)$ in Equation (8) is an RKHS with reproducing kernel in Equation (1) like what we have mentioned before.

Then, if it comes to Kepler ball $\mathbb{B}$, we consider the holomorphic functions which are square integrable on $\mathbb{B}$ with $\frac{|\omega^n|}{n!} \exp -\nu\phi$ as its measure. Upmeier [20] has found the reproducing kernel for this space

$$K_\nu(t, e) = \mathcal{D}_r\, {}_2F_1. \tag{13}$$

where ${}_2F_1$ is a Gauss hypergeometric function on Ω in Equation (10).

Thus, the function space $H^2(\mathbb{B}, \frac{|\omega^n|}{n!} \exp -\nu\phi)$ is an RKHS with reproducing kernel in Equation (13).

Finally, we consider the function space in Equation (11) on $h\mathbb{B}$, and it is an RKHS with reproducing kernel in Equation (12).

4. Weak Maximal Selection Principle

Before proving the weak maximal selection principle, we propose following lemmas.

Proposition 1. *$\{k_{a_i}\}_{a_i \in \mathring{Z}_r}$ is a linearly independent set.*

Proof. Considering

$$\sum_{i=1}^{n} c_i k_{a_i} = 0, \tag{14}$$

our aim is to prove $c_i = 0$ for $i = 1, 2, \ldots, n$.

Taking the inner product with $f \in H^2(\mathring{Z}_r)$ on both sides in Equation (14), we have

$$\langle f, \sum_{i=1}^{n} c_i k_{a_i} \rangle = \sum_{i=1}^{n} \overline{c_i} \langle f, k_{a_i} \rangle = 0. \tag{15}$$

By reproducing the property in Definition 2, Equation (15) reduces to

$$\langle f, \sum_{i=1}^{n} c_i k_{a_i} \rangle = \sum_{i=1}^{n} \overline{c_i} f(a_i) = 0. \tag{16}$$

Letting $f(z) = \exp t\langle a_i, z\rangle$, Equation (16) reduces to

$$\sum_{i=1}^{n} \overline{c_i} \exp t\langle a_i, a_i \rangle = \sum_{i=1}^{n} \overline{c_i} \exp ||a_i||^2 t = 0.$$

Due to the linear independence of $\exp t$, we have $\overline{c_i} = 0$.

Thus, $c_i = 0$. We complete the proof. □

Due to Proposition 1, one can apply Gram–Schmidt orthonormalization method to obtain $\{\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n\}$, where $\mathcal{B}_m = \frac{k_{a_m} - \sum_{i=1}^{m-1} \langle k_{a_m}, \mathcal{B}_i \rangle \mathcal{B}_i}{||k_{a_m} - \sum_{i=1}^{m-1} \langle k_{a_m}, \mathcal{B}_i \rangle \mathcal{B}_i||}$.

Lemma 1. *$\langle \mathcal{B}_n, \mathcal{B}_i \rangle = \langle \mathcal{B}_n^b, \mathcal{B}_i \rangle = 0$, where*

$$\mathcal{B}_n = \frac{k_{a_n} - \sum_{i=1}^{n-1} \langle k_{a_n}, \mathcal{B}_i \rangle \mathcal{B}_i}{||k_{a_n} - \sum_{i=1}^{n-1} \langle k_{a_n}, \mathcal{B}_i \rangle \mathcal{B}_i||}$$

and

$$\mathcal{B}_n^b = \frac{k_b - \sum_{i=1}^{n-1} (k_b, \mathcal{B}_i) \mathcal{B}_i}{||k_b - \sum_{i=1}^{n-1} \langle k_b, \mathcal{B}_i \rangle \mathcal{B}_i||}.$$

Proof. Since

$$\begin{aligned} \langle \mathcal{B}_n, \mathcal{B}_i \rangle &= \langle \frac{k_{a_n} - \sum_{i=1}^{n-1} \langle k_{a_n}, \mathcal{B}_i \rangle \mathcal{B}_i}{||k_{a_n} - \sum_{i=1}^{n-1} \langle k_{a_n}, \mathcal{B}_i \rangle \mathcal{B}_i||}, \mathcal{B}_i \rangle \\ &= \frac{1}{||k_{a_n} - \sum_{i=1}^{n-1} \langle k_{a_n}, \mathcal{B}_i \rangle \mathcal{B}_i||} \{\langle k_{a_n}, \mathcal{B}_i \rangle - \langle k_{a_n}, \mathcal{B}_i \rangle\} \\ &= 0 \end{aligned}$$

