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Higher-Order Asymptotic Numerical Solutions for Singularly Perturbed Problems with Variable Coefficients

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Abstract: For the purpose of solving a second-order singularly perturbed problem (SPP) with variable coefficients, a mth-order asymptotic-numerical method was developed, which decomposes the solutions into two independent sub-problems: a reduced first-order linear problem with a left-end boundary condition; and a linear second-order problem with the boundary conditions given at two ends. These are coupled through a left-end boundary condition. Traditionally, the asymptotic solution within the boundary layer is carried out in the stretched coordinates by either analytic or numerical method. The present paper executes the mth-order asymptotic series solution in terms of the original coordinates. After introducing $2(m+1)$ new variables, the outer and inner problems are transformed together to a set of $3(m+1)$ first-order initial value problems with the given zero initial conditions; then, the Runge–Kutta method is applied to integrate the differential equations to determine the $2(m+1)$ unknown terminal values of the new variables until they are convergent. The asymptotic-numerical solution exactly satisfies the boundary conditions, which are different from the conventional asymptotic solution. Several examples demonstrated that the newly proposed method can achieve a better asymptotic solution. For all values of the perturbing parameter, the method not only preserves the inherent asymptotic property within the boundary layer but also improves the accuracy of the solution in the entire domain. We derive the sufficient conditions, which terminate the series of asymptotic solutions for inner and outer problems of the SPP without having the spring term. For a specific case, we can derive a closed-form asymptotic solution, which is also the exact solution of the considered SPP.

Keywords: linear singularly perturbed problem; higher-order asymptotic-numerical method; initial value problem method; iterative method; modified asymptotic solution

MSC: 65L11

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1. Introduction

Inside the singularly perturbed problem (SPP) is a second-order derivative term multiplied by a small parameter, whose perturbation operates over a thin region across which the solution varies rapidly. This phenomenon happens for the boundary layer in fluid mechanics, the edge layer in solid mechanics, and the skin layer in electronics. To properly simulate this sort of thin-layer behavior, some special numerical methods have been developed in [1–12] by taking the singularity for the SPP into account.

The most common asymptotic approximation to the SPP is the matched asymptotic expansion method, which includes the solutions to outer and inner problems and their matching technique [13,14]. However, the asymptotic solution only satisfies a one-side boundary condition within the boundary layer, and it does not exactly match the boundary condition on another side. A modified asymptotic approximation can improve the conventional asymptotic solution. Some researchers have solved the SPP by dividing the domain of the problem into non-overlapping outer and inner regions with a terminal point near the boundary layer [15–18]. Within each region, a different type of governing equation is given with two boundary conditions being attached.

Instead of the process of determining the outer and inner expansions, matching them and then performing a composite expansion, many authors directly decomposed the asymptotic solution as the superposition of an outer solution in terms of the original variable and an inner solution in terms of stretched variable. The decomposition methods [19–22] have been broadly used to find an asymptotic expansion of the SPP due to its advantages toward the asymptotic analysis by resolving two sub-problems, which modified the original problem into a reduced problem and a boundary layer correction problem. There exists no discrepancy about the reduced problem to find the outer solution; however, there are different techniques to construct the boundary layer correction problems for the inner solutions. Padmaja and Reddy [23], according to the idea of [19], developed a numerical patching method with the Padé approximation to solve the linear SPP.

In the text books [24,25], there are many different methods, such as the WKB, a method to eliminate the first derivative term and then use the exponential phase function to approximate the singular solution, and the reproducing kernel method to reconstruct the solution in the Hilbert space. Recently, Xu et al. [26] have applied the reproducing kernel method to solve some BVPs with the optimal convergence rate.

There are different initial value methods appearing in the literature [27–32]. Some methods are replacing the original SPP by an asymptotically equivalent first-order differential equations system and solving them as the initial value problem. Reddy and Chakravarthy [30] have factorized the original problem into three first-order initial value problems. These are different from our approaches outlined above. In [33], the method of the reduction of order was proposed for solving SPP, which is replaced by a pair of initial-value problems. The integration of these initial-value problems goes in the opposite direction and the second problem can be solved only if the solution of the first one is known.

In this paper, we propose a mth-order asymptotic numerical method to treat the linear SPP by decomposing the numerical process into a coupled outer solution to the inner solution. The latter problem satisfies the derived boundary conditions. Inspired by the previous works in [34], a novel initial value problem method is developed which guarantees that all boundary conditions are satisfied. Consequently, we need to solve $3(m+1)$ first-order problems with the given zero initial values and integrate them in the same direction. The method of the reduction of order is distinguished by the fact that the original problem is replaced by initial value problems, which are easy implementations to compute.

We arrange the paper as follows. The mathematical backgrounds are given in Section 2, prescribing the basic ingredients in the asymptotic analysis for a certain example. In Section 3, we decompose the SPP into finding an inner solution and an outer solution in the newly proposed boundary layer correction problem, and introduce a transformation of the independent variable, such that the second-order SPP in the new coordinate is less sharpened within the boundary layer. Here, a higher-order asymptotic expansion method is depicted. In Section 4, we derive two functions for automatically preserving the boundary conditions, and the SPP is transformed into the initial value problems (IVPs) for two new variables. A mth-order iterative algorithm is developed to determine the unknown right-end values of the new variables, and thus, the modified asymptotic solution can be successfully determined with a few iterations. Some numerical examples are solved in Section 5 by the proposed asymptotic-numerical algorithm. A special type SPP without having the spring term is considered in Section 6, where the three main results are proven and three examples are given. For a specific relation of the damping coefficient and the forcing term, a closed-form asymptotic solution can be derived. Finally, the conclusions are drawn in Section 7.

2. Mathematical Backgrounds

We consider a second-order linear SPP with variable coefficients:

$$\varepsilon u''(x) + p(x)u'(x) + q(x)u(x) = r(x), \;\; 0 < x < 1, \tag{1}$$
$$u(0) = \alpha, \;\; u(1) = \beta. \tag{2}$$

The exhibition of boundary layers at one or both ends of the interval depends on the property of $p(x)$. Under the assumption $p(x) > 0$, the boundary layer is attached to the left end. As customarily used in the mechanical vibration problem, $r(x)$ is a forcing term, $p(x)u'(x)$ is a damping term with $p(x)$ a damping coefficient, and $q(x)u(x)$ is a spring term with $q(x)$ as a spring coefficient.

Before embarking on the higher-order asymptotic numerical solution of Equations (1) and (2), we demonstrate some basic ingredients of the first-order asymptotic analysis demonstrated via the following example:

$$\varepsilon u''(x) + u'(x) + u(x) = 0, \tag{3}$$
$$u(0) = \alpha, \;\; u(1) = \beta, \tag{4}$$

where $\varepsilon > 0$ is a sufficiently small perturbing parameter. The exact solution is

$$u_e(x) = \frac{1}{e^{a_2} - e^{a_1}}[(\alpha e^{a_2} - \beta)e^{a_1 x} + (\beta - \alpha e^{a_1})e^{a_2 x}], \tag{5}$$

where

$$a_1 = \frac{-1 + \sqrt{1 - 4\varepsilon}}{2\varepsilon}, \;\; a_2 = \frac{-1 - \sqrt{1 - 4\varepsilon}}{2\varepsilon}, \tag{6}$$

and $0 < \varepsilon < 0.25$ is the admissible range of ε.

