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Statistical Inference of Wiener Constant-Stress Accelerated Degradation Model with Random Effects

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Abstract: In the field of reliability analysis, the constant-stress accelerated degradation test is one of the most commonly used methods to evaluate a product's reliability as degradation data are provided. In this paper, a constant-stress accelerated degradation test model of the Wiener process with random effects is proposed. First, the generalized confidence intervals of the model parameters are developed by constructing generalized pivotal quantities. Second, utilizing the substitution method, the generalized confidence intervals for the reliability function of lifetime, mean time to failure and the generalized prediction intervals for the degradation characteristic at the normal operating condition are also developed. Simulation studies are conducted to investigate the performances of the proposed generalized confidence intervals and prediction intervals. The simulation results reveal that the proposed generalized confidence intervals and prediction intervals work well in terms of the coverage percentage. In particular, a comparative analysis is made with the traditional bootstrap confidence intervals. At last, the proposed procedures are used for a real data analysis.

Keywords: accelerated degradation test; wiener process; random effects; generalized confidence interval; generalized prediction interval

MSC: 62F30

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1. Introduction

With the rapid development of science, technology and manufacturing, more and more high-reliability products appear. For high-reliability products, it is difficult for mechanical engineers to acquire plenty of failure data in the short period of a test duration and to effectively evaluate the products' reliability metrics, such as the reliability function and the mean time to failure (MTTF). The accelerated life test (ALT) procedures are widely used to obtain enough failure data for a reliability analysis in the area. However, for some high-reliability products, it is hard to fail in a short period of time, even subject to a high stress level. In this case, the traditional ALT techniques are not enough for the reliability assessment of products, and the degradation data analysis provides a new way to solve this problem [1].

For some extreme highly reliable products, such as aerospace and nuclear power plants, the degradation tests implemented under normal using conditions still take a long time to obtain enough degradation data for use. Similar to implementing an ALT for more failure data, we can use the accelerated degradation test (ADT) to quickly acquire more degradation data and make some statistic inference for the products' lifetime. ADTs are able to greatly reduce the test time and save the test cost, so the research of an ADT model became popular in the last few decades [2–4]. Two main kinds of models are used for an ADT data analysis. The first kind of model is called the stochastic process model, which is used to capture the time-dependent structure of the degradation over time. The constant-stress accelerated degradation test (CSADT) is one of the most widely used ADTs in the field of reliability. In a CSADT, the products are divided into several groups, and

each group of products are subjected to a very severe stress condition to test and collect the degradation data. The second kind is the general path model [5,6] in which a degradation regression model is assumed and estimated.

Product performance degradation is an irreversible process; when the degradation exceeds a given threshold the product will fail. Compared with a classical failure time data analysis, a degradation data analysis aims to describe the underlying failure process and requires fewer test units to obtain the same estimation accuracy [7]. In real applications, the performance degradation is often modeled by a stochastic process $\{X(t); t \geq 0\}$ to account for inherent randomness. Based on the assumption that the degradation is additive, three kinds of degradation processes have been deep exploited, i.e., the Gamma process [8–10], the Wiener process [11–13] and the inverse Gaussian process [14–17].

In many cases, the products' degradation processes are influenced by some observable environmental factors, such as the temperate, humidity and usage rates. The effects on the degradation process can be treated as covariates. Refs. [18–20] provide different ways of incorporating covariates in the Wiener process models. On the other hand, when unobservable factors, e.g., the size of an internal defect and the unobservable field-use conditions, influence product degradation, their effects are often represented by incorporating a random effect into the degradation model. Random-effects variants of the Wiener process can be found in [13,21]. Pan and Balakrishnan [22] discussed the multiple-step step-stress accelerated degradation modeling based on the Wiener and Gamma processes. Wang et al. [23] mainly study the accurate reliability inference based on the Wiener degradation model with a random drift parameter. They developed an exact test method to test whether there exists population heterogeneity. Guan et al. [24] used the Bayesian method to study the Wiener ADT model. Wang et al. [25] discussed the reliability analysis for accelerated degradation data based on the Wiener process with random effects. Pan et al. [26] studied a reliability estimation approach via the Wiener degradation model with measurement errors. Jiang et al. [27] proposed a Wiener CSADT model and obtained the exact confidence intervals of the model parameters. In addition, they also developed an optimal plan from the perspective of degradation prediction for the proposed model. Notice that existing interval estimation methods mainly depend on the large sample results and a Bayesian approach in many cases, such as the approximation normality of the maximum-likelihood estimation (MLE), the bootstrap confidence interval and the Wald confidence interval. The performances of these large sample confidence intervals may be poor in small and moderate sample size cases. In degradation tests, the sample size is generally small due to the limitations of the test time, cost, etc. Hence, it is meaning to develop small sample interval estimation procedures for the Wiener CSADT model with random degradation rates.

In this study, we propose a Wiener CSADT model with random degradation rates based on the invariance of degradation mechanism. The relationship between the degradation rate and accelerated stress is linear, and the diffusion parameter is free of accelerated stress. The main objective of this study is to investigate the interval estimation of the model parameters and some reliability metrics under a small sample size. Obviously, it is very difficult to obtain the exact interval estimations of them. Thus, we focus on developing the generalized confidence intervals (GCIs) and generalized prediction intervals (GPIs) of the model parameters and some reliability metrics. The innovation of this paper is that an effective interval estimation method for model parameters and some reliability metrics is developed under a small sample case. It should be emphasized that it is not easy to construct the generalized pivot quantity (GPQ) of model parameters a and b, which is the foundation and core of our whole research work. In addition, a simulation study is conducted to assess the effectiveness of the GCIs/GPIs and compare them to the bootstrap-CIs/PIs in terms of coverage percentage.

The remainder of the article is organized as follows. In Section 2, we describe the Wiener CSADT model with random effects. In Section 3, the point estimations of the model parameters are obtained. The GCIs and GPIs of the model parameters and some reliability metrics are developed through the generalized pivotal quantity (GPQ) method.

In Section 4, a simulation study is conducted to evaluate the performance of the proposed intervals. An example is provided to illustrate the proposed procedures. Finally, we give some final conclusions in Section 5.

2. Wiener CSADT Model with Random Effects

2.1. Model Descriptions and Assumptions

In practice, each unit can undergo various operating conditions, which causes the degradation process of different units to demonstrate different degradation rates. In addition, especially for an immature product, the degradation may exhibit some heterogeneities due to such reasons as difference in the raw materials, production environment factor and variations in the manufacturing process. It is a natural consideration to incorporate unit-to-unit heterogeneity in the degradation model. Under this circumstance, random-effects models may be helpful to capture the heterogeneities [28]. Thus, we consider the degradation rate δ as a random parameter to describe unit-to-unit variability and σ^2 to be a fixed parameter. For the convenience of subsequent modeling and mathematical processing, we assume that the prior distribution of δ follows a normal distribution and is statistically independent of σ^2. The ideas of random effects and Gaussian assumptions are widely adopted in the degradation modeling literature; some recent references can be seen [29–31]. For notational ease, we describe this random-effects Wiener process as

$$X(t) = \delta t + \sigma B(t), \tag{1}$$

where $\delta \sim \mathcal{N}(\mu, \omega^2)$ and σ^2 is a fixed parameter. Condition on δ, the degradation path $X(t) \sim \mathcal{N}(\delta t, \sigma^2 t)$. Unconditionally, the degradation path $X(t)$ follows the normal distribution $\mathcal{N}(\mu t, \sigma^2 t + \omega^2 t^2)$. The lifetime T of a unit is defined as the first hitting time which the cumulative degradation $X(t)$ exceeds a failure threshold $L(> 0)$. Given the failure threshold L, using the total probability formula, the cumulative distribution function (CDF) $F_T(t)$ of the lifetime T can be obtained by first computing $P(T < t|\delta)$ and then marginal integrating δ out, which is given by

$$F_T(t) = \Phi\left(\frac{\mu t - L}{\sqrt{\sigma^2 t + \omega^2 t^2}}\right) + \exp\left(\frac{2\mu L}{\sigma^2} + \frac{2\omega^2 L^2}{\sigma^4}\right)\Phi\left(-\frac{2\omega^2 L t + \sigma^2(\mu t + L)}{\sigma^2\sqrt{\sigma^2 t + \omega^2 t^2}}\right), t > 0. \tag{2}$$

It is worth noting that for ADT the most important issue is to ensure that the degradation mechanism remains consistent and invariant under different stress levels [32]. Degradation mechanism refers to a product's physical or chemical changes caused by accelerated stress, and the relationship between stress and the degradation process is usually modeled by a physically reasonable statistical function which is used for extrapolation. No function can give a correct and reasonable extrapolation when the degradation mechanism changes at different stress levels [33]. Based on the principle of the degradation mechanism invariance, statistical inference of the Wiener CSADT model with random effects is usually based on the following assumptions:

A1. The CSADT is conducted by one single stress, which has K levels: $\xi_1 < \xi_2 < \cdots < \xi_K$. ξ_0 and ξ_K are the normal using stress level and the highest stress level used in the ADT, respectively.