and

$$\langle \mathcal{B}_n^b, \mathcal{B}_i \rangle = \frac{1}{||k_b - \sum_{i=1}^{n-1} \langle k_b, \mathcal{B}_i \rangle \mathcal{B}_i||} \{\langle k_b, \mathcal{B}_i \rangle - \langle k_b, \mathcal{B}_i \rangle\} = 0,$$

we complete the proof. □

Lemma 2. *Letting* $f_n = f_{n-1} - \langle f_{n-1}, \mathcal{B}_{n-1}\rangle \mathcal{B}_{n-1}, f_1 = f$, *it holds that:*

$$\langle f_n, \mathcal{B}_n\rangle = \langle f_1, \mathcal{B}_n\rangle$$

and

$$\langle f_n, \mathcal{B}_n^b\rangle = \langle f_1, \mathcal{B}_n^b\rangle.$$

Proof. By the direct calculation and recurrence method, we have
$\langle f_n, \mathcal{B}_n\rangle = \langle f_{n-1} - \langle f_{n-1}, \mathcal{B}_{n-1}\rangle \mathcal{B}_{n-1}, \mathcal{B}_n\rangle = \langle f_{n-1}, \mathcal{B}_n\rangle = \cdots = \langle f_1, \mathcal{B}_n\rangle$;
$\langle f_n, \mathcal{B}_n^b\rangle = \langle f_{n-1} - \langle f_{n-1}, \mathcal{B}_{n-1}\rangle \mathcal{B}_{n-1}, \mathcal{B}_n^b\rangle = \langle f_{n-1}, \mathcal{B}_n^b\rangle = \cdots = \langle f_1, \mathcal{B}_n^b\rangle$. □

Then, we define

$$g_n(b) = |\langle f_n, \mathcal{B}_n^b\rangle| \tag{17}$$

for $b \in \mathring{Z}_r \setminus \{a_1, \ldots, a_{i-1}\}$, which is similar to Equation (3) and a supremum

$$S = \sup\left\{ g_n(b) \in \mathbb{R} \middle| b \in \mathring{Z}_r \setminus \{a_1, \ldots, a_{n-1}\} \right\}. \tag{18}$$

The following lemma and proposition show that S can be reached.

Lemma 3. *Let* $g_n(b)$ *be defined in Equation (17), and* $g_n(b)$ *is continuous on* $\mathring{Z}_r$.

Proof. $\forall \epsilon > 0, \exists \delta > 0, \forall a$ satisfying $|b - a| < \delta$, we have

$$\begin{aligned} |g_n(b) - g_n(a)| &= |\langle f_n, \mathcal{B}_n^b\rangle - \langle f_n, \mathcal{B}_n^a\rangle| \\ &= |\langle f_n, \mathcal{B}_n^b - \mathcal{B}_n^a\rangle| \\ &\leq ||f_n||||\mathcal{B}_n^b - \mathcal{B}_n^a||. \end{aligned} \tag{19}$$

Since $\lim\limits_{b\to a} \mathcal{B}_n^b = \mathcal{B}_n^a$, then, when $|b - a| < \delta$, we have $||\mathcal{B}_n^b - \mathcal{B}_n^a|| < \frac{\epsilon}{M}$. Because $||f||$ is bounded; i.e., $||f|| \leq M$, Equation (19) reduces to $|g_n(b) - g_n(a)| < \epsilon$.

Therefore, we complete the proof. □

Corollary 1. *In the case* $n = 1$, *we have that* $f_1 = f$ *and* $\mathcal{B}_1^b = \frac{k_b}{||k_b||}$. *Under these conditions, we have that* $g_1(b) = |\langle f, \frac{k_b}{||k_b||}\rangle|$ *is continuous.*

Proposition 2. *Let* $g_n(b)$ *and* S *be defined in Equations (17) and (18), respectively. There exists a point* $c \in \mathring{Z}_r \setminus \{a_1, \ldots, a_{i-1}\}$, *such that* $g_n(c) = S$.