2.1. Conventional Asymptotic Match Method

We demonstrate the first-order asymptotic matched method to approximate Equations (3) and (4). The outer solution is

$$u_o(x) = y_0(x) + \varepsilon y_1(x) + \ldots. \tag{7}$$

Inserting it into Equation (3) and by equating the coefficients preceding $\varepsilon^0 = 1$ and ε, we have

$$\begin{cases} y_0'(x) + y_0(x) = 0, \; y_0(1) = \beta, \\ y_1'(x) + y_1(x) = -y_0''(x), \; y_1(1) = 0. \end{cases} \tag{8}$$

Hence, we can derive the first-order outer solution:

$$u_o(x) = \beta[1 + \varepsilon(1 - x)]e^{1-x} + \mathcal{O}(\varepsilon^2). \tag{9}$$

To seek the inner solution $u_i(x)$ of Equations (3) and (4), a stretched coordinate is considered:

$$\zeta := \frac{x}{\varepsilon}, \tag{10}$$

such that

$$\frac{du(x)}{dx} = \frac{1}{\varepsilon}\frac{du(\zeta)}{d\zeta}, \;\; \frac{d^2u(x)}{dx^2} = \frac{1}{\varepsilon^2}\frac{d^2u(\zeta)}{d\zeta^2}. \tag{11}$$

Inserting them into Equation (3) and multiplying the resultant by ε yields

$$\frac{d^2u(\zeta)}{d\zeta^2} + \frac{du(\zeta)}{d\zeta} + \varepsilon u(\zeta) = 0. \tag{12}$$

The inner solution reads as

$$u_i(x) = w_0(x) + \varepsilon w_1(x) + \ldots, \tag{13}$$

which, as it is inserted into Equation (12) and by equating the coefficients preceding $\varepsilon^0 = 1$ and ε, generates

$$\begin{cases} w_0''(\zeta) + w_0'(\zeta) = 0, \ w_0(0) = \alpha, \\ w_1''(\zeta) + w_1'(\zeta) = -w_0(\zeta), \ w_1(0) = 0. \end{cases} \tag{14}$$

Solving Equation (14), the first-order inner solution is given by

$$u_i(\zeta) = \alpha - c_1(1 - e^{-\zeta}) + \varepsilon\{c_2(1 - e^{-\zeta}) - [\alpha - c_1(1 + e^{-\zeta})]\zeta\} + \mathcal{O}(\varepsilon^2), \tag{15}$$

where c_1 and c_2 are integration constants, determined by the matching principle [35]:

$$u_i^o := \lim_{\zeta \to \infty} u_i = \lim_{x \to 0} u_o =: u_o^i, \tag{16}$$

which leads to $c_1 = \alpha - e\beta$ and $c_2 = e\beta$. Hence, the first-order inner solution is given by

$$u_i(\zeta) = e\beta + (\alpha - e\beta)e^{-\zeta} + \varepsilon\{e\beta(1 - e^{-\zeta}) - [e\beta - (\alpha - e\beta)e^{-\zeta}]\zeta\}. \tag{17}$$

Finally, the first-order uniform asymptotic solution denoted by $u_a(x)$ can be obtained by a composition technique [35]:

$$u_a(x) = u_o + u_i - u_o^i = u_o + u_i - u_i^o = \beta[1 + \varepsilon(1 - x)]e^{1-x} + [(\alpha - e\beta)(1 + x) - e\beta\varepsilon]e^{-x/\varepsilon}. \tag{18}$$

We can observe that

$$u_a(0) = \alpha, \ u_a(1) = \beta + [2(\alpha - e\beta) - e\beta\varepsilon]e^{-1/\varepsilon} \neq \beta. \tag{19}$$

This means that the asymptotic solution $u_a(x)$ does not match the right-end boundary condition in Equation (4), which has an absolute error $|2(\alpha - e\beta) - e\beta\varepsilon|e^{-1/\varepsilon}$. When ε is small, the error is negligible; however, when ε is a moderate value, the error cannot be neglected. Thus, it may induce a large error of the original asymptotic solution (18) in the entire domain.

2.2. A New Asymptotic Method

In order to improve the drawback of the conventional asymptotic method, we propose a new asymptotic method to approximate Equations (3) and (4). We express Equation (14) in terms of x with the aid of Equation (11):

$$\begin{cases} \varepsilon^2 w_0''(x) + \varepsilon w_0'(x) = 0, \\ \varepsilon^2 w_1''(x) + \varepsilon w_1'(x) = -w_0(x), \end{cases}$$

and then, we obtain

$$\varepsilon w_0''(x) + w_0'(x) = 0, \tag{20}$$

$$\varepsilon^2 w_1''(x) + \varepsilon w_1'(x) = -w_0(x), \tag{21}$$

where the prime denotes the differential with respect to x. Letting $z_j = \varepsilon^j w_j, \ j = 0, 1$, Equation (21) is changed to

$$\varepsilon z_1''(x) + z_1'(x) = -z_0(x), \tag{22}$$

and the inner solution (13) becomes

$$u_i(x) = z_0(x) + z_1(x) + \ldots. \tag{23}$$

Now, we solve

$$\begin{cases} \varepsilon z_0''(x) + z_0'(x) = 0, \; z_0(0) = \alpha - u_o(0), \; z_0(1) = 0, \\ \varepsilon z_1''(x) + z_1'(x) = -z_0(x), \; z_1(0) = z_1(1) = 0, \end{cases} \tag{24}$$

and

$$u(x) = u_o(x) + u_i(x) = u_o(x) + z_0(x) + z_1(x) \tag{25}$$

represents a new first-order asymptotic solution of Equations (3) and (4), where $u_o(x)$ is still given by Equation (9) with

$$u_o(0) = \beta e(1+\varepsilon). \tag{26}$$

Inserting it into Equation (24), $z_0(0) = \alpha - \beta e(1+\varepsilon)$ is obtained.

Instead of considering the left-end condition in Equation (14), using the matching method to determine the integration constants c_1 and c_2 and then finding the composition solution (18), we directly subject $u_i(x)$ to the boundary conditions in Equation (24) and employ the direct sum in Equation (25) to determine the new asymptotic solution.

Through some operations on Equation (24), we can derive the new first-order asymptotic solution:

$$u(x) = \beta[1+\varepsilon(1-x)]e^{1-x} + A + Be^{-x/\varepsilon} - Ax + Bxe^{-x/\varepsilon} + \frac{2A[1-e^{-x/\varepsilon}]}{1-e^{-1/\varepsilon}}, \tag{27}$$

where

$$A := \frac{[e\beta(1+\varepsilon)-\alpha]e^{-1/\varepsilon}}{1-e^{-1/\varepsilon}}, \;\; B := \frac{\alpha - e\beta(1+\varepsilon)}{1-e^{-1/\varepsilon}}. \tag{28}$$

Here, $u(x)$ in Equation (27) exactly satisfies the boundary conditions $u(0) = \alpha$ and $u(1) = \beta$ in Equation (4).

Given $\varepsilon = 0.245$, $\alpha = 0$ and $\beta = 1$, Figure 1 compares u_e, u_a and the present result u in Equation (27), where the maximum error (ME) of $|u_e - u_a|$ is 1.41×10^{-1} and the ME of $|u_e - u|$ is 6.85×10^{-2}. The present u in Equation (27) is closer to the exact solution than that of $u_a(x)$ in Equation (18). In Table 1, we compare the ME1 of $|u_e - u|$ and the ME2 of $|u_e - u_a|$ for different values of ε, which are close when $\varepsilon \le 0.1$. For $\varepsilon \ge 0.2$, ME1 is smaller than ME2. Equations (27) and (18) possess the same asymptotic property, since

$$A \to 0, \; 1 - e^{-1/\varepsilon} \to 1, \;\; \text{when } \varepsilon \to 0.$$

In this situation,

$$B \to \alpha - e\beta, \text{ or } B \to \alpha - e\beta(1+\varepsilon),$$

and Equations (27) and (18) are the same.

