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On the Residual Lifetime and Inactivity Time in Mixtures

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Abstract: In this paper we study the aging characteristics in mixtures of distributions, providing characterizations for their derivatives that explain the smooth behavior of the mixture. The classical preservation results for the reversed hazard rate, mean residual life and mean inactivity time are derived under a different approach than in previous studies. We focus on the variance of both the residual life and inactivity time in mixtures, obtaining some preservation properties. We also state conditions for weak and strong bending properties for the variance of the residual life and the inactivity time in mixtures.

Keywords: mixture; residual life variance; inactivity time variance; aging class; bending property

MSC: 62N05; 60E15; 60K10

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1. Introduction

Systems fail due to age, and the way this process occurs is described by the aging characteristics. The failure rate, indicating the proneness to failure, and the residual life measuring the remaining time span of a system that has not yet failed, are traditionally used as indicators of the system state. Therefore they can be used to assess the appropriateness of carrying out some types of preventive maintenance. It can be observed that both the failure rate and the residual life are calculated at a given time x for individuals or units that have survived up to x, and thus we can define them as "forward age characteristics" since they reflect, respectively, the probability of failure in the imminent future and the random time from x until it occurs.

Sometimes the research subject of interest emerges when the failure has already occurred. For example, when a failure is unrevealed, that is, it is detected only by inspection, the maintainer would like to estimate the probability that it occurred sometime between two consecutive inspections if, for example, he suspects that failures are induced by inspections [1], or he may wish to estimate the losses incurred up to a given moment due to the downtime. This aspect is also particularly relevant in epidemiological studies when people are diagnosed as being infected by a virus and a retrospective analysis is required [2].

Given a non-negative random variable X, the reversed hazard rate $q_X(x)$ and the inactivity time $\nu_X(x)$ provide information for this type of analysis. Both can be interpreted as "backward age characteristics" since they are defined conditionally for a failure that has occurred in $[0, x]$. Thus, $q_X(x)dx$ is interpreted in [3] as the conditional probability of failure for an object in $(x - dx, x]$, whereas the inactivity time represents the time elapsed from the failure until x. Both concepts are relevant for maintenance models if a downtime cost is assumed. When dealing with people diagnosed with a disease, they represent the probability of having been infected just before the infection is detected and the period from infection to detection, respectively. The connection between the two characteristics is studied in [4]. In this study, we aim at shedding light on both the expectation and variance of the residual life and inactivity time.

Often the time to failure of a system is described by a mixture of distributions. This is so, for example, when there is a subpopulation of "bad" units mixed with the normal ones. Studies of aging characteristics in mixtures lead to actual applications in maintenance modeling. The work in [5] uses a mixture to model the case of progressive wear in metal cutting tools when a proportion of the supply is affected by hidden defects. The mixing random variable is usually an unobserved random variable (frailty) representing, for example, users with different levels of expertise or the changing environmental conditions under which the system operates. The frailty is used to introduce random effects to account for such heterogeneities caused by distinct risks when there are no observable covariates.

An amazing outcome is that the aging characteristics are observed to improve in mixed populations compared to those corresponding to the distributions in the mixture. The best known preservation property is that mixtures of DFR (decreasing failure rate) populations are also DFR. In addition, a bending behavior is observed for some mixtures of IFR populations, which are first increasing and then asymptotically DFR [6]. This means that mixtures tend to transform a positive aging of the failure rate into a negative one. The research in [7] illustrates this property with a number of actual examples ranging from social issues (divorce) to entomology (mortality in fruit flies) and health (mortality due to cancer). In doing so, the author develops a nice intuition regarding the meaning of frailty and its effects.

Regarding the reversed hazard rate, mean residual life and mean inactivity time, the following properties are preserved by mixtures: increasing reversed hazard [8], increasing mean residual life [9] and decreasing mean inactivity time [8]. All these properties imply that the system reliability increases with time and therefore preventive maintenance at the beginning of the useful life may be profitable, to prevent early failures [10]. The bending properties imply that the aging characteristics in a mixture of populations take greater values than the mean value of the corresponding aging characteristics of the subpopulations in the mixture. Therefore, the study of bending properties provides additional insight into condition-based maintenance.

The residual life variance and inactivity time variance are useful in many areas of statistics, including biometry, actuarial science and reliability theory. In addition, there is an increasing interest in the study of the corresponding stochastic orders and their associated aging classes at a fixed time. A number of papers [11–22] contain relevant results on this issue. Recent research concerning stochastic orders of discrete random variables can be found in [23].

The preservation of the increasing residual life variance under mixtures was addressed in [24]. As far as we know, a similar study for the decreasing inactivity time variance has not yet been undertaken. In this paper, the corresponding preservation properties are derived, generalizing the property in [24], since we relax the assumptions.

Our description of the residual life variance and inactivity time variance follows the approach in [25] for the failure rate and that in [26] for the reversed hazard rate in mixtures. Both studies present a Bayesian perspective, based on the conditional distribution of the frailty given the data, which is different from that in [2–4,6–8,11–24]. Hence, the frailty cannot be observed but it can be updated. Highlighting the differences between this work and previous research, we must also point out that the authors in [2,7] provide data-driven properties, whereas this paper focuses on theoretical results with pending applicability in empirical studies.

The bending properties of the failure rate in mixtures were originally studied in [25]. The authors in [27] extended this analysis to the reversed hazard rate, mean residual life and mean inactivity time. The properties of mixtures under the proportional reversed hazard rate were considered in [28]. Recently, the discrete case of the reversed hazard rate was studied in [29].

This paper is organized as follows. Notation, basic results and the representation assumed for mixtures are presented in Section 2. Section 3 is devoted to the preservation of aging characteristics under mixtures, whereas the bending properties are the central topic of Section 4. Section 5 contains the main conclusions of this paper.

2. Preliminaries

In what follows, we present the basics of aging properties, mixtures and stochastic orders. Almost all of these have been obtained in previous research, and therefore the corresponding proofs can be found in the cited references. For the readability of the results without extending the section with known results, some of them are developed in more detail in Appendix A. The following notation is used throughout the paper.

A random variable X $(X > 0)$ can be specified in different ways. The more popular ways are the density function f_X, the cumulative distribution function $F_X(x) = P(X \leq x)$, the reliability or survival function $\overline{F}_X(x) = P(X > x)$ and the hazard rate $r_X(x)$, defined as follows:

$$r_X(x) = \lim_{\Delta x \to 0} \frac{P(x \leq X < x + \Delta | X > x)}{\Delta x} = \frac{f(x)}{\overline{F}_X(x)}.$$

Observe that $r_X(x)\Delta x = \frac{f(x)\Delta x}{\overline{F}_X(x)}$ can be interpreted as the probability of imminent failure in $[x, x + \Delta x)$.

The reversed hazard rate $q_X(x)$, defined below, is another aging characteristic:

$$q_X(x) = \lim_{\Delta x \to 0} \frac{P(x - \Delta x < X \leq x | X \leq x)}{\Delta x} = \frac{f_X(x)}{F_X(x)}.$$

It follows that $q_X(x)\Delta x = \frac{f_X(x)\Delta x}{F_X(x)}$ is the probability that the failure has just occurred when time x arrives. Consider, for example, that a tumor is diagnosed at x, then $q_X(x)\Delta x$ is the probability that tumor appeared during the time $(x - \Delta x, x]$. An important result states that there exists no non-negative random variable with support in $(0, \infty)$ having an increasing reversed hazard rate function [3]. From a practical point of view, this makes perfect sense. Observe that if such a variable were possible, then under the previous context a tumor diagnosed at time x_1 would be less likely to be formed in $(x_1 - \Delta x_1, x_1]$ than in $(x_2 - \Delta x_2, x_2]$ in the case where the diagnosis time is x_2 with $x_2 > x_1$. In other words, the longer the time until the tumor is detected, the greater the probability that it had just occurred and, thus, the lower the time elapsed since then. This contradicts actual knowledge of this disease, which points to early detection as one of the keys for cure. The importance of the reversed hazard rate in the estimation of the survival function under left-censored observations was highlighted in [2,3].