Proof. Due to Proposition A3 in Appendix B and the fact that $f_n \in H^2(\mathring{Z}_r, \mathbf{d}\rho)$, we have

$$g_n(b) = |\langle f_n, \mathcal{B}_n^b\rangle| \leq ||f_n||||\mathcal{B}_n^b|| = ||f_n|| < \infty.$$

Thus, $\{g_n(b) \in \mathbb{R} | b \in \mathring{Z}_r \setminus \{a_1, \ldots, a_{n-1}\}\}$ is bounded.

By Theorem A1, we have that S exists.

By Lemma 3 and the extreme value theorem, we obtain that there exists a point c such that

$$g_n(c) = \sup g_n(b) = \max g_n(b).$$

Therefore, $g_n(c) = S$. □

Corollary 2. *In the case* $n = 1$, *there exists a point* $a \in \mathring{Z}_r$, *such that*

$$g_1(a) = S = \sup\{g_1(b) \in \mathbb{R} | b \in \mathring{Z}_r\}.$$

Next, we summarize the weak maximal selection principle from Section 2.

Theorem 1 (Weak maximal selection principle). *For any function $f \in H^2(\mathring{Z}_r, \mathbf{d}\rho)$ and sequence $\{\rho_i\}_{i\geq 0}$ with $0 < \rho_0 \leq \rho_i < 1$ for $i \geq 1$, there exists a sequence $\{a_i\}_{i\geq 1}$ of distinct points in $\mathring{Z}_r$ such that*

$$|\langle f, \mathcal{B}_m\rangle| \geq \rho_m \sup\left\{ |\langle f, \mathcal{B}_m^b\rangle| \middle| b \in \mathring{Z}_r \setminus \{a_1, \ldots, a_{m-1}\}\right\},$$

where $m = 1, 2, \ldots, n$, $\mathcal{B}_m = \frac{k_{a_m} - \sum_{i=1}^{m-1}\langle k_{a_m}, \mathcal{B}_i\rangle \mathcal{B}_i}{||k_{a_m} - \sum_{i=1}^{m-1}\langle k_{a_m}, \mathcal{B}_i\rangle \mathcal{B}_i||}$ and

$$\mathcal{B}_m^b = \frac{k_b - \sum_{i=1}^{m-1}(k_b, \mathcal{B}_i)\mathcal{B}_i}{||k_b - \sum_{i=1}^{m-1}\langle k_b, \mathcal{B}_i\rangle \mathcal{B}_i||}.$$

Proof of Theorem 1. Mathematical induction is used for this proof. Additionally, there are two steps including the base case and induction step in the procedure of mathematical induction.

Firstly, one can consider the m-th residual function f_m(same as f_n in Lemma 2) as follows:

$$f_m = f_{m-1} - \langle f_{m-1}, \mathcal{B}_{m-1}\rangle \mathcal{B}_{m-1},$$

where $m = 2, \ldots, n$ and $f_1 = f$.

Secondly, one can validate the base case $m = 1$.

In the case $m = 1$, by Corollaries 1 and 2, there exist $a_1 \in \mathring{Z}_r$ and $\mathcal{B}_1 = \frac{k_{a_1}}{||k_{a_1}||}$ such that

$$|\langle f, \mathcal{B}_1\rangle| = |\langle f, \frac{k_{a_1}}{||k_{a_1}||}\rangle| = \sup\left\{ |\langle f, \frac{k_b}{||k_b||}\rangle| \middle| b \in \mathring{Z}_r\right\}.$$

Then, for $0 < \rho_1 < 1$, we have

$$|\langle f, \mathcal{B}_1\rangle| \geq \rho_1 \sup\left\{ |\langle f, \frac{k_b}{||k_b||}\rangle| \middle| b \in \mathring{Z}_r\right\},$$

since $|\langle f, \mathcal{B}_1\rangle|$ is non-negative.

To be specific, if $|\langle f, \mathcal{B}_1\rangle| = 0$, we have that $\sup\left\{ |\langle f, \frac{k_b}{||k_b||}\rangle| \middle| b \in \mathring{Z}_r\right\} = 0$. Then, for $b \in \mathring{Z}_r \setminus \{a_1, \ldots, a_{m-1}\}$, $|\langle f, \mathcal{B}_m^b\rangle| = 0$. It is obvious that Theorem 1 holds since $|\langle f, \mathcal{B}_m\rangle| \geq 0$.

Thirdly, for other cases $m \neq 1$ and points $\{a_i\}_{i\geq 1}$, the induction step is implemented as follows.