Table 1. Comparing ME1 and ME2 obtained from the present solution and the original asymptotic solution to the exact one with different ε.

ε	ME1	ME2
0.24	6.566×10^{-2}	1.311×10^{-1}
0.2	4.693×10^{-2}	7.033×10^{-2}
0.1	2.485×10^{-2}	2.198×10^{-2}
0.01	5.931×10^{-4}	5.895×10^{-4}
0.001	6.664×10^{-6}	6.661×10^{-6}
0.0001	6.775×10^{-8}	6.774×10^{-8}

In summary, we can say that the new asymptotic solution (27) not only preserves the same asymptotic behavior as that of the original asymptotic solution (18), but also enhances the accuracy in the entire domain. The present method is easier to work with than the original asymptotic matching method and is suitable for the linear SPP with all the values of the perturbing parameter ε.

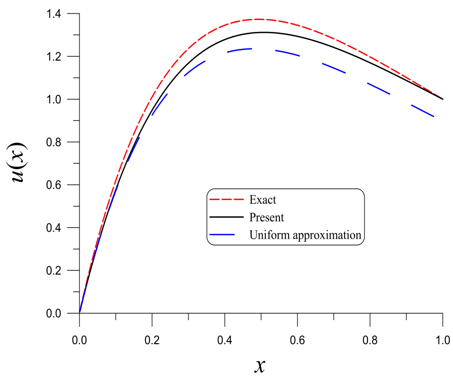

Figure 1. For a given example, comparing the exact solution, a uniform approximation and the present solution.

3. Higher-Order Asymptotic Expansion Method

Motivated by the analysis in Section 2.2, we extend it to a higher-order asymptotic expansion method by assuming

$$u(x) = u_o(x) + u_i(x) = \sum_{j=0}^{m} \varepsilon^j y_j(x) + \sum_{j=0}^{m} \varepsilon^j w_j(x) = \sum_{j=0}^{m} \varepsilon^j y_j(x) + \sum_{j=0}^{m} z_j(x), \tag{29}$$

where m is the order of asymptotic approximation, and

$$z_j(x) := \varepsilon^j w_j(x),\ j = 0, 1, \ldots, m. \tag{30}$$

Inserting $u_o(x)$ into Equation (1) and equating the coefficients preceding ε^j, $j = 0, 1, \ldots, m$, we can derive

$$\begin{cases} y_0'(x) = \frac{r(x)}{p(x)} - \frac{q(x)}{p(x)} y_0(x),\ y_0(1) = \beta, \\ y_j'(x) = -\frac{q(x)y_j(x)}{p(x)} - \frac{y_{j-1}''(x)}{p(x)},\ y_j(1) = 0,\ j = 1, \ldots, m. \end{cases} \tag{31}$$

Then, we derive the governing equation for the inner solution $u_i(x)$ of $u(x)$. In Equation (1), the nonhomogeneous term $r(x)$ was already taken into account by Equation (31) in the outer solution and thus, we consider a homogeneous ODE for the inner solution:

$$\varepsilon u_i''(x) + p(x)u_i'(x) + q(x)u_i(x) = 0, \tag{32}$$

$$u_i(0) = \alpha - u_o(0),\ \ u_i(1) = 0. \tag{33}$$

In terms of ζ in Equation (10), Equation (32) changes to

$$u_i''(\zeta) + p(\varepsilon\zeta)u_i'(\zeta) + \varepsilon q(\varepsilon\zeta)u_i(\zeta) = 0. \tag{34}$$

We assume

$$u_i(\zeta) = \sum_{j=0}^{m} \varepsilon^j w_j(\zeta), \tag{35}$$

which is inserted into Equation (34) and by equating the coefficients preceding ε^j, $j = 0, 1, \ldots, m$, we have

$$\begin{cases} w_0''(\zeta) + p(\varepsilon\zeta)w_0'(\zeta) = 0,\ w_0(0) = \alpha - u_o(0),\ w_0(1) = 0, \\ w_j''(\zeta) + p(\varepsilon\zeta)w_j'(\zeta) + q(\varepsilon\zeta)w_{j-1}(\zeta) = 0,\ w_j(0) = w_j(1) = 0,\ j = 1, \ldots, m. \end{cases} \tag{36}$$

Now, it is crucial that we can express Equation (36) in terms of x with the aid of Equation (11):

$$\varepsilon w_0''(x) + p(x)w_0'(x) = 0,\ w_0(0) = \alpha - u_o(0),\ w_0(1) = 0, \tag{37}$$
$$\varepsilon^2 w_j''(x) + \varepsilon p(x)w_j'(x) + q(x)w_{j-1}(x) = 0,\ w_j(0) = w_j(1) = 0,\ j = 1, \ldots, m. \tag{38}$$

Multiplying Equation (38) by ε^{j-1}, yields

$$\varepsilon^{j+1} w_j''(x) + \varepsilon^j p(x)w_j'(x) + q(x)\varepsilon^{j-1}w_{j-1}(x) = 0,\ w_j(0) = 0,\ w_j(1) = 0,\ j = 1, \ldots, m. \tag{39}$$

Then, resorting to the definition (30), Equations (37) and (39) change to

$$\begin{cases} \varepsilon z_0''(x) + p(x)z_0'(x) = 0,\ z_0(0) = \alpha - u_o(0),\ \ z_0(1) = 0, \\ \varepsilon z_j''(x) + p(x)z_j'(x) + q(x)z_{j-1}(x) = 0,\ z_j(0) = z_j(1) = 0,\ j = 1, \ldots, m. \end{cases} \tag{40}$$

When $0 < \varepsilon \ll 1$, the SPP (1) is very stiff within the boundary layer. In order to integrate the differential Equations (31) and (40), the following transformation between the independent variables x and t is considered:

$$x(t) = 1 - \frac{\tanh[\lambda(1-t)]}{\tanh\lambda},\ x(0) = 0,\ x(1) = 1. \tag{41}$$

It follows from Equations (31), (40) and (41) that

$$\dot{y}_0(t) = f_0(t, y_0) := \frac{\lambda e(t)r(t)}{p(t)} - \frac{\lambda e(t)q(t)y_0}{p(t)},\ \ y_0(1) = \beta, \tag{42}$$
$$\dot{y}_j(t) = f_j(t, y_j, y_{j-1}'') := -\lambda e(t)\left[\frac{q(t)y_j}{p(t)} + \frac{y_{j-1}''}{p(t)}\right],\ \ y_j(1) = 0,\ j = 1, \ldots, m, \tag{43}$$
$$\ddot{z}_0(t) = F_0(t, \dot{z}_0) := \left[2\lambda\tanh[\lambda(1-t)] - \frac{\lambda e(t)}{\varepsilon}p(t)\right]\dot{z}_0(t),$$
$$z_0(0) = \alpha - u_o(0),\ \ z_0(1) = 0, \tag{44}$$
$$\ddot{z}_j(t) = F_j(t, \dot{z}_j, z_{j-1}) := \left[2\lambda\tanh[\lambda(1-t)] - \frac{\lambda e(t)}{\varepsilon}p(t)\right]\dot{z}_j(t) - \frac{\lambda^2 e^2(t)}{\varepsilon}q(t)z_{j-1},$$
$$z_j(0) = z_j(1) = 0,\ j = 1, \ldots, m, \tag{45}$$

where

$$e(t) := \frac{1 - \tanh^2[\lambda(1-t)]}{\tanh\lambda}. \tag{46}$$

For saving notations, $p(t)$ means that $p(x(t))$ and others are similar. The term y_{j-1}'' in Equation (31) can be expressed as a function of $y_0, \ldots, y_{j-1}$, which, however, needs a tedious work when m is increased.