The residual life and inactivity time are random variables closely related to the hazard rate and reversed hazard rate, respectively. Thus, the residual life of a unit that has survived up to x is the remaining life until the unit fails, that is, $X - x | X > x$. $P(X - x > t | X > x)$ is the probability that such a unit survives t additional units of time. This is important for maintenance scheduling since the larger the previous probability for a given t, the less urgent the replacement of the system.

The inactivity time refers to the elapsed time from failure on condition that it occurred before a given time x, $x - X | X \leq x$. The importance of this variable appears, for example, if we wish to obtain information about the time since a tumor appeared when it is diagnosed at x. Now, $P(x - X > t | X \leq x)$ determines the probability that the tumor was formed more than t units ago. Therefore a high probability is an adverse result, since the more likely it has remained hidden, the lower the chance of recovery. Both the expectation and the variance of the random variables are always relevant, and so we focus on the the mean residual life $m_X(x) = E[X - x | X > x]$, the mean inactivity time $\nu_X(x) = E[x - X | X \leq x]$, the residual life variance $\sigma^2_X(x) = Var[X - x | X > x]$ and the inactivity time variance $\overline{\sigma}^2_X(x) = Var[x - X | X \leq x]$. The hazard rate and the mean residual life are dual functions; when the former increases, the latter decreases. The reversed hazard rate and the mean inactivity time are coupled in the same way.

Z denotes the frailty random variable, and we assume without loss of generality that it is a continuous non-negative random variable with probability density function (pdf) $\pi(z)$. In addition, X^* represents the mixture of random variables and $X^*|Z = z$ is the distribution of the mixture given that the conditions take a particular value z, that is, if $Z = z$.

At this point, we must highlight a crucial difference in Z when the reversed hazard rate and inactivity time are under study. The analysis in [7] concerning the heterogeneity of individuals sheds light on the effect of non-susceptible subgroups, that is, those people either immune to or cured of an illness. The time to failure of long-term survivors is represented by a defective distribution [30]. In [31], the concept of resilience is used as an alternative to frailty when the survival function increases with the mixing random variable. Therefore, and for estimation purposes, the possibility of immune persons in that particular problem cannot be neglected. However, in the case of the reversed hazard rate and inactivity time, this situation no longer applies. Observe that both are "backward age characteristics" and hence are defined once the event has occurred. It follows that non-susceptible (immune) individuals make no sense when both measures are involved, and hence the distributions for the frailty cannot include defective distributions.

The aging characteristics of the population in the mixture conditional to a given value of the frailty $Z = z$ are represented by $F(x,z)$ (distribution function), $\overline{F}(x,z)$ (reliability function), $f(x,z)$ (density function), $r(x,z)$ (failure rate), $q(x,z)$ (reversed hazard rate), $m(x,z)$ (mean residual time), $\nu(x,z)$ (mean inactivity time), $\sigma^2(x,z)$ (residual life variance) and $\overline{\sigma}^2(x,z)$ (inactivity time variance). The corresponding aging characteristics of the mixture are, respectively, denoted by $F^*(x)$, $\overline{F}^*(x)$, $f^*(x)$, $r^*(x)$, $q^*(x)$, $m^*(x)$, $\nu^*(x)$, $\sigma^{*2}(x)$ and $\overline{\sigma}^{*2}(x)$.

The comparison between random variables emerges naturally in reliability or survival analysis. For example, does the disease-free time increase under a new treatment? Are two vaccines equally effective? Does the interval between failures in a machine depend on the working conditions? Stochastic orders answer questions like these.

X is less or equal than Y ($X \leq Y$) under a specific stochastic order if the corresponding properties given in the following hold:

- Usual stochastic order *(st)*:
  $\overline{F}_X(x) \leq \overline{F}_Y(x)$. Equivalently, $E[h(X)] \leq E[h(Y)]$ for an increasing function h;
- Hazard rate order (*hr*): $r_X(x) \geq r_Y(x)$;
- Reversed hazard rate order (*rhr*): $q_X(x) \leq q_Y(x)$;
- Likelihood ratio order (*lr*): $f_X(y)f_Y(x) \leq f_X(x)f_Y(y)$, $x \leq y$;
- Mean residual life order (*mrl*): $m_X(x) \leq m_Y(x)$;
- Residual life variance order (*rlv*): $\sigma^2_X(x) \leq \sigma^2_Y(x)$;
- Mean inactivity time order (*mit*): $\nu_X(x) \geq \nu_Y(x)$;
- Inactivity time variance order (*itv*): $\overline{\sigma}^2_X(x) \geq \overline{\sigma}^2_Y(x)$;

given that x, y are in the support set of X and Y.

Note that $X \leq Y$ under all the previous stochastic orders except for the *rlv*, indicates that X is worse than Y in different senses: a greater hazard rate or a smaller mean residual life. A lower reversed hazard rate, and therefore a larger mean inactivity time, also represent worse conditions.

Some of the previous properties are stronger than others. A thorough study of this subject can be found in [32–34]. The chain of implications between orders is well known, and these results are shown in Appendix A.

Since the frailty is used to reflect different operating conditions in a system or biological heterogeneity between individuals [7], then it is relevant to analyze its effect on the aging characteristics. As an alternative to models with covariates, the following definitions aim at describing the effect of these unobservable variations.

Remark 1. *$X^*|Z = z$ is increasing in z under a particular order if, for all $z_1 < z_2$, the corresponding properties below hold:*

- *Usual stochastic order:* $\overline{F}(x, z_1) \leq \overline{F}(x, z_2)$*;*
- *Hazard rate order:* $r(x, z_1) \geq r(x, z_2)$*;*
- *Reversed hazard rate order:* $q(x, z_1) \leq q(x, z_2)$*;*
- *Mean residual life order:* $m(x, z_1) \leq m(x, z_2)$*;*
- *Mean inactivity time order:* $\nu(x, z_1) \geq \nu(x, z_2)$*;*
- *Residual life variance order:* $\sigma^2(x, z_1) \leq \sigma^2(x, z_2)$*;*
- *Inactivity variance time order:* $\overline{\sigma}^2(x, z_1) \geq \overline{\sigma}^2(x, z_2)$*.*

Therefore, an increasing z implies a beneficial effect of the frailty, since all the aging characteristics reveal an improvement in the system with the exception of the residual life variance.

Given the chain of implications in Appendix A, if the failure rate $r(x, z)$ is decreasing in z, then $\overline{F}(x, z)$, $m(x, z)$ and $\sigma^2(x, z)$ are increasing in z, and if $q(x, z)$ is increasing in z, then $F(x, z)$, $\nu(x, z)$ and $\overline{\sigma}^2(x, z)$ are decreasing in z.

The following property is relevant for the forthcoming results. Its proof can be found in [35,36].

Lemma 1. *Let X be a random variable and $h(x)$, $g(x)$ two real functions.*

(a) *If both $h(x)$ and $g(x)$ are simultaneously increasing or decreasing, then*

$$Cov(h(X), g(X)) \geq 0,$$

(b) *If $h(x)$ is increasing and $g(x)$ is decreasing, then*

$$Cov(h(X), g(X)) \leq 0.$$

The conditions for Lemma 1 to hold are relaxed in the next result in Remark 2.