One can assume that Theorem 1 holds in the case $m = n$; then, there are n points obtained satisfying

$$|\langle f_m, \mathcal{B}_m\rangle| \geq \rho_m \sup\left\{ |\langle f_m, \mathcal{B}_m^b\rangle| \middle| b \in \mathring{Z}_r \setminus \{a_1, \ldots, a_{m-1}\}\right\}, \tag{20}$$

where $m = 1, 2, \ldots, n$.

Then, the aim is to show that Theorem 1 holds in the case $m = n + 1$ under the previous assumption, i.e., Equation (20).

For the case $m = n + 1$, there are two possibilities:

- If there exists one $f_m = 0$ for $m = 1, 2, \ldots, n$, then Theorem 1 holds since $|\langle f_m, \mathcal{B}_m\rangle| \geq 0$.
- Otherwise, for $0 < \rho_0 < \rho_{n+1} < 1$, by Proposition 2, one can obtain that there exists a point $a_{n+1} \in \mathring{Z}_r$ such that

$$|\langle f_{n+1}, \mathcal{B}_{n+1}^b\rangle| \geq \rho_{n+1} \ \sup\left\{ |\langle f_{n+1}, \mathcal{B}_{n+1}^b\rangle| \middle| b \in \mathring{Z}_r \setminus \{a_1, \ldots, a_n\}\right\}, \tag{21}$$

where $\mathcal{B}_{n+1} = \frac{k_{a_{n+1}} - \sum_{j=1}^{n}(k_{a_{n+1}}, \mathcal{B}_j)\mathcal{B}_j}{||k_{a_{n+1}} - \sum_{j=1}^{n}(k_{a_{n+1}}, \mathcal{B}_j)\mathcal{B}_j||}$ and

$$\mathcal{B}^b_{n+1} = \frac{k_b - \sum_{j=1}^{n}(k_b, \mathcal{B}_j)\mathcal{B}_j}{||k_b - \sum_{j=1}^{n}(k_b, \mathcal{B}_j)\mathcal{B}_j||}.$$

By Lemmas 1 and 2, Equation (21) reduces to

$$|\langle f, \mathcal{B}^b_{n+1}\rangle| \geq \rho_{n+1} \sup\left\{ |\langle f, \mathcal{B}^b_{n+1}\rangle| \middle| b \in \mathring{Z}_r \setminus \{a_1, \ldots, a_n\} \right\}, \tag{22}$$

where $\mathcal{B}_{n+1} = \frac{k_{a_{n+1}} - \sum_{j=1}^{n}(k_{a_{n+1}}, \mathcal{B}_j)\mathcal{B}_j}{||k_{a_{n+1}} - \sum_{j=1}^{n}(k_{a_{n+1}}, \mathcal{B}_j)\mathcal{B}_j||}$ and

$$\mathcal{B}^b_{n+1} = \frac{k_b - \sum_{j=1}^{n}(k_b, \mathcal{B}_j)\mathcal{B}_j}{||k_b - \sum_{j=1}^{n}(k_b, \mathcal{B}_j)\mathcal{B}_j||}.$$

One can set that the final point a_{n+1} is the point $b \in \mathring{Z}_r \setminus \{a_1, \ldots, a_n\}$; then, it follows

$$|\langle f, \mathcal{B}^b_{n+1}\rangle| = |\langle f, \mathcal{B}_{n+1}\rangle|. \tag{23}$$

By Equation (23), Equation (22) reduces to

$$|\langle f, \mathcal{B}_{n+1}\rangle| \geq \rho_{n+1} \sup\left\{ |\langle f, \mathcal{B}^b_{n+1}\rangle| \middle| b \in \mathring{Z}_r \setminus \{a_1, \ldots, a_n\} \right\}.$$

Therefore, the case $m = n + 1$ is showed. By Proposition A1 in Appendix A.1, the weak maximal selection principle is proved. □

Remark 5. *In this proof, we use the fact that, if $|\langle f, \mathcal{B}_i\rangle| = 0$, the decomposition will come to the end.*