Remark 1. *For the sake of comparison, the higher-order formulas developed by Kaushik et al.* [36] *are listed as follows:*

$$u(x) = v(x) + w(\zeta), \tag{47}$$

$$v = v_0 + \varepsilon v_1 + \ldots + \varepsilon^k v_k + \varepsilon^{k+1} V, \tag{48}$$

$$p(x)v_0' + q(x)v_0 = r(x), \ \ v_0(1) = \beta, \tag{49}$$

$$p(x)v_i'(x) + q(x)v_i = -v_{i-1}'', \ v_i(1) = 0, \ i = 1, \ldots, k, \tag{50}$$

$$\varepsilon V'' + p(x)V'(x) + q(x)V = -v_k'', \ V(0) = V(1) = 0, \tag{51}$$

$$w = w_0 + \varepsilon w_1 + \ldots + \varepsilon^k w_k, \tag{52}$$

$$w_0'' + p(0)w_0' = 0, \ w_0(0) = \alpha - v_0(0), \ \lim_{\zeta\to\infty} w_0(\zeta) = 0, \tag{53}$$

$$\begin{aligned} w_i'' + p(0)w_i'(x) &= -\sum_{j=1}^{i}\left[\frac{p^{(j)}(0)}{j!}\zeta^j w_{i-j}' + \frac{q^{(j-1)}(0)}{(j-1)!}\zeta^{j-1} w_{i-j}\right], \\ w_i(0) &= -v_i(0), \ \lim_{\zeta\to\infty} w_i(\zeta) = 0, \ i = 1, \ldots, k. \end{aligned} \tag{54}$$

Equations (47)–(54) are more complicated than Equations (31) and (40). The right boundary conditions $\lim_{\zeta\to\infty} w_0(\zeta) = 0$ and $\lim_{\zeta\to\infty} w_i(\zeta) = 0$ are not easily realized by numerical method. Fortunately, Kaushik et al. [36] have derived the formulas:

$$w_0(x) = [\alpha - v_0(0)] \exp \frac{-p(0)x}{\varepsilon}, \tag{55}$$

$$\begin{aligned} w_1(x) = &\left(\frac{p'(0)[\alpha - v_0(0)]}{p^2(0)} - v_1(0) + \frac{b(0)[\alpha - v_0(0)]x}{p(0)\varepsilon}\right) \exp \frac{-p(0)x}{\varepsilon} \\ &- p'(0)[\alpha - v_0(0)]\left(\frac{x^2}{2\varepsilon^2} + \frac{x}{p(0)\varepsilon} + \frac{1}{p^2(0)}\right) \exp \frac{-p(0)x}{\varepsilon}. \end{aligned} \tag{56}$$

Examples 5 and 7 will be given in Sections 6.2.1 and 6.2.3 to show that the accuracy of the above method is worse than the method in Equations (31) and (40). These two methods in Equations (40), (53) and (54) are different in four aspects: the coordinates x and ζ, the coefficients $p(x), q(x)$ and $p(0), q(0)$, the left boundary conditions $z_i(0) = 0$ and $w_i(0) = -v_i(0)$, and the right boundary conditions $z_i(1) = 0$ and $\lim_{\zeta\to\infty} w_i(\zeta) = 0$.

4. A Novel *m*th-Order Asymptotic-Numerical Method

4.1. Two Free Functions

When p and q are nonlinear functions of x, the analytic asymptotic solution is not easy to obtain from the exact solutions of Equations (31) and (40). Instead, we developed a novel numerical method to find the asymptotic-numerical solution. Before deriving a novel iterative method to solve Equations (42)–(45), we cite the following results [37,38].

Theorem 1. *For any free function $Y(t) \in C[0,1]$, the function*

$$y(t) = Y(t) - G(t) \tag{57}$$

satisfies $y(1) = b$, where

$$G(t) := e^{t-1}[Y(1) - b]. \tag{58}$$

Proof. It is obvious that

$$y(1) = Y(1) - G(1) = Y(1) - e^{1-1}[Y(1) - b] = Y(1) - [Y(1) - b] = b;$$

hence, we prove that $y(t)$ in Equation (57) satisfies the right-end boundary condition $y(1) = b$. □

Theorem 2. *For any free function $Z(t) \in \mathcal{C}[0,1]$, the function*

$$z(t) = Z(t) - H(t) \tag{59}$$

satisfies the boundary conditions $z(0) = a$ and $z(1) = 0$, where

$$H(t) := (1-t)[Z(0) - a] + tZ(1). \tag{60}$$

Proof. In Equations (59) and (60), we insert $t = 0$ to obtain

$$z(0) = Z(0) - H(0) = Z(0) - [Z(0) - a] = a. \tag{61}$$

In Equations (59) and (60), we insert $t = 1$ to obtain

$$z(1) = Z(1) - H(1) = Z(1) - Z(1) = 0. \tag{62}$$

Thus, we end the proof. □

4.2. Transforming to the Initial Value Problem

Theorems 1 and 2 can be applied in the asymptotic numerical solution of the linear SPP. For Equations (42)–(45), we consider the following transformations of variables:

$$y_j(t) = Y_j(t) - G_j(t) = Y_j(t) - e^{t-1}[Y_j(1) - b_j],\ j = 0,1,\ldots,m, \tag{63}$$

$$z_j(t) = Z_j(t) - H_j(t) = Z_j(t) - (1-t)[Z_j(0) - a_j] - tZ_j(1),\ j = 0,1,\ldots,m, \tag{64}$$

where

$$b_0 = \beta,\ b_j = 0,\ j = 1,\ldots,m, \tag{65}$$

$$a_0 = \alpha - u_o(0),\ a_j = 0,\ j = 1,\ldots,m, \tag{66}$$

in which

$$u_o(0) = \sum_{j=0}^{m} \varepsilon^j y_j(0) = \sum_{j=0}^{m} \varepsilon^j \{Y_j(0) - e^{-1}[Y_j(1) - b_j]\}. \tag{67}$$

Letting $Y = Y_j$, $Z = Z_j$, $y = y_j$ and $z = z_j$ with $Y_j(t) \in \mathcal{C}^1[0,1]$ and $Z_j(t) \in \mathcal{C}^2[0,1]$ and by Theorems 1 and 2, y_j satisfies the right-end boundary condition $y_j(1) = b_j$, and z_j satisfies the boundary conditions $z_j(0) = a_j$ and $z_j(1) = 0$, automatically.

Inserting Equations (63) and (64) into Equations (42)–(45), we can derive

$$\dot{Y}_0(t) = \dot{G}_0 + f_0(t, Y_0 - G_0), \tag{68}$$

$$\dot{Y}_j(t) = \dot{G}_j + f_j(t, Y_j - G_j, \ddot{Y}_{j-1} - \ddot{G}_{j-1}),\ j = 1,\ldots,m, \tag{69}$$

$$\ddot{Z}_0(t) = F_0(t, \dot{Z}_0 - \dot{H}_0), \tag{70}$$

$$\ddot{Z}_j(t) = F_j(t, \dot{Z}_j - \dot{H}_j, Z_{j-1} - H_{j-1}),\ j = 1,\ldots,m. \tag{71}$$

In Equations (68)–(71), we take the initial values to be $Y_j(0) = Z_j(0) = \dot{Z}_j(0) = 0$, $j = 0,1,\ldots,m$ for saving parameters, while the unknown terminal values $Y_j(1)$ and $Z_j(1)$, $j = 0,1,\ldots,m$ are to be determined.