A2. Under stress level ξ_i, the degradation process of the jth test unit can be described as a Wiener process
$$X_{i,j}(t) = \delta_{i,j} t + \sigma B(t), \delta_{i,j} \sim \mathcal{N}(\mu_i, \omega^2).$$

A3. At stress levels $\xi_i (i = 1, 2, \cdots, K)$ and ξ_0, the degradation process has the same degradation mechanism. This means that the diffusion parameters σ^2 and ω^2 are not affected by the stress level ξ_i, but the drift parameter μ is affected by it.

A4. The product's degradation process is affected by the stress through the parameter–stress relationship
$$\mu_i = a + b\xi_i,$$

where a and $b(>0)$ are unknown parameters. The degradation rate under normal using condition can be given by $\mu_0 = a + b\xi_0$.

A5. At each stress level, the test units have the same test duration, but the measurement intervals and the measurement times are different for each unit. Generally, at different stress levels, the testing durations of units are different.

2.2. CSADT and the Data

Suppose that n_i units are tested under the stress level ξ_i, and these n_i units have the same test duration $\mathcal{T}_i$, but the measurement intervals may be different for each unit. The measurement times are denoted by $\mathbf{t}_{i,j} = \{t_{i,j,k}; j = 1,2,\ldots,n_i, k = 0,1,\ldots,r_{i,j}\}$, where $t_{i,j,0} = 0, t_{i,j,r_{i,j}} = \mathcal{T}_i$, and $r_{i,j}$ is the number of measurements for the jth testing unit. Generally, for different stress levels, the testing durations may be different, that is, $\mathcal{T}_i \neq \mathcal{T}_j$ when $i \neq j$. Let $\mathbf{X}_{i,j} = \{X_{i,j}(t_{i,j,k}); k = 0,1,\ldots,r_{i,j}\}$ be the observed degradation data for the jth testing unit under stress level ξ_i, for $j = 1,2,\ldots,n_i$. $\Delta X_{i,j,k}$ denotes degradation increment of the jth testing unit at the ith stress level, $\Delta t_{i,j,k}$ denotes the inspection time interval between the $(k-1)$th and kth of the jth test unit at the ith stress level, i.e., $\Delta X_{i,j,k} \hat{=} X_{i,j}(t_{i,j,k}) - X_{i,j}(t_{i,j,k-1})$, $\Delta t_{i,j,k} \hat{=} t_{i,j,k} - t_{i,j,k-1}$, for $i = 1,2,\ldots,K; j = 1,2,\ldots,n_i; k = 1,2,\ldots,r_{i,j}$.

The data collected from stress level ξ_i are $\mathbb{D}_i = \{(\mathbf{t}_{i,j}, \mathbf{X}_{i,j}); j = 1,2,\ldots,n_i\}$, and the data from the CSADT are $\mathbb{D} = \bigcup_{i=1}^{K} \mathbb{D}_i$. Let $N = \sum_{i=1}^{K} n_i$ be the total number of testing units and $M_i = \sum_{j=1}^{n_i} r_{i,j}$ be the total number of measurements under stress level ξ_i. Further, denote $M = \sum_{i=1}^{K} M_i$ as the total number of measurements in the CSADT and $\mathcal{T} = \sum_{i=1}^{K} n_i \mathcal{T}_i$ as the total time on test.

3. Point and Interval Estimations

3.1. Point Estimations for Model Parameters

For the proposed random-effects Wiener CSADT model, we assumed that the diffusion parameters ω^2 and σ^2 are fixed, and the drift parameter μ is affected by the stress as $\mu = a + b\xi$. Given the degradation data $\mathbb{D}_i$, let $\delta_{i,j}$ be the degradation rate of the jth unit under stress level ξ_i, $j = 1,2,\ldots,n_i$, then $\delta_{i,j} \sim \mathcal{N}(\mu_i, \omega^2)$, $\mu_i = a + b\xi_i$. Based on the degradation data from the jth unit, denote $\mathcal{X}_{i,j} \hat{=} \Sigma_{k=1}^{r_i} \Delta X_{i,j,k}$, $\mathcal{T}_i \hat{=} \Sigma_{k=1}^{r_i} \Delta t_{i,j,k}$, the estimates of $\delta_{i,j}$ and σ^2 are given by

$$\widehat{\delta}_{i,j} = \frac{\sum_{k=1}^{r_{i,j}} \Delta X_{i,j,k}}{\sum_{k=1}^{r_{i,j}} \Delta t_{i,j,k}} \hat{=} \frac{\mathcal{X}_{i,j}}{\mathcal{T}_i}, S_{i,j}^2 = \frac{1}{r_{i,j}-1} \sum_{k=1}^{r_{i,j}} \frac{(\Delta X_{i,j,k} - \widehat{\delta}_{i,j} \Delta t_{i,j,k})^2}{\Delta t_{i,j,k}}. \tag{3}$$

It is easy to show that the two estimates are independent. Condition on $\delta_{i,j}$, $\widehat{\delta}_{i,j} \sim \mathcal{N}(\delta_{i,j}, \sigma^2/\mathcal{T}_i)$. Unconditionally, $\widehat{\delta}_{i,j} \sim \mathcal{N}(\mu_i, \mathcal{A}_i^2)$, where $\mathcal{A}_i^2 \hat{=} \sigma^2/\mathcal{T}_i + \omega^2$ is unknown. Based on $\mathbb{D}_i$, estimates of μ_i and σ^2 are given by

$$\widehat{\mu}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} \widehat{\delta}_{i,j} \sim \mathcal{N}\left(\mu_i, \frac{\mathcal{A}_i^2}{n_i}\right), \tag{4}$$

and

$$S_i^2 = \frac{1}{M_i - n_i} \sum_{j=1}^{n_i} \sum_{k=1}^{r_{i,j}} \frac{(\Delta X_{i,j,k} - \widehat{\delta}_{i,j} \Delta t_{i,j,k})^2}{\Delta t_{i,j,k}},$$

respectively.

Because $(r_{i,j}-1)S_{i,j}^2/\sigma^2 \sim \chi^2(r_{i,j}-1)$, and $S_{i,j}^2$ are mutually independent for $j = 1,2,\ldots,n_i$, utilizing the additivity of χ^2 distribution, so $(M_i - n_i)S_i^2/\sigma^2 \sim \chi^2(M_i - n_i)$. Based on the unconditional distribution of $\widehat{\delta}_{i,j}$, an estimate of $\mathcal{A}_i^2$ is given by $W_i^2 =$

$\frac{1}{n_i-1}\sum_{j=1}^{n_i}(\widehat{\delta}_{i,j}-\widehat{\mu}_i)^2$, and this estimate is independent of $\widehat{\mu}_i$. Moreover, we can prove that the quantity $(n_i-1)W_i^2/\mathcal{A}_i^2$ follows a χ^2 distribution with free degrees n_i-1. Therefore, an estimate for ω^2 is given by

$$\widehat{\omega}_i^2=\frac{1}{n_i-1}\sum_{j=1}^{n_i}(\widehat{\delta}_{i,j}-\widehat{\mu}_i)^2-\frac{S_i^2}{\mathcal{T}_i}.$$

Recall that S_i^2 are mutually independent for different stress level ξ_i. Similarly, W_i^2 are also mutually independent for $i=1,2,\ldots,K$. Based on the degradation data $\mathbb{D}$, utilizing the additivity of χ^2 distribution, we have the following facts

$$\frac{1}{\sigma^2}\sum_{i=1}^{K}\sum_{j=1}^{n_i}\sum_{k=1}^{r_{i,j}}\frac{(\Delta X_{i,j,k}-\widehat{\delta}_{i,j}\Delta t_{i,j,k})^2}{\Delta t_{i,j,k}}\sim\chi^2(M-N). \tag{5}$$

According to Equation (5), the point estimate of σ^2 is obtained by

$$S^2=\frac{1}{M-N}\sum_{i=1}^{K}\sum_{j=1}^{n_i}\sum_{k=1}^{r_{i,j}}\frac{(\Delta X_{i,j,k}-\widehat{\delta}_{i,j}\Delta t_{i,j,k})^2}{\Delta t_{i,j,k}}. \tag{6}$$

Notice that $\omega^2=\mathcal{A}_i^2-\sigma^2/\mathcal{T}_i$, based on the degradation data $\mathbb{D}$, the estimate of parameter ω^2 is given as

$$W^2=\frac{1}{K}\sum_{i=1}^{K}W_i^2-\frac{1}{K}\sum_{i=1}^{K}\frac{S_i^2}{\mathcal{T}_i}. \tag{7}$$

to avoid negative values, we may define the estimate of ω^2 is max $(0,W^2)$ instead.

In order to obtain the estimates of parameters a and b, we consider the following regression model

$$\widehat{\mu}_i=a+b\xi_i+\varepsilon_i,\quad \varepsilon_i\sim\mathcal{N}(0,\frac{\mathcal{A}_i^2}{n_i}),i=1,2,\ldots,K. \tag{8}$$

Theorem 1. *Given the degradation data $\mathbb{D}$, for the linear regression model (8), then:*

(1) The estimates of parameters a and b are given as

$$\widehat{a}=\frac{GH-IM}{FG-I^2},\widehat{b}=\frac{FM-IH}{FG-I^2}. \tag{9}$$

where

$$F=\sum_{i=1}^{K}\frac{n_i}{\mathcal{A}_i^2},G=\sum_{i=1}^{K}\frac{n_i}{\mathcal{A}_i^2}{\xi_i}^2,I=\sum_{i=1}^{K}\frac{n_i}{\mathcal{A}_i^2}\xi_i,H=\sum_{i=1}^{K}\frac{n_i}{\mathcal{A}_i^2}\widehat{\mu}_i,M=\sum_{i=1}^{K}\frac{n_i}{\mathcal{A}_i^2}\widehat{\mu}_i\xi_i.$$

(2) The estimates $\widehat{a}$ and $\widehat{b}$ are unbiased, that is, $E(\widehat{a})=a,E(\widehat{b})=b$.