Remark 2. *Case (a) [(b)] follows, provided that $(h(x) - h(y))(g(x) - g(y)) \geq [\leq] 0$ for all x and y in the support of X.*

Observe that

$$\begin{aligned} &(h(x) - h(y))(g(x) - g(y)) \geq [\leq]\, 0 \Leftrightarrow \\ &(h(x) - h(y))(g(x) - g(y)) f_X(x) f_X(y) \geq [\leq]\, 0 \end{aligned}$$

where f_X is the density function of X. Therefore, the condition in Lemma 1 is verified:

$$\int_0^\infty \int_0^\infty (h(x) - h(y))(g(x) - g(y)) f_X(x) f_X(y) dx dy = 2Cov(h(X), g(X)).$$

Hence, $h(x)$ and $g(x)$ having the same monotonicity behavior is a sufficient but not a necessary condition for Lemma 1 to hold.

Models with covariates are basically concerned with estimating risks or predicting new values under different observable conditions. Hence, there is no uncertainty about the latter. In contrast, frailty models allow the variation caused by unobserved environments to be taken into account, and therefore a Bayesian analysis emerges as a natural way to study the frailty distribution once the data are observed [37]. Thus, right-censored and left-censored data can provide relevant information about the frailty. This is considered next.

The pdfs of the conditional distributions $Z|X^* > x$ and $Z|X^* \leq x$ are given as follows:

$$f_{Z|X^*>x}(z) = \frac{\overline{F}(x, z)\pi(z)}{\int_0^\infty \overline{F}(x, z)\pi(z) dz}, \quad z \geq 0 \tag{1}$$

and

$$f_{Z|X^* \le x}(z) = \frac{F(x,z)\pi(z)}{\int_0^\infty F(x,z)\pi(z)dz}, \quad z \ge 0 \tag{2}$$

Similar formulae for the the conditional reliability and the cumulative failure rate can also be obtained. The corresponding expressions are derived in Appendix B.

The expressions below for the mixture failure rate and reversed hazard rate can be found, respectively, in [25,26].

$$r^*(x) = E[r(x,Z)|X^* > x], \quad q^*(x) = E[q(x,Z)|X^* \le x]. \tag{3}$$

By using the same techniques as in [25,26], the following identities for the mean residual life and mean inactivity time in mixtures can be derived:

$$m^*(x) = E[m(x,Z)|X^* > x], \quad \nu^*(x) = E[\nu(x,Z)|X^* \le x]. \tag{4}$$

The expressions in (3) and (4) not only provide nice representations of the aging characteristics of the mixture but will also be useful in forthcoming results.

Given a random variable X with a distribution function $F_X(x)$ and x such that $F_X(x) > 0$ and $\overline{F}_X(x) > 0$, the following definitions apply:

The mean residual life is

$$m_X(x) = E[X - x|X > x] = \frac{\int_x^\infty \overline{F}_X(y)dy}{\overline{F}_X(x)}. \tag{5}$$

The residual life variance is

$$\sigma_X^2(x) = Var[X - x|X > x] = 2\frac{\int_x^\infty \int_y^\infty \overline{F}_X(u)dudy}{\overline{F}_X(x)} - m_X^2(x). \tag{6}$$

The mean inactivity time is

$$\nu_X(x) = E[x - X|X \le x] = \frac{\int_0^x F_X(y)dy}{F_X(x)}. \tag{7}$$

The inactivity time variance is

$$\overline{\sigma}_X^2(x) = Var[x - X|X \le x] = 2\frac{\int_0^x \int_0^y F_X(u)dudy}{F_X(x)} - \nu_X^2(x). \tag{8}$$

These representations are repeatedly used in this paper and, although they are well known, we derive them for the mean residual life and residual life variance in Appendix C, aiming at producing a self-contained text. We omit the proofs corresponding to the mean inactivity time and inactivity time variance since they are similar.

The main properties of the mean residual lifetime (inactivity time) and the residual life (inactivity time) variance can be found in the references mentioned in the Introduction. Next, we focus on their monotonicity properties, which are determinant in maintenance decision-making.

A random variable X shows increasing (decreasing) residual life variance if $\sigma_X^2(x)$ is increasing (decreasing) in x. Similar definitions regarding $\overline{\sigma}_X^2(x)$ lead to the class increasing (decreasing) inactivity time variance.

Next, well-known relations between the aging characteristics are recalled:

$$r_X(x) = \frac{\frac{dm_X(x)}{dx} + 1}{m_X(x)}. \tag{9}$$

$$q_X(x) = \frac{1 - \frac{d\nu_X(x)}{dx}}{\nu_X(x)}. \tag{10}$$

$$\frac{d\sigma_X^2(x)}{dx} = r_X(x)(\sigma_X^2(x) - m_X^2(x)). \tag{11}$$

$$\frac{d\overline{\sigma}_X^2(x)}{dx} = q_X(x)(\nu_X^2(x) - \overline{\sigma}_X^2(x)). \tag{12}$$

In the next remark, conditions for the monotonicity of the aging characteristics are stated.

Remark 3. *From (11), increasing (decreasing) residual life variance is equivalent to*

$$\sigma_X^2(x) \geq (\leq) m_X^2(x)$$

for all x.

From (12), decreasing (increasing) residual life variance is equivalent to

$$\overline{\sigma}_X^2(x) \geq (\leq) \nu_X^2(x)$$

for all x.

The following lemma contains the key representations for the residual life variance and inactivity time variance of the mixture.

Lemma 2.

$$\sigma^{*2}(x) = E[\sigma^2(x,Z)|X^* > x] + Var[m(x,Z)|X^* > x]; \tag{13}$$

$$\overline{\sigma}^{*2}(x) = E[\overline{\sigma}^2(x,Z)|X^* \leq x] + Var[\nu(x,Z)|X^* \leq x]. \tag{14}$$

Proof of Lemma 2. From (6), it follows that

$$\begin{aligned}
\sigma^{*2}(x) &= 2\frac{\int_x^\infty \int_y^\infty \overline{F}^*(u)dudy}{\overline{F}^*(x)} - m^{*2}(x) \\
&= \int_0^\infty \frac{2\int_x^\infty \int_y^\infty \overline{F}(u,z)dudy\pi(z)dz}{\int_0^\infty \overline{F}(x,z)\pi(z)dz} - m^{*2}(x) \\
&= \int_0^\infty \left(\frac{2\int_x^\infty \int_y^\infty \overline{F}(u,z)dudy}{\overline{F}(x,z)} - m^2(x,z) \right) f_{Z|X^*>x}(z)dz \\
&+ \int_0^\infty m^2(x,z) f_{Z|X^*>x}(z)dz - \left(\int_0^\infty m(x,z) f_{Z|X^*>x}(z)dz \right)^2 \\
&= \int_0^\infty \sigma^2(x,z) f_{Z|X^*>x}(z)dz + \int_0^\infty m^2(x,z) f_{Z|X^*>x}(z)dz - \left(\int_0^\infty m(x,z) f_{Z|X^*>x}(z)dz \right)^2.
\end{aligned}$$

From (8), we have

$$\begin{aligned}
\overline{\sigma}^{*2}(x) &= 2\frac{\int_0^x \int_0^y F^*(u)dudy}{F^*(x)} - \nu^{*2}(x) \\
&= \int_0^\infty \frac{2\int_0^x \int_0^y F(u,z)dudy\pi(z)dz}{\int_0^\infty F(x,z)\pi(z)dz} - \nu^{*2}(x) \\
&= \int_0^\infty \overline{\sigma}^2(x,z) f_{Z|X^*\leq x}(z)dz + \int_0^\infty \nu^2(x,z) f_{Z|X^*\leq x}(z)dz - \left(\int_0^\infty \nu(x,z) f_{Z|X\leq x}(z)dz \right)^2.
\end{aligned}$$

□

3. Preservation of Aging Classes under Mixtures

When defining a maintenance policy, the monotonicity of the failure rate plays a central role. If it is an increasing function, preventive replacement can be carried out, avoiding the cost derived from failure, which is usually higher than that incurred due to the replacement itself. However, it is well known that the optimum replacement time is infinite in the case of non-increasing failure rates as the exponential. This is so, either for revealed failures [38] or unrevealed failures, when inspections to detect them are free from false-negative outcomes [39]. The recent work carried out in [40] introduces a more general concept, i.e., the deviation cost per unit time between replacement and failure, so that age replacement policies can be valid for the exponential distribution. The forthcoming results aim at studying the aging in frailty models.