4.3. The Iterative Algorithm

We denote the unknown values $Y_j(1)$ and $Z_j(1)$ by

$$c_j = Y_j(1),\ d_j = Z_j(1),\ j = 0,1,\ldots,m.$$

To find the mth-order asymptotic numerical solution of u, the current method is: (i) giving m, $c_j^0 = d_j^0 = 0$, $j = 0,1,\ldots,m$, ϵ, and N; and (ii) repeating $k = 0,1,2,\ldots$ until convergence, integrating Equations (68)–(71) by using the RK4 with N steps from $t = 0$ to $t = 1$, and taking

$$c_j^{k+1} = Y_j(1),\ d_j^{k+1} = Z_j(1),\ j = 0, 1, \ldots, m.$$

If

$$\sqrt{\sum_{j=0}^{m}(c_j^{k+1} - c_j^k)^2 + \sum_{j=0}^{m}(d_j^{k+1} - d_j^k)^2} < \epsilon$$

is satisfied, then the iterations are terminated. The asymptotic numerical solution $u(t)$ is given by

$$u(t) = \sum_{j=0}^{m}[\varepsilon^j y_j + z_j] = \sum_{j=0}^{m}[\varepsilon^j (Y_j - G_j) + Z_j - H_j], \tag{72}$$

where

$$G_j = e^{t-1}[c_j^k - b_j],\ H_j = t d_j^k - (1-t)a_j, \tag{73}$$

in which c_j^k and d_j^k are the convergent values of the sequences c_j^k and d_j^k, $k = 1, 2, \ldots$.

5. Numerical Examples

For most linear SPPs with variable coefficients, there exist no closed-form solutions. Here, we will apply the initial value problem method (IVPM) developed in [34] to compute the solutions, which are used as the referenced "exact" solutions, if the "truly exact" solution is not available.

5.1. Example 1

We consider Equations (3) and (5) again with $\varepsilon = 0.01$, $\alpha = 0$ and $\beta = 1$. Now, we solve it by using the IVPM developed in [34], and upon comparing it to the exact solution (5), the ME is found to be 7.096×10^{-10}. We give $m = 1$, $\lambda = 3.5$, $N = 1000$, and $\epsilon = 10^{-10}$ and apply the iterative algorithm in Section 4.3 to find the first-order asymptotic numerical solution of Equation (3), whose ME is 5.9305485×10^{-4}. On the other hand, the ME of the asymptotic solution (27) is 5.9305476×10^{-4}, which is very close to the asymptotic numerical solution. In Figure 2, we compare those four solutions, which are very close.

As an extension of the method in Section 2.1 to the second-order asymptotic approximation, we can derive

$$\begin{aligned} u_a(x) = {}& \beta e^{1-x}\left[1 + \varepsilon(1-x) + \varepsilon^2\left(\frac{1}{2}(1-x)^2 + 2(1-x)\right)\right] \\ & + \left[(\alpha - e\beta)\left(1 + x + \frac{x^2}{2}\right) - \varepsilon[e\beta - (\alpha - 2e\beta)x] - \frac{5}{2}e\beta\varepsilon^2\right]e^{-x/\varepsilon}. \end{aligned} \tag{74}$$

On the other hand, m is raised to $m = 2$ and we apply the iterative algorithm in Section 4.3 to find the second-order asymptotic numerical solution of Equation (3), whose ME is 1.68×10^{-5}, which is comparable to the solution (74), whose ME is 1.66×10^{-5}. Upon comparison with the first-order solutions, the improvement of accuracy is one-order.

5.2. Example 2

Consider [30,32,36]

$$\varepsilon u''(x) + u'(x) - u(x) = 0,\ u(0) = 1,\ \ u(1) = 1, \tag{75}$$

whose solution is

$$u(x) = \frac{1}{e^{a_2} - e^{a_1}}[(e^{a_2} - 1)e^{a_1 x} + (1 - e^{a_1})e^{a_2 x}], \tag{76}$$

where

$$a_1 = \frac{-1 + \sqrt{1 + 4\varepsilon}}{2\varepsilon},\ \ a_2 = \frac{-1 - \sqrt{1 + 4\varepsilon}}{2\varepsilon}. \tag{77}$$

Given $m = 2$, $\lambda = 2.9$, $N = 1000$, and $\epsilon = 10^{-7}$, the present asymptotic numerical method for the solution of Equation (75) with $\varepsilon = 0.001$ converges within 725 iterations. In Table 2, we compare the numerical results to that obtained by Reddy and Chakravarthy [30] and El-Zahar [32]. Obviously, our errors with ε^2, as expected, are smaller than other solutions by approximately two orders.

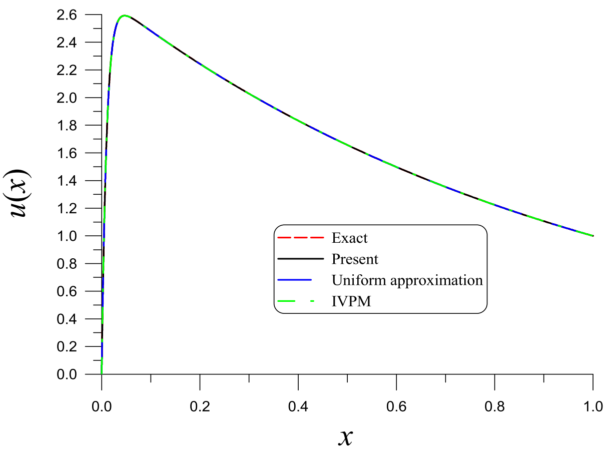

Figure 2. For example 1, comparing the exact solution, a uniform approximation, the present asymptotic-numerical solution and the IVPM solution.

Table 2. For example 2 with $\varepsilon = 0.001$, comparing the numerical solutions at different x with an exact solution and other solutions.

x	Present	[30]	[32]	Exact
0.01	0.3719750	0.3712379	0.3716054	0.3719724
0.02	0.3756809	0.3749439	0.3753111	0.3756784
0.03	0.3794527	0.3787160	0.3790831	0.3794502
0.04	0.3832624	0.3825260	0.3828929	0.3832599
0.05	0.3871104	0.3863742	0.3867410	0.38710787
0.10	0.4069374	0.4062043	0.4065697	0.4069350
0.50	0.6068350	0.6062278	0.6065307	0.6068334
0.90	0.9049277	0.9047471	0.9048374	0.9049277

5.3. Example 3

Consider a variable coefficient SPP [39]:

$$\varepsilon u''(x) + \left(1 - \frac{x}{2}\right)u'(x) - \frac{1}{2}u(x) = 0,\ u(0) = 0,\ \ u(1) = 1, \tag{78}$$

whose asymptotic solution is given [8]:

$$u_a(x) = \frac{1}{2-x} - \frac{1}{2}\exp\left(\frac{x^2/4 - x}{\varepsilon}\right). \tag{79}$$

It is obvious that

$$u_a(0) = 0,\ \ u_a(1) = 1 - \frac{1}{2}\exp\left(\frac{-3}{4\varepsilon}\right) < 1, \tag{80}$$

so that Equation (79) does not exactly satisfy the right boundary condition.

