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Tail Asymptotics for a Retrial Queue with Bernoulli Schedule

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Abstract: In this paper, we study the asymptotic behaviour of the tail probability of the number of customers in the steady-state $M/G/1$ retrial queue with Bernoulli schedule, under the assumption that the service time distribution has a regularly varying tail. Detailed tail asymptotic properties are obtained for the conditional probability of the number of customers in the (priority) queue and orbit, respectively, in terms of the recently proposed exhaustive stochastic decomposition approach. Numerical examples are presented to show the impacts of system parameters on the tail asymptotic probabilities.

Keywords: $M/G/1$ retrial queue; Bernoulli schedule; number of customers; asymptotic tail probability; regularly varying distribution

MSC: 60K25; 60G50; 90B22

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1. Introduction

As one of the important types of queueing systems, retrial queues have been extensively studied for more than 40 years and hundreds of literature publications have been produced. Research on retrial queues is still very active due to continuously emerging new challenges. A general picture of retrial models, together with their applications in various areas, and basic results on retrial queues can be acquired from books or recent surveys, such as Falin and Templeton [1], Artajelo and Gómez-Corral [2], Choi and Chang [3], Kim and Kim [4], Phung-Duc [5], among possible others.

Tail asymptotic analysis of retrial queueing systems, especially asymptotic properties in tail stationary probabilities for a stable retrial queue, has been a focus of the investigation in the past 10 years or so, due to two main reasons: first, for most of retrial queues, it is not expected to have explicit non-transform solutions for their stationary distributions, and, second, tail asymptotic properties often lead to approximations to performance metrics and numerical algorithms. Both light-tailed and heavy-tailed properties have been obtained for a number of retrial queues, including the following incomplete list: Kim, Kim and Ko [6], Liu and Zhao [7], Kim, Kim and Kim [8], Liu, Wang and Zhao [9], Kim and Kim [10], Artalejo and Phung-Duc [11], Walraevens, Claeys and Phung-Duc [12], Kim, Kim and Kim [13], Yamamuro [14], Masuyama [15], Liu, Min and Zhao [16], and Liu and Zhao [17].

In this paper, we consider an $M/G/1$ retrial queue with a Bernoulli schedule. This model was first proposed and studied by Choi and Park [18], but tail asymptotic behaviour was not a focus of the study. This $M/G/1$ retrial queueing system consists of a (priority) queue of infinite waiting capacity and an orbit. External customers arrive to this system according to a Poisson process with rate λ. There is a single server in this system. If the server is idle upon the arrival of a customer, the customer receives the service immediately and leaves the system after the completion of the service. Otherwise, if the server is busy, the arriving customer would join the queue with probability q, becoming a priority customer waiting for the service according to the first-in-first-out discipline; or the orbit with probability $p = 1 - q$, becoming a repeated customer who will retry later for receiving its service, which is referred to as the Bernoulli scheduling. A priority customer has priority over a repeated customer for receiving the service. This implies that, upon the completion of a service, if there is a customer in the queue, the server will serve the customer at the head of the queue; otherwise, the server becomes idle. Each of the repeated customers in the orbit independently repeatedly tries to receive service, according to a Poisson process with the retrial rate μ, until an idle server is found, and then immediately receives its service. Upon the completion of the service, the customer leaves the system. Service times for all customers are i.i.d. random variables. By T_β, we denote the generic service time whose distribution $F_\beta(x)$ with $F_\beta(0) = 0$ is assumed to have a finite mean β_1. The Laplace–Stieltjes transform (LST) of the distribution function $F_\beta(x)$ is denoted by $\beta(s)$. Our interest in this paper is the tail stationary asymptotic behaviour for this system with the following assumption on the heavy-tailed service time:

Assumption 1. *The service time T_β has tail probability $P\{T_\beta > t\} \sim t^{-a}L(t)$ as $t \to \infty$, where $a > 1$, and $L(t)$ is a slowly varying function at ∞ (see Definition A1) .*

It should be noted that this assumption is different from that made in [17], where the low priority customers have the tail probabilities lighter than the high priority customers. Therefore, the study carried out in this paper is not overlapped with that in [17].

We also mention some literature studies on similar models to that considered in [18], such as Falin, Artalejo, and Martin [19], in which a model with two independent (primary) Poisson arrival streams was considered. The priority customers, when blocked upon arrival, are queued and waiting for service, while the non-priority customers, when blocked, join the orbit and retry for service later; Li and Yang [20], in which the discrete-time counterpart to the model studied in this paper, or a Geo/G/1 retrial queue with Bernoulli schedule, was considered; and Atencia and Moreno [21], in which the $M/G/1$ retrial queueing system with Bernoulli schedule has a general retrial time, but only the customer at the head of the orbit is allowed to retry for service, or retrials with a constant rate. Once again, our focus and also the method in the study are different from those in the above mentioned studies.

Let $\lambda_1 = \lambda q$, $\lambda_2 = \lambda p$, $\rho_1 = \lambda_1\beta_1$, $\rho_2 = \lambda_2\beta_1$ and $\rho = \rho_1 + \rho_2 = \lambda\beta_1$. It follows from [18] that the system considered in this paper is stable if and only if (iff) $\rho < 1$, which is assumed to hold throughout the paper. For obtaining asymptotic properties in various tail stationary probabilities of this $M/G/1$ retrial queue with Bernoulli schedule, which is the focus of this paper, we start with two expressions for probability transformations obtained in [18]. The method employed in our analysis is the exhaustive stochastic decomposition, recently proposed in [17]. By assuming a regular varying tail in the service time distribution, as made in Assumption A, we obtain asymptotic properties for the conditional tail probabilities of customers:

(1) in the orbit, given that the server is idle (Section 4.1);
(2) in the queue, given that the server is busy (Section 4.2); and
(3) in the orbit, given that the server is busy (Section 4.3).

Numerical curves are presented to demonstrate how system parameters, say the arrival rate, the expected service time or the retrial rate, impact the tail asymptotic probabilities.

The rest of this paper is organized as follows: preliminary results are provided in Section 2; exhaustive stochastic decompositions are obtained in Section 3; and the main results on asymptotic properties for tail probabilities are derived in Section 4; numerical examples are presented in Section 5; and conclusions are made in the final section.