(3) The variance and covariance of the estimates $\widehat{a}$ and $\widehat{b}$ are given by

$$Var(\widehat{a})=\frac{G}{FG-I^2},Var(\widehat{b})=\frac{F}{FG-I^2},Cov(\widehat{a},\widehat{b})=\frac{-I}{FG-I^2}.$$

Proof. Let $V=diag(\frac{\mathcal{A}_1^2}{n_1},\cdots,\frac{\mathcal{A}_K^2}{n_K}),Y=(\widehat{\mu}_1,\cdots,\widehat{\mu}_K)^T$ and

$$Z=\begin{pmatrix}1&1&\cdots&1&\cdots&1\\ \xi_1&\xi_2&\cdots&\xi_i&\cdots&\xi_K\end{pmatrix}^T$$

then the estimates $(\widehat{a}, \widehat{b})$ are given by

$$\begin{pmatrix} \widehat{a} \\ \widehat{b} \end{pmatrix} = (Z^T V^{-1} Z)^{-1} Z^T V^{-1} Y = \frac{1}{FG - I^2} \begin{pmatrix} GH - IM \\ FM - IH \end{pmatrix}$$

so we have the expectations of $\widehat{a}$ and $\widehat{b}$

$$\begin{aligned} E\begin{pmatrix} \widehat{a} \\ \widehat{b} \end{pmatrix} &= (Z^T V^{-1} Z)^{-1} Z^T V^{-1} E(Y) \\ &= \frac{1}{FG - I^2} \begin{pmatrix} GE(H) - IE(M) \\ FE(M) - IE(H) \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \end{aligned}$$

and furthermore, the covariance matrix of the estimators $(\widehat{a}, \widehat{b})$ can be calculated and expressed as

$$\begin{aligned} Var\begin{pmatrix} \widehat{a} \\ \widehat{b} \end{pmatrix} &= Var\left[(Z^T V^{-1} Z)^{-1} Z^T V^{-1} Y\right] \\ &= \left[(Z^T V^{-1} Z)^{-1} Z^T V^{-1}\right] Var(Y) \left[(Z^T V^{-1} Z)^{-1} Z^T V^{-1}\right]^T \\ &= (Z^T V^{-1} Z)^{-1} = \frac{1}{FG - I^2} \begin{pmatrix} G & -I \\ -I & F \end{pmatrix} \end{aligned}$$

so the variance and covariance of the estimators $\widehat{a}$ and $\widehat{b}$ are obtained by

$$Var(\widehat{a}) = \frac{G}{FG - I^2}, Var(\widehat{b}) = \frac{F}{FG - I^2}, Cov(\widehat{a}, \widehat{b}) = \frac{-I}{FG - I^2}$$

□

Therefore, the degradation rate μ_0 at normal using stress level ξ_0 can be estimated by $\widehat{\mu}_0 = \widehat{a} + \widehat{b}\xi_0$. In addition, the estimate $\widehat{\mu}_0$ is also unbiased and has the variance $Var(\widehat{\mu}_0) = (FG - I^2)^{-1}(G + F{\xi_0}^2 - 2I\xi_0)$. However, (σ^2, ω^2) are unknown, so $\mathcal{A}_i^2, i = 1, 2, \ldots, K$ are also unknown. However, we can use its estimate $W_i^2 = \frac{1}{n_i - 1} \sum_{j=1}^{n_i} (\widehat{\delta}_{i,j} - \widehat{\mu}_i)^2$ substitute for $\mathcal{A}_i^2$.

3.2. GCIs for Model Parameters a, b, σ^2 and ω^2

For random-effects Wiener CSADT model, we are interested in the interval estimations of model parameters and some reliability metrics, such as the reliability function of lifetime, the MTTF and the degradation characteristic at normal using condition. Unfortunately, it is difficult to obtain the exact confidence intervals for model parameters and reliability metrics. Hence, we will develop the GCIs for them by constructing GPQs. However, it is not an easy task to obtain the GPQs of model parameters, especially for parameters a and b.

In this subsection, we first give the GPQs of σ^2, ω^2 and $\mathcal{A}_i^2$, then derive the GPQs of parameters a and b. Based on dataset $\{W_i^2, S_i^2, \widehat{\mu}_i, i = 1, 2, \ldots, K\}$, an estimate of $\mathcal{A}_i^2$ is given by $W_i^2 = \frac{1}{n_i - 1} \sum_{j=1}^{n_i} (\widehat{\delta}_{i,j} - \widehat{\mu}_i)^2$, and this estimate is independent of $\widehat{\mu}_i$. Moreover, we can easily prove the following fact.

$$Q_{0,i} \hat{=} \frac{(n_i - 1) W_i^2}{\mathcal{A}_i^2} \sim \chi^2(n_i - 1). \tag{10}$$

generate a copy $Q_{0,i}^*$ from the $\chi^2(n_i-1)$ distribution, then the GPQ of $\mathcal{A}_i^2$ can be given as

$$\mathcal{G}_{0,i} = \frac{(n_i-1)W_i^2}{Q_{0,i}^*} \tag{11}$$

Notice that

$$S^2 = \frac{1}{M-N}\sum_{i=1}^{K}\sum_{j=1}^{n_i}\sum_{k=1}^{r_{i,j}}\frac{(\Delta X_{i,j,k} - \widehat{\delta}_{i,j}\Delta t_{i,j,k})^2}{\Delta t_{i,j,k}}$$

and

$$Q_1 \hat{=} \frac{(M-N)S^2}{\sigma^2} \sim \chi^2(M-N). \tag{12}$$

generate a copy Q_1^* from the $\chi^2(M-N)$ distribution, then the GPQ of σ^2 can be obtained by

$$\mathcal{G}_1 = \frac{(M-N)S^2}{Q_1^*} \tag{13}$$

On the other hand,

$$Q_2 \hat{=} \sum_{i=1}^{K}\sum_{j=1}^{n_i}\frac{(\widehat{\delta}_{i,j} - \widehat{\mu}_i)^2}{\sigma^2/\mathcal{T}_i + \omega^2} \sim \chi^2(N-K). \tag{14}$$

Given σ^2, we find that the quantity Q_2 is monotone decreasing with respect to ω^2. Therefore, given a copy Q_2^* from the $\chi^2(N-K)$ distribution, for (14), there exists a unique solution for ω^2 when

$$Q_2^* \le \sum_{i=1}^{K}\sum_{j=1}^{n_i}\frac{(\widehat{\delta}_{i,j} - \widehat{\mu}_i)^2}{\sigma^2/\mathcal{T}_i}. \tag{15}$$

Let $\mathcal{G}_2(\sigma^2, Q_2^*)$ be the unique solution of ω^2 when (15) is satisfied, and $\mathcal{G}_2(\sigma^2, Q_2^*) = 0$ otherwise. Thus, the GPQ of ω^2 is given as

$$\mathcal{G}_2 = \begin{cases} \mathcal{G}_2(\mathcal{G}_1, Q_2^*), & Q_2^* \le \sum_{i=1}^{K}\sum_{j=1}^{n_i}\frac{(\widehat{\delta}_{i,j} - \widehat{\mu}_i)^2}{\mathcal{G}_1/\mathcal{T}_i}; \\ 0, & Q_2^* > \sum_{i=1}^{K}\sum_{j=1}^{n_i}\frac{(\widehat{\delta}_{i,j} - \widehat{\mu}_i)^2}{\mathcal{G}_1/\mathcal{T}_i}. \end{cases} \tag{16}$$

Based on model (8), we consider the following weighted sum of squares:

$$Q(a,b) = \sum_{i=1}^{K}\frac{n_i}{\mathcal{G}_{0,i}}(\widehat{\mu}_i - a - b\xi_i)^2 \tag{17}$$

By minimizing Equation (17), we have

$$\bar{a} = \frac{\tilde{G}\tilde{H} - \tilde{I}\tilde{M}}{\tilde{F}\tilde{G} - \tilde{I}^2}, \quad \bar{b} = \frac{\tilde{F}\tilde{M} - \tilde{I}\tilde{H}}{\tilde{F}\tilde{G} - \tilde{I}^2}$$

where

$$\tilde{F} = \sum_{i=1}^{K}\frac{n_i}{\mathcal{G}_{0,i}}, \tilde{G} = \sum_{i=1}^{K}\frac{n_i}{\mathcal{G}_{0,i}}\xi_i{}^2, \tilde{I} = \sum_{i=1}^{K}\frac{n_i}{\mathcal{G}_{0,i}}\xi_i, \tilde{H} = \sum_{i=1}^{K}\frac{n_i}{\mathcal{G}_{0,i}}\widehat{\mu}_i, \tilde{M} = \sum_{i=1}^{K}\frac{n_i}{\mathcal{G}_{0,i}}\widehat{\mu}_i\xi_i.$$

