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# Sufficient Conditions of 6-Cycles Make Planar Graphs DP-4-Colorable

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**Abstract:** In simple graphs, DP-coloring is a generalization of list coloring and thus many results of DP-coloring generalize those of list coloring. Xu and Wu proved that every planar graph without 5-cycles adjacent simultaneously to 3-cycles and 4-cycles is 4-choosable. Later, Sittitrai and Nakprasit showed that if a planar graph has no pairwise adjacent 3-, 4-, and 5-cycles, then it is DP-4-colorable, which is a generalization of the result of Xu and Wu. In this paper, we extend the results on 3-, 4-, 5-, and 6-cycles by showing that every planar graph without 6-cycles simultaneously adjacent to 3-cycles, 4-cycles, and 5-cycles is DP-4-colorable, which is also a generalization of previous studies as follows: every planar graph $G$ is DP-4-colorable if $G$ has no 6-cycles adjacent to $i$-cycles where $i \in \{3, 4, 5\}$.

**Keywords:** DP-coloring; list coloring; planar graph; cycle

**MSC:** 05C10; 05C15



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## 1. Introduction

The concept of list coloring was introduced independently by Vizing [1] and Erdős et al. [2]. A *k-list assignment* $L$ of a graph $G$ assigns for each vertex $v$ in $G$ a list $L(v)$ of $k$ colors. An *L-coloring* is a proper coloring $c$ such that $c(v) \in L(v)$ for each $v$ in $V(G)$. A graph $G$ is *L-colorable* if $G$ has an $L$-coloring. If $G$ is $L$-colorable for any $k$-list assignment $L$, then $G$ is said to be *k-choosable*.

DP-coloring is a generalization of list coloring. Dvořák and Postle [3] introduced the concept of DP-coloring and they called it correspondence coloring. Later on, it is called DP-coloring by Bernshteyn et al. [4].

Assume $L$ is an assignment of a graph $G$. $H$ is a *cover* of $G$ if it admits all the following properties:

(i) Its vertex set $V(H)$ is $\bigcup_{v \in V(G)}(\{v\} \times L(v)) = \{(v, c) : v \in V(G), c \in L(v)\}$;
(ii) $H[\{v\} \times L(v)]$ is a complete graph for every $v \in V(G)$;
(iii) The set $E_H(\{u\} \times L(u), \{v\} \times L(v))$ is a matching (empty matching is allowable) for each $uv \in E(G)$.
(iv) If $uv \notin E(G)$, then there are no edges of $H$ connect $\{u\} \times L(u)$ and $\{v\} \times L(v)$.

An independent set in a cover $H$ of a graph $G$ with size $|V(G)|$ is called an *$(H, L)$-coloring* of $G$. If every cover $H$ with any *k-assignment* $L$ of a graph $G$ admits an $(H, L)$-coloring for $G$, then we say that $G$ is DP-*k-colorable*. The minimum $k$ in which a graph $G$ is DP-$k$-colorable is called the DP-chromatic number of $G$ and denoted by $\chi_{DP}(G)$.

If edges on $H$ are defined to match exactly identical colors between $L(u)$ and $L(v)$ for each $uv \in E(G)$, then $G$ admits an $(H, L)$-coloring is equivalent to $G$ is $L$-colorable. Consequently, DP-coloring is a generalization of list coloring. Furthermore, this implies that $\chi_{DP}(G) \geq \chi_l(G)$.

Dvořák and Postle [3] proved that for every planar graph $G$, $\chi_{DP}(G) \leq 5$, which extends a seminal result by Thomassen [5] on list coloring. Meanwhile, Voigt [6] constructed an example of a non-4-choosable planar graph (and thus, not DP-4-colorable). It motivates the investigation to obtain sufficient conditions for being DP-4-colorable of planar graphs. Kim and Ozeki [7] proved that every planar graph is DP-4-colorable if it does not contain $k$-cycles for each $k = 3, 4, 5, 6$. Kim et al. [8] proved that every planar graph is DP-4-colorable if it contains neither 7-cycles nor butterflies. In [9], Kim and Yu proved that every planar graph is DP-4-colorable if it does not contain triangles adjacent to 4-cycles, which extends the result on 3- and 4-cycles. In 2019, Liu and Li [10] improved the previous result of Kim and Yu [9] by relaxing the condition of one triangle into two triangles. Chen et al. [11] showed that every planar graph that contains no 4-cycles adjacent to $k$-cycles where $k = 5, 6$ is DP-4-colorable. Liu et al. [12] extended the result of Kim and Ozeki [7] on 3-, 5-, and 6-cycles by proving that every planar graph contains no $k$-cycles adjacent to triangles is DP-4-colorable. Xu and Wu [13] proved that every planar graph, which contains no 5-cycles adjacent simultaneously to 3-cycles and 4-cycles is 4-choosable. Recently, Sittitrai and Nakprasit [14] showed that every planar graph that contains no pairwise adjacent 3-, 4-, and 5-cycle is DP-4-colorable which generalizes the result of Xu and Wu [13].

In this work, the results on 3-, 4-, 5-, and 6-cycles are extended by the result on Theorem 1, which generalizes the aforementioned results by Chen et al. [11] and Liu et al. [12].

**Theorem 1.** *Every planar graph without 6-cycles simultaneously adjacent to 3-cycles, 4-cycles, and 5-cycles is DP-4-colorable.*

Then we have the following two Corollaries. Moreover, some results on [11,12] are some part of Corollary 1 for $i = 4$ and $i = 3$, respectively.

**Corollary 1.** *Every planar graph without 6-cycles adjacent to i-cycles is DP-4-colorable for each* $i \in \{3, 4, 5\}$.

**Corollary 2.** *Every planar graph without 6-cycles simultaneously adjacent to i-cycles and j-cycles is DP-4-colorable for each* $i, j \in \{3, 4, 5\}$ *and* $i \neq j$.

## 2. Preliminaries

First, some notations and definitions are introduced in this section. Let $G$ be a plane graph. The vertex set, edge set, and face set of the graph $G$ are denoted, respectively, by $V(G)$, $E(G)$, and $F(G)$. We use $B(f)$ to denote the boundary of a face $f$. Two faces $f$ and $g$ are adjacent if $B(f)$ and $B(g)$ are adjacent. A *wheel* $W_n$ is a graph of $n$ vertices formed by connecting all vertices of an $(n-1)$-cycle (these vertices are called *external vertices*) to a single vertex (*hub*). A $k$-vertex, $k^+$-vertex, and $k^-$-vertex is a vertex of degree $k$, at least $k$, and at most $k$, respectively. Similar notation is applied to cycles and faces.

Note that some faces may appear several times in the order. If a face is incident to at least two $5^+$-vertices (respectively, exactly one $5^+$-vertex, no $5^+$-vertices), it is called *rich* (*semi-rich*, *poor*, respectively).

A semi-rich 5-face is a *proper semi-rich* 5-face if each incident edge with two endpoints of degree 4 is on the boundary of a 3-face, otherwise it is called an *improper semi-rich* 5-face.

A bounded face is an *extreme* face if it has a vertex incident to the unbounded face. An *inner* face is a bounded face but is not an extreme face.