2. Preliminaries

Assume that the system is in steady state. Let R_{que} be the number of priority customers in the queue, *excluding* the possible one in the service, let R_{orb} be the number of repeated customers in the orbit, and let $I_{ser} = 1$ or 0, whenever the server is busy or idle, respectively. Let R_0 be a random variable (r.v.) whose distribution coincides with the conditional distribution of R_{orb}, given that $I_{ser} = 0$, and let (R_{11}, R_{12}) be a two-dimensional r.v. whose distribution coincides with the conditional distribution of (R_{que}, R_{orb}), given that $I_{ser} = 1$. Precisely, R_0, R_{11}, and R_{12} are all nonnegative integer-valued r.v.s; and R_0 has the probability generating function (PGF) $R_0(z_2) = E(z_2^{R_0}) \stackrel{\text{def}}{=} E(z_2^{R_{orb}} | I_{ser} = 0)$ and (R_{11}, R_{12}) has the PGF $R_1(z_1, z_2) = E(z_1^{R_{11}} z_2^{R_{12}}) \stackrel{\text{def}}{=} E(z_1^{R_{que}} z_2^{R_{orb}} | I_{ser} = 1)$.

Our starting point for tail asymptotic analysis is based on the expressions for $R_0(z_2)$ and $R_1(z_1, z_2)$. Following the discussions in [18], let

$$M_a(z_1, z_2) = \frac{1}{\rho} \cdot \frac{1 - \beta(\lambda - \lambda_1 z_1 - \lambda_2 z_2)}{1 - p z_2 - q z_1}, \tag{1}$$

$$M_b(z_1, z_2) = (1 - \rho_1) \cdot \frac{h(z_2) - z_1}{\beta(\lambda - \lambda_1 z_1 - \lambda_2 z_2) - z_1}, \tag{2}$$

$$M_c(z_2) = \frac{1 - \rho}{1 - \rho_1} \cdot \frac{1 - z_2}{h(z_2) - z_2}, \tag{3}$$

where $h(\cdot)$ is determined uniquely by the following equation:

$$h(z) = \beta(\lambda - \lambda_1 h(z) - \lambda_2 z). \tag{4}$$

Since $P\{I_{ser} = 0\} = 1 - \rho$ and $P\{I_{ser} = 1\} = \rho$, obtained in [18], we have the following expressions immediately from Equations (12) and (13) in [18]:

$$R_0(z_2) = \exp\left\{ -\frac{\lambda}{\mu} \int_{z_2}^{1} \frac{1 - h(u)}{h(u) - u} du \right\}, \tag{5}$$

$$R_1(z_1, z_2) = M_a(z_1, z_2) \cdot M_b(z_1, z_2) \cdot M_c(z_2) \cdot R_0(z_2). \tag{6}$$

Next, we provide a probabilistic interpretation for $h(z)$ in (4). Let T_α be the busy period of the standard $M/G/1$ queue (without retrial) with arrival rate λ_1 and service time T_β. By $F_\alpha(x)$, we denote the probability distribution function of T_α, and by $\alpha(s)$, the LST of $F_\alpha(x)$. The following are classic results on the busy period of this standard $M/G/1$ queue (referring, e.g., to [22]):

$$\alpha(s) = \beta(s + \lambda_1 - \lambda_1 \alpha(s)), \tag{7}$$

$$\alpha_1 \stackrel{\text{def}}{=} E(T_\alpha) = \beta_1 / (1 - \rho_1). \tag{8}$$

Throughout this paper, we use the notation $N_b(t)$ to represent the number of Poisson arrivals, with rate b, within the time interval $(0, t]$, and $N_{\lambda_2}(T_\alpha)$ to represent the number of arrivals of a Poisson process, with arrival rate λ_2, within the independent random time T_α. The PGF of $N_{\lambda_2}(T_\alpha)$ is easily obtained as follows:

$$E(z^{N_{\lambda_2}(T_\alpha)}) = \int_0^\infty \sum_{n=0}^\infty z^n \frac{(\lambda_2 x)^n}{n!} e^{-\lambda_2 x} dF_\alpha(x) = \alpha(\lambda_2 - \lambda_2 z). \tag{9}$$

It follows from (7) that

$$\alpha(\lambda_2 - \lambda_2 z) = \beta(\lambda - \lambda_1 \alpha(\lambda_2 - \lambda_2 z) - \lambda_2 z). \tag{10}$$

By comparing (4) and (10), and noticing the uniqueness of $h(z)$, we immediately have

$$h(z) = \alpha(\lambda_2 - \lambda_2 z) = E(z^{N_{\lambda_2}(T_\alpha)}). \tag{11}$$

Remark 1. *$h(z)$ is the PGF of the number of arrivals of a Poisson process with arrival rate λ_2 within an independent random time T_α, where T_α has the same probability distribution as that for the busy period of the standard $M/G/1$ queue with arrival rate λ_1 and service time T_β.*

Since T_α is the busy period of the standard $M/G/1$ queue with arrival rate λ_1 and the service time T_β, its asymptotic tail probability is regularly varying according to de Meyer and Teugels [22] (see Lemma A1 in Appendix A).

3. Exhaustive Stochastic Decompositions

In this section, we verify that $R_0(z_2)$ can be viewed as the PGF of a r.v., which is written in a form of stochastic decompositions. The result will be used for the asymptotic analysis in later sections.

Based on the definition of R_0 given earlier, this is a r.v. whose distribution coincides with the conditional distribution of R_{orb}, given that $I_{ser} = 0$. In this section, we provide a new probabilistic interpretation for R_0, which is useful for our tail asymptotic analysis.