Let $V_1 = \bar{a} - a$, $V_2 = \bar{b} - b$, then V_1 and V_2 can be rewritten as

$$V_1 = \frac{\tilde{G}(\sum_{i=1}^{K} \frac{Z_i\sqrt{n_i}}{\mathcal{G}_{0,i}}(\frac{\sigma^2}{\mathcal{T}_i} + \omega^2)^{\frac{1}{2}}) - \tilde{I}(\sum_{i=1}^{K} \frac{Z_i\xi_i\sqrt{n_i}}{\mathcal{G}_{0,i}}(\frac{\sigma^2}{\mathcal{T}_i} + \omega^2)^{\frac{1}{2}})}{\tilde{F}\tilde{G} - \tilde{I}^2} \tag{18}$$

$$V_2 = \frac{\tilde{F}(\sum_{i=1}^{K} \frac{Z_i\xi_i\sqrt{n_i}}{\mathcal{G}_{0,i}}(\frac{\sigma^2}{\mathcal{T}_i} + \omega^2)^{\frac{1}{2}}) - \tilde{I}(\sum_{i=1}^{K} \frac{Z_i\sqrt{n_i}}{\mathcal{G}_{0,i}}(\frac{\sigma^2}{\mathcal{T}_i} + \omega^2)^{\frac{1}{2}})}{\tilde{F}\tilde{G} - \tilde{I}^2} \tag{19}$$

where $Z_i \hat{=} [\widehat{\mu}_i - (a + b\xi_i)] / \sqrt{\frac{1}{n_i}(\frac{\sigma^2}{\mathcal{T}_i} + \omega^2)} \sim \mathcal{N}(0,1)$. From (18) and (19), we know that V_1 and V_2 depend on the unknown parameters σ^2 and ω^2. According to the substitute method in Weerahandi [34,35], we replace σ^2 and ω^2 with their GPQs $\mathcal{G}_1$ and $\mathcal{G}_2$ in V_1, V_2, respectively. Then, the GPQs of parameters a and b are obtained as follows:

$$\mathcal{G}_3 = \bar{a} - \tilde{V}_1, \quad \mathcal{G}_4 = \bar{b} - \tilde{V}_2. \tag{20}$$

where

$$\tilde{V}_1 = \frac{\tilde{G}[\sum_{i=1}^{K} \frac{Z_i\sqrt{n_i}}{\mathcal{G}_{0,i}}(\frac{\mathcal{G}_1}{\mathcal{T}_i} + \mathcal{G}_2)^{\frac{1}{2}}] - \tilde{I}[\sum_{i=1}^{K} \frac{Z_i\sqrt{n_i}}{\mathcal{G}_{0,i}}(\frac{\mathcal{G}_1}{\mathcal{T}_i} + \mathcal{G}_2)^{\frac{1}{2}}\xi_i]}{\tilde{F}\tilde{G} - \tilde{I}^2};$$

$$\tilde{V}_2 = \frac{\tilde{F}[\sum_{i=1}^{K} \frac{Z_i\sqrt{n_i}}{\mathcal{G}_{0,i}}(\frac{\mathcal{G}_1}{\mathcal{T}_i} + \mathcal{G}_2)^{\frac{1}{2}}\xi_i] - \tilde{I}[\sum_{i=1}^{K} \frac{Z_i\sqrt{n_i}}{\mathcal{G}_{0,i}}(\frac{\mathcal{G}_1}{\mathcal{T}_i} + \mathcal{G}_2)^{\frac{1}{2}}]}{\tilde{F}\tilde{G} - \tilde{I}^2}.$$

Let $\mathcal{G}_{i,\gamma}$ be the γ percentile of $\mathcal{G}_i$, then $[\mathcal{G}_{i,\gamma/2}, \mathcal{G}_{i,1-\gamma/2}], i = 1,2,3,4$ are the $1-\gamma$-level GCIs for parameters σ^2, ω^2, a and b, respectively. The percentiles of $\mathcal{G}_i, i = 1,2,3,4$ can be obtained by the following Monte Carlo simulation Algorithm 1.

Algorithm 1 Percentiles for model parameters a, b, σ^2 and ω^2.

1: For given data set $\{(\Delta x_{i,j,k}, \Delta t_{i,j,k}, \xi_i), i = 1,\ldots,K; j = 1,\ldots,n_i; k = 1,\ldots,r_{i,j}\}$, compute the quantities $\widehat{\delta}_{i,j}, \widehat{\mu}_i, W_i^2, S^2$ and W^2.
2: Generate $Q_1 \sim \chi^2(M-N)$, then obtain $\mathcal{G}_1$ from Equation (13).
3: Generate $Q_2 \sim \chi^2(N-K)$, based on $\mathcal{G}_1$, then compute $\mathcal{G}_2$.
4: Generate a series of $Z_i \sim \mathcal{N}(0,1)$, based on $\mathcal{G}_1$, $\mathcal{G}_2$, using Equation (20) compute $\mathcal{G}_3$ and $\mathcal{G}_4$.
5: Repeat steps (2)–(4) B times, get B values of $\mathcal{G}_i, i = 1,2,3,4$, respectively.
6: Arrange all $\mathcal{G}_i$ values in ascending order: $\mathcal{G}_{i,(1)} < \mathcal{G}_{i,(2)} < \cdots < \mathcal{G}_{i,(B)}$, then the γ percentile of $\mathcal{G}_i$ is estimated by $\mathcal{G}_{i,(\tau B)}$.

Remark 1. *It is important to emphasize that the GCI obtained by $\mathcal{G}_1$ is an exact confidence interval of parameter σ^2. In fact, there is another way to obtain the exact confidence interval of σ^2. Based on Equation (12), an exact confidence interval of σ^2 can be given by $[\frac{(M-N)S^2}{\chi^2_{\frac{\gamma}{2}}(M-N)}, \frac{(M-N)S^2}{\chi^2_{1-\frac{\gamma}{2}}(M-N)}]$. Here, $\chi^2_\gamma(n)$ is the upper γ percentile of χ^2 distribution with free degrees n.*

3.3. GCIs for μ_0, $R(t_0)$ and MTTF

In practical applications, some important quantities such as the degradation rate μ_0, the reliability function $R(t)$ and the MTTF are the ones we are more interested in and more focused on than the model parameters $(a, b, \sigma^2, \omega^2)$. However, because the expressions for these quantities are complicated, their interval estimations tend to be very difficult. Similar to the case of parameters $(a, b, \sigma^2, \omega^2)$, we can develop the generalized interval estimation procedures for these quantities.

Notice that the degradation rate μ_0, the reliability function $R(t_0)$ and the MTTF are given by $\mu_0 = a + b\xi_0$, $R(t_0) = 1 - F_T(t_0|\sigma^2, \omega^2, \mu_0)$ and $\int_0^{+\infty} R(t)dt$, respectively. Then, according to the substitution method given by [34], parameters $(\sigma^2, \omega^2, a, b)$ in the

expressions of μ_0, $R(t_0)$ and MTTF are substituted by $(\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4)$. Thus, the GPQs of μ_0, $R(t_0)$ and MTTF are given by

$$\mathcal{G}_5 = \mathcal{G}_3 + \mathcal{G}_4\xi_0, \tag{21}$$

$$\mathcal{G}_6 = \Phi\left(\frac{L - \mathcal{G}_5 t}{\sqrt{\mathcal{G}_1 t + \mathcal{G}_2 t^2}}\right) - \exp\left(\frac{2\mathcal{G}_5 L}{\mathcal{G}_1} + \frac{2\mathcal{G}_2 L^2}{\mathcal{G}_1^2}\right)\Phi\left(-\frac{2\mathcal{G}_2 L t + \mathcal{G}_1(\mathcal{G}_5 t + L)}{\mathcal{G}_1\sqrt{\mathcal{G}_1 t + \mathcal{G}_2 t^2}}\right), \tag{22}$$

$$\mathcal{G}_7 = \int_0^{+\infty}\left[\Phi\left(\frac{L - \mathcal{G}_5 t}{\sqrt{\mathcal{G}_1 t + \mathcal{G}_2 t^2}}\right) - \exp\left(\frac{2\mathcal{G}_5 L}{\mathcal{G}_1} + \frac{2\mathcal{G}_2 L^2}{\mathcal{G}_1^2}\right)\Phi\left(-\frac{2\mathcal{G}_2 L t + \mathcal{G}_1(\mathcal{G}_5 t + L)}{\mathcal{G}_1\sqrt{\mathcal{G}_1 t + \mathcal{G}_2 t^2}}\right)\right]dt, \tag{23}$$

respectively.

Let $\mathcal{G}_{i,\gamma}$ denote the γ percentiles of $\mathcal{G}_i$. Then, $[\mathcal{G}_{i,\gamma/2}, \mathcal{G}_{i,1-\gamma/2}], i = 5, 6, 7$ are the $1-\gamma$-level GCIs for μ_0, $R(t_0)$ and MTTF, respectively. The percentiles of $\mathcal{G}_i, i = 5, 6, 7$ can be obtained by the following Monte Carlo simulation Algorithm 2.