An edge $uv$ is a *chord* in an embedding cycle $C$ if $u, v \in V(C)$ but $uv$ is not in $E(C)$. If a chord is inside $C$, then it is called an *internal chord*, otherwise it is called an *external chord*. A graph $C(m, n)$ is obtained from a cycle $x_1x_2 \ldots x_{m+n-2}$ with an internal chord $x_1x_m$. For example, cycles $uvw$ and $vwxyz$ form $C(3, 5)$. A graph $C(l, m, n)$ is obtained from a cycle $x_1x_2 \ldots x_{l+m+n-4}$ with internal chords $x_1x_l$ and $x_1x_{l+m-2}$. The previous definition can be extended similarly to a graph $C(m, n, p, q)$. The graphs $int(C)$ and $ext(C)$ are induced by vertices inside and outside a cycle $C$, respectively. A *separating cycle* $C$ is a cycle with non-empty $int(C)$ and $ext(C)$.

Let $\mathcal{A}$ denote the family of planar graphs without 6−cycle simultaneously adjacent 3-, 4-, and 5-cycle.

To prove that every planar graph without 6-cycles simultaneously adjacent to 3-cycles, 4-cycles, and 5-cycles is DP-4-colorable, we prove a stronger result as follows.

**Theorem 2.** *If $G \in \mathcal{A}$ with a precolored 3-cycle, then the precoloring can be extended to be a DP-4-coloring of $G$.*

## 3. Structures

Let $G$ be a minimal counterexample to Theorem 2 with respect to the order $|V(G)|$. Then, (i) $G \in \mathcal{A}$ and (ii) $G$ is a minimal graph with a precoloring of a 3-cycle that cannot be extended to be a DP-4-coloring in $G$. Some tools in [14] are used to deal with graphs satisfying (ii). We assume that $G$ contains a 3-cycle since every planar graph without 3-cycles is DP-4-colorable [9].

Thus we let $C_0$ be a 3-cycle in $G$ that is precolored.

**Lemma 1** (Lemma 3.1 in [14])**.** *$G$ has no separating 3-cycles (See the proof in Lemma A1).*

It follows from Lemma 1 that we may assume $C_0$ to be the boundary of the unbounded face of $G$.

**Lemma 2** (Lemma 3.3 in [14])**.** *Each vertex in $int(C_0)$ has degree at least four (See the proof in Lemma A3).*

**Lemma 3.** *The following statements hold.*

(i) *A bounded $6^-$-face has its boundary as a cycle.*

(ii) *If a bounded $k_1$-face $f$ and a bounded $k_2$-face $g$ with $k_1 + k_2 \leq 8$ are adjacent, then $B(f) \cup B(g) = C(k_1, k_2)$.*

(iii) *Let a bounded 3-face $f$ and a bounded 4-face $g$ be adjacent. If $f$ or $g$ is adjacent to a bounded 3-face $h$, then $B(f) \cup B(g) \cup B(h)$ is a 6-cycle with two internal chords.*

**Proof.**

(i) Clearly, a boundary of a $5^-$-face is a cycle. Consider a bounded 6-face $f$. A boundary closed walk is in a form of $uvwxywu$ if $B(f)$ is not a cycle. By Lemma 2, $u$ or $x$ has degree at least 4. It follows that $uvw$ or $xyw$ is a separating 3-cycle, contrary to Lemma 1.

(ii) It suffices to show that $B(f)$ and $B(g)$ share exactly two vertices.
If $B(f) = uvw, B(g) = vwx$ and $u = x$, then $f$ or $g$ is the unbounded face, a contradiction.
If $B(f) = uvw, B(g) = vwxy$ and $u = x$ or $y$, then $d(w) = 2$ or $d(v) = 2$, which contradicts Lemma 2.
If $B(f) = uvw, B(g) = vwxyz$ and $u = x$ or $z$, then $d(w) = 2$ or $d(v) = 2$, which contradicts Lemma 2. If $B(f) = uvw, B(g) = vwxyz$ and $u = y$, then $vyz$ or $wxy$ is a separating 3-cycle, which contradicts Lemma 1.
If $B(f) = stuv, B(g) = uvwx$ and $s = w$, then $d(v) = 2$, which contradicts Lemma 2.
If $B(f) = stuv, B(g) = uvwx$ and $s = x$, then $utx$ or $vwx$ is a separating 3-cycle, which contradicts Lemma 1. The remaining cases are similar.

(iii) Lemma 3 (ii) yields that $B(f) \cup B(g)$ is a 5-cycle with one chord. Similar to the proof of Lemma 3 (ii), one can show that $B(h)$ and $B(f) \cup B(g)$ share exactly two vertices. This yields a desired result.

□

**Lemma 4.** *If C is a 6-cycle and has a triangular chord, then C has only one chord. Moreover, every 6-cycle has at most one internal chord.*

**Proof.** Let $C$ be a 6-cycle $tuvxyz$ and let $tv$ be its triangular chord. Suppose to the contrary that $C$ has at least two chords. Since $C$ is adjacent to a 3-cycle $tuv$ and a 5-cycle $uvxyz$, it suffices to show that $C$ is adjacent to a 4-cycle. By symmetry, we assume another chord $e$ of $C$ is $ux, uy, tx, ty$, or $xz$.

If $e = ux$, then $C$ is adjacent to a 4-cycle $tuxv$.

If $e = uy$, then $C$ is adjacent to a 4-cycle $uvxy$.

If $e = tx$, then $C$ is adjacent to a 4-cycle $tuvx$.

If $e = ty$, then $C$ is adjacent to a 4-cycle $vxyt$.

If $e = xz$, then $C$ is adjacent to a 4-cycle $tvxz$.

Thus, $C$ has exactly one chord. Note that $C$ has a triangular chord if $C$ has at least two internal chords. It follows that every 6-cycle has at most one internal chord. □

A *cluster* in a plane graph $G$ is a subgraph of $G$ consisting of 3-cycles from a minimal set of bounded 3-faces such that they are not adjacent to other bounded 3-faces outside the set. A *k-cluster* is formed by $k$ bounded 3-faces. An *adjacent face* of an $i$-cluster $H_i$ is a face that is adjacent to some bounded 3-face in $H_i$. Since $G \in \mathcal{A}$, one can observe that every cluster in $G$ is a $4^-$-cluster where a 4-cluster is isomorphic to $W_5$.

**Lemma 5.** *The following statements hold.*

(i) *If a 4-face $f$ is adjacent to an inner 3-face $g$, then $f$ is not adjacent to other inner 3-faces and $f$ is not adjacent to any 4-faces.*

(ii) *If an inner 3-face $f$ is adjacent to a 5-face $g$, then $f$ and $g$ are not adjacent to any 4-faces.*

(iii) *Every adjacent face of a 2-cluster is a $6^+$-face or the unbounded 3-face $D$.*

(iv) *Every adjacent face of a $3^+$-cluster is a $7^+$-face or the unbounded 3-face $D$.*

**Proof.**

(i) Let $f$ be a 4-face adjacent to an inner 3-face $g$ and another face $h$.
Suppose to the contrary that $h$ is an inner 3-face or a 4-face.
If $h$ is an inner 3-face, then $B(f) \cup B(g) \cup B(h)$ is a 6-cycle with two internal chords by Lemma 3 (iii), contrary to Lemma 4.
If $h$ is a 4-face, then Lemma 3 (ii) yields a 6-cycle from $B(f) \cup B(h)$, which is adjacent to a 5-cycle from $B(f) \cup B(g)$, a 4-cycle from $B(f)$, and a 3-cycle from $B(g)$, contrary to $G \in \mathcal{A}$.