The analysis in [8] provides bounds for $\frac{dq^*(x)}{dx}$ and $\frac{dv^*(x)}{dx}$, whereas the corresponding ones for $\frac{dr^*(x)}{dx}$ and $\frac{dm^*(x)}{dx}$ can be found in [41]. These results indicate that the change in the aging characteristic in the mixture is not completely arbitrary but is under control of both the distributions in the mixture and the frailty. Theorem 1 contains more precise expressions, since they are new representations of the derivatives of the aging characteristic in the mixture. Moreover, these results provide the exact difference between the derivative of the aging characteristics of the mixture and the conditional expectation of the derivatives of the aging characteristic corresponding to the distributions in the mixture given the updated frailty. Hence, they constitute a formal approach to the improvement observed in mixtures of populations compared with individuals.

Theorem 1. *The following properties hold, provided the derivatives exist and can be interchanged with the corresponding integrals.*

(a)

$$\frac{dr^*(x)}{dx} = E\left[\frac{dr(x,Z)}{dx}|X^* > x\right] - Var(r(x,Z)|X^* > x)$$

(b)

$$\frac{dq^*(x)}{dx} = E\left[\frac{dq(x,Z)}{dx}|X^* \le x\right] + Var(q(x,Z)|X^* \le x)$$

(c)

$$\frac{dm^*(x)}{dx} = E\left[\frac{dm(x,Z)}{dx}|X^* > x\right] - Cov(r(x,Z), m(x,Z)|X^* > x)$$

(d)

$$\frac{dv^*(x)}{dx} = E\left[\frac{dv(x,Z)}{dx}|X^* \le x\right] + Cov(v(x,z), q(x,z)|X^* \le x)$$

Proof of Theorem 1. The result in *(a)* has been proven in [42].

(b) Straightforward derivatives in (1) and (2) lead, respectively, to

$$\frac{df_{Z|X^*>x}(z)}{dx} = f_{Z|X^*>x}(z)(-r(x,z) + r^*(x)) \tag{15}$$

$$\frac{df_{Z|X^*\le x}(z)}{dx} = f_{Z|X^*\le x}(z)(q(x,z) - q^*(x)) \tag{16}$$

Taking the derivative in the expression of the reversed hazard rate in (3), and given the assumption of possible exchange between the derivative and the integral, it follows that

$$\frac{dq^*(x)}{dx} = \int_0^\infty \frac{dq(x,z)}{dx} f_{Z|X^*\le x}(z)dz + \int_0^\infty q(x,z)\frac{df_{Z|X^*\le x}(z)}{dx}dz,$$

and then, based on (16), *(b)* is proven.

Consider now the derivatives in (4), again with the corresponding assumption of possible exchange with the integrals. Then,

$$\frac{dm^*(x)}{dx} = \int_0^\infty \frac{dm(x,z)}{dx} f_{Z|X^*>x}(z)dz + \int_0^\infty m(x,z)\frac{df_{Z|X^*>x}(z)}{dx}dz \tag{17}$$

$$\frac{dv^*(x)}{dx} = \int_0^\infty \frac{dv(x,z)}{dx} f_{Z|X^*\leq x}(z)dz + \int_0^\infty \nu(x,z)\frac{df_{Z|X^*\leq x}(z)}{dx}dz \tag{18}$$

The results in *(c)* and *(d)* follow from (15) and (16), respectively. □

In the case that the individuals in the mixture present a deteriorating state with time expressed as $r_X(x)$ increases, $q_X(x)$ decreases, $m_X(x)$ decreases or $\nu_X(x)$ increases, then the second term on the right-hand side in (a)–(d) is, respectively, negative, positive, positive or negative. Therefore, the mixture presents a smoother behavior than the distributions that compound the mixture. When individuals with increasing failure rate are mixed, the effect is that the derivative of the failure rate of the mixture is below the mean of the derivatives, and therefore a decreasing behavior could even be observed given the negative term on the right-hand side of the equality in (a). Similar comments apply for individuals with a decreasing reversed hazard rate, decreasing mean residual life or increasing inactivity time. In all the cases, the derivative in the mixture is below the mean of the derivatives, and reversed behaviors can also occur. This explains the noticeable improvement in the mixture compared with the distributions therein.

The preservation under mixtures of the classes decreasing hazard rate, increasing reversed hazard rate, increasing mean residual life and decreasing mean inactivity time are known properties. The next corollary presents them as a straightforward consequence of Theorem 1.

Corollary 1.

(a) *Preservation, under mixtures of both the decreasing failure rate and the increasing reversed hazard rate [8], follows from Theorem 1 (a) and (b), respectively.*

(b) *Preservation of the increasing mean residual life holds, provided that the term on the right-hand side of Theorem 1 (c) is positive. Consider $z_1 \leq z_2$ in the support of the frailty Z and the roots of the equation $r(y,z_1) - r(y,z_2) = 0$. Then, for a given x, there exist two roots y_0 and y_1 ($y_0 \leq y_1$) of the previous equation such that $x \in [y_0, y_1]$ and $r(y,z_1) - r(y,z_2)$ are non-negative or non-positive in $[y_0, y_1]$.*

Let us assume that $r(y,z_1) - r(y,z_2) \leq 0$. We define the failure rates $\lambda_1(x)$, $\lambda_2(x)$ as follows:

$$\lambda_1(y) = \begin{cases} r(y,z_1), & y_0 \leq y \leq y_1 \\ r(y_1,z_1), & y > y_1 \end{cases}$$

$$\lambda_2(y) = \begin{cases} r(y,z_2), & y_0 \leq y \leq y_1 \\ r(y_1,z_2), & y > y_1 \end{cases}$$

Observe that the assumption $r(y,z_1) - r(y,z_2) \leq 0$ leads to $\lambda_1(y) \leq \lambda_2(y)$. In addition, $\lambda_1(y)$ and $\lambda_2(y)$ are constant values for $y \geq y_1$, with $\lambda_1(y) = \lambda_2(y)$. Denoting by X_1 and X_2 the random variables with failure rates $\lambda_1(y)$ and $\lambda_2(y)$, respectively, then, $X_1 \geq_{hr} X_2$.

Consider next the mean residual lives $m_1(y)$ and $m_2(y)$ such that $m_1(y) = m(y,z_1)$ and $m_2(y) = m(y,z_2)$ for $y \in [y_0, y_1]$, and taking the constant values indicated below, otherwise:

$$\begin{cases} m_1(y) = m_2(y) = \frac{1}{r(y_0,z_1)}, & y \leq y_0 \\ m_1(y) = m_2(y) = \frac{1}{r(y_1,z_1)}, & y \geq y_1 \end{cases}$$

From (9), it follows that the mean residual life of X_i is $m_i(x)$, $i = 1, 2$, and Remark 1 implies $X_1 \geq_{mrl} X_2$. Therefore, $(r(x,z_1) - r(x,z_2))(m(x,z_1) - m(x,z_2)) \leq 0$, and then $Cov(r(x,Z), m(x,Z)) \leq 0$ follows from both Lemma 1 and Remark 2.