Algorithm 2 Percentiles for quantities μ_0, $R(t_0))$ and MTTF.

1: For given data set $\{(\Delta x_{i,j,k}, \Delta t_{i,j,k}, \xi_i), i = 1, \ldots, K; j = 1, \ldots, n_i; k = 1, \ldots, r_{i,j}\}$, compute the quantities $\widehat{\delta}_{i,j}, \widehat{\mu}_i, W_i^2, S^2$ and W^2.
2: Generate $Q_1 \sim \chi^2(M - N)$, then obtain $\mathcal{G}_1$ from Equation (13).
3: Generate $Q_2 \sim \chi^2(N - K)$, based on $\mathcal{G}_1$, then compute $\mathcal{G}_2$.
4: Generate a series of $Z_i \sim \mathcal{N}(0, 1)$, based on $\mathcal{G}_1$, $\mathcal{G}_2$, using Equation (20) compute $\mathcal{G}_3$ and $\mathcal{G}_4$.
5: Based on $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$ and $\mathcal{G}_4$, through Equations (21)–(23) compute $\mathcal{G}_5$, $\mathcal{G}_6$ and $\mathcal{G}_7$.
6: Repeat steps (2)–(5) B times, get B values of $\mathcal{G}_i, i = 5, 6, 7$, respectively.
7: Arrange all $\mathcal{G}_i$ values in ascending order: $\mathcal{G}_{i,(1)} < \mathcal{G}_{i,(2)} < \cdots < \mathcal{G}_{i,(B)}$, then the γ percentile of $\mathcal{G}_i$ is estimated by $\mathcal{G}_{i,(\tau B)}$.

3.4. GPI for Degradation Characteristic $X_0(t)$

In practical applications, the prediction interval for the degradation characteristic under normal operating condition may be more practical and interesting for the product designer and user. Thus, it is important and meaningful to discuss the prediction interval of degradation characteristic $X_0(t)$. However, for the Wiener CSADT model with random effects, given a future time point t, it is difficult to obtain the exact prediction interval of $X_0(t)$, so we develop the GPI for it.

At the normal using stress level ξ_0, the degradation characteristic $X_0(t)$ can be presented as

$$X_0(t) = \delta t + \sigma B(t), \delta \sim \mathcal{N}(\mu_0, \omega^2), \mu_0 = a + b\xi_0$$

based on GPQs $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_5$, using the substitution method given by [33], the generalized prediction pivotal quantity (GPPQ) is obtained by

$$\mathcal{G}_8 = \mathcal{G}_5 t + \sqrt{\mathcal{G}_1 t + \mathcal{G}_2 t^2}Z, \quad Z \sim \mathcal{N}(0, 1) \tag{24}$$

Let $\mathcal{G}_{8,\gamma}$ be the γ percentile of $\mathcal{G}_8$, then $[\mathcal{G}_{8,\gamma/2}, \mathcal{G}_{8,1-\gamma/2}]$ is the $1-\gamma$-level GPI for degradation characteristic $X_0(t)$. The percentile of $\mathcal{G}_8$ can be obtained by the following Monte Carlo simulation Algorithm 3.

Algorithm 3 Percentile for degradation characteristic $X_0(t)$.

1: For given data set $\{(\Delta x_{i,j,k}, \Delta t_{i,j,k}, \xi_i), i = 1, \ldots, K; j = 1, \ldots, n_i; k = 1, \ldots, r_{i,j}\}$, compute the quantities $\widehat{\delta}_{i,j}, \widehat{\mu}_i, W_i^2, S^2$ and W^2.
2: Generate $Q_1 \sim \chi^2(M - N)$, then obtain $\mathcal{G}_1$ from Equation (13).
3: Generate $Q_2 \sim \chi^2(N - K)$, based on $\mathcal{G}_1$ compute $\mathcal{G}_2$.
4: Generate a series of $Z_i \sim \mathcal{N}(0, 1)$, based on $\mathcal{G}_1$, $\mathcal{G}_2$, using Equation (20) compute $\mathcal{G}_3$ and $\mathcal{G}_4$.
5: Based on $\mathcal{G}_3$ and $\mathcal{G}_4$, through Equation (21) compute $\mathcal{G}_5$.
6: Generate $Z \sim \mathcal{N}(0, 1)$, based on $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_5$, through Equation (24) compute $\mathcal{G}_8$.
7: Repeat steps (2)–(6) B times, get B values of $\mathcal{G}_8$.
8: Arrange all $\mathcal{G}_8$ values in ascending order: $\mathcal{G}_{8,(1)} < \mathcal{G}_{8,(2)} < \cdots < \mathcal{G}_{8,(B)}$, then the γ percentile of $\mathcal{G}_8$ is estimated by $\mathcal{G}_{8,(\tau B)}$.

4. Simulation Study and Data Analysis

4.1. Simulation Study

To assess the performance of the proposed GCIs/GPIs for the Wiener CSADT model with random effects, a Monte Carlo simulation study is implemented to examine the coverage percentages (CPs) and average interval lengths (AILs) of the proposed GCIs/GPIs. In order to comprehensively evaluate the performance of the GPQ procedures, we also consider the upper confidence limit (UCL) and the lower confidence limit (LCL) of the model parameters and some other quantities based on the CPs and AILs. Without the loss of generality, we present four different parameter settings in Table 1 for the simulation study. The four accelerated stress levels are given as $\xi_1 = 1.0, \xi_2 = 1.5, \xi_3 = 2.0$ and $\xi_4 = 2.5$, and the normal using stress level is assumed to be $\xi_0 = 0.5$. For convenience, the values of $n_i, r_{i,j}$ and $\Delta t_{i,j,k}$ are chosen to be $n_1 = \cdots = n_K \hat{=} n = 6, 9, 12$, $r_{i,j} \hat{=} r = 6, 8, 10$ and $\Delta t_{i,j,k} = 1$. We take $B = 10{,}000$ in the simulation study. All the results are based on 5000 replications. The simulation results are given in Tables 2 and 3.

Table 1. Parameter settings used in the simulation study.

Parameter	a	b	σ^2	ω^2	L
I	−0.90	2.00	1.00	0.50	6.30
II	−0.40	1.40	1.21	0.64	7.92
III	−1.10	2.60	1.50	1.00	8.75
IV	0.20	0.80	1.96	1.21	11.55

It is observed from Tables 2 and 3 that the CPs of the proposed GCIs/GPIs are quite close to the nominal levels, even for small sample sizes. Based on the normal approximation to the binomial distribution, CPs between 89% and 91% and also 94% and 96% are considered appropriate for the 90% and 95% confidence intervals. Tables 2 and 3 also indicate that, for the fixed parameter settings, as the value of n and r increase, the AILs decrease. These findings show that the proposed GCIs work well.

Table 2. The CPs and AILs (in parentheses) of parameters a, b, σ^2 and ω^2 for nominal levels 0.9, 0.95, based on 5000 replications.

Case	(n, r)	0.9	0.95	0.9	0.95
		a		b	
II	$(6, 6)$	0.9020 (2.2480)	0.9524 (2.7191)	0.9025 (1.2222)	0.9530 (1.4783)
	$(9, 8)$	0.9034 (1.7128)	0.9512 (2.0598)	0.8982 (0.9315)	0.9508 (1.1203)
	$(12, 10)$	0.9002 (1.4283)	0.9496 (1.7133)	0.9000 (0.7767)	0.9486 (0.9317)
III	$(6, 6)$	0.9026 (2.7391)	0.9528 (3.3134)	0.9036 (1.4894)	0.9532 (1.8013)
	$(9, 8)$	0.8962 (2.1017)	0.9468 (2.5265)	0.8968 (1.1415)	0.9465 (1.3727)
	$(12, 10)$	0.9020 (1.7548)	0.9464 (2.1049)	0.9028 (0.9535)	0.9474 (1.1438)
IV	$(6, 6)$	0.9022(3.0371)	0.9532 (3.6738)	0.9034 (1.6514)	0.9528 (1.9972)
	$(9, 8)$	0.9022 (2.3226)	0.9510 (2.7931)	0.8978 (1.2630)	0.9502 (1.5190)
	$(12, 10)$	0.8996 (1.9413)	0.9494 (2.3286)	0.9000 (1.0557)	0.9488 (1.2663)
		σ^2		ω^2	
II	$(6, 6)$	0.8960 (0.5277)	0.9478 (0.6329)	0.8955 (1.0175)	0.9462 (1.2647)
	$(9, 8)$	0.9014 (0.3588)	0.9516 (0.4288)	0.8988 (0.7180)	0.9472 (0.8780)
	$(12, 10)$	0.8996 (0.2730)	0.9462 (0.3259)	0.8966 (0.5719)	0.9494 (0.6943)
III	$(6, 6)$	0.8965 (0.6542)	0.9478 (0.7846)	0.8964 (1.5096)	0.9460 (1.8765)
	$(9, 8)$	0.8952 (0.4433)	0.9466 (0.5298)	0.8970 (1.0757)	0.9474 (1.3155)
	$(12, 10)$	0.9034 (0.3382)	0.9508 (0.4037)	0.9032 (0.8624)	0.9534 (1.0469)
IV	$(6, 6)$	0.8970 (0.8548)	0.9478 (1.0253)	0.8968 (1.8565)	0.9466 (2.3076)
	$(9, 8)$	0.9014 (0.5811)	0.9516 (0.6946)	0.8994 (1.3196)	0.9468 (1.6135)
	$(12, 10)$	0.8996 (0.4423)	0.9462 (0.5279)	0.8976 (1.0563)	0.9498 (1.2823)

Table 3. The CPs and AILs (in parentheses) of quantities μ_0, $R(5)$, MTTF and $X_0(8)$ for nominal levels 0.9, 0.95, based on 5000 replications.