(ii) Let an inner 3-face $f$ and a 5-face $g$ be adjacent. Lemma 3 (ii) yields that $B(f) \cup B(g)$ contains a 6-cycle. Thus, $f$ or $g$ is not adjacent to any 4-faces since $G \in \mathcal{A}$.

(iii) Let $f$ and $g$ be bounded 3-faces in a 2-cluster $H_2$ and let $h$ be a bounded face adjacent to $f$. By the definition, $h$ is not a bounded 3-face.
If $h$ is a 4-face, then Lemma 3 (iii) yields that $B(f) \cup B(h) \cup B(g)$ contains a 6-cycle with two internal chords, contrary to Lemma 4.
If $h$ is a 5-face, then it follows from Lemmas 3 (i) and (ii) that a 6-cycle from $B(f) \cup B(h)$ is adjacent to a 5-cycle from $B(h)$, a 4-cycle from $B(f) \cup B(g)$, and 3-cycle from $B(f)$, contrary to $G \in \mathcal{A}$.
Thus, $h$ is a $6^+$-face or the unbounded face.

(iv) Let $f_1, f_2$, and $f_3$ be the bounded 3-faces of $3^+$-cluster $H_3$ in a consecutive order.
By similar arguments as in the proof of (iii), it follows that $H_3$ cannot be adjacent to a bounded $5^-$-face.
Let $H_3$ be adjacent to a 6-face $f_4$. By Lemma 3 (ii) and an argument similar to its proof, one can show that $H_3$ is a 5-cycle with two chords. Since $B(f_4)$ is a 6-cycle by Lemma 3 (i), we have a 6-cycle adjacent to a 3-, a 4-, and a 5-cycle in $H_3$, contrary to $G \in \mathcal{A}$.

If $H_3$ is adjacent to a 6-face $f_4$, then by Lemma 3 (ii), a 6-cycle $B(f_4)$ is adjacent to a 3-, a 4-, and a 5-cycle, which are in $H_3$, contrary to $G \in \mathcal{A}$.

□

For Corollary 3 (i), it is proved by the fact that every $5^+$-vertex is not adjacent to four consecutive bounded 3-faces. Thus, each $5^+$-vertex has at least two $4^+$-faces. For Corollary 3 (ii), it is proved by Lemmas 5 (iii) and (iv) that each 3-face in $H_2^+$ is not adjacent to a 5-face. Thus, each $5^+$-vertex has at least three $4^+$-faces.

**Corollary 3.** *Let $v$ be a k-vertex in G where $v \notin V(C_0)$ and $k \geq 5$. It follows that:*

(i) *$v$ is incident to at most $k-2$ bounded 3-faces;*
(ii) *$v$ is incident to at most $k-3$ bounded 3-faces, if $v$ has an incident 5-face.*

**Proof.** If $v$ is incident to $k-1$ bounded 3-faces, then there are four consecutive bounded faces forming a 4-cluster that is not a wheel, contrary to $G \in \mathcal{A}$. This proves (i). It follows from Lemmas 5 (iii) and (iv) that each 3-face in a $2^+$-cluster is not adjacent to a 5-face. Thus, each $5^+$-vertex incident to a 5-face must be incident to at least three $4^+$-faces. This proves (ii). □

**Lemma 6** (Lemma 3.6 in [14]). *$C(l_1, \ldots, l_k)$ is defined to be a cycle $C = x_1 \ldots x_m$ with k internal chords such that $x_1$ is their common endpoint and $V(C) \cap V(C_0) = \emptyset$. Suppose $x_2$ or $x_m$ is not the endpoint of any chords in C. If $d(x_1) \leq k+3$, then some $i \in \{2, 3, \ldots, m\}$ satisfies $d(x_i) \geq 5$ (See the proof in Lemma A4).*

**Lemma 7.** *Let a 4-vertex $v$ be incident to bounded faces $f_1, \ldots, f_4$ in cyclic order and let $F = B(f_1) \cup B(f_2)$, where $V(F) \cap V(C_0) = \emptyset$. If $(d(f_1), d(f_2)) = (3,3)$ or $(3,5)$, then there is a vertex $w \in V(F) - \{v\}$ such that $d(w) \geq 5$.*

**Proof.** If $(d(f_1), d(f_2)) = (3,3)$, it follows from Lemma 3 (ii) that $F = C(3,3)$. Moreover, $F$ has exactly one chord, otherwise there is a separating 3-cycle, which contradicts Lemma 1.

If $(d(f_1), d(f_2)) = (3,5)$, it follows from Lemma 3 (ii) that $F = C(3,5)$. Moreover, $F$ has exactly one chord by Lemma 4.

The proof is complete by Lemma 6. □

**Lemma 8.** *Let $v$ be a 5-vertex with incident bounded faces $f_1, \ldots, f_5$ in a cyclic order. Let $F = B_1 \cup B_2 \cup B_3$ where $B_i$ denote $B(f_i)$ and $V(F) \cap V(C_0) = \emptyset$. If $(d(f_1), d(f_2), d(f_3)) = (5,3,5)$, then there exists $w \in V(F) - \{v\}$ such that $d(w) \geq 5$.*

**Proof.** Let $B_1 = x_1x_2x_3x_4x_5$, $B_2 = x_1x_5x_6$, and $B_3 = x_1x_6x_7x_8x_9$, where $x_1 = v$. It follows from Lemma 3 (ii) that $B_1 \cup B_2$ is a $C(3,5)$ and $B_2 \cup B_3$ is a $C(3,5)$. Suppose to the contrary that $F$ is not a $C(5,3,5)$. Then, there is $i \in \{2,3,4\}$ and $j \in \{7,8,9\}$ such that $x_i = x_j$. If $i = 2$, then a 6-cyclce $x_1x_5x_6x_7x_8x_9$ has a triangular chord $x_1x_6$ and a chord $x_1x_j$, contrary to Lemma 4. If $i = 2$, then a 6-cyclce $x_1x_5x_6x_7x_8x_9$, has a triangular chord $x_1x_6$ and a chord $x_5x_j$, contrary to Lemma 4.

Suppose that $i = 3$. Note that a 6-cycle $C = x_1x_5x_6x_7x_8x_9$ is adjacent to a 3-cycle $x_1x_5x_6$ and a 5-cycle $x_1x_6x_7x_8x_9$. It suffices to show that $C$ is adjacent to a 4-cycle to get a contradiction. If $x_3 = x_7$, then $C$ is adjacent to a 4-cycle $x_1x_2x_7x_6$. If $x_3 = x_8$, then $C$ is adjacent to a 4-cycle $x_1x_2x_8x_9$. If $x_3 = x_9$, then $C$ is adjacent to a 4-cycle $x_1x_5x_4x_9$.