We give $\varepsilon = 0.1$, $m = 1$, $\lambda = 2$, $N = 1000$ and $\epsilon = 10^{-10}$ and apply the iterative algorithm in Section 4.3 to find the first-order asymptotic numerical solution of Equation (78), which converges within 23 iterations, as shown in Figure 3a. In Figure 3b, we compare the asymptotic numerical solution to the asymptotic solution in Equation (79). We can observe that the improvement is achieved by using the asymptotic numerical solution, where ME1: $= \max |u_e - u| = 2.62 \times 10^{-2}$ and ME2: $= |u_e - u_a| = 4.94 \times 10^{-2}$.

We give $\varepsilon = 0.01$, $m = 2$, $\lambda = 3.5$, $N = 1000$ and $\epsilon = 10^{-10}$ and apply the iterative algorithm in Section 4.3 to find the second-order asymptotic numerical solution of Equation (78), which converges with 24 iterations. We compare the second-order asymptotic numerical solution to the asymptotic solution in Equation (79), and ME1:$= \max |u_e - u| = 4.903 \times 10^{-5}$ is much smaller than ME2:$= |u_e - u_a| = 7.28 \times 10^{-3}$.

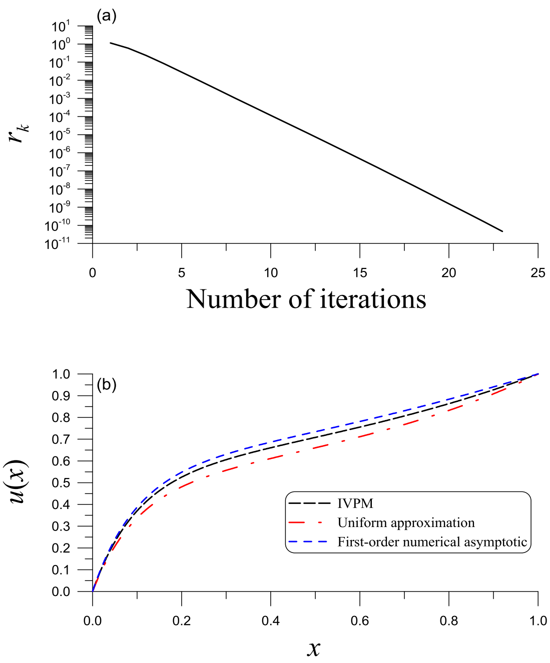

Figure 3. For example 3: (**a**) showing the convergence of iterations; and (**b**) comparing the exact solution, a uniform approximation and the present first-order solution.

5.4. Example 4

Consider

$$\varepsilon u''(x) + (2x+1)u'(x) + 2u(x) = 0,\ u(0) = 1,\ \ u(1) = 1, \tag{81}$$

whose asymptotic solution is given by

$$u_a(x) = \frac{3}{2x+1} - 2\exp\left(\frac{-x^2 - x}{\varepsilon}\right) + \varepsilon\left[\frac{6}{(2x+1)^3} - \frac{2}{3(2x+1)} - \frac{16}{3}\exp\left(\frac{-x^2 - x}{\varepsilon}\right)\right]. \tag{82}$$

We give $\varepsilon = 0.005$, $m = 1$, $\lambda = 4$, $N = 1000$ and $\epsilon = 10^{-10}$ and apply the iterative algorithm in Section 4.3 to find the first-order asymptotic numerical solution of Equation (81), which converges within 207 iterations, as shown in Figure 4a. In Figure 4b, we compare the asymptotic numerical solution to the asymptotic solution in Equation (82). We can observe that the improvement is achieved by using the asymptotic numerical solution, where ME1: $= \max |u_e - u| = 6.87 \times 10^{-4}$ and ME2: $= |u_e - u_a| = 7.34 \times 10^{-4}$.

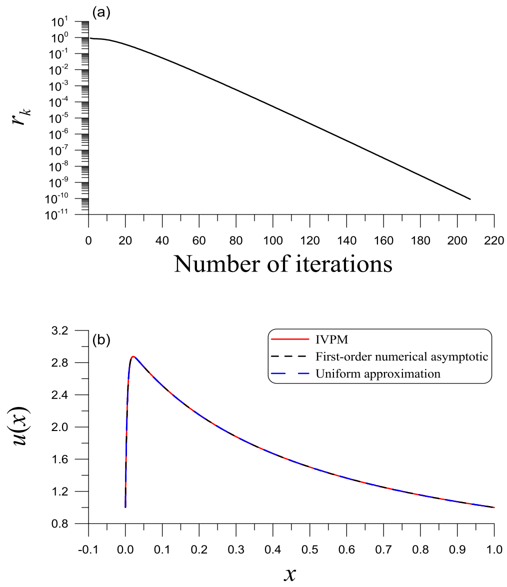

Figure 4. For example 4: (**a**) showing the convergence of iterations; and (**b**) comparing the exact solution, a uniform approximation and the present first-order solution.

6. Special Case with $q(x) = 0$

For the special case with $q(x) = 0$, Equations (1) and (2) reduce to

$$\varepsilon u''(x) + p(x)u'(x) = r(x), \quad 0 < x < 1, \tag{83}$$

$$u(0) = \alpha, \quad u(1) = \beta. \tag{84}$$

6.1. Main Results

Theorem 3. *For Equations (83) and (84), the series $z_j(x)$ in Equation (40) for the inner solution terminates with*

$$z_j(x) = 0, \; j \geq 1. \tag{85}$$

Proof. Inserting $q(x) = 0$ into Equation (40) yields

$$\varepsilon z_j''(x) + p(x)z_j'(x) = 0, \; j \geq 1. \tag{86}$$

Let

$$I(x) := \exp\left(\int_0^x \frac{p(\xi)}{\varepsilon} d\xi\right) \tag{87}$$

be the integrating factor. Then, Equation (86) can be written as

$$\frac{d}{dx}[I(x)z_j'(x)] = 0, \; j \geq 1, \tag{88}$$

and then we have

$$z_j(x) = k_1 \int_0^x \frac{d\xi}{I(\xi)} + k_2, \; j \geq 1, \tag{89}$$

where k_1 and k_2 are integration constants. Using the conditions $z_j(0) = z_j(1) = 0$, $j \geq 1$, we can derive $k_1 = k_2 = 0$. This ends the proof. □

Theorem 4. *For Equations (83) and (84), if $p(x)$ and $r(x)$ satisfy*

$$\frac{d}{dx}\frac{r(x)}{p(x)} = kp(x), \tag{90}$$

where $k \neq 0$ is a constant, then the series $y_j(x)$ in Equation (31) for the outer solution terminates with

$$y_j(x) = 0,\ j \geq 2. \tag{91}$$

Proof. Inserting $q(x) = 0$ into Equation (31) yields

$$y_0'(x) = \frac{r(x)}{p(x)},\ y_1'(x) = -\frac{y_0''(x)}{p(x)} = -\frac{1}{p(x)}\frac{d}{dx}\frac{r(x)}{p(x)}. \tag{92}$$

Because of Equation (90), the latter one reduces to

$$y_1'(x) = -k. \tag{93}$$

Inserting $q(x) = 0$ and $j = 2$ into Equation (31) and using the above equation yields

$$y_2'(x) = -\frac{y_1''(x)}{p(x)} = 0, \tag{94}$$

which implies $y_2(x) = 0$ due to $y_2(1) = 0$. Then, $y_j = 0,\ j > 3$ easily follows by using

$$y_j'(x) = -\frac{y_{j-1}''(x)}{p(x)} = 0,\ y_j(1) = 0. \tag{95}$$

This ends the proof. □

As a continuation of Theorem 4, we can prove the following result.