Substituting (11) into (5), we have

$$R_0(z_2) = \exp\left\{-\frac{\lambda}{\mu}\int_{z_2}^{1} \frac{1-\alpha(\lambda_2-\lambda_2 u)}{\alpha(\lambda_2-\lambda_2 u)-u}du\right\}. \tag{12}$$

In order to rewrite (12), we let

$$\psi = \frac{\rho}{\mu(1-\rho)}, \tag{13}$$

$$\kappa(s) = \frac{1-\rho}{\beta_1}\cdot\frac{1-\alpha(s)}{s-\lambda_2+\lambda_2\alpha(s)}, \tag{14}$$

$$\omega(s) = 1-\int_0^s \kappa(u)du, \tag{15}$$

$$\tau(s) = \exp\{\psi\omega(s)-\psi\}. \tag{16}$$

From (12)–(16),

$$\begin{aligned} R_0(z_2) &= \exp\left\{-\frac{\lambda}{\mu}\int_0^{\lambda_2-\lambda_2 z_2}\frac{1-\alpha(s)}{s-\lambda_2+\lambda_2\alpha(s)}ds\right\} \\ &= \exp\left\{-\psi\int_0^{\lambda_2-\lambda_2 z_2}\kappa(u)du\right\} \\ &= \tau(\lambda_2-\lambda_2 z_2). \end{aligned} \tag{17}$$

In the following three remarks, we assert that $\kappa(s)$, $\omega(s)$ and $\tau(s)$ are the LSTs of three probability distribution functions on $[0,\infty)$, respectively. For the first assertion, let $F_\alpha^{(e)}(x)$ be the equilibrium distribution of $F_\alpha(x)$, which is defined as $F_\alpha^{(e)}(x) = \alpha_1^{-1}\int_0^x(1-F_\alpha(t))dt$, where $\alpha_1 = E(T_\alpha)$ is given in (8). The LST of $F_\alpha^{(e)}(x)$ can be written as $\alpha^{(e)}(s) = (1-\alpha(s))/(\alpha_1 s)$. From (14), we have

$$\kappa(s) = \frac{(1-\vartheta)\alpha^{(e)}(s)}{1-\vartheta\alpha^{(e)}(s)} = \sum_{k=1}^{\infty}(1-\vartheta)\vartheta^{k-1}(\alpha^{(e)}(s))^k, \tag{18}$$

where

$$\vartheta = \lambda_2\alpha_1 = \rho_2/(1-\rho_1) < 1. \tag{19}$$

Remark 2. *Immediately from (18), $\kappa(s)$ can be viewed as the LST of the distribution function of a r.v. T_κ, whose distribution function is denoted by $F_\kappa(x)$. More precisely, $T_\kappa \stackrel{d}{=} T_{\alpha,1}^{(e)} + T_{\alpha,2}^{(e)} + \cdots + T_{\alpha,J}^{(e)}$, where $T_{\alpha,j}^{(e)}$, $j \ge 1$ are i.i.d. r.v.s, each with the distribution $F_\alpha^{(e)}(x)$, $P(J=j) = (1-\vartheta)\vartheta^{j-1}$, $j \ge 1$, and J is independent of $T_{\alpha,j}^{(e)}$ for $j \ge 1$. In the above expression, and also later, the notation "$\stackrel{d}{=}$" stands for equality in probability distribution.*

For the second assertion, we will use Theorem 1 in Feller [23] (see p. 439): A function $\varphi(s)$ is the LST of a probability distribution function iff $\varphi(0) = 1$ and $\varphi(s)$ is completely monotone, i.e., $\varphi(s)$ possesses derivatives $\varphi^{(n)}(s)$ of all orders such that $(-1)^n\varphi^{(n)}(s) \geq 0$ for $s > 0$.

Remark 3. *Immediately from (15), $\omega(0) = 1$ and $(-1)^n\omega^{(n)}(s) = (-1)^{n-1}\kappa^{(n-1)}(s) \geq 0$ for $s > 0$, which implies that $\omega(s)$ is the LST of the probability distribution function of a r.v. T_ω, whose distribution is denoted by $F_\omega(x)$.*

For the third assertion, we write (16) as

$$\tau(s) = \sum_{k=0}^{\infty} \frac{\psi^k}{k!} e^{-\psi} (\omega(s))^k. \tag{20}$$

Remark 4. *Immediately from (20), $\tau(s)$ can be viewed as the LST of the distribution function of a r.v. T_τ, whose distribution function is denoted by $F_\tau(x)$. More precisely, $T_\tau \stackrel{\mathrm{d}}{=} \sum_{j=0}^{J} T_{\omega,j}$, where $T_{\omega,j}$, $j \geq 1$, are i.i.d. r.v.s each with the distribution $F_\omega(x)$, and $P(J = j) = \frac{\psi^j}{j!}e^{-\psi}$, $j \geq 0$, where J is independent of $T_{\omega,j}$ for $j \geq 0$.*

The following remark provides a detailed interpretation on (17).

Remark 5. *R_0 can be regarded as the number of Poisson arrivals with rate λ_2 within an independent random time T_τ, i.e., $R_0 \stackrel{\mathrm{d}}{=} N_{\lambda_2}(T_\tau)$.*

4. Tail Asymptotics

In this section, we will study the asymptotic behaviour for the tail probabilities $P\{R_{orb} > j | I_{ser} = 0\}$, $P\{R_{que} > j | I_{ser} = 1\}$ and $P\{R_{orb} > j | I_{ser} = 1\}$, as $j \to \infty$, respectively.

4.1. Asymptotic Tail Probability for $P\{R_{orb} > j | I_{ser} = 0\}$

To study the asymptotic behaviour for the tail probability $P\{R_{orb} > j | I_{ser} = 0\} \equiv P\{R_0 > j\}$, let us first study the asymptotic properties of the tail probabilities for T_κ, T_ω and T_τ, respectively.

By Lemma A1, we know that $P\{T_\alpha > t\} \sim (1-\rho_1)^{-a-1}t^{-a}L(t)$ as $t \to \infty$, where the r.v. T_α is the busy period defined in Section 2. Applying Karamata's theorem (e.g., p. 28 in [24]), we have $\int_t^\infty (1 - F_\alpha(x))dx \sim (a-1)^{-1}(1-\rho_1)^{-a-1}t^{-a+1}L(t)$, which implies that $1 - F_\alpha^{(e)}(t) \sim ((a-1)\alpha_1)^{-1}(1-\rho_1)^{-a-1}t^{-a+1}L(t)$, $t \to \infty$. By Remark 2 and applying Lemma A2, we obtain the following tail asymptotic property for T_κ.