A similar proof applies when $r(y,z_1) - r(y,z_2) \geq 0$. If so, $X_1 \leq_{hr} X_2$ and $X_1 \leq_{mrl} X_2$.

(c) *The preservation of decreasing mean inactivity time under mixtures [8] can also be derived from Theorem 1 (d), once the second term on the right side of (d) is proven to be negative, using a similar strategy to the previous one in (b). Now, when defining the mean inactivity times, $\nu_1(y)$ and $\nu_2(y)$, we assume $\nu_1(y) = \nu_2(y)$ for $y \leq y_0$. The remaining details are omitted for brevity.*

The preservation results under mixtures are considered to be important properties, given their practical relevance, for example in maintenance modeling. The next theorem generalizes the preservation results, since the assumption of non-crossing distributions in the mixture appearing in previous research has been dropped.

Theorem 2. *Distributions with increasing (decreasing) residual life (inactivity time) variance distributions are preserved under mixtures.*

Proof of Theorem 2. The result is proved for the residual life variance. Since the variance is non-negative, by Equation (13) we have that

$$\sigma^{*2}(x) \geq E[\sigma^2(x,Z)|X^* > x] \geq E[m^2(x,Z)|X^* > x] \geq E^2[m(x,Z)|X^* > x] = m^*(x)^2$$

where the second inequality follows given that $\sigma^2(x,z)$ is increasing in x (see Remark 3) and the third inequality is derived from Jensen's inequality applied to the convex function x^2. Therefore, the claim is proven by Remark 3. The details for the inactivity time variance are omitted, since the result is obtained by using a similar series of inequalities as in (14) and Remark 3. □

4. Bending Properties

When dealing with a random variable, its expectation and variance are usually considered to be key values. This is so because it is enlightening to know its average behavior and whether an observation lies far from the mean, becoming an outlier. In addition, in order to obtain good estimators of both the mean and the variance, only a large enough sample is required. This idea also applies for mixtures. Let X be a random variable with L = (r_X, q_X, m_X, σ^2_X, ν_X, $\overline{\sigma}^2_X$) being an specific aging characteristic of X. A bending property for a mixture of distributions ($\{\overline{F}(x,z)\}$) with frailty Z is a comparison between the mixture aging characteristic $L^*(x)$ and its mean value $L_E(x) = E[L(x,Z)]$. Previous studies state that $L_E(x)$ retains the monotonicity conditions of the mixture $L^*(x)$ ([25]). In [27], bending properties for the mean residual life and mean inactivity time are studied. In what follows, we extend this analysis to the residual lifetime variance and inactivity time variance.

Regarding the residual life (inactivity time) variance, the following properties are defined:

- The weak bending up property if $\sigma^{*2}(x) \geq \sigma^2_E(x)$ ($\overline{\sigma}^{*2}(x) \geq \overline{\sigma}^2_E(x)$).
- The strong bending up property if $\sigma^{*2}(x) - \sigma^2_E(x)$ ($\overline{\sigma}^{*2}(x) - \overline{\sigma}^2_E(x)$) is increasing (decreasing) in x.

Observe that under both definitions, the variability of the residual life and inactivity time of the mixture is larger than in the individuals. Large residual lives are associated with long-term survivors and small ones with those corresponding to high failure rates. Regarding the inactivity time in failed units, a proportion of them will have just entered the failed state at the moment it is detected, whereas the failure will have remained undiscovered for longer periods in other cases. Thus, the previous definitions follow the idea that when considering the overall data, strong individuals are mixed with weak ones. They also give a theoretical support for the interpretation in [43] regarding the reversal of increasing failure rates under mixtures. The authors in [43] state that the long-term survivors, although they represent a small proportion of the pooled data, determine the behavior of the failure rate in the long run, resembling outliers in a regression analysis.

Theorem 3. *In the case of a mixture of distributions $\{\overline{F}(x,z)\}$ with frailty Z, the following properties apply:*

(a) *If $\overline{F}(x,z)$ is increasing (decreasing) in z and $\sigma^2(x,z)$ is increasing (decreasing) in z, then the weak bending up property holds for the residual lifetime variance.*
(b) *If $\overline{F}(x,z)$ is increasing (decreasing) in z and $\overline{\sigma}^2(x,z)$ is decreasing (increasing) in z, then the weak bending up property holds for the inactivity time variance.*

Proof of Theorem 3.

(a) The following inequalities apply:

$$\sigma^{*2}(x) \geq E[\sigma^2(x,Z)|X^* > x] \geq E[\sigma^2(x,Z)]$$

where the first inequality is derived by (13). The following steps lead to the second inequality:

1. From (1), we have that $\frac{\overline{F}(x,z)}{\int_0^\infty \overline{F}(x,z)\pi(z)dz} = \frac{f_{Z|X^*>x}(z)}{\pi(z)}$. Since the left-hand side of this equality is assumed to be increasing (decreasing) in z, so is the term on the right-hand side.
2. From the previous point and $z_1 \leq z_2$, it follows that

$$f_{Z|X^*>x}(z_1)\pi(z_2) \leq (\geq) f_{Z|X^*>x}(z_2)\pi_(z_2)$$

and thus, $Z|X^* > x \geq_{\text{lr}} (\leq_{\text{lr}}) Z$.
3. The implications in Remark 1 (which follow the chain in Appendix A) imply that $Z|X^* > x \geq_{\text{st}} (\leq_{\text{st}}) Z$.
4. The equivalent definition for the usual stochastic order when the expectation for an increasing function is considered, leads to the second inequality, since $\sigma^2(x,z)$ is assumed to be increasing (decreasing) in z.

(b) The following inequalities hold:

$$\overline{\sigma}^{*2}(x) \geq E[\overline{\sigma}^2(x,Z)|X^* \leq x] \geq E[\overline{\sigma}^2(x,Z)]$$

The result in (14) leads to the first inequality. The steps to prove the second one are as follows:

1. From (2), we have that $\frac{F(x,z)}{\int_0^\infty F(x,z)\pi(z)dz} = \frac{f_{Z|X^*\leq x}(z)}{\pi(z)}$. Since the left-hand side of this equality is assumed to be decreasing (increasing) in z, so is the term on the right-hand side.
2. From the previous point and $z_1 \leq z_2$, it follows that

$$f_{Z|X^*\leq x}(z_1)\pi(z_2) \geq (\leq) f_{Z|X^*\leq x}(z_2)\pi_(z_2)$$

and thus $Z|X^* \leq x \leq_{\text{lr}} (\geq_{\text{lr}}) Z$.
3. The implications in Remark 1 (which follow the chain in Appendix A) imply that $Z|X^* \leq x \leq_{\text{st}} (\geq_{\text{st}}) Z$.
4. The assumption that $\overline{\sigma}^2(x,z)$ is decreasing (increasing) leads to the second inequality.

□

The results in Theorem 3 are consistent with the negative aging that mixtures tend to show. In case (a), if $\overline{F}(x,z)$ is increasing in z, then the larger Z is, the better the effect on the survival function. The situation is just the opposite if $\overline{F}(x,z)$ is decreasing in z. In both cases, the distributions for which the random effect Z is good correspond to a long-term survivor. According to [7], Z represents a biological advantage or an individual propensity, and with the assumptions in Theorem 3, these individuals also present the greatest variance. Therefore it makes perfect sense that the variance of the residual life of the mixture is greater than the mean of the variances.

In case (b), if the values of Z corresponding to the shortest survival times are also those with larger inactivity time variances, then the variance of the inactivity time of the mixture is greater than the mean of the variances. Following the interpretation in [43], the mixture shows the effect of outliers.