Thus, $F = C(5,3,5)$. By Lemma 6, it remains to show that $x_2$ or $x_m$ is not an endpoint to a chord in $C$, say $x_1x_2 \ldots x_9$. Suppose $C$ has a chord $e = x_2x_i$, otherwise the desired condition is obtained. If $x_2x_9 \in E(G)$, then we have separating 3-cycle $x_1x_2x_9$, contrary to Lemma 1. By Lemma 4, we have $i \notin \{4,5,6\}$. Then, $x_i = x_7$ or $x_8$. By Lemma 4, $x_9$ is not adjacent to $x_6$ or $x_7$. Thus, a chord of $C'$ cannot have $x_9$ as its endpoint. □

**Corollary 4.** *Let $v$ be a 4-vertex incident to bounded faces $f_1, \ldots, f_4$ in cyclic order, where $f_1$ is an inner 5-face, $f_2$ is an inner 3-face, $f_3$ is an inner 5-face, and $f_4$ is an arbitrary face. If $f_3$ is a poor 5-face, then $f_1$ is a rich 5-face or an improper semi-rich 5-face.*

**Proof.** Let $B_1 = x_1x_2x_3x_4x_5$, $B_2 = x_1x_5x_6$, and $B_3 = x_1x_6x_7x_8x_9$, where $x_1 = v$. Let $f_3$ be a poor 5-face. Then, $x_1, x_6, x_7, x_8$, and $x_9$ are 4-vertices. By Lemma 7, $x_5$ is a $5^+$-vertex. If $x_2, x_3$, or $x_4$ is a $5^+$-vertex, then $f_1$ is a rich 5-face. Now suppose that $x_2, x_3$, and $x_4$ are 4-vertices. If $f_4$ is a not a 3-face, then $f_1$ is an improper semi-rich 5-face. If $f_4$ is a 3-face, then $x_2$ is a $5^+$-vertex by considering $f_1$ and $f_4$ into Lemma 7, a contradiction. □

## 4. Discharging Process

In this section, we use the discharging procedure to get a contradiction and complete the proof of Theorem 2.

For each vertex and bounded face $x \in V(G) \cup F(G)$, let an initial charge of $x$ be $\mu(x) = d(x) - 4$ and let $\mu(D) = d(D) + 4 = 7$ where $D$ is the unbounded face. By Euler's Formula, $\sum_{x \in V \cup F} \mu(x) = 0$. Let $\mu^*(x)$ be the charge after the discharge procedure of $x \in V \cup F$. To get a contradiction, we prove that $\mu^*(x) \geq 0$ for each $x \in V(G) \cup F(G)$ and $\mu^*(D) > 0$.

Let $w(x \to f)$ be the transferred charge from $x$ to a face $f$ where $x$ is a vertex or a face.

**The discharging rules:**

**(R1)** Let $v$ be a 5-vertex where $v \notin V(C_0)$ and $f$ be an incident 3-face of $v$.

$$w(v \to f) = \begin{cases} \frac{1}{2}, & \text{if } v \text{ is incident to some 5-faces,} \\ \frac{1}{7}, & \text{if } v \text{ is not incident to any 5-faces and} \\ & f \text{ is not adjacent to any incident 3-faces of } v, \\ \frac{3}{7}, & \text{if } v \text{ is not incident to any 5-faces and} \\ & f \text{ is adjacent to exactly one incident 3-face of } v. \end{cases}$$

**(R2)** Let $v$ be a $6^+$-vertex where $v \notin V(C_0)$ and $f$ be an incident 3-face of $v$.

$$w(v \to f) = \begin{cases} \frac{2}{3}, & \text{if } v \text{ is incident to some 5-faces,} \\ \frac{1}{2}, & \text{if } v \text{ is not incident to any 5-faces.} \end{cases}$$

Let $g$ be a $k$-face with $k$ incident vertices, say $v_1, v_2, \ldots, v_k$ in cyclic order, and with $k$ adjacent faces, say $f_1, f_2, \ldots, f_k$ in cyclic order. Let $f_i$ be incident to $v_i$ and $v_{i+1}$ ($i$ is taken modulo $k$).

**(R3)** Let $g$ be a 4-face.
$w(g \to f_i) = \frac{1}{3}$ if $f_i$ is an inner 3-face.

**(R4)** Let $g$ be a 5-face.
$w(g \to f_i) = \frac{1}{5}$ if $f_i$ is a 4-face.

- Let $g$ be an inner poor 5-face.
  $w(g \to f_i) = \frac{1}{5}$ if $f_i$ is an inner 3-face.
- Let $g$ be an inner proper semi-rich 5-face.
  $w(g \to f_i) = \frac{1}{3}$ if $f_i$ is an inner 3-face where both $v_i$ and $v_{i+1}$ are 4-vertices.
- Let $g$ be an inner rich 5-face or an inner improper semi-rich 5-face.
  $$w(g \to f_i) = \begin{cases} \frac{1}{6}, & \text{if } f_i \text{ is an inner 3-face where exactly one of } v_i \text{ and } v_{i+1} \text{ is a 4-vertex.} \\ \frac{1}{3}, & \text{if } f_i \text{ is an inner 3-face where both } v_i \text{ and } v_{i+1} \text{ are 4-vertices.} \end{cases}$$
- Let $g$ be an extreme 5-face.
  $w(g \to f_i) = \frac{2}{3}$ if $f_i$ is an inner 3-face.

**(R5)** Let $g$ be a $k$-face where $k \geq 6$.

$$w(g \to f_i) = \begin{cases} \theta(f_i) + \chi(f_{i+1})\theta(f_{i+1}) + \chi(f_{i-1})\theta(f_{i-1}), & \text{if } f_i \text{ is a } 4^- \text{-face}, \\ 0, & \text{otherwise}. \end{cases}$$ where

$$\chi(f_i) = \begin{cases} \frac{1}{2}, & \text{if } f_i \text{ is not a } 4^- \text{-face}, \\ 0, & \text{otherwise}. \end{cases}$$

and $\theta(f_i) = \frac{d(g)-4}{d(g)}$ for each $i \in \{1, 2, \ldots, n\}$.

**(R6)** The unbounded face $D$ incident to a vertex $v$ receives charge $\mu(v)$ from $v$ but gives 1 to each of its intersecting 3-faces and 5-faces.

It follows from (R6) that $\mu^*(v) = 0$ for every $v \in V(C_0)$. By this, we consider only a vertex $v$ such that $v \notin V(C_0)$.

**CASE 1:** $v$ is a 5-vertex.

- $v$ is incident to some 5-faces.
  Then, $v$ has at most two incident 3-faces by Corollary 3. Thus, $\mu^*(v) \geq \mu(v) - 2 \times \frac{1}{2} = 0$ by (R1).
- $v$ is not incident to any 5-faces.
  It follows from Corollary 3 that $v$ is incident to at most three 3-faces. Then, $v$ has at most two incident 3-faces, which are adjacent to exactly one incident 3-face of $v$. Thus, $\mu^*(v) \geq \mu(v) - 2 \times \frac{3}{7} - \frac{1}{7} = 0$ by (R1).