Lemma 1.

$$P\{T_\kappa > t\} \quad \sim \quad c_\kappa \cdot t^{-a+1}L(t), \quad t \to \infty, \tag{21}$$

where

$$c_\kappa = \frac{1}{\alpha_1(1-\vartheta)(a-1)(1-\rho_1)^{a+1}} = \frac{1}{\beta_1(1-\rho)(a-1)(1-\rho_1)^{a-1}}. \tag{22}$$

In the following lemma, we present the asymptotic tail probability of T_ω.

Lemma 2.

$$P\{T_\omega > t\} \sim (1 - 1/a)c_\kappa \cdot t^{-a}L(t), \quad t \to \infty. \tag{23}$$

Proof. Recall that T_ω has the distribution function $F_\omega(x)$, defined in terms of its LST $\omega(s)$ in (15), which is determined by the LST $\kappa(s)$ of the distribution function of T_κ. We divide the proof into two parts, depending on whether a is an integer or not.

Case 1: Non-integer $a > 1$.

Suppose that $m < a < m+1$, $m \in \{1,2,\ldots\}$. Since $P\{T_\kappa > t\} \sim c_\kappa \cdot t^{-a+1}L(t)$, we have its moments $\kappa_{m-1} < \infty$ and $\kappa_m = \infty$. Define $\kappa_{m-1}(s)$ in a manner similar to that in (A3). Then, $\kappa(s) = \sum_{k=0}^{m-1} \frac{\kappa_k}{k!}(-s)^k + (-1)^m \kappa_{m-1}(s)$. By Lemma A5, we know that

$$\kappa_{m-1}(s) \sim \left[\Gamma(a-m)\Gamma(m+1-a)/\Gamma(a-1)\right] \cdot c_\kappa s^{a-1} L(1/s), \quad s \downarrow 0. \tag{24}$$

From (15), there are constants $\{v_k;\ k = 0,1,2,\ldots,m\}$ satisfying $\omega(s) = \sum_{k=0}^{m} v_k(-s)^k + (-1)^{m+1}\int_0^s \kappa_{m-1}(u)du$. Define $\omega_m(s)$ in a manner similar to that in (A3). Then,

$$\omega_m(s) = \int_0^s \kappa_{m-1}(u)du \sim \left[\Gamma(a-m)\Gamma(m+1-a)/\Gamma(a)\right] \cdot \frac{a-1}{a} c_\kappa s^a L(1/s), \quad s \downarrow 0, \tag{25}$$

where we have used (24) and Karamata's theorem (p. 28 in [24]). Applying Lemma A5, we complete the proof of Lemma 2 for non-integer $a > 1$.

Case 2: Integer $a > 1$.

Suppose that $a = m \in \{2,3,\ldots\}$. Since $P\{T_\kappa > t\} \sim c_\kappa \cdot t^{-m+1}L(t)$ implies that T_κ has its moment $\kappa_{m-2} < \infty$, we can define $\kappa_{m-2}(s)$ and $\widehat{\kappa}_{m-2}(s)$ in a manner similar to that in (A3) and (A4), respectively. Then, $\kappa(s) = \sum_{k=0}^{m-2} \frac{\kappa_k}{k!}(-s)^k + (-1)^{m-1}\kappa_{m-2}(s)$. By Lemma A6, we obtain

$$\widehat{\kappa}_{m-2}(xu) - \widehat{\kappa}_{m-2}(u) \sim -(\log x) c_\kappa L(1/u)/(m-2)! \quad \text{as } u \downarrow 0. \tag{26}$$

From (15), there exist constants $\{v_k;\ k = 0,1,2,\ldots,m-1\}$ satisfying $\omega(s) = \sum_{k=0}^{m-1} v_k(-s)^k + (-1)^m \int_0^s \kappa_{m-2}(u)du$. Define $\widehat{\omega}_{m-1}(s)$ in a manner similar to that in (A4). Then, we have

$$\widehat{\omega}_{m-1}(s) = \frac{1}{s^m}\int_0^s u^{m-1}\widehat{\kappa}_{m-2}(u)du, \tag{27}$$

which immediately gives us $\widehat{\omega}_{m-1}(xs) = \frac{1}{s^m}\int_0^s u^{m-1}\widehat{\kappa}_{m-2}(xu)du$. It follows that

$$\begin{aligned} \widehat{\omega}_{m-1}(xs) - \widehat{\omega}_{m-1}(s) &= \frac{1}{s^m}\int_0^s u^{m-1}\left[\widehat{\kappa}_{m-2}(xu) - \widehat{\kappa}_{m-2}(u)\right]du \\ &\sim -\frac{m-1}{m} c_\kappa \log x \frac{L(1/s)}{(m-1)!} \quad \text{as } s \downarrow 0, \end{aligned} \tag{28}$$

where we have used (26) and Karamata's theorem (p. 28 in [24]). By applying Lemma A6, we complete the proof of Lemma 2 for integer $a = m \in \{2,3,\ldots\}$. □

For the tail asymptotic property of T_τ, recall Remark 4, with $T_\tau \stackrel{d}{=} \sum_{j=0}^{J} T_{\omega,j}$, where J has a Poisson distribution with parameter ψ, $T_{\omega,j}$, $j \geq 1$, are i.i.d. r.v.s with the distribution function $F_\omega(x)$, and J is independent of $T_{\omega,j}$, $j \geq 1$. Then, by Lemmas A2 and 2, we have the following property.