Theorem 5. *For Equations (83) and (84), if $p(x)$ and $r(x)$ satisfy Equation (90) with $k \neq 0$, then the asymptotic solution is given by*

$$\begin{aligned} u(x) &= y_0(x) + \varepsilon y_1(x) + z_0(x) \\ &= \alpha - k\varepsilon x + \int_0^x \frac{r(\xi)}{p(\xi)} d\xi + \frac{1}{B}[\beta - \alpha + k\varepsilon - A] \int_0^x \frac{d\xi}{I(\xi)}, \end{aligned} \tag{96}$$

which is identical to the exact solution where

$$A := \int_0^1 \frac{r(\xi)}{p(\xi)} d\xi,\ B := \int_0^1 \frac{d\xi}{I(\xi)}, \tag{97}$$

and $I(x)$ is defined by Equation (87).

Proof. According to Theorems 3 and 4, we merely consider $y_0(x)$, $y_1(x)$ and $z_0(x)$ by

$$y_0'(x) = \frac{r(x)}{p(x)},\ y_0(1) = \beta, \tag{98}$$

$$y_1'(x) = -k,\ y_1(1) = 0, \tag{99}$$

$$\varepsilon z_0''(x) + p(x) z_0'(x) = 0,\ z_0(0) = \alpha - [y_0(0) + \varepsilon y_1(0)],\ z_0(1) = 0. \tag{100}$$

The first two ODEs are derived from Equations (92) and (90), and the last ODE is derived from Equation (86) with $j = 0$.

It follows from Equations (98)–(100) that

$$y_0(x) = \int_0^x \frac{r(\xi)}{p(\xi)} d\xi + \beta - A, \tag{101}$$

$$y_1(x) = k - kx, \tag{102}$$

$$z_0(x) = c_1 \int_0^x \frac{d\xi}{I(\xi)} + c_2, \tag{103}$$

$$c_1 = \frac{1}{B}[\beta - \alpha + k\varepsilon - A],\ c_2 = \alpha - \beta - k\varepsilon + A. \tag{104}$$

Inserting $y_0(x)$, $y_1(x)$ and $z_0(x)$ into

$$u(x) = y_0(x) + \varepsilon y_1(x) + z_0(x),$$

we can derive Equation (96).

Multiplying Equation (83) by $I(x)$, it changes to

$$\frac{d}{dx}[I(x)u'(x)] = \frac{I(x)r(x)}{\varepsilon}. \tag{105}$$

Upon using $I'/p = I/\varepsilon$ in

$$\begin{aligned}\int_0^x \frac{I(\xi)r(\xi)}{\varepsilon} d\xi &= \int_0^x \frac{I'(\xi)r(\xi)}{p(\xi)} d\xi = \frac{I(\xi)r(\xi)}{p(\xi)}\bigg|_0^x - \int_0^x I(\xi)\frac{d}{d\xi}\frac{r(\xi)}{p(\xi)} d\xi \\ &= \frac{I(\xi)r(\xi)}{p(\xi)}\bigg|_0^x - k\int_0^x I(\xi)p(\xi)d\xi = \frac{I(\xi)r(\xi)}{p(\xi)}\bigg|_0^x - k\varepsilon[I(x) - 1],\end{aligned} \tag{106}$$

we can deduce

$$I(x)u'(x) = \frac{I(x)r(x)}{p(x)} - k\varepsilon[I(x) - 1] + k_1, \tag{107}$$

where Equation (90) was taken into account and k_1 is an integration constant. The constant $I(0)r(0)/p(0)$ was absorbed into k_1.

Further using the condition $u(0) = \alpha$ and from Equation (107), we can derive

$$\begin{aligned}u(x) &= \alpha + \int_0^x \frac{r(\xi)}{p(\xi)} d\xi - k\varepsilon \int_0^x \frac{1}{I(\xi)}[I(\xi) - 1] d\xi + k_1 \int_0^x \frac{d\xi}{I(\xi)} \\ &= \alpha + \int_0^x \frac{r(\xi)}{p(\xi)} d\xi + k_1 \int_0^x \frac{d\xi}{I(\xi)} + k\varepsilon \int_0^x \left[\frac{1}{I(\xi)} - 1\right] d\xi.\end{aligned} \tag{108}$$

Imposing another condition $u(1) = \beta$ generates

$$\beta = \alpha + A + k_1 B + k\varepsilon(B - 1). \tag{109}$$

Solving k_1 and inserting it into Equation (108), we can again derive Equation (96). This ends the proof. □

6.2. Examples

6.2.1. Example 5

Consider [40]:

$$\varepsilon u''(x) + u'(x) = 1 + 2x,\ u(0) = 0,\ \ u(1) = 1, \tag{110}$$

whose exact solution is

$$u_e(x) = x(x + 1 - 2\varepsilon) + (2\varepsilon - 1)\frac{1 - e^{-x/\varepsilon}}{1 - e^{-1/\varepsilon}}. \tag{111}$$

Equation (110) satisfies Equations (85) and (91) as a special case with $p = 1$ and $r = 1 + 2x$. Equations (31) and (40) lead to

$$y_0(x) = x + x^2 - 1,\ y_1(x) = 2 - 2x, \tag{112}$$

$$z_0(x) = (2\varepsilon - 1)\frac{e^{-1/\varepsilon} - e^{-x/\varepsilon}}{1 - e^{-1/\varepsilon}}. \tag{113}$$

Hence,

$$u(x) = y_0(x) + \varepsilon y_1(x) + z_0(x) = x + x^2 - 1 + \varepsilon(2 - 2x) + (2\varepsilon - 1)\frac{e^{-1/\varepsilon} - e^{-x/\varepsilon}}{1 - e^{-1/\varepsilon}}, \tag{114}$$

which is just the exact solution (111).

For this example, according to the method developed by Kaushik et al. [36], we can derive

$$u_k(x) = x + x^2 - 1 + \varepsilon(2 - 2x) + (1 - 2\varepsilon - 1)e^{-x/\varepsilon}, \tag{115}$$

which does not satisfy the right boundary condition because of $u_k(1) = 1 + (1 - 2\varepsilon - 1)e^{-1/\varepsilon} \neq 1$.

In Table 3, we compare ME1:= $\max|u_e - u|$ and ME2:= $\max|u_e - u_k|$ for different values of ε, which are close when $\varepsilon \leq 0.01$. For $\varepsilon \geq 0.05$, ME1 is smaller than ME2. Equations (114) and (115) possess the same asymptotic property, because of

$$e^{-1/\varepsilon} \to 0, \text{ when } \varepsilon \to 0.$$

Table 3. For example 5, comparing ME1 and ME2 for different ε.