Case	(n, r)	0.9	0.95	0.9	0.95
		μ_0		$R(5)$	
II	$(6, 6)$	0.9026 (1.6778)	0.9510 (2.0295)	0.9008 (0.3429)	0.9510 (0.4117)
	$(9, 8)$	0.9030 (1.2783)	0.9498 (1.5372)	0.9024 (0.2622)	0.9530 (0.3156)
	$(12, 10)$	0.9022 (0.7888)	0.9512 (0.9144)	0.9036 (0.2037)	0.9548 (0.2464)
III	$(6, 6)$	0.9030 (1.2064)	0.9522 (1.4303)	0.8955 (0.3205)	0.9462 (0.3525)
	$(9, 8)$	0.8966 (0.9632)	0.9472 (1.1328)	0.8968 (0.2416)	0.9464 (0.2939)
	$(12, 10)$	0.9000 (0.8405)	0.9464 (0.9819)	0.9048 (0.1999)	0.9466 (0.2428)
IV	$(6, 6)$	0.9025 (1.6519)	0.9516 (1.9286)	0.9028 (*0.2632*)	0.9538 (0.3054)
	$(9, 8)$	0.9026 (1.3713)	0.9500 (1.5889)	0.9036 (0.2556)	0.9526 (0.3086)
	$(12, 10)$	0.9022 (1.2030)	0.9510 (1.3941)	0.9038 (0.2121)	0.9528 (0.2557)
		MTTF		$X_0(t)$	
II	$(6, 6)$	0.9022 (5.8332×10^4)	0.9532 (6.6712×10^4)	0.9002 (27.6940)	0.9500 (33.5407)
	$(9, 8)$	0.9018 (4.8620×10^4)	0.9500 (5.6348×10^4)	0.8992 (26.0592)	0.9466 (31.3321)
	$(12, 10)$	0.9002 (2.9923×10^4)	0.9502 (3.3542×10^4)	0.9032 (24.7582)	0.9534 (29.6830)
III	$(6, 6)$	0.9032 (3.3393×10^4)	0.9536 (3.7233×10^4)	0.8964 (31.7443)	0.9508 (38.4358)
	$(9, 8)$	0.8968 (2.9464×10^4)	0.9464 (3.3212×10^4)	0.8966 (30.6825)	0.9518 (36.8906)
	$(12, 10)$	0.9014 (2.7190×10^4)	0.9478 (3.0730×10^4)	0.8952 (30.1169)	0.9460 (36.1111)
IV	$(6, 6)$	0.9032 (3.6767×10^4)	0.9528 (4.0445×10^4)	0.8990 (35.8711)	0.9514 (43.4082)
	$(9, 8)$	0.9012 (3.3915×10^4)	0.9520 (3.7657×10^4)	0.8976 (34.5438)	0.9478 (41.5150)
	$(12, 10)$	0.8982 (3.1433×10^4)	0.9500 (3.5266×10^4)	0.9032 (33.7932)	0.9530 (40.5162)

As everyone knows, the parametric bootstrap procedure is a commonly used method to obtain confidence intervals of model parameters. To evaluate the performances of the GCIs/GPIs, we consider the bootstrap CIs for the Wiener CSADT model with random effects. For comparison, we also consider confidence limits (CLs), such as the lower confidence limit (LCL) and upper confidence limit (UCL), for the model parameters and reliability metrics. We performed a comparative analysis of the CIs, LCLs and UCLs obtained by the GPQ method and the parametric bootstrap method. For saving space, we

only give the simulation results under the parameter setting I. The simulation results of the CIs, LCLs and UCLs in the GPQ and bootstrap-p methods under the parameter setting I are provided in Tables 4–6. The bootstrap-p procedure is also based on 5000 bootstrap samples.

Tables 4–6 show that the CPs of the CIs, LCLs and UCLs obtained by the GPQ method are all very close to the nominal levels, even for small sample sizes. However, the CPs of the CIs, LCLs and UCLs obtained by the bootstrap-p method are not close to the nominal levels when sample size n is small. In particular, from Tables 5 and 6, we find that the LCLs and UCLs obtained in the bootstrap-p method perform badly. For example, the CPs of the model parameters σ^2, ω^2 and the reliability function $R(t_0)$ are far away from the nominal levels.

As sample size n increases, the CPs of the bootstrap-p CIs/PIs approach the nominal levels. Tables 4–6 also indicate that, for the fixed parameter settings, as sample size n increases, the AILs become shorter, the LCLs become larger and the UCLs become smaller for both the GPQ method and the bootstrap-p method, as expected. These findings indicate that the CIs, LCLs and UCLs obtained in the GPQ method perform better than the corresponding bootstrap-p one for all cases, according to the CP. Therefore, we recommend the proposed CIs, LCLs and UCLs in the GPQ method for the proposed Wiener CSADT model, especially in a small sample case.

Table 4. The CPs and AILs (in parentheses) of different CIs under parameter setting I for nominal levels 0.9, 0.95, based on 5000 replications.

(n, r)	Parameter	GCI		Bootstrap-p CI	
		0.9	**0.95**	**0.9**	**0.95**
$(6, 6)$	a	0.9024 (2.0007)	0.9522 (2.4200)	**0.8872** (1.9162)	**0.9368** (2.2915)
	b	0.9022 (1.0878)	0.9526 (1.3157)	0.8912 (1.0407)	0.9414 (1.2441)
	σ^2	0.8970 (0.4361)	0.9478 (0.5231)	0.8938 (0.4252)	0.9428 (0.5070)
	ω^2	0.8978 (0.8062)	0.9472 (1.0019)	**0.8736** (0.6920)	**0.9256** (0.8266)
	μ_0	0.9028 (1.4934)	0.9508 (1.8063)	0.8930 (1.4308)	0.9418 (1.7116)
	$R(5)$	0.9006 (0.3413)	0.9498 (0.4098)	0.8920 (0.3224)	0.9406 (0.3861)
	MTTF	0.9024 (6.1217×10^4)	0.9522 (6.9653×10^4)	0.8920 (6.3831×10^4)	0.9424 (7.3118×10^4)
	$X_0(8)$	0.8994 (24.6218)	0.9504 (29.8223)	**0.8878** (23.5011)	**0.9394** (28.2697)
$(9, 8)$	a	0.8998 (1.5255)	0.9536 (1.8345)	**0.8872** (1.4767)	**0.9374** (1.7621)
	b	0.8998 (0.8285)	0.9512 (0.9963)	0.8910 (0.8025)	0.9414 (0.9575)
	σ^2	0.8978 (0.2969)	0.9464 (0.3549)	0.9072 (0.2925)	0.9560 (0.3486)
	ω^2	0.9028 (0.5667)	0.9536 (0.6930)	**0.8864** (0.5107)	**0.9290** (0.6097)
	μ_0	0.9018 (1.1387)	0.9520 (1.3693)	**0.8878** (1.1021)	0.9430 (1.3152)
	$R(5)$	0.8974 (0.2599)	0.9486 (0.3129)	0.8902 (0.2481)	0.9412 (0.2969)
	MTTF	0.9036 (5.1554×10^4)	0.9530 (5.9501×10^4)	**0.8870** (5.2715×10^4)	0.9424 (6.1244×10^4)
	$X_0(8)$	0.8990 (23.1594)	0.9490 (27.8547)	**0.8850** (22.3982)	**0.9384** (26.8549)
$(12, 10)$	a	0.8974 (1.2686)	0.9482 (1.5215)	0.8940 (1.2431)	0.9436 (1.4824)
	b	0.8964 (0.6895)	0.9472 (0.8271)	0.8922 (0.6758)	0.9446 (0.8058)
	σ^2	0.9008 (0.2252)	0.9512 (0.2688)	0.8938 (0.2237)	0.9444 (0.2666)
	ω^2	0.9028 (0.4509)	0.9490 (0.54739)	**0.8892** (0.4204)	**0.9336** (0.5016)
	μ_0	0.8976 (0.9466)	0.9492 (1.1354)	0.8942 (0.9275)	0.9472 (1.1061)
	$R(5)$	0.8994 (0.2133)	0.9462 (0.2568)	0.8906 (0.2087)	0.9432 (0.2496)
	MTTF	0.8958 (4.5046×10^4)	0.9500 (5.2422×10^4)	0.8936 (4.5832×10^4)	0.9444 (5.3604×10^4)
	$X_0(8)$	0.8996 (22.4528)	0.9484 (26.9199)	**0.8888** (21.9932)	0.9446 (26.3262)

The number in bold indicates that the corresponding CP is significantly off the nominal levels.

Table 5. The CPs and average values (in parentheses) of different LCLs for model parameters and some quantities under parameter setting I for nominal levels 0.9, 0.95, based on 5000 replications.