Lemma 3.

$$P\{T_\tau > t\} \sim \psi \cdot P\{T_\omega > t\} \sim (1 - 1/a)c_\kappa \psi \cdot t^{-a}L(t), \quad t \to \infty. \tag{29}$$

Now, the asymptotic tail probability for $P\{R_{orb} > j | I_{ser} = 0\}$ can be obtained based on Lemma 3. From Remark 5, we know that $R_0 = N_{\lambda_2}(T_\tau)$. Lemma A3 leads to:

$$P\{R_{orb} > j | I_{ser} = 0\} = P\{R_0 > j\} \sim P\{T_\tau > j/\lambda_2\} \sim \frac{\lambda\lambda_2^a}{a\mu(1-\rho)^2(1-\rho_1)^{a-1}} \cdot j^{-a}L(j), \quad j \to \infty. \tag{30}$$

4.2. Asymptotic Tail Probability for $P\{R_{que} > j | I_{ser} = 1\}$

In this subsection, we will study the asymptotic behaviour of the tail probability $P\{R_{que} > j | I_{ser} = 1\} \equiv P\{R_{11} > j\}$ as $j \to \infty$. For this purpose, we examine the generating function $E(z_1^{R_{11}})$ of R_{11}. Note that $E(z_1^{R_{11}}) = R_1(z_1, 1)$. Taking $z_2 \to 1$ in (3), (5) and (6) and using the fact that $\frac{d}{dz}h(z)|_{z=1} = \lambda_2\alpha_1 = \rho_2/(1-\rho_1)$, we immediately have $M_c(1) = 1$, $R_0(1) = 1$ and

$$E(z_1^{R_{11}}) = M_a(z_1, 1)M_b(z_1, 1). \tag{31}$$

It follows from (1) and (2) that

$$M_a(z_1, 1) = \frac{1}{\rho_1} \cdot \frac{1 - \beta(\lambda_1 - \lambda_1 z_1)}{1 - z_1}, \tag{32}$$

$$M_b(z_1, 1) = \frac{(1-\rho_1)(1-z_1)}{\beta(\lambda_1 - \lambda_1 z_1) - z_1}. \tag{33}$$

Denote $F_\beta^{(e)}(x)$ to be the equilibrium distribution of $F_\beta(x)$, that is, $F_\beta^{(e)}(x) = \beta_1^{-1}\int_0^x (1 - F_\beta(t))dt$. The LST of $F_\beta^{(e)}(x)$ can be written as $\beta^{(e)}(s) = (1-\beta(s))/(\beta_1 s)$. We now have

$$M_a(z_1, 1) = \beta^{(e)}(\lambda_1 - \lambda_1 z_1), \tag{34}$$

$$M_b(z_1, 1) = \frac{1-\rho_1}{1 - \rho_1\beta^{(e)}(\lambda_1 - \lambda_1 z_1)} = \sum_{n=0}^{\infty}(1-\rho_1)\rho_1^n\Big(\beta^{(e)}(\lambda_1 - \lambda_1 z_1)\Big)^n. \tag{35}$$

Substituting (34) and (35) into (31), we have

$$E(z_1^{R_{11}}) = \xi(\lambda_1 - \lambda_1 z_1), \tag{36}$$

where

$$\xi(s) = \sum_{n=1}^{\infty}(1-\rho_1)\rho_1^{n-1}(\beta^{(e)}(s))^n. \tag{37}$$

Remark 6. *Immediately from (37), $\xi(s)$ can be viewed as the LST of the distribution function of the r.v. $T_\xi \stackrel{d}{=} T_{\beta,1}^{(e)} + T_{\beta,2}^{(e)} + \cdots + T_{\beta,J}^{(e)}$, where $T_{\beta,j}^{(e)}$, $j \geq 1$, are i.i.d. r.v.s. with a common distribution $F_\beta^{(e)}(x)$, $P(J = j) = (1-\rho_1)\rho_1^{j-1}$, $j \geq 1$, and J is independent of $T_{\beta,j}^{(e)}$, $j \geq 1$.*

Under Assumption A, by Karamata's theorem (e.g., p. 28 in [24]), we have $\int_t^\infty (1 - F_\beta(x))dx \sim (a-1)^{-1}t^{-a+1}L(t)$, which implies that $1 - F_\beta^{(e)}(t) \sim ((a-1)\beta_1)^{-1}t^{-a+1}L(t)$, $t \to \infty$. By Remark 6 and applying Lemma A2, we have

$$P\{T_\xi > t\} \sim \frac{1}{(1-\rho_1)(a-1)\beta_1} \cdot t^{-a+1}L(t), \quad t \to \infty. \tag{38}$$

Remark 7. *With (36), one can interpret R_{11} as the number of Poisson arrivals with rate λ_1 within the independent random time T_ξ, i.e., $R_{11} \stackrel{d}{=} N_{\lambda_1}(T_\xi)$.*

By Remark 7 and applying Lemma A3, we have

$$P\{R_{que} > j | I_{ser} = 1\} = P\{R_{11} > j\} \sim P\{T_\xi > j/\lambda_1\} \sim \frac{\lambda_1^{a-1}}{(1-\rho_1)(a-1)\beta_1} \cdot j^{-a+1}L(j), \quad j \to \infty. \tag{39}$$

4.3. Asymptotic Tail Probability for $P\{R_{orb} > j | I_{ser} = 1\}$

In this subsection, we will study the asymptotic behaviour of the tail probability $P\{R_{orb} > j | I_{ser} = 1\} \equiv P\{R_{12} > j\}$ as $j \to \infty$. By (6),

$$E(z_2^{R_{12}}) = R_1(1,z_2) = M_a(1,z_2) \cdot M_b(1,z_2) \cdot M_c(z_2) \cdot R_0(z_2). \quad (40)$$

Taking $z_1 \to 1$ in (1)–(3), we have

$$\begin{aligned} M_a(1,z_2) \cdot M_b(1,z_2) \cdot M_c(z_2) &= \frac{1-\rho}{\rho_2} \cdot \frac{1-h(z_2)}{h(z_2)-z_2}. \quad (41) \\ &= \frac{1-\rho}{\rho_2} \cdot \frac{1-\alpha(\lambda_2-\lambda_2 z_2)}{\alpha(\lambda_2-\lambda_2 z_2)-z_2} \quad (\text{ by } (11)) \\ &= \kappa(\lambda_2-\lambda_2 z_2), \quad (42) \end{aligned}$$

where the last equality follows from (14).