Corollary 3 (Kepler ball case). *For any function $f \in H^2(\mathbb{B}, \frac{|\omega^n|}{n!} \exp -\nu\phi)$ and sequence $\{\rho_i\}_{i\geq 0}$ with $0 < \rho_0 \leq \rho_i < 1$ for $i \geq 1$, there exists a sequence $\{a_i\}_{i\geq 1}$ of distinct points in $\mathbb{B}$ such that*

$$|\langle f, \mathcal{B}_m\rangle| \geq \rho_m \sup\left\{ |\langle f, \mathcal{B}^b_m\rangle| \middle| b \in \mathbb{B} \setminus \{a_1, \ldots, a_{m-1}\} \right\},$$

where $\mathcal{B}_m = \frac{k_{a_m} - \sum_{i=1}^{m-1}\langle k_{a_m}, \mathcal{B}_i\rangle\mathcal{B}_i}{||k_{a_m} - \sum_{i=1}^{m-1}\langle k_{a_m}, \mathcal{B}_i\rangle\mathcal{B}_i||}$ and

$$\mathcal{B}^b_m = \frac{k_b - \sum_{i=1}^{m-1}(k_b, \mathcal{B}_i)\mathcal{B}_i}{||k_b - \sum_{i=1}^{m-1}\langle k_b, \mathcal{B}_i\rangle\mathcal{B}_i||}.$$

Proof. The proof is straightforward by Theorem 1 when we constrained functions on $\mathbb{B}$ with a reproducing kernel in Equation (13). □

Corollary 4 (Rotation measure case). *For any function $f \in H^2(h\mathbb{B}, \mathbf{d}(\rho \otimes \mu))$ and sequence $\{\rho_i\}_{i\geq 0}$ with $0 < \rho_0 \leq \rho_i < 1$ for $i \geq 1$, there exists a sequence $\{a_i\}_{i\geq 1}$ of distinct points in $\mathring{Z}_r$ such that*

$$|\langle f, \mathcal{B}_m\rangle| \geq \rho_m \sup\left\{ |\langle f, \mathcal{B}^b_m\rangle| | b \in h\mathbb{B} \setminus \{a_1, \ldots, a_{m-1}\} \right\},$$

where $\mathcal{B}_m = \frac{k_{a_m} - \sum_{i=1}^{m-1}\langle k_{a_m}, \mathcal{B}_i\rangle\mathcal{B}_i}{||k_{a_m} - \sum_{i=1}^{m-1}\langle k_{a_m}, \mathcal{B}_i\rangle\mathcal{B}_i||}$ and

$$\mathcal{B}^b_m = \frac{k_b - \sum_{i=1}^{m-1}(k_b, \mathcal{B}_i)\mathcal{B}_i}{||k_b - \sum_{i=1}^{m-1}\langle k_b, \mathcal{B}_i\rangle\mathcal{B}_i||}.$$

Proof. The proof is straightforward by Theorem 1 when we constrained functions on $H^2(h\mathbb{B}, \mathbf{d}(\rho \otimes \mu))$ with a reproducing kernel in Equation (12). □

5. Convergence of WPOAFD

Theorem 2 (Covergence theorem)**.** *For any function $f \in H^2(\mathring{Z}_r, \mathbf{d}\rho)$, f can be represented as follows:*

$$f = \sum_{i\geq 1} \langle f, \mathcal{B}_i \rangle \mathcal{B}_i,$$

where $\mathcal{B}_i = \frac{k_b - \sum_{m=1}^{i-1} \langle k_b, \mathcal{B}_m \rangle \mathcal{B}_m}{||k_b - \sum_{m=1}^{i-1} \langle k_b, \mathcal{B}_m \rangle \mathcal{B}_m||}$, and $\langle f, \mathcal{B}_i \rangle$ can be obtained by Theorem 1.

Proof of Theorem 2. Proof by contradiction is used in this proof by assuming that

$$h = f - \sum_{i\geq 1} \langle f, \mathcal{B}_i \rangle \mathcal{B}_i \neq 0.$$

Then, the sequence $\{a_i\}$ is not finite. Since h is non-zero, there exists an open ball $B(\epsilon)$ with $\epsilon > 0$, such that $|h(z)| > 0$ on the ball.

Due to $\sum_{i=1}^{\infty} |\langle f, \mathcal{B}_i \rangle|^2 < \infty$, then there exists $N > 0$ such that, for all $n > N$,

$$\sum_{i=n}^{\infty} |\langle f, \mathcal{B}_i \rangle^2| < (\frac{\rho_0 C_0}{2})^2.$$

Then, it is obvious that $|\langle f_n, \mathcal{B}_n \rangle| = |\langle f, \mathcal{B}_n \rangle| < \frac{\rho_0 C_0}{2}$.