**CASE 2:** $v$ is a $6^+$-vertex.

- $v$ is incident to some 5-faces.
  It follows from Corollary 3 that $v$ is incident to not more than $d(v) - 3$ of 3-faces. Thus, $\mu^*(v) \geq \mu(v) - (d(v) - 3) \times \frac{2}{3} = (d(v) - 4) - (\frac{2d(v)}{3} - 2) = \frac{d(v)}{3} - 2 \geq 0$ by (R2) and $d(v) \geq 6$.
- $v$ is not incident to any 5-faces.
  It follows from Corollary 3 that $v$ is incident to at most $d(v) - 2$ 3-faces. Thus, $\mu^*(v) \geq \mu(v) - (d(v) - 2) \times \frac{1}{2} = (d(v) - 4) - (\frac{d(v)}{2} - 1) = \frac{d(v)}{2} - 3 \geq 0$ by (R2) and $d(v) \geq 6$.

For a 3-face in an $i$-cluster $H_i$, we consider the total of charges in the same cluster. That is $\mu(H_i) = -i$ and we show that $\mu^*(H_i) \geq 0$ instead.

**CASE 3:** $f$ is a 3-face in an $i$-cluster, say $H_i$ where $|V(H_i) \cap V(C_0)| \geq 1$.

- If $|V(H_1) \cap V(C_0)| \geq 1$, then $\mu^*(H_1) \geq \mu(H_1) + 1 = 0$ by (R6).
- If $|V(H_2) \cap V(C_0)| = 1$, then each adjacent face of $H_2$ is a $6^+$-face by Lemma 5 (iii). Thus, $\mu^*(H_2) \geq \mu(H_2) + 1 + 4 \times \frac{1}{3} > 0$ by (R5) and (R6).
- If $|V(H_2) \cap V(C_0)| \geq 2$, then each 3-face in $H_2$ is an extreme 3-face. Thus, $\mu^*(H_2) \geq \mu(H_2) + 2 \times 1 = 0$ by (R6).
- If $|V(H_3) \cap V(C_0)| = 1$, then each adjacent face of $H_3$ is a $7^+$-face by Lemma 5 (iv). Thus, $\mu^*(H_3) \geq \mu(H_3) + 1 + 5 \times \frac{3}{7} > 0$ by (R5) and (R6).
- If $|V(H_3) \cap V(C_0)| = 2$, then $H_3$ is adjacent to at least four $7^+$-faces by Lemma 5 (iv). Moreover, there are at least two extreme 3-faces in $H_3$. Thus, $\mu^*(H_3) \geq \mu(H_3) + 2 + 4 \times \frac{3}{7} > 0$ by (R5) and (R6).
- If $|V(H_3) \cap V(C_0)| = 3$, then each 3-face in $H_3$ is an extreme 3-face. Thus, $\mu^*(H_3) \geq \mu(H_3) + 3 \times 1 = 0$ by (R6).
- If $|V(H_4) \cap V(C_0)| = 1$, then there are two extreme 3-faces in $H_4$ and each adjacent face of $H_4$ is a $7^+$-face by Lemma 5 (iv). If each vertex in $V(H_4) - V(C_0)$ is a 4-vertex, we have $\mu^*(H_4) \geq \mu(H_4) + 2 + 2 \times \frac{9}{14} + 2 \times \frac{6}{7} > 0$ by (R5) and (R6). Otherwise, there is a vertex in $V(H_4) - V(C_0)$, which is not a 4-vertex, then we have $\mu^*(H_4) \geq \mu(H_4) + 2 + 6 \times \frac{3}{7} > 0$ by (R1), (R2), (R5), and (R6).
- If $|V(H_4) \cap V(C_0)| = 2$, then there are at least three extreme 3-faces in $H_4$. Moreover, $H_3$ is adjacent to at least three $7^+$-faces by Lemma 5 (iv). Thus, $\mu^*(H_4) \geq \mu(H_4) + 3 \times 1 + 3 \times \frac{3}{7} > 0$ by (R5) and (R6).
- If $|V(H_4) \cap V(C_0)| = 3$, then each 3-face in $H_4$ is an extreme 3-face. Thus, $\mu^*(H_4) \geq \mu(H_4) + 4 \times 1 = 0$ by (R6).

**CASE 4:** $f$ is an inner 3-face in a 1-cluster.

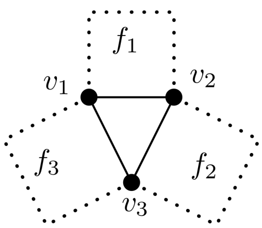
Let $v_1, v_2, v_3$ be three incident vertices in cyclic order and $f_1$, $f_2$, $f_3$ be three adjacent faces in cyclic order. Moreover, let $f_i$ be incident to $v_i$ and $v_{i+1}$ ($i$ is taken modulo 3) (See Figure 1).

**Subcase 4.1:** $f$ is not adjacent to any 5-faces.

Thus, $\mu^*(f) \geq \mu(f) + 3 \times \frac{1}{3} = 0$ by (R3) and (R5).

Next, we consider that $f$ is adjacent to some 5-faces in Subcases 4.2 to 4.5. It follows from Lemma 5 (ii) and the assumption of Case 4 that $f$ is not adjacent to a $4^-$-faces.

**Subcase 4.2:** An inner 3-face $f$ is adjacent to some extreme 5-faces.




CASE 4

**Figure 1.** The configuration in CASE 4.

WLOG, let $f_1$ be an extreme 5-face. Then, $w(f_1 \to f) = \frac{2}{3}$ by (R4).

- $f_i$ is not an inner 5-face where $i = 2$ or 3.
  Then, $f_i$ is an extreme 5-face or a $6^+$-face. Thus, $w(f_i \to f) \geq \frac{1}{3}$ by (R4) and (R5). Therefore, $\mu^*(f) \geq \mu(f) + \frac{2}{3} + \frac{1}{3} = 0$.
- $f_2$ and $f_3$ are inner 5-faces.
  - If $v_i$ is a $5^+$-vertex for some $i \in \{1,2,3\}$, then $w(v_i \to f) = \frac{1}{2}$ by (R1) and (R2). Thus, $\mu^*(f) \geq \mu(f) + \frac{2}{3} + \frac{1}{2} > 0$.
  - If $v_i$ is a 4-vertex for each $i \in \{1,2,3\}$, then $w(f_2 \to f) \geq \frac{1}{5}$ and $w(f_3 \to f) \geq \frac{1}{5}$ by (R4). Thus, $\mu^*(f) \geq \mu(f) + \frac{2}{3} + 2 \times \frac{1}{5} > 0$.

We now consider the cases that each adjacent 5-face of $f$ is not an extreme 5-face.

**Subcase 4.3:** $f$ is a poor 3-face.

It follows from Lemma 7 that $f_i$ is not a poor 5-face for each $i \in \{1,2,3\}$. Thus, $\mu^*(f) \geq \mu(f) + 3 \times \frac{1}{3} = 0$ by (R4) and (R5).

**Subcase 4.4:** $f$ is a semi-rich 3-face.

Let $v_1$ be a $5^+$-vertex. By symmetry, we only consider two following cases.