ε	**0.6**	**0.4**	**0.3**	**0.1**	**0.05**
ME1	0	0	0	0	0
ME2	3.78×10^{-2}	1.64×10^{-2}	1.43×10^{-2}	3.63×10^{-5}	1.86×10^{-9}

6.2.2. Example 6

Consider

$$\varepsilon u''(x) + \frac{1}{\sqrt{2x + 2}}u'(x) = 1,\ u(0) = 0,\ \ u(1) = 1, \tag{116}$$

which is a special case satisfying Equation (90) with $k = 1$. The exact and analytic asymptotic solution is given by

$$u_{aa}(x) = \frac{1}{3}[(2x + 2)^{3/2} - 2^{3/2}] - \varepsilon x + \frac{1}{B}[1 + \varepsilon - A]\int_0^x \frac{d\xi}{\exp[\{\sqrt{2\xi + 2} - \sqrt{2}\}/\varepsilon]}, \tag{117}$$

$$A = \frac{1}{3}[8 - 2^{3/2}],\ B = \int_0^1 \frac{d\xi}{\exp[\{\sqrt{2\xi + 2} - \sqrt{2}\}/\varepsilon]}. \tag{118}$$

We give $\varepsilon = 0.01$, $m = 1$, $\lambda = 3$, $N = 1000$, and $\epsilon = 10^{-10}$ and apply the iterative algorithm in Section 4.3 to find the first-order asymptotic numerical solution of Equation (116), which converges within 54 iterations as shown in Figure 5a. In Figure 5b, we compare the asymptotic numerical solution, the analytic asymptotic solution and the exact solution obtained from the IVPM. We can observe that they are almost coincident with ME1:= $\max|u_e - u| = 2.28 \times 10^{-10}$ and ME2:= $\max|u_e - u_{aa}| = 1.19 \times 10^{-7}$.

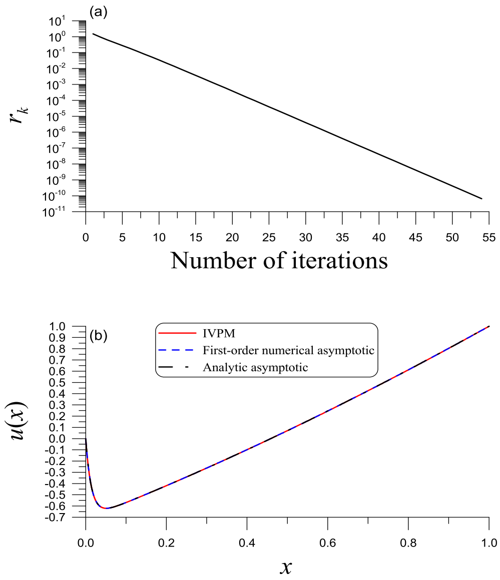

Figure 5. For example 6: (**a**) showing the convergence of iterations; and (**b**) comparing exact solution and the present first-order solution.

6.2.3. Example 7

Consider

$$\begin{aligned} &\varepsilon u''(x) + (2 - \sin 2\pi x)u'(x) = (2 - \sin 2\pi x)\left(2x + \frac{1}{2\pi}\cos 2\pi x\right), \\ &u(0) = 1, \ \ u(1) = 1, \end{aligned} \tag{119}$$

which is a special case satisfying Equation (90) with $k = 1$. Then, the exact and analytic asymptotic solution is given by

$$u_{aa}(x) = 1 - \varepsilon x + \int_0^x \left(2\xi + \frac{1}{2\pi}\cos 2\pi\xi\right) d\xi + \frac{1}{B}[\varepsilon - A] \int_0^x \frac{d\xi}{\exp[\{2\xi + (\cos 2\pi\xi - 1)/(2\pi)\}/\varepsilon]}, \tag{120}$$

$$A = \int_0^1 \left(2\xi + \frac{1}{2\pi}\cos 2\pi\xi\right) d\xi, \ B = \int_0^1 \frac{d\xi}{\exp[\{2\xi + (\cos 2\pi\xi - 1)/(2\pi)\}/\varepsilon]}. \tag{121}$$

We give $\varepsilon = 0.01$, $m = 1$, $\lambda = 3$, $N = 1000$, and $\epsilon = 10^{-10}$ and apply the iterative algorithm in Section 4.3 to find the first-order asymptotic numerical solution of Equation (119), which converges within 125 iterations, as shown in Figure 6a. In Figure 6b, we compare the asymptotic numerical solution to the exact solution obtained from the IVPM, and we can observe that they are almost coincident with ME: $= \max |u_e - u| = 4.17 \times 10^{-9}$. Notice that ME: $= \max |u_e - u_{aa}| = 4.08 \times 10^{-9}$.

For this example, according to the method developed by Kaushik et al. [36], we can derive

$$u_k(x) = x^2 + \frac{1}{4\pi^2}\sin 2\pi x + \varepsilon(x - 1) + \left(1 + \varepsilon + \pi x + \frac{\pi x^2}{\varepsilon}\right)e^{-2x/\varepsilon}, \tag{122}$$

which does not satisfy the right boundary condition because of $u_k(1) = 1 + (1 + \varepsilon + \pi + \pi/\varepsilon)e^{-2/\varepsilon} > 1$.

In Table 4, we compare ME1:$= \max |u_e - u_{aa}|$ and ME2:$= \max |u_k - u_{aa}|$ for different values of ε. ME1 is much smaller than ME2.

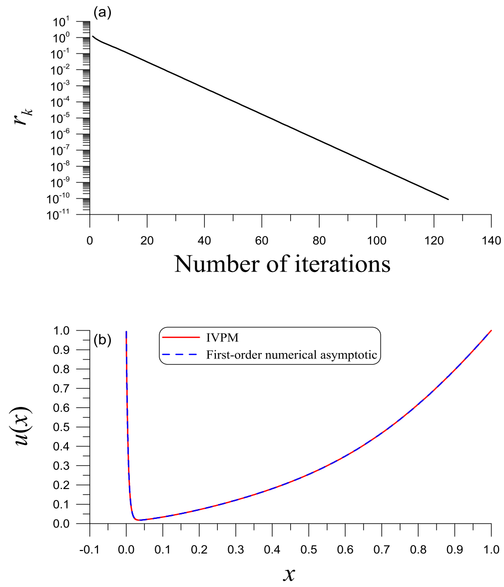

Figure 6. For example 7: (**a**) showing the convergence of iterations; and (**b**) comparing exact solution and the present first-order solution.

Table 4. For example 7, comparing ME1 and ME2 for different ε.

ε	**0.2**	**0.1**	**0.05**	**0.03**	**0.01**
ME1	1.79×10^{-11}	6.54×10^{-11}	2.78×10^{-11}	5.82×10^{-11}	1.19×10^{-10}
ME2	2.17×10^{-1}	1.84×10^{-1}	1.00×10^{-1}	5.95×10^{-2}	1.97×10^{-2}

7. Conclusions

It is of utmost importance that an asymptotic-numerical solution for second-order variable coefficients linear SPP can match the boundary conditions exactly. For the linear SPP, we proposed a novel boundary layer correction problem in the original coordinates which can accurately capture the asymptotic behavior within the boundary layer and at the same time preserve the boundary conditions. Therefore, the new mth-order asymptotic solution is an improvement of the conventional asymptotic solution. Resorted on the free functions in Theorems 1 and 2 as being the new variables, we exactly transformed the linear SPP to the initial value problems for the $2(m+1)$ new variables with the given zero initial conditions and thus a newly developed iterative algorithm is converging very fast to determine the $2(m+1)$ unknown right-end values of the new variables and to find the singularly perturbed asymptotic solution very quickly. Based on the new idea, we provided a modification for the conventional asymptotic solution of the linear SPP, such that the new asymptotic-numerical solution exactly satisfies the boundary conditions. In doing so, the accuracy of the asymptotic-numerical solution was raised and the applicable range of the perturbing parameter in the modified asymptotic solution can be extended to a moderate value. More importantly, the modified asymptotic solution possesses the same asymptotic behavior with the conventional asymptotic solution. For the special case with $q(x) = 0$, we derived the sufficient conditions for the termination of the asymptotic series solutions of outer and inner problems, and found that it may lead to the exact solution, if $d(r/p)/dx = kp$ holds for a nonzero value of k.

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