One can note that $e_b = \frac{k_b}{||k_b||}$. According to Theorem 1,

$$|\langle f_n, \mathcal{B}_n \rangle| \geq \rho_n \sup\left\{ |\langle f_n, \mathcal{B}_n^z \rangle| \middle| z \in \mathring{Z}_r \setminus \{a_1, a_2, \ldots, a_{N-1}\} \right\}$$

$$\begin{aligned} |\langle f_n, e_b \rangle| &= |\langle f_n, \frac{k_b}{||k_b||} \rangle| \\ &= \frac{|f_n, k_b - \sum_{i=1}^{n-1} \langle k_b, \mathcal{B}_i \rangle \mathcal{B}_i|}{||k_b||} \\ &\leq \frac{|f_n, k_b - \sum_{i=1}^{n-1} \langle k_b, \mathcal{B}_i \rangle \mathcal{B}_i|}{||k_b - \sum_{i=1}^{n-1} \langle k_b, \mathcal{B}_i \rangle \mathcal{B}_i||} \\ &= |\langle f_n, \mathcal{B}_n^b \rangle| \\ &\leq \frac{1}{\rho_0} |\langle f_n, \mathcal{B}_n \rangle| \\ &< \frac{C_0}{2}. \end{aligned} \tag{24}$$

Let $f_n = f - \sum_{i=1}^{n-1} \langle f, \mathcal{B}_i \rangle \mathcal{B}_i = \sum_{i=n}^{\infty} \langle f, \mathcal{B}_i \rangle \mathcal{B}_i + h$; then,

$$\begin{aligned}|\langle f_n, e_b\rangle| &= |\langle h + \sum_{i=n}^{\infty}\langle f, \mathcal{B}_i\rangle \mathcal{B}_i, e_b\rangle| \\ &\geq |\frac{h(b)}{K(b,b)}| - |\langle \sum_{i=n}^{\infty}\langle f, \mathcal{B}_i\rangle \mathcal{B}_i, e_b\rangle| \\ &\geq C_0 - \sqrt{\sum_{i=n}^{\infty}|\langle f, \mathcal{B}_i\rangle|^2} \\ &> \frac{C_0}{2}. \end{aligned} \tag{25}$$

It is obvious that there is a contradiction between Equations (24) and (25). By Proposition A2 in Appendix A.2, $h = 0$.

Therefore, the proof is complete. □

Corollary 5 (Kepler ball case). *For any function $f \in H^2(\mathbb{B}, \frac{|\omega^n|}{n!}\exp -\nu\phi)$, f can be represented as follows:*

$$f = \sum_{i\geq 1}\langle f, \mathcal{B}_i\rangle \mathcal{B}_i,$$

where $\mathcal{B}_i = \frac{k_b - \sum_{m=1}^{i-1}\langle k_b, \mathcal{B}_m\rangle \mathcal{B}_m}{||k_b - \sum_{m=1}^{i-1}\langle k_b, \mathcal{B}_m\rangle \mathcal{B}_m||}$, and $\langle f, \mathcal{B}_i\rangle$ can be obtained by Corollary 3.

Proof. Same idea as Corollary 4. □

Corollary 6 (Rotation measure case). *For any function $f \in H^2(h\mathbb{B}, \mathbf{d}(\rho \otimes \mu))$, f can be represented as follows:*

$$f = \sum_{i\geq 1}\langle f, \mathcal{B}_i\rangle \mathcal{B}_i,$$

where $\mathcal{B}_i = \frac{k_b - \sum_{m=1}^{i-1}\langle k_b, \mathcal{B}_m\rangle \mathcal{B}_m}{||k_b - \sum_{m=1}^{i-1}\langle k_b, \mathcal{B}_m\rangle \mathcal{B}_m||}$, and $\langle f, \mathcal{B}_i\rangle$ can be obtained by Corollary 2.

Proof. Same idea as Corollary 2. □

6. Application

When it comes to the application of the main results in this paper, one can consider the closure of Kepler ball $\overline{\mathbb{B}} = \mathbb{B} \cup \partial\mathbb{B}$ [18], where $\partial\mathbb{B} = \left\{z \in \mathring{Z}_r \middle| ||z|| = 1\right\}$. It is an important research objective associated with Kepler ball. From the view of this paper, one can do WPOAFD directly. In [18], by the change of variables, f can be approximated to $L(1 + bL\log L + aL + \cdots), b \neq 0$. Thus, it is reasonable to use this expansion to do WPOAFD as in Theorems 1 and 2.