- $f_2$ is a poor 5-face.
  Then $w(f_2 \to f) \geq \frac{1}{5}$ by (R4). Note that if $f_i$ is an improper semi-rich 5-face, a rich 5-face, or a $6^+$-face where $i \in \{1,3\}$, then $w(f_i \to f) \geq \frac{1}{6}$ by (R4) and (R5).
  - If $f_i$ is a 5-face for $i = 1$ or 3, then $f_i$ is an improper semi-rich 5-face or a rich 5-face by Corollary 4. It follows that $w(v_1 \to f) \geq \frac{1}{2}$ by (R1) and (R2). Thus, $\mu^*(f) \geq \mu(f) + 2 \times \frac{1}{6} + \frac{1}{5} + \frac{1}{2} > 0$.
  - If $f_1$ and $f_3$ are $6^+$-faces, then $w(v_1 \to f) \geq \frac{1}{7}$ by (R1) and (R2). Thus, $\mu^*(f) \geq \mu(f) + 2 \times \frac{1}{3} + \frac{1}{5} + \frac{1}{7} > 0$.
- $f_2$ is a $5^+$-face but not a poor 5-face.
  Then $w(f_2 \to f) \geq \frac{1}{3}$ by (R4) and (R5). If $f_1$ and $f_3$ are $6^+$-faces, then $\mu^*(f) \geq \mu(f) + 3 \times \frac{1}{3} = 0$ by (R5). If $v_1$ is a $6^+$-vertex and $f_1$ or $f_3$ is a 5-face, then $\mu^*(f) \geq \mu(f) + \frac{1}{3} + \frac{2}{3} = 0$ by (R2). Thus, it remains to check the case that $f_1$ or $f_3$ is a 5-face and $v_1$ is a 5-vertex. Note that $w(v_1 \to f) \geq \frac{1}{2}$ by (R1).
  - If $f_1$ and $f_3$ are 5-faces, then $f_i$ is a rich 5-face for $i = 1$ or 3 by Lemma 8. It follows that $w(f_i \to f) = \frac{1}{6}$ by (R4). Thus, $\mu^*(f) \geq \mu(f) + \frac{1}{3} + \frac{1}{2} + \frac{1}{6} = 0$.

- If $f_1$ is a 5-face and $f_3$ is a $6^+$-face, then $w(f_3 \to f) \geq \frac{1}{3}$ by (R5). Thus, $\mu^*(f) \geq \mu(f) + 2 \times \frac{1}{3} + \frac{1}{2} = 0$.

**Subcase 4.5:** $f$ is an inner rich 3-face.

Let $v_1$ and $v_2$ be $5^+$-vertices. Recall that $f_1, f_2$, and $f_3$ are inner $5^+$-faces and at least one of them is a 5-face. By symmetry, we only consider two following cases.

- $f_1$ is a 5-face or $f_2$ and $f_3$ are 5-faces.
  That makes $v_1$ and $v_2$ incident to some 5-faces. Then, $w(v_1 \to f) \geq \frac{1}{2}$ and $w(v_2 \to f) \geq \frac{1}{2}$ by (R1) and (R2). Thus, $\mu^*(f) \geq \mu(f) + 2 \times \frac{1}{2} = 0$.
- $f_1$ is a $6^+$-face and either $f_2$ or $f_3$ is a $6^+$-face.
  WLOG, let $f_2$ be a 5-face. That makes $v_2$ incident to some 5-faces. Then $w(f_1 \to f) \geq \frac{1}{3}$ and $w(f_3 \to f) \geq \frac{1}{3}$ by (R5) and $w(v_2 \to f) \geq \frac{1}{2}$ by (R1) and (R2). Thus, $\mu^*(f) \geq \mu(f) + 2 \times \frac{1}{3} + \frac{1}{2} > 0$.

**CASE 5:** $f$ is a 3-face in a 2-cluster $H_2$ where $|V(H_2) \cap V(D)| = 0$.

Let $H_2$ be a 4-cycle $v_1v_2v_3v_4$ with a chord $v_1v_3$. Let $f_1$, $f_2, f_3, f_4$ be four adjacent faces of $H_2$ in cyclic order. Moreover, let $f_i$ be incident to $v_i$ and $v_{i+1}$ ($i$ is taken modulo 4) (See Figure 2). It follows from Lemma 5 (iii) that $f_1$, $f_2, f_3$, and $f_4$ are $6^+$-faces. By symmetry, we only consider two following cases.

- $v_1$ and $v_3$ are 4-vertices.
  Then $w(f_i \to H_2) \geq \frac{1}{2}$ for $i \in \{1,2,3,4\}$ by (R5). Thus, $\mu^*(H_2) \geq \mu(H_2) + 4 \times \frac{1}{2} = 0$.
- $v_1$ is a $5^+$-vertex and $v_3$ is a $4^+$-vertex.
  Then $w(v_1 \to H_2) \geq 2 \times \frac{3}{7}$ by (R1) and (R2), and $w(f_i \to H_2) \geq \frac{1}{3}$ for $i \in \{1,2,3,4\}$ by (R5). Thus, $\mu^*(H_2) \geq \mu(H_2) + 4 \times \frac{1}{3} + 2 \times \frac{3}{7} > 0$.

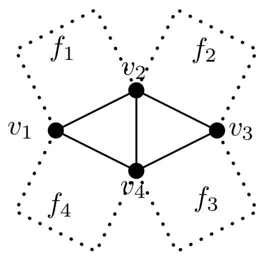



**Figure 2.** The configuration in CASE 5.

**CASE 6:** $f$ is a 3-face in a 3-cluster $H_3$ where $|V(H_3) \cap V(D)| = 0$.

Let $H_3$ be a 5-cycle $v_1v_2v_3v_4v_5$ with two chords $v_1v_3$ and $v_1v_4$. Let $f_1$, $f_2, f_3, f_4, f_5$ be five adjacent faces of $H_3$ in cyclic order. Moreover, let $f_i$ be incident to $v_i$ and $v_{i+1}$ ($i$ is taken modulo 5). Note that $f_1$ and $f_5$ may be the same face (See Figure 3). It follows from Lemma 5 (iv) that $f_1$, $f_2, f_3, f_4$, and $f_5$ are $7^+$-faces. By symmetry, we only consider the two following cases.

- $v_3$ and $v_4$ are 4-vertices.
  Then $w(f_1 \to H_3) \geq \frac{3}{7}$ and $w(f_5 \to H_3) \geq \frac{3}{7}$ by (R5), $w(f_2 \to H_3) \geq \frac{9}{14}$ and $w(f_4 \to H_3) \geq \frac{9}{14}$ by (R5), and $w(f_3 \to H_3) = \frac{6}{7}$ by (R5). Thus, $\mu^*(H_3) \geq \mu(H_3) + 2 \times \frac{3}{7} + 2 \times \frac{9}{14} + \frac{6}{7} = 0$.
- $v_3$ is a $5^+$-vertex and $v_4$ is a $4^+$-vertex.
  Then $w(v_3 \to H_3) \geq 2 \times \frac{3}{7}$ by (R1) and (R2), and $w(f_i \to H_3) \geq \frac{3}{7}$ for $i \in \{1,2,3,4,5\}$ by (R5). Thus, $\mu^*(H_3) \geq \mu(H_3) + 7 \times \frac{3}{7} = 0$.