Substituting (42) and (17) into (40), we have

$$E(z_2^{R_{12}}) = \kappa(\lambda_2-\lambda_2 z_2) \cdot \tau(\lambda_2-\lambda_2 z_2). \quad (43)$$

Remark 8. *With (43), one can interpret R_{12} as the number of Poisson arrivals with rate λ_2 within an independent random time $T_\kappa + T_\tau$, i.e., $R_{12} \stackrel{d}{=} N_{\lambda_2}(T_\kappa + T_\tau)$, where T_κ and T_τ are assumed to be independent.*

By (21) and (29), and applying Lemma A4, we have

$$P\{T_\kappa + T_\tau > t\} \sim P\{T_\kappa > t\} \sim c_\kappa \cdot t^{-a+1} L(t), \quad t \to \infty. \quad (44)$$

Applying Remark 8 and Lemma A3, we have

$$\begin{aligned} P\{R_{orb} > j | I_{ser} = 1\} =& P\{R_{12} > j\} \sim P\{T_\kappa + T_\tau > j/\lambda_2\} \\ &\sim \frac{\lambda_2^{a-1}}{\beta_1(1-\rho)(a-1)(1-\rho_1)^{a-1}} \cdot j^{-a+1} L(j), \quad j \to \infty. \quad (45) \end{aligned}$$

5. Numerical Examples

In this section, using numerical examples, we demonstrate how system parameters, like the arrival, service and retrial rates, impact the tail decay rate. According to the tail asymptotic expressions obtained in (30), (39) and (45), it is expected that the tail becomes fatter (or the tail probability becomes bigger) as the arrival rate (λ) increases while all other parameter values remain the same for all three types of conditional probabilities. Similar results are expected as the service rate (β_1^{-1}) or the retrial rate (μ) decreases, respectively. The above claim has been supported by our extensive numerical tests using various slowly varying functions and a broad range of parameter values. For a quantitative pictorial example, we take the conditional tail probability $P\{R_{orb} > j | I_{ser} = 0\}$ as expressed in (30).

Assume that the service time T_β follows a Pareto distribution with shape parameter $a > 1$ and scale parameter $b > 0$, i.e.,

$$F_\beta(x) = 1 - (1 + x/b)^{-a}, \quad \text{for } x \ge 0, \quad (46)$$

for convenience, since, in this case, we can easily compute the mean service time, given as $\beta_1 = E(T_\beta) = b/(a-1)$ and the traffic intensity, given as $\rho = \lambda\beta_1 = \lambda b/(a-1)$.

We provide three figures to demonstrate the quantitative changes of the tail probability as a function of the arrival rate λ, the expected service time β_1, and the retrial rate μ, respectively. In Figure 1, set $a = 3/2$, $b = 1/5$, $q = 1/2$ (Bernoulli probability of joining the queue), and $\mu = 1$. Then, the tail asymptotic probability is a function of the arrival rate λ. Four curves in different colours are given to show the changes as λ changes, corresponding to $\lambda = 3/4, 1, 5/4, 3/2$, respectively.

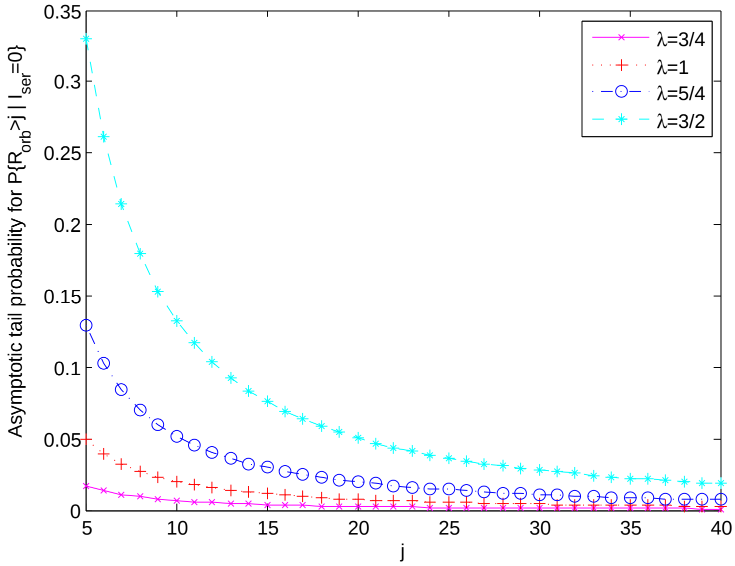

Figure 1. Impact of the arrival rate λ on the asymptotic tail $P\{R_{orb} > j|I_{ser} = 0\}$.

In Figure 2, we set $\lambda = 1.2$, $q = 1/2$ and $\mu = 1$. Then, the tail asymptotic probability is a function of the expected service time β_1. By further assuming $b = 1/5$, the tail asymptotic probability is simply a function of a (see Assumption 1). In order to see the impact of the service time distribution $F_\beta(x)$ on the asymptotic tail $P\{R_{orb} > j|I_{ser} = 0\}$, four curves in different colours are given, corresponding to $\beta_1 = 1/2, 2/5, 1/3, 1/4$, respectively (or, correspondingly, $a = 1 + b/\beta_1 = 1.4, 1.5, 1.6, 1.8$).

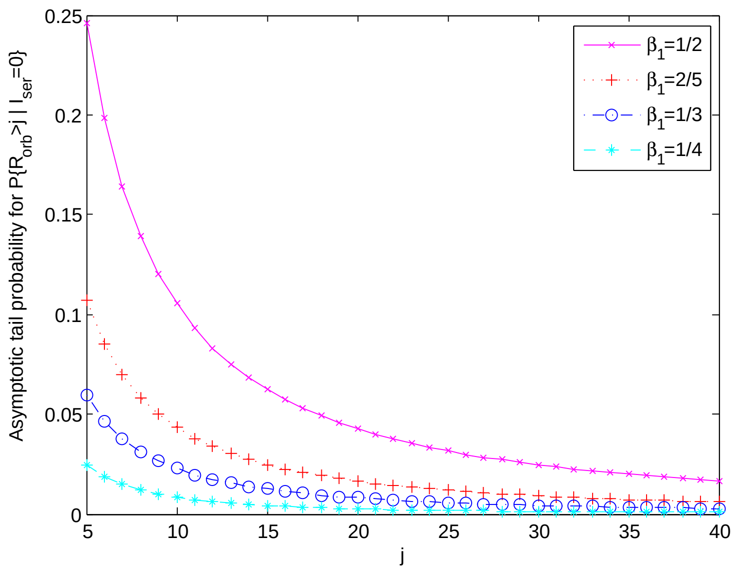

Figure 2. Impact of the expected service time β_1 on the asymptotic tail $P\{R_{orb} > j|I_{ser} = 0\}$.