The advantages of this form are as follows:

1. By the change of variables, the points in Kepler ball are transformed to an interval;
2. To implement the weak maximal principle is much easier because the computational cost is lower.

Moreover, one is supposed to find that the boundary vanishing property (BVP) holds on $\mathbb{B}$. Thus, POAFD, which is a stronger version than WPOAFD, can be implemented. In addition, maximal selection is proposed replacing the weak form.

BVP is presented as follows.

Lemma 4 (BVP). *For any $f \in H^2(\mathbb{B})$, the function $g(a) = \langle f, e_a\rangle$ defined on $\mathbb{B}$ satisfies $\lim_{z\to b_0} g(z) = 0$ for any $b_0 \in \partial\mathbb{B}$.*

Proof. Let $f \in H^2(\mathbb{B})$ because the span of $\{k_a | a \in \mathbb{B}\}$ is dense, and there exists a polynomial g such that $\|f - g\| < \varepsilon$. By Proposition A3 in Appendix B, one can obtain

$$|\langle f, e_z \rangle| = |\langle f - g + g, e_z \rangle| \le |\langle f - g, e_z \rangle| + |\langle g, e_z \rangle| \le \varepsilon + |\langle g, e_z \rangle| = (1 - |z|^2) g(z) + \varepsilon.$$

Let $z \to b_0$; $|z|^2 \to 1$, and one can obtain $|\langle f, e_z \rangle| \to 0$. This implies that $\lim_{z \to b_0} g(z) = 0$. Therefore, we complete the proof. □

By Lemma 4, the domain of the function g can be extended from $\mathbb{B}$ to $\overline{\mathbb{B}}$. Then, we propose the maximal selection principle on $\overline{\mathbb{B}}$.

Theorem 3 (Maximum Selection Principle). *For any $f \in H^2(\mathbb{B})$, there exists $a \in \mathbb{B}$ such that*

$$|\langle f, e_a \rangle| = \sup \left\{ |\langle f, e_b \rangle| \;\middle|\; b \in \mathbb{B} \right\}.$$

Proof. One can define the function g as in Lemma 4. Because $K(z, a) = \frac{1}{1 - \bar{a}z}$ is a reproducing kernel for H^2 [10], one can calculate that

$$\begin{aligned}
\langle k_z, k_a \rangle &= k_z(a) = \frac{1}{1 - \bar{z}a}; \\
\|k_a\|^2 &= \langle k_a, k_a \rangle = k_a(a) = \frac{1}{1 - |a|^2}; \\
\|e_z - e_a\|^2 &= \|e_z\|^2 + \|e_a\|^2 - 2\mathrm{Re}\langle e_z, e_a \rangle \\
&= 1 + 1 - 2\mathrm{Re}\left(\frac{\langle k_z, k_a \rangle}{\|k_a\| \|k_z\|} \right) \\
&= 2 - 2\mathrm{Re}\left(\frac{\sqrt{1 - |a|^2}\sqrt{1 - |z|^2}}{1 - \bar{z}a} \right); \\
\lim_{z \to a} \|e_z - e_a\|^2 &= \lim_{z \to a} \left(2 - 2\mathrm{Re} \frac{\sqrt{(1 - |a|^2)(1 - |z|^2)}}{1 - \bar{z}a} \right) \\
&= 2 - 2\mathrm{Re} \frac{\sqrt{(1 - |a|^2)(1 - |a|^2)}}{1 - \bar{a}a} = 2 - 2 = 0.
\end{aligned}$$

By Proposition A3, one can obtain that

$$\begin{aligned}
g(z) = |\langle f, e_z \rangle| &= |\langle f, e_z - e_a \rangle + \langle f, e_a \rangle| \\
&\le |\langle f, e_z - e_a \rangle| + |\langle f, e_a \rangle| \\
&\le \|f\| \cdot \|e_z - e_a\| + g(a).
\end{aligned}$$

Because the status of a and z is equivalent, $g(a) \le \|f\| \cdot \|e_a - e_z\| + g(z)$ holds. Due to $|g(z) - g(a)| \le \|f\| \cdot \|e_a - e_z\|$, one can know that g is continuous on $\mathbb{B}$.