**CASE 7:** $f$ is a 3-face in a 4-cluster $H_4$ where $|V(H_4) \cap V(D)| = 0$.

Let $H_4$ be the wheel $W_5$ where $v_5$ is a hub and $v_1$,$v_2$, $v_3$, and $v_4$ are external vertices in cyclic order. Let $f_1$, $f_2, f_3, f_4$ be four adjacent faces of $H_4$ in cyclic order. Moreover, let $f_i$ be incident to $v_i$ and $v_{i+1}$ ($i$ is taken modulo 4) (See Figure 4). By Lemma 5 (iv), $f_1$, $f_2, f_3$, and $f_4$ are $7^+$-faces. Moreover, at least two vertices in $\{v_1, v_2, v_3, v_4\}$ are $5^+$-vertices by Lemma 7. By symmetry, we only consider the three following cases.

- $v_1$ and $v_2$ are $5^+$-vertices and $v_3$ and $v_4$ are 4-vertices.
  Then $w(f_1 \to H_4) \geq \frac{3}{7}$, $w(f_2 \to H_4) \geq \frac{9}{14}$, $w(f_4 \to H_4) \geq \frac{9}{14}$, $w(f_3 \to H_4) = \frac{6}{7}$ by (R5), $w(v_1 \to H_4) \geq 2 \times \frac{3}{7}$ and $w(v_2 \to H_4) \geq 2 \times \frac{3}{7}$ by (R1) and (R2). Thus, $\mu^*(H_4) \geq \mu(H_4) + 2 \times \frac{9}{14} + \frac{6}{7} + 5 \times \frac{3}{7} > 0$.
- $v_1$ and $v_3$ are $5^+$-vertices and $v_2$ and $v_4$ are 4-vertices.
  Then $w(f_i \to H_4) \geq \frac{9}{14}$ for $i \in \{1, 2, 3, 4\}$ by (R5), and $w(v_1 \to H_4) \geq 2 \times \frac{3}{7}$ and $w(v_3 \to H_4) \geq 2 \times \frac{3}{7}$ by (R1) and (R2). Thus, $\mu^*(H_4) \geq \mu(H_4) + 4 \times \frac{9}{14} + 4 \times \frac{3}{7} > 0$.
- $v_1, v_2$, and $v_3$ are $5^+$-vertices and $v_4$ is a $4^+$-vertex.
  Then $w(f_i \to H_4) \geq \frac{3}{7}$ for $i \in \{1, 2, 3, 4\}$ by (R5) and $w(v_i \to H_4) \geq 2 \times \frac{3}{7}$ for $i \in \{1, 2, 3\}$ by (R1) and (R2). Thus, $\mu^*(H_4) \geq \mu(H_4) + 10 \times \frac{3}{7} > 0$.

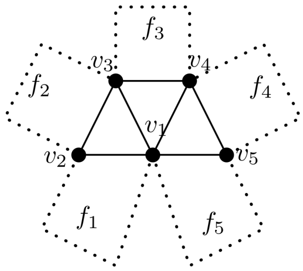



**Figure 3.** The configuration in CASE 6.

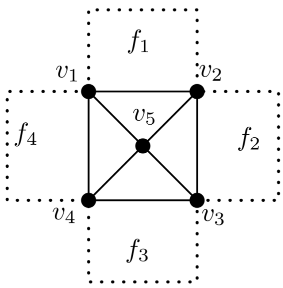



**Figure 4.** The configuration in CASE 7.

**CASE 8:** $f$ is a 4-face adjacent to an inner 3-face, say $h$.

Since $h$ is an inner 3-face, we have $|B(f) \cap B(D)| \leq 2$ where $D$ is the unbounded 3-face. Consequently, there are at least two adjacent faces of $f$, which are not $h$ and $D$. Moreover, they are $5^+$-faces by Lemma 5 (i). Thus $\mu^*(f) \geq \mu(f) - \frac{1}{3} + 2 \times \frac{1}{5} > 0$ by (R3), (R4), and (R5).

**CASE 9:** $f$ is a 5-face.

- Let $f$ be adjacent to some 4-faces.
  Then $f$ is not adjacent to any 3-faces by Lemma 5 (ii). Thus, $\mu^*(f) \geq \mu(f) - 5 \times \frac{1}{5} = 0$ by (R4).
- Let $f$ be an inner poor 5-face.
  Then $\mu^*(f) \geq \mu(f) - 5 \times \frac{1}{5} = 0$ by (R4).
- Let $f$ be an inner semi-rich 5-face.
  - If $f$ is a proper semi-rich 5-face, then $B(f)$ has three edges with two 4-endpoints. Thus, $\mu^*(f) \geq \mu(f) - 3 \times \frac{1}{3} = 0$ by (R4).
  - If $f$ an improper semi-rich 5-face, then $B(f)$ has at most two edges with two 4-

endpoints and at most two edges with exactly one $5^+$-endpoint. Thus, $\mu^*(f) \geq \mu(f) - 2 \times \frac{1}{3} - 2 \times \frac{1}{6} = 0$ by (R4).

- Let $f$ be an inner rich 5-face.
  Then $f$ has at least two incident $5^+$-vertices.
  If two incident $5^+$-vertices are not adjacent in $B(f)$, then $B(f)$ has at most one edge with two 4-endpoints. Thus, $\mu^*(f) \geq \mu(f) - \frac{1}{3} - 4 \times \frac{1}{6} = 0$ by (R4). It remains to consider the case that $f$ has exactly two incident $5^+$-vertices and they are adjacent in $B(f)$. Then $B(f)$ has two edges with two 4-endpoints and two edges with exactly one $5^+$-endpoint. Thus, $\mu^*(f) \geq \mu(f) - 2 \times \frac{1}{3} - 2 \times \frac{1}{6} = 0$ by (R4).
- Let $f$ be an extreme 5-face.
  Then $f$ has at most an adjacent inner 3-face. Thus, $\mu^*(f) \geq \mu(f) + 1 - 3 \times \frac{2}{3} = 0$ by (R4) and (R6).

**CASE 10:** $f$ is an $m$-face where $m \geq 6$.

Then, by (R5) we have $w(f \to f_i) \leq (1 - 2\chi(f_i))\theta(f_i) + \chi(f_{i+1})\theta(f_{i+1}) + \chi(f_{i-1})\theta(f_{i-1})$.

$$\begin{aligned}
\mu^*(f) &= \mu(f) - \sum_{i=1}^{m} w(f \to f_i) \\
&\geq \mu(f) - \sum_{i=1}^{m} ((1 - 2\chi(f_i))\theta(f_i) + \chi(f_{i+1})\theta(f_{i+1}) + \chi(f_{i-1})\theta(f_{i-1})) \\
&= \mu(f) - \sum_{i=1}^{m} (\theta(f_i) - 2\chi(f_i)\theta(f_i) + 2\chi(f_i)\theta(f_i)) \\
&= m - 4 - m(\frac{m-4}{m}) \\
&= 0.
\end{aligned}$$

**CASE 11:** The unbounded face $D$.

Let the number of intersecting 3-faces and 5-faces of $D$ be denoted by $f'$. Let $E(C_0, V(G) - C_0)$ denote the set of edges between $V(G) - C_0$ and $C_0$ where this set has size $e(C_0, V(G) - C_0)$. Then by (R6),

$$\begin{aligned}
\mu^*(D) &= 3 + 4 + \sum_{v \in C_0} (d(v) - 4) - f' \\
&= 1 + \sum_{v \in C_0} (d(v) - 2) - f' \\
&= 1 + e(C_0, V(G) - C_0) - f'.
\end{aligned}$$

So we may consider that $D$ sends charge 1 to each edge $e \in E(C_0, V(G) - C_0)$. So each intersecting 3-face and 5-face contains at least two edges in $E(C_0, V(G) - C_0)$. It follows that $e(C_0, V(G) - C_0) - f' \geq 0$. Thus, $\mu^*(D) > 0$.