Finally, we show, in Figure 3, the trend of the change for the tail asymptotic probability $P\{R_{orb} > j|I_{ser} = 0\}$, as a function of the retrial rate μ. Specifically, we set $a = 3/2$, $b = 1/5$, $\lambda = 1.2$ and $q = 1/2$. The four curves correspond to the following four different retrial rates: $\mu = 1/5, 2/5, 4/5, 8/5$, respectively.

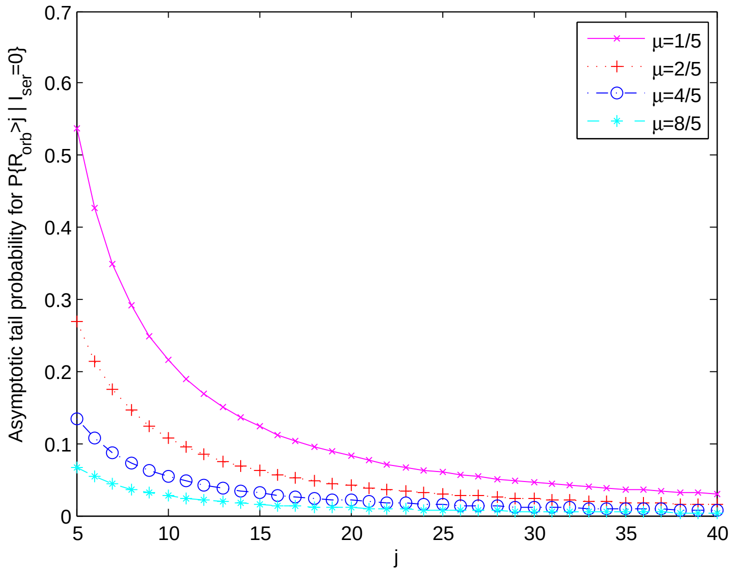

Figure 3. Impact of the retrial rate μ on the asymptotic tail $P\{R_{orb} > j|I_{ser} = 0\}$.

6. Conclusions

In this paper, the exhaustive version of the stochastic decomposition method was used to decompose the generating function $R_0(z_2)$ (or corresponding r.v. R_0) into detailed components, each of which has probabilistic interpretations (see Remark 5). This decomposition leads to a tail asymptotic expression for $P\{R_{orb} > j|I_{ser} = 0\}$, given in (30). By applying the same decomposition method to the generating function $E(z_1^{R_{11}})$ (see Equation (31) and Remark 7), we obtained the tail asymptotic expression for $P\{R_{que} > j|I_{ser} = 1\}$ given in (39), and by applying the exhaustive version of decompositions to the generating function $E(z_2^{R_{12}})$ (see Equation (40) and Remark 8), we obtained the tail asymptotic expression for $P\{R_{orb} > j|I_{ser} = 1\}$ given in (45).

This exhaustive version of the stochastic decomposition method has also been applied to a priority queueing system in [17] and a retrial queue with batch arrivals in [16]. We expect that the same method can be applied to many other queueing models.

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Appendix A

Appendix A.1. Definitions and Useful Results from the Literature

Definition A1 (Bingham, Goldie and Teugels [24])**.** *A measurable function* $U : (0, \infty) \to (0, \infty)$ *is regularly varying at* ∞ *with index* $\sigma \in (-\infty, \infty)$ *(written* $U \in \mathcal{R}_\sigma$*) iff* $\lim_{t\to\infty} U(xt)/$

$U(t) = x^{\sigma}$ for all $x > 0$. If $\sigma = 0$, we call U slowly varying, i.e., $\lim_{t\to\infty} U(xt)/U(t) = 1$ for all $x > 0$.

Definition A2 (Foss, Korshunov and Zachary [25]). *A distribution F on $(0, \infty)$ belongs to the class of subexponential distribution (written $F \in \mathcal{S}$) if $\lim_{t\to\infty} \overline{F^{*2}}(t)/\overline{F}(t) = 2$, where $\overline{F} = 1 - F$ and F^{*2} denotes the second convolution of F.*

It is well known that, for a distribution F on $(0, \infty)$, if $\overline{F}$ is regularly varying with index $-\sigma$, $\sigma \geq 0$ or $\overline{F} \in \mathcal{R}_{-\sigma}$, then F is subexponnetial or $F \in \mathcal{S}$ (see, e.g., Embrechts et al. [26]).

Lemma A1 (de Meyer and Teugels [22]). *Under Assumption 1,*

$$P\{T_{\alpha} > t\} \sim \frac{1}{(1-\rho_1)^{a+1}} \cdot t^{-a} L(t) \quad \text{as } t \to \infty. \tag{A1}$$

The result (A1) is straightforward due to the main theorem in [22].

Lemma A2 (pp. 580–581 in [26]). *Let N be a discrete non-negative integer-valued r.v. with $p_k = P\{N = k\}$ such that for some $\varepsilon > 0$, $\sum_{k=0}^{\infty} p_k (1+\varepsilon)^k < \infty$, and let $\{Y_k\}_{k=1}^{\infty}$ be a sequence of non-negative, i.i.d. r.v.s having a common subexponential distribution F. Define $S_n = \sum_{k=1}^{n} Y_k$. Then, $P\{S_N > t\} \sim E(N) \cdot (1 - F(t))$ as $t \to \infty$.*

Two special cases:

(1) if $p_k = (1-\sigma)\sigma^{k-1}$, $0 < \sigma < 1$, $k \geq 1$, then $P\{S_N > t\} \sim (1 - F(t))/(1-\sigma)$ as $t \to \infty$;

(2) if $p_k = \frac{\sigma^k}{k!} e^{-\sigma}$, $\sigma > 0$, $k \geq 0$, then $P\{S_N > t\} \sim \sigma \cdot (1 - F(t))$ as $t \to \infty$.