By Lemma 4, one is supposed to have that g is continuous on $\overline{\mathbb{B}}$. Thus, $\sup g \ge 0$ can be reached on $\mathbb{B}$. The equality holds when $g \equiv 0$ on $\mathbb{B}$. Therefore, there exists $a \in \mathbb{B}$ such that

$$|\langle f, e_a \rangle| = \sup \left\{ |\langle f, e_b \rangle| \;\middle|\; b \in \mathbb{B} \right\}.$$

Therefore, we complete the proof. □

Then, one can use Theorem 3 to obtain the convergence theorem on $\mathbb{B}$. In addition, the proof is similar to Corollary 5. Although the convergence theorems of POAFD and WPOAFD are quite similar, the selection principles are different. The maximum selection principle is easier than the weak maximum selection principle due to the fact that the supremum can be reached directly. POAFD can only be used to the case where BVP holds, whereas WPOAFD can be applied to any case regardless of whether BVP holds or not.

7. Conclusions

In this paper, we propose a procedure of WPOAFD in H^2 on the Kepler manifold $\mathring{Z}_r$ in great detail and prove the convergence of this approximation. Two corollaries are also obtained. Without BVP, we still have the weak maximal selection principle, which plays an important role in proving the convergence theorem. The connection between the main results and Bergman space is that a procedure is proposed to present the general form of any function in a weighted Bergman space on the Kepler manifold. Previous work covers mainly other forms of the reproducing kernel and their Tian–Yau–Zelditch expansion (TYZ expansion) [19,21]. The future work will be done in the following two parts:

1. Use other descriptions of reproducing kernel to establish the WPOAFD procedure, and study its relation to the result of TYZ expansions;
2. Exploring n-best WPOAFD [22,23] in function spaces studied in this paper.

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Abbreviations

The following abbreviations are used in this manuscript:

RKHS	Reproducing kernel Hilbert space
AFD	Adaptive Fourier decomposition
POAFD	Pre-orthogonal adaptive Fourier decomposition
WPOAFD	Weak Pre-orthogonal adaptive Fourier decomposition
BVP	Boundary vanishing property
TYZ expansion	Tian–Yau–Zelditch expansion

Appendix A

Appendix A.1. Mathematical Induction

We review the basic conception of mathematical induction, which is used in the proof of Theorem 1 in Appendix A.1.

Mathematical induction is a well-known method in mathematical proofs. The motivation is obvious in [24], like a list of natural numbers, if one starts at the beginning 1 and continues to reach $2, 3, \cdots$ one by one, any fixed number can be reached. Therefore, if one can show the statement involving n holds when $n = 1$, and the truth of n implies the truth of $n + 1$, then the statement is true for all n.

The strict expression of mathematical induction is presented as follows [25]:

Proposition A1. *Letting $P_1, P_2 \cdots P_n, \cdots$ be statements depends on n, which are true or false, and one can suppose that*

1. *P_1 is true;*
2. *$P_n \Rightarrow P_{n+1}$.*

   Then, $P_1, P_2 \cdots P_n, \cdots$ are all true.

Therefore, to prove a statement P_n, one can validate P_1 (the base case). Then, if one can obtain $\mathrm{P}_n \Rightarrow \mathrm{P}_{n+1}$ (induction step), the statement holds for all n.

Appendix A.2. Proof by Contradiction

Proof by contradiction is a traditional mathematical method based on the assumption that the statement is false. If one can show that, under such assumption, it will lead to a contradiction, the statement is true. In addition, we use it in the proof of Theorem 2.

Proposition A2 ([26])**.** *To prove P, assume $\neg$P and derive absurdity.*

Thus, the mode of proving statement P using proof by contradiction is to assume $\neg$P first and then obtain the contradiction.

Appendix B

In Appendix B, supremum and infimum principle and Cauchy–Schwarz inequality used in this paper are reviewed.

Theorem A1 (Supremum and infimum principle)**.** *If a set S is bounded, its supremum or infimum are supposed to exist. To be specific, if S has an upper bound, its supremum will exist; if S has a lower bound, its infimum will exist.*

Proposition A3 (Cauchy–Schwarz inequality)**.** $|\langle a, b\rangle| \leq ||a||||b||$, *where a, b are vectors in Hilbert space and $\langle \cdot, \cdot \rangle$ is an inner product in the Hilbert space.*

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