In the last result, we revisit the proportional mean residual life model [44]. This is an alternative to Cox's proportional hazard model for describing the effect of the frailty. A mixture of distributions $\{\overline{F}(x,z)\}$ with the frailty random variable Z follows the proportional mean residual life if the mean residual life of the mixture when $Z=z$, $m(x,z)$ verifies:

$$m(x,z) = z m_X(x), \quad 0 < z \leq 1$$

where $m_X(x)$ is a baseline mean residual life.

If we assume that $m_X(x)$ accounts for the mean residual life when the system operates under "normal" conditions, the proportional mean residual life is useful for representing more adverse environments that accelerate the failure. The function $m(x,z)$ is increasing with z, and so a higher value of the frailty implies a better operating condition for the system.

The failure rate of the mixture conditional to $Z=z$ is derived from (9) as

$$r(x,z) = r_X(x) + \frac{1-z}{z}\frac{1}{m_X(x)} \tag{19}$$

where $r(x,z)$ is decreasing with z.

The mean residual life of the mixture is

$$m^*(x) = E[Z|X^* > x] m_X(x) \tag{20}$$

The next theorem states the conditions where there is an increasing difference between the variance of the residual life of the mixture and the average variance of the residual lives of the subpopulations. In other words, the effect of the frailty is stronger with time. The outlier subpopulations tend to appear more different from the rest of the distributions as the time increases.

Theorem 4. *Consider a mixture of distributions $\{\overline{F}(x,z)\}$ satisfying the proportional mean residual life. If $m_X(x)$ is an increasing function, then the strong bending up property holds for the residual lifetime variance.*

Proof of Theorem 4. By Equation (13) we have

$$\sigma^{*2}(x) - E[\sigma^2(x,Z)] = \int_0^\infty \sigma^2(x,z) f_{Z|X^*>x}(z)dz - E[\sigma^2(x,Z)]$$
$$+ \int_0^\infty m^2(x,z) f_{Z|X^*>x}(z)dz - \left(\int_0^\infty m(x,z) f_{Z|X^*>x}(z)dz\right)^2.$$

The derivative with respect to x in the foregoing expression can be written as follows after exchanging the derivative and integral and taking into account the result in (15):

$$\begin{aligned}
& \int_0^\infty \frac{d\sigma^2(x,z)}{dx} f_{Z|X^*>x}(z)dz - E\left[\frac{d\sigma^2(x,Z)}{dx}\right] + \int_0^\infty \sigma^2(x,z)\frac{df_{Z|X^*>x}(z)}{dx}dz \\
+ \; & 2\int_0^\infty m(x,z)\frac{dm(x,z)}{dx} f_{Z|X^*>x}(z)dz + \int_0^\infty m^2(x,z)\frac{df_{Z|X^*>x}(z)dz}{dx} \\
- \; & 2\int_0^\infty m(x,z) f_{Z|X^*>x}(z)dz\left(\int_0^\infty \frac{dm(x,z)}{dx} f_{Z|X^*>x}(z)dz + \int_0^\infty m(x,z)\frac{df_{Z|X^*>x}(z)}{dx}dz\right) \\
= \; & \int_0^\infty \frac{d\sigma^2(x,z)}{dx} f_{Z|X^*>x}(z)dz - E\left[\frac{d\sigma^2(x,Z)}{dx}\right] \\
+ \; & \int_0^\infty (\sigma^2(x,z) + m^2(x,z))(r^*(x) - r(x,z)) f_{Z|X^*>x}(z)dz \\
+ \; & 2\int_0^\infty m(x,z)\frac{dm(x,z)}{dx} f_{Z|X^*>x}(z)dz \\
- \; & 2m^*(x)\int_0^\infty \left(\frac{dm(x,z)}{dx} + m(x,z)(r^*(x) - r(x,z))\right) f_{Z|X^*>x}(z)dz \\
= \; & E\left[\frac{d\sigma^2(x,Z)}{dx}|X^* > x\right] - E\left[\frac{d\sigma^2(x,Z)}{dx}\right] && (21) \\
- \; & Cov\left(r(x,Z), \sigma^2(x,Z) + m^2(x,Z)|X^* > x\right) && (22) \\
+ \; & 2Cov\left(m(x,Z), \frac{dm(x,Z)}{dx}|X^* > x\right) && (23) \\
+ \; & 2m^*(x)Cov(r(x,Z), m(x,Z)|X^* > x). && (24)
\end{aligned}$$

The following steps aim at checking that all the previous terms are positive.

We must take into account the fact that that $\frac{d\sigma^2(x,z)}{dx}$ is increasing in z. The proof is in Remark A2 in Appendix C.

Equation (21) is non-negative.

According to the assumptions, $r(x,z)$ is decreasing in z. Then, from Remark 1, it follows that $\overline{F}(x,z)$ is increasing in z and so is $\frac{f_{Z|X^*>x}(z)}{\pi(z)}$, following the same algebra as that in step 1 of Theorem 3 (a). Hence, $Z|X^* > x \geq_{\text{st}} Z$, which in addition to $\frac{d\sigma^2(x,z)}{dx}$ being increasing in z implies the positiveness of the term.

With the assumption that $m_X(x)$ is increasing, Equation (23) is also non-negative:

$$2Cov\left(m(x,Z), \frac{dm(x,Z)}{dx}|X^* > x\right) = 2m_X(x)\frac{dm_X(x)}{dx}Cov(Z,Z) =$$
$$2m_X(x)\frac{dm_X(x)}{dx}Var[Z|X^* > x].$$

The expressions in (22) and (24) can be alternatively expressed as:

$$-Cov\left(r(x,Z), \sigma^2(x,Z) - m^2(x,Z)|X^* > x\right) - 2Cov\left(r(x,Z), m^2(x,Z)|X^* > x\right)$$
$$+2m^*(x)Cov(r(x,Z), m(x,Z)|X^* > x).$$

Next, we focus on the right-hand side of the first term, $A(z) = \sigma^2(x,Z) - m^2(x,Z)$, which is increasing in z. The proof is in Remark A3 in Appendix C. Since $r(x,z)$ is decreasing with z, Lemma 1 implies that

$$-Cov\left(r(x,Z), \sigma^2(x,Z) - m^2(x,Z)|X^* > x\right) > 0.$$

The remaining two terms verify

$$
\begin{aligned}
& -2Cov(r(x,Z), m^2(x,Z)|X^* > x) + 2m^*(x)Cov(r(x,Z), m(x,Z)|X^* > x) \\
= & -2Cov\left(r(x) + \frac{1-Z}{Z}\frac{1}{m_X(x)}, Z^2 m_X^2(x)|X^* > x\right) \\
& +2m_X(x)E[Z|X^* > x]Cov\left(r(x) + \frac{1-Z}{Z}\frac{1}{m_X(x)}, Zm(x)|X^* > x\right) \\
= & -2m_X(x)Cov\left(\frac{1-Z}{Z}, Z^2|X^* > x\right) + 2m_X(x)E[Z|X^* > x]Cov\left(\frac{1-Z}{Z}, Z\right) \\
= & \ 2m_X(x)\left(E\left[\frac{1-Z}{Z}|X^* > x\right]E[Z^2|X^* > x] - E[Z(1-Z)|X^* > x]\right) \\
+ & \ 2m_X(x)\left(E[Z|X^* > x]E[1-Z|X^* > x] - E^2[Z|X^* > x]E\left[\frac{1-Z}{Z}|X^* > x\right]\right) \\
= & \ 2\left(E\left[\frac{1-Z}{Z}|X^* > x\right]Var[Z|X^* > x] - Cov(Z, 1-Z|X^* > x])\right).
\end{aligned}
$$

$E\left[\frac{1-Z}{Z}|X^* > x\right]$ is positive since it is the expectation of a positive random variable. According to Lemma 1, $Cov(Z, 1-Z|X^* > x) < 0$. Given the positiveness of all the terms, then $\sigma^{*2}(x) - \sigma_E^2(x)$ is increasing and the strong bending up property holds. □