(n, r)	Parameter	LCL in GPQ Method		LCL in Bootstrap-p Method	
		0.9	**0.95**	**0.9**	**0.95**
$(6, 6)$	a	0.8968 (-1.6616)	0.9512 (-1.8934)	0.9020 (-1.6542)	0.9506 (-1.8682)
	b	0.9100 (1.5743)	0.9534 (1.4481)	0.8974 (1.6012)	0.9452 (1.4850)
	σ^2	0.8978 (0.8577)	0.9432 (0.8207)	**0.9246** (0.8412)	**0.9682** (0.7999)
	ω^2	0.9018 (0.2964)	0.9502 (0.2507)	**0.9546** (0.2483)	**0.9862** (0.1945)
	$R(5)$	0.8976 (0.696)	0.9518 (0.6311)	**0.8766** (0.7162)	**0.9236** (0.6564)
	MTTF	0.9044 (1.7927×10^4)	0.9538 (1.3248×10^4)	0.9052 (1.7236×10^4)	0.9508 (1.2075×10^4)
$(9, 8)$	a	0.9020 (-1.4870)	0.9460 (-1.6608)	0.8982 (-1.4757)	0.9470 (-1.6396)
	b	0.8986 (1.6783)	0.9520 (1.5840)	0.8922 (1.6854)	0.9458 (1.5964)
	σ^2	0.8946 (0.8978)	0.9432 (0.8704)	**0.9200** (0.8869)	**0.9674** (0.8573)
	ω^2	0.8998 (0.3442)	0.9508 (0.3071)	**0.9498** (0.3086)	**0.9824** (0.2655)
	$R(5)$	0.9012 (0.7535)	0.9554 (0.7057)	**0.8708** (0.7660)	**0.9276** (0.7226)
	MTTF	0.9056 (2.1389×10^4)	0.9548 (1.7046×10^4)	0.8952 (2.0895×10^4)	0.9438 (1.6264×10^4)
$(12, 10)$	a	0.9090 (-1.3988)	0.9558 (-1.5421)	0.8936 (-1.3809)	0.9460 (-1.5185)
	b	0.8936 (1.7357)	0.9424 (1.6578)	0.8962 (1.7361)	0.9486 (1.6612)
	σ^2	0.8998 (0.9186)	0.9512 (0.8969)	0.9094 (0.9138)	0.9596 (0.8906)
	ω^2	0.9016 (0.3682)	0.9504 (0.3361)	**0.9344** (0.3441)	**0.9750** (0.3070)
	$R(5)$	0.8978 (0.7857)	0.9466 (0.7479)	**0.8830** (0.7901)	**0.9352** (0.7552)
	MTTF	0.8926 (2.3986×10^4)	0.9386 (1.9975×10^4)	0.9060 (2.3172×10^4)	0.9540 (1.8956×10^4)

The number in bold indicates that the corresponding CP is significantly off the nominal levels.

Table 6. The CPs and average values (in parentheses) of different UCLs for model parameters and some quantities under parameter setting I for nominal levels 0.9, 0.95, based on 5000 replications.

(n, r)	Parameter	UCL in GPQ Method		UCL in Bootstrap-p Method	
		0.9	**0.95**	**0.9**	**0.95**
$(6, 6)$	a	0.9024 (-0.1242)	0.9542 (0.1074)	0.8956 (-0.1656)	0.9446 (0.0481)
	b	0.8950 (2.4102)	0.9518 (2.5360)	0.9012 (2.4099)	0.9500 (2.5257)
	σ^2	0.8980 (1.1954)	0.9498 (1.2568)	**0.8816** (1.1723)	**0.9306** (1.2252)
	ω^2	0.8870 (0.9001)	0.9416 (1.0569)	**0.8308** (0.7860)	**0.8874** (0.8866)
	$R(5)$	0.8978 (0.9596)	0.9488 (0.9723)	**0.9256** (0.9663)	**0.9684** (0.9789)
	MTTF	0.8980 (6.7811×10^4)	0.9516 (7.4465×10^4)	0.8934 (6.8816×10^4)	0.9432 (7.5906×10^4)
$(9, 8)$	a	0.9046 (-0.3087)	0.9538 (-0.1353)	0.8916 (-0.3265)	0.9402 (-0.1629)
	b	0.9004 (2.3185)	0.9478 (2.4125)	0.8980 (2.3101)	0.9452 (2.3989)
	σ^2	0.9010 (1.1285)	0.9526 (1.1673)	**0.8862** (1.1148)	**0.9368** (1.1499)
	ω^2	0.9016 (0.7751)	0.9520 (0.8737)	**0.8440** (0.7058)	**0.9040** (0.7762)
	$R(5)$	0.8944 (0.9537)	0.9420 (0.9656)	**0.9182** (0.9585)	**0.9626** (0.9707)
	MTTF	0.8998 (6.2798×10^4)	0.9488 (6.8600×10^4)	0.8914 (6.2941×10^4)	0.9432 (6.8979×10^4)
$(12, 10)$	a	0.8934 (-0.4164)	0.9416 (-0.2734)	0.9006 (-0.4130)	0.9510 (-0.2754)
	b	0.9002 (2.2697)	0.9510 (2.3473)	0.8942 (2.2623)	0.9436 (2.3371)
	σ^2	0.8998 (1.0938)	0.9496 (1.1221)	**0.8828** (1.0880)	**0.9342** (1.1143)
	ω^2	0.8992 (0.7133)	0.9524 (0.7871)	**0.8552** (0.6712)	**0.9142** (0.7274)
	$R(5)$	0.9012 (0.9502)	0.9528 (0.9613)	0.9042 (0.9522)	0.9554 (0.9639)
	MTTF	0.9102 (5.9889×10^4)	0.9532 (6.5021×10^4)	0.8932 (5.9508×10^4)	0.9426 (6.4788×10^4)

The number in bold indicates that the corresponding CP is significantly off the nominal levels.

4.2. Real Data Analysis

In this section, an illustrative example is provided to evaluate the performance of the proposed GPQ methods based on the degradation data. The degradation data are obtained from Table C.3 of Meeker and Escobar [7] in which three ADTs was used for a reliability analysis of a carbon-film resistor at three different temperature levels. Figure 1 shows the degradation paths for the test units. Figures 2–4 show the Q-Q plot for the degradation increments based on the carbon-film resistors' degradation data at different temperature

levels. The points scatter around the line nicely, except for a few points at temperature level 173 °C. This is probably due to the change in the degradation mechanism caused by a high accelerated temperature for individual products. Thus, it is reasonable to use the Wiener degradation process to fit the degradation paths of carbon-film resistor units. The degradation path $X(t)$ is assumed to follow a Wiener process at each temperature level. The maximum test duration allowed is 8084 h and the number of measurements for each unit is 4. The normal using temperature level S_0 is specified as 50 °C and the maximum test temperature level is $S_3 = 173$ °C. The other test temperature levels are $S_1 = 83$ °C and $S_2 = 133$ °C, respectively. The carbon-film resistor is assumed to fail when the resistance increases by 5% from its initial value.

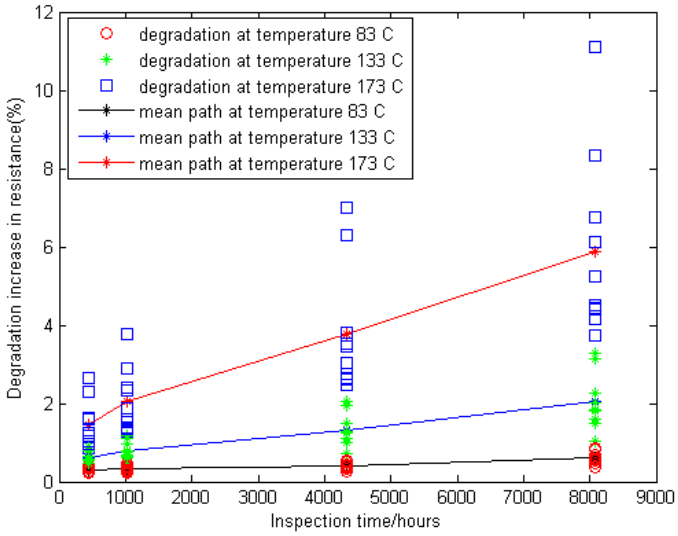

Figure 1. Sample degradation data at different temperatures.