Lemma A3 (Proposition 3.1 in [27]). *Let $N_{\lambda}(t)$ be a Poison process with rate λ and let T be a positive r.v. with distribution F, which is independent of $N_{\lambda}(t)$. If $\bar{F}(t) = P\{T > t\}$ is heavier than $e^{-\sqrt{t}}$ as $t \to \infty$, then $P(N_{\lambda}(T) > j) \sim P\{T > j/\lambda\}$ as $j \to \infty$.*

Lemma A3 holds for any distribution F with a regularly varying tail because it is heavier than $e^{-\sqrt{t}}$ as $t \to \infty$.

Lemma A4 (p. 48 in [25]). *Let F, F_1 and F_2 be distribution functions. Suppose that $F \in \mathcal{S}$. If $\bar{F}_i(t)/\bar{F}(t) \to c_i$ as $t \to \infty$ for some $c_i \geq 0$, $i = 1, 2$, then $\overline{F_1 * F_2}(t)/\bar{F}(t) \to c_1 + c_2$ as $t \to \infty$, where the symbol $\bar{F} \overset{\text{def}}{=} 1 - F$ and "$F_1 * F_2$" stands for the convolution of F_1 and F_2.*

To prove Lemma 2, we list some notations and results. Let $F(x)$ be any distribution on $[0, \infty)$ with the LST $\phi(s)$. We denote the nth moment of $F(x)$ by ϕ_n, $n \geq 0$. It is well known that $\phi_n < \infty$ iff

$$\phi(s) = \sum_{k=0}^{n} \frac{\phi_k}{k!} (-s)^k + o(s^n), \quad n \geq 0. \tag{A2}$$

Next, if $\phi_n < \infty$, we introduce the notation $\phi_n(s)$ and $\widehat{\phi}_n(s)$, defined by

$$\phi_n(s) \overset{\text{def}}{=} (-1)^{n+1} \left\{ \phi(s) - \sum_{k=0}^{n} \frac{\phi_k}{k!} (-s)^k \right\}, \quad n \geq 0, \tag{A3}$$

$$\widehat{\phi}_n(s) \overset{\text{def}}{=} \phi_n(s)/s^{n+1}, \quad n \geq 0. \tag{A4}$$

Note that, if $\phi_n < \infty$, then $\lim_{s\downarrow 0} \widehat{\phi}_{n-1}(s) = \phi_n/n!$ and $s\widehat{\phi}_n(s) = \phi_n/n! - \widehat{\phi}_{n-1}(s)$ for $n \geq 1$. In addition, if $\phi_n < \infty$, one can define a sequence of functions F_k recursively by: $F_1(t) = F(t)$ and $1 - F_{k+1}(t) \overset{\text{def}}{=} \int_t^{\infty} (1 - F_k(x)) dx$, $k = 1, 2, \ldots, n$. It is not difficult to check that $1 - F_{k+1}(0) = \phi_k/k!$ and $\int_0^t (1 - F_k(x))dx$ has the LST $\widehat{\phi}_{k-1}(s)$. Namely, for $k = 1, 2, \ldots, n$,

$$\begin{aligned}
\int_0^\infty e^{-st}(1 - F_k(t))dt &= \frac{\phi_{k-1}}{(k-1)!}\frac{1}{s} - \frac{1}{s}\int_0^\infty e^{-st} dF_k(t) \\
&= \frac{\phi_{k-1}}{(k-1)!}\frac{1}{s} - \frac{1}{s}\int_0^\infty e^{-st}(1 - F_{k-1}(t))dt \\
&= \frac{\phi_{k-1}}{(k-1)!}\frac{1}{s} - \frac{\phi_{k-2}}{(k-2)!}\frac{1}{s^2} + \cdots + (-1)^{k-1}\frac{1}{s^k} + (-1)^k \frac{1}{s^k}\int_0^\infty e^{-st} dF_1(t) \\
&= \widehat{\phi}_{k-1}(s). \qquad \text{(A5)}
\end{aligned}$$

Lemma A5 (pp. 333–334 in [24])**.** *Assume that* $n < d < n + 1$, $n \in \{0, 1, 2, \ldots\}$. *Then, the following are equivalent:*

$$1 - F(t) \sim t^{-d}L(t), \quad t \to \infty, \qquad \text{(A6)}$$
$$\phi_n(s) \sim \big[\Gamma(d-n)\Gamma(n+1-d)/\Gamma(d)\big] \cdot s^d L(1/s), \quad s \downarrow 0. \qquad \text{(A7)}$$

Definition A3 (e.g., Bingham et al. (1989) [24], p. 128)**.** *A function* $F : (0, \infty) \to (0, \infty)$ *belongs to the de Haan class* Π *at* ∞ *if there exists a function* $H : (0, \infty) \to (0, \infty)$ *such that* $\lim_{t\uparrow\infty} \frac{F(xt)-F(t)}{H(t)} = \log x$ *for all* $x > 0$*, where the function* H *is called the auxiliary function of* F.

Lemma A6. *Assume that* $n \in \{1, 2, \ldots\}$. *Then, the following two statements are equivalent:*

$$1 - F(t) \sim t^{-n}L(t), \quad t \to \infty; \qquad \text{(A8)}$$
$$\lim_{s\downarrow 0} \frac{\widehat{\phi}_{n-1}(xs) - \widehat{\phi}_{n-1}(s)}{L(1/s)/(n-1)!} = -\log x, \quad \textit{for all } x > 0. \qquad \text{(A9)}$$

Proof. Recall the definition of $F_k(t)$. Repeatedly using Karamata's theorem (p. 27 in [24]) and the monotone density theorem (p. 39 in [24]), we know that $1 - F(t) \sim t^{-n}L(t)$ is equivalent to $1 - F_n(t) \sim t^{-1}L(t)/(n-1)!$, which in turn is equivalent to $\int_0^t (1 - F_n(x))dx \in \Pi$ with an auxiliary function that can be taken as $L(t)/(n-1)!$ (see, e.g., p. 335 in [24]). By (A5), $\int_0^t (1 - F_n(x))dx$ has the LST $\widehat{\phi}_{n-1}(s)$. Applying Theorem 3.9.1 in [24] (pp. 172–173), we complete the proof. □

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