Example 1. *Consider the proportional mean residual life $m(x,z) = zm_X(x)$, where $m_X(x)$ is a baseline mean residual life and $0 < z \leq 1$. The relation in (19) implies that $r(x,z)$ is decreasing in z and $\overline{F}(x,z)$ and $\sigma^2(x,z)$ are also increasing (Remark 1). Therefore, the assumptions in Theorem 3 (a) hold and so does the weak bending up property for the residual life variance, when the assumption that $m_X(x)$ is increasing is dropped.*

Example 2.

$$m(x,z) = z(1+x), \quad 0 < z \leq 1, \quad x \geq 0.$$

In this case

$$\overline{F}(x,z) = (1+x)^{-\left(\frac{1}{z}+1\right)}, \quad 0 < z \leq 1, \quad x \geq 0.$$

Straightforward algebra yields

$$\sigma^2(x,z) = (x+1)^2 z^2 \frac{1+z}{1-z}, \quad 0 < z < 1, \quad x \geq 0.$$

As the baseline mean residual life $(1+x)$ is increasing, conditions for both the weak and strong bending up property in the case of the residual life variance are fulfilled. We assume now that the frailty Z follows a beta distribution with parameters $a, b > 0$. The corresponding density function is

$$f_Z(z) = \frac{1}{B(a,b)} z^{a-1}(1-z)^{b-1}, \quad 0 < z < 1$$

where $B(a,b)$ is the standard beta function. Thus,

$$f_{Z|X^*>x}(z) = \frac{(1+x)^{-\left(\frac{1}{z}+1\right)} z^{a-1}(1-z)^{b-1}}{\int_0^1 (1+x)^{-\left(\frac{1}{v}+1\right)} v^{a-1}(1-v)^{b-1} dv}.$$

It follows that

$$E[\sigma^2(x,Z)] = (x+1)^2 \int_0^1 z^2 \frac{1+z}{1-z} \frac{1}{B(a,b)} z^{a-1}(1-z)^{b-1} dz$$

and

$$\begin{aligned}\sigma^{*2}(x) &= E[\sigma^2(x,Z)|X^* > x] + Var[(1+x)Z|X^* > x] \\ &= (x+1)^2 \int_0^1 z^2 \frac{1+z}{1-z} \frac{(1+x)^{-(\frac{1}{z}+1)} z^{a-1}(1-z)^{b-1}}{\int_0^1 (1+x)^{-(\frac{1}{v}+1)} v^{a-1}(1-v)^{b-1} dv} \\ &+ (x+1)^2 \left(\int_0^1 z^2 f_{Z|X^*>x}(z) dz - \left(\int_0^1 z f_{Z|X^*>x}(z) dz \right)^2 \right).\end{aligned}$$

Figure 1 represents $\sigma^{*2}(x) - E[\sigma^2(x,Z)]$ with Z following a beta random variable under different values of a and b, with $m(x,z) = z(1+x)$. The function $\sigma^{*2}(x) - E[\sigma^2(x,Z)]$ is non-negative and therefore the weak bending up property is verified. Furthermore, the strong bending up property also holds, since $\sigma^{*2}(x) - E[\sigma^2(x,Z)]$ is increasing.

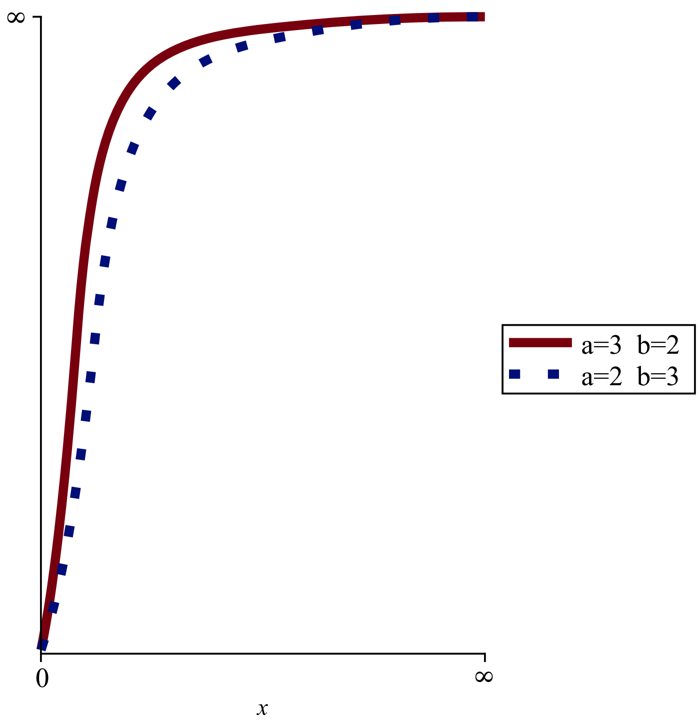

Figure 1. The difference against time between the residual life variance of the mixture and the average variance of the residual lives of the subpopulations, $\sigma^*(x) - E[\sigma^2(x,Z)]$. The mixture follows the mean proportional residual life model with the baseline $m_X(x) = 1 + x$, and Z is a beta random variable with parameters a and b.

5. Conclusions

The study of the residual lifetime and inactivity time in mixtures is crucial in reliability, since the behavior of the former has implications in maintenance modeling and the latter in retrospective analysis. When systems are affected by a changing environment but there are no observable covariates, the changes are described by a random variable (frailty), and mixtures emerge to describe the time to failure of the whole population.

This paper focused on the residual life and inactivity time of a non-negative random variable, representing a time to failure. The former is useful for maintenance purposes, to decide whether to replace a non-failed system. The latter provides information in retrospective studies, such as when a disease is detected and the time elapsed since the person was infected is relevant to understanding the infection process and even determining contacts with high risks of exposure.

Aging properties are used to describe and summarize random variables. In this paper, we analyzed the residual life variance and inactivity time, in contrast to previous papers dealing with the corresponding expectations. The results were obtained by using the approach in [25,26], based on the conditional distribution of the frailty. In so doing, we provided new representations of the residual life variance and inactivity time variance, extending the preservation of the classes increasing residual life variance and decreasing inactivity time variance for an arbitrary mixture.

We provided new characterizations for the derivatives of the aging characteristics in mixtures in terms of the expectations of the corresponding derivatives of the distributions in the mixture which, in turn, included the information from the data. The results provided in this paper explain the improvement of the aging characteristics with time in actual systems affected by random effects. When the distributions present a decreasing reliability, the mixture shows smoother changes with less adverse aging than the mean of the distributions in the mixture.

This paper was also concerned with bending properties for comparing the residual time variance and the inactivity time variance of the mixture with the corresponding means of the residual time variances and inactivity time variances of the distributions therein. The former are greater when subpopulations that are more different from the rest due to the frailty match those with larger variances. This effect of strong components determining the behavior of the mixture has been reported many times in both theoretical and empirical studies. Some authors refer to them as outliers.

Regarding forthcoming work, this methodology based on the conditional distribution of the frailty seems to be a promising way to obtain new bending properties involving, for example, the inactivity time variance. Once we have data at hand, the verification of these properties via actual problems would enhance the interest of this research. At present, the use of simulations seems likely to be more affordable in the near future.