This completes the proof.

## 5. Conclusions

We prove that every planar graph without 6-cycles simultaneously adjacent to 3-cycles, 4-cycles, and 5-cycles is DP-4-colorable. This result is a special case of two following open problems.

1. Every planar graph without $i$-cycles simultaneously adjacent to $j$-cycles, $k$-cycles, and $l$-cycles is DP-4-colorable for $\{i, j, k, l\} = \{3, 4, 5, 6\}$.

2. Every planar graph without 3-, 4-, 5-, and 6-cycles that are pairwise adjacent is DP-4-colorable.

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**Conflicts of Interest:** The authors declare no conflict of interest.

## Appendix A

**Lemma A1** ([14]). *$G$ has no separating 3-cycles.*

**Proof.** Suppose to the contrary that $G$ contains $C_0$, which is a separating 3-cycle. Consider a 3-cycle $C$, which is precolored. Note that $C$ and $C_0$ may be different. By symmetry, one may assume $V(C) \subseteq V(C_0) \cup int(C_0)$. By minimality, a precoloring can be extended from $C$ to $V(C_0) \cup int(C_0)$. After $C_0$ is colored, one can extend the coloring of $C_0$ to $ext(C_0)$. In this way, we obtain a DP-4-coloring of $G$, a contradiction. □

To prove Lemmas A3 and A4, Ref. [14] gave the definition of residual list assignment and Lemma A2 as follows.

Let $G$ be a graph with a list assignment $L$ and let $H$ be its cover. Let $F$ be an induced subgraph of $G$ and $G' = G - F$. A *restriction of* $L$ on $G'$ is a list assignment, say $L'$ such that $L'(u) = L(u)$ for every vertex $u$ in $G'$.

If a graph $H' = H[\{\{v\} \times L(v) : v \in V(G')\}]$, then we say $H'$ is a *restriction of* $H$ on $G'$. Assume $G'$ has an $(H', L')$-coloring such that $I'$ is an independent set in $H'$ with $|I'| = |V(G)| - |V(F)|$.

Define a *residual list assignment* $L^*$ of $F$ to be

$$L^*(x) = L(x) - \bigcup_{ux \in E(G)} \{c' \in L(x) : (u,c)(x,c') \in E(H) \text{ and } (u,c) \in I'\}$$

for every $x \in V(F)$.

Define *residual cover* $H^*$ to be $H[\{\{x\} \times L^*(x) : x \in V(F)\}]$.

**Lemma A2.** *Let $I'$ be an $(H', L')$-coloring of $G'$. It follows that a residual cover $H^*$ becomes a cover of $F$ with a list assignment $L^*$. Additionallay, $F$ is $(H^*, L^*)$-colorable implies $G$ is $(H, L)$-colorable.*

**Proof.** The first part follows immediately from the definitions of a cover and a residual cover.

Suppose that $F$ is $(H^*, L^*)$-colorable. Consequently, $H^*$ has an independent set $I^*$ with the size $|I^*| = |F|$. The definition of residual cover implies that no edges connect between $H^*$ and $I'$. Furthermore, $I'$ and $I^*$ are disjoint. Put them together, we have $I = I' \cup I^*$ is an independent set of $H$ such that $|I| = (|V(G)| - |V(F)|) + |V(F)| = |V(G)|$. So we can conclude that $G$ is $(H, L)$-colorable as desired. □

**Lemma A3** ([14])**.** *Each vertex in* $int(C_0)$ *has degree of at least four.*

**Proof.** Suppose otherwise that $G$ has a vertex $v$ of degree less than 4. Let $L$ be a 4-assignment in $G$ and $H$ be a cover of $G$ in which $G$ has no $(H, L)$-coloring. By minimality, we have $G' = G - x$ with an $(H', L')$-coloring where $L'$ (respectively, $H'$) is a restriction of $L$ (respectively, $H$) on $G'$. Thus, there is an independent set $I'$ with $|I'| = |G'|$ in $H'$. Let $L^*$ be a residual list assignment. Since $d(x) \leq 3$ and $|L(v)| = 4$, it follows that $|L^*(v)| \geq 1$. It is obvious that $\{(v, c)\}$ with $c \in L^*(v)$ is an independent set in $G[\{v\}]$. It follows that $G[\{v\}]$ is $(H^*, L^*)$-colorable. Lemma A2 yields that $G$ is $(H, L)$-colorable. This contradiction completes the proof. □

**Lemma A4.** *Assume* $C(l_1, \ldots, l_k)$ *is a cycle* $C = v_1 \ldots v_m$ *with* $k$ *internal chords that share an endpoint* $v_1$ *with* $V(C) \cap V(C_0) = \emptyset$*. Suppose* $v_m$ *is not an endpoint of a chords in* $C$*. If* $d(v_1) \leq k + 3$*, then there exists* $v_i \in V(C) - \{v_1\}$ *such that* $d(v_i) \geq 5$*.*

**Proof.** Let $v_m$ be not an endpoint of a chord in $C$. Suppose otherwise that $d(v_i) \leq 4$ for each $v_i \in V(C) - \{v_1\}$. Assume $G$ has a 4-assignment $L$ with a cover $H$ in which $G$ has no $(H, L)$-coloring. By minimality, $G' = G - \{v_1, \ldots, v_m\}$ has an $(H', L')$-coloring where $L'$ (respectively, $H'$) is a restriction of $L$ (respectively, $H$) in $G'$. Thus an independent set $I'$ in $H'$ with $|I'| = |G'|$ exists.

Let $L^*$ be a residual list assignment on $F$. From $|L(v)| = 4$ for every $v \in V(G)$, it follows that $|L^*(v_1)| \geq 3$ and $|L^*(v)| \geq 3$ for each vertex $v \in V(C)$ such that $v_1v$ is an edge whereas $|L^*(v_i)| \geq 2$ for each remaining vertex $v_i$ in $V(C)$. Assume $H^*$ is a residual cover of $F$. Recall that $v_m$ is not an endpoint of a chord in $C$. It follows that there exists a color $c$ in $L^*(v_1)$ with $|L^*(v_m) - \{c' : (v_1, c)(v_m, c') \in E(H^*)\}| \geq 2$. Greedily coloring $v_2, v_3, \ldots, v_m$ sequently, we have an independent set $I^*$ where its size $|I^*| = m = |F|$. It follows that $F$ is $(H^*, L^*)$-colorable. By Lemma A2, we have $G$ is $(H, L)$-colorable, which is a contradiction. □

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