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Appendix A

The chain of implications between stochastic orders is as follows:

$$
\begin{array}{ccccccc}
 & & & & \text{mrl} & \longrightarrow & \text{rlv} \\
 & & \text{hr} & \begin{smallmatrix}\nearrow\\ \searrow\end{smallmatrix} & & & \\
\text{lr} & \begin{smallmatrix}\nearrow\\ \searrow\end{smallmatrix} & & & \text{st} & & \\
 & & \text{rhr} & \begin{smallmatrix}\nearrow\\ \searrow\end{smallmatrix} & & & \\
 & & & & \text{mit} & \longrightarrow & \text{itv}
\end{array}
$$

In this chain of implications, the stronger the stochastic order, the closer its position to the left-hand side. Observe that the likelihood ratio order is the strongest condition, whereas the residual life variance and inactivity time variance are the weakest ones. In addition, when two orders are connected by an arrow, it means that if two random variables present the stochastic order on the left, then they also verify the order indicated by the arrowhead. The reverse implication does not necessarily hold. When two stochastic orders are not connected, each of them can hold, independently of the other.

$X \leq_{lr} Y$ implies that given two values, for example two survival times x and y with $x \leq y$, it is more likely that the larger value y belongs to the population Y and the smaller value x to X than the other way around. Hence, when X and Y represent times to failure, the latter constitutes the tough group and the other two conditions follow: $X \leq_{hr} Y$ and $X \leq_{rhr} Y$. Therefore, X presents a higher failure rate than Y and a lower reversed hazard rate. Thus, when a failure or a disease is detected, the probability that it has just occurred is lower in the weaker population X. If $X \leq_{hr} Y$, then an item from population X has a lower chance of remaining without failure from a given time x onward than another item belonging to Y, provided that neither of them had failed at t. Hence, $X \leq_{mrl} Y$. In a similar way, $X \leq_{rhr} Y$ implies that when a disease is revealed, it is more likely that group Y was infected later than X, and thus $X \leq_{mit} Y$.

Appendix B

Next, some aging properties of the frailty, conditional to data, are provided.

The conditional reliability function is:

$$\bar{F}_{Z|X^*>x}(z) = P(Z \geq z|X^* > x) = \int_z^\infty f_{Z|X^*>x}(t)dt = \frac{\int_z^\infty \overline{F}(x,t)\pi(t)dt}{\int_0^\infty \overline{F}(x,t)\pi(t)dt}, \quad z \geq 0$$

where the last equality was obtained after (1).

The conditional failure rate is:

$$r_{Z|X^*>x}(z) = \frac{f_{Z|X^*>x}(z)}{\bar{F}_{Z|X^*>x}(z)} = \frac{\overline{F}(x,z)\pi(z)}{\int_z^\infty \overline{F}(x,t)\pi(t)dt}.$$

The conditional cumulative failure rate is:

$$\Delta_{Z|X^*>x}(z) = \int_0^z r_{Z|X^*>x}(t)dt = \int_0^z \frac{\overline{F}(x,t)\pi(t)dt}{\int_t^\infty \overline{F}(x,u)\pi(u)du}.$$

Appendix C

The expectations of the residual residual life and its square are:

$$E[X-x, X>x] =$$
$$\int_x^\infty (t-x)f_X(t)dt = -(t-x)\bar{F}_X(t)|_x^\infty + \int_x^\infty \bar{F}_X(t)dt = \int_x^\infty \bar{F}_X(t)dt.$$

The previous result is derived by using integration by parts with $(t-x) = u$ and $dv = f_X(t)dt$. Therefore,

$$m_X(x) = \frac{\int_x^\infty \overline{F}_X(y)dy}{\overline{F}_X(x)}$$

$$E[(X-x)^2, X>x] =$$
$$\int_x^\infty (t-x)^2 f_X(t)dt = -(t-x)^2\bar{F}_X(t)|_x^\infty + 2\int_x^\infty (t-x)\bar{F}_X(t)dt = 2\int_x^\infty (t-x)\bar{F}_X(t)dt$$

where a new integration by parts with $(t-x)^2 = u$ and $dv = f_X(t)dt$ has been applied.

$$2\int_x^\infty (t-x)\bar{F}_X(t)dt = 2\int_x^\infty \left(\int_x^t du\right)\bar{F}_X(t)dt = 2\int_x^\infty \int_u^\infty \bar{F}_X(t)dtdu.$$

The last integral is obtained using Fubini's theorem. Hence,

$$\sigma_X^2(x) = 2\frac{\int_x^\infty \int_y^\infty \overline{F}_X(u)dudy}{\overline{F}_X(x)} - m_X^2(x)$$

The next remarks are used in the proof of Theorem 4.

Remark A1. *The failure rate, $r(x,z)$ and the reliability function $\bar{F}(x,z)$ verify*

$$\bar{F}(x,z) = e^{-\int_0^x r(u,z)du}$$

Remark A2. *Under the proportional mean residual life $m(x,z) = zm_X(x)$, with $m_X(x)$ as an increasing function, then $\frac{d\sigma^2(x,z)}{dx}$ is increasing in z.*

Consider the proportional mean residual life $m(x,z) = zm_X(x)$, where $m_X(x)$ is a baseline mean residual life and $0 < z \leq 1$. Following (6) we obtain

$$\begin{aligned}
\sigma^2(x,z) &= \frac{2\int_x^\infty \int_y^\infty \overline{F}(u,z)dudy}{\overline{F}(x,z)} - m^2(x,z) \\
&= \frac{2\int_x^\infty \overline{F}(y,z)m(y,z)dy}{\overline{F}(x,z)} - m^2(x,z) \\
&= \frac{2\int_x^\infty \overline{F}(y,z)(m(y,z) - m(x,z))dy}{\overline{F}(x,z)} + m^2(x,z)
\end{aligned}$$

where the last equality follows from (5).

Combining the previous identity and (11), we obtain

$$\begin{aligned}
\frac{d\sigma^2(x,z)}{dx} &= r(x,z)(\sigma^2(x,z) - m^2(x,z)) = \left(zr_X(x) + (1-z)\frac{1}{m_X(x)}\right) \\
&\times \; 2\int_x^\infty (m_X(y) - m_X(x))e^{-\int_x^y r(u,z)du}dy.
\end{aligned}$$

Given the formula in (9), the derivative of the foregoing expression with respect to z is

$$\begin{aligned}
&\frac{\frac{dm_X(x)}{dx}}{m_X(x)}2\int_x^\infty (m_X(y) - m_X(x))e^{-\int_x^y r(u,z)du}dy \\
&- \; (zr_X(x) + (1-z)\frac{1}{m_X(x)})2\int_x^\infty (m_X(y) - m_X(x))\int_x^y \left(\frac{dr(u,z)}{dz}du\right)e^{-\int_x^y r(u,z)du}dy \geq 0.
\end{aligned}$$

The first term is non-negative since $m_X(x)$ is increasing. The second term is also non-negative because $r(x,z)$ decreasing in z results in $\frac{dr(u,z)}{dz}$ being negative. Hence, $\frac{d\sigma^2(x,z)}{dx}$ is increasing in z.

Remark A3. *Under the proportional mean residual life $m(x,z) = zm_X(x)$, where $m_X(x)$ is an increasing function, then $A(z) = \sigma^2(x,Z) - m^2(x,Z)$ is an increasing function.*

From Remark A2, we can write

$$\begin{aligned}
&A(z) = \sigma^2(x,z) - m^2(x,z) = \frac{2\int_x^\infty \overline{F}(y,z)(m(y,z) - m(x,z))dy}{\overline{F}(x,z)} \\
&= 2z\int_x^\infty e^{-\int_x^y r(u,z)du}(m_X(y) - m_X(x))dy.
\end{aligned}$$

The last term in the previous formula is obtained by using the expression of the proportional mean residual life together with the relation between $r(x,z)$ and $\bar{F}(x,z)$ given in Remark A1. In addition, $A(z)$ is increasing in z, since $(m_X(y) - m_X(x))$ is a positive term and $r(x,z)$ is decreasing in z. Both result in $A(z)$ being increasing in z.

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