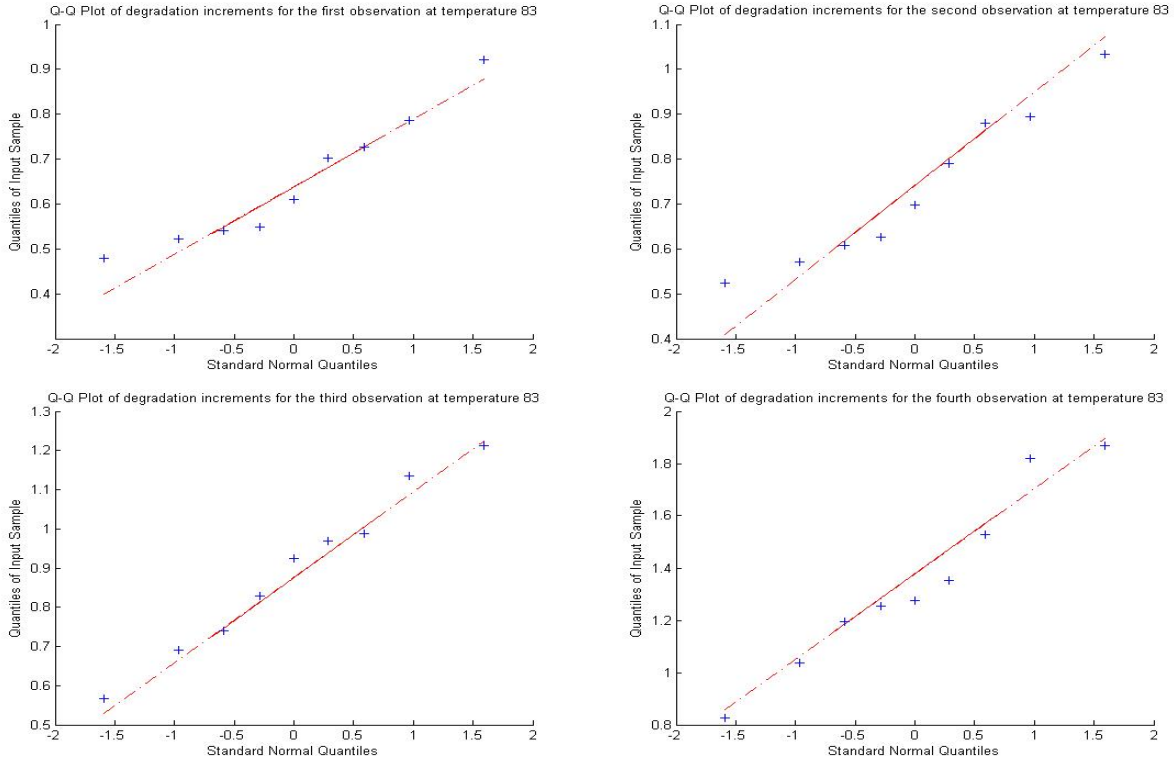

Figure 2. The Q-Q plot for degradation data at temperature 83 °C.

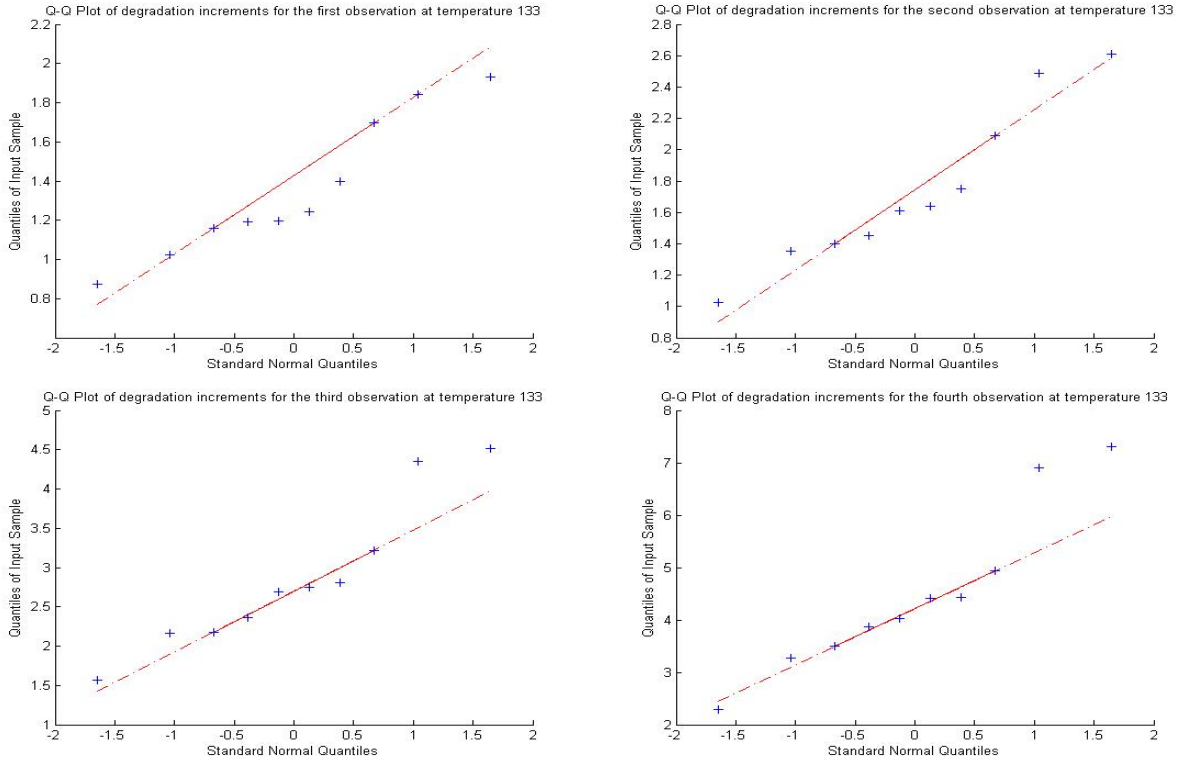

Figure 3. The Q-Q plot for degradation data at temperature 133 °C.

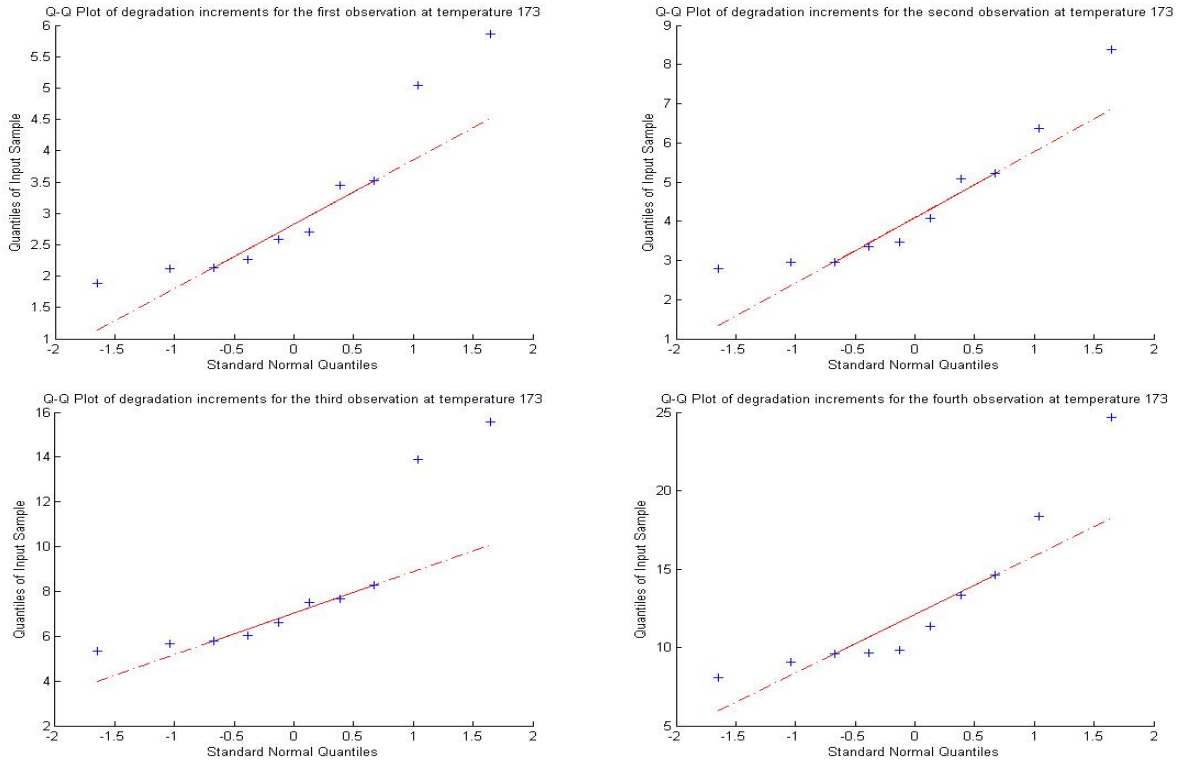

Figure 4. The Q-Q plot for degradation data at temperature 173 °C.

An Arrhenius model is used as the accelerating relation between the drift parameter and the accelerated temperature level. Following Escobar and Meeker [31], let $\varphi(S_i) =$

$\exp(-11605/(S_i + 273.15))$. Similar to Ye et al. [15] and Lim and Yum [36], we can standardize the temperature levels by selecting

$$\zeta_i = \frac{\varphi(S_i) - \varphi(S_0)}{\varphi(S_3) - \varphi(S_0)}, i = 0, 1, 2, 3.$$

so the standardized temperature levels are given as: $\zeta_0 = 0, \zeta_1 = 0.0013, \zeta_2 = 0.0771$, $\zeta_3 = 1.0$. The relationship between the drift parameter μ_i and the standardized temperature level ζ_i is given by $\mu_i = a + b\,\zeta_i$. Notice that $\zeta_0 = 0$, so the degradation rate at normal using condition is the same as parameter a, i.e., $\mu_0 = a$.

Using Equations (6), (7) and (9) (let $\mathcal{A}_i^2$ be replaced by W_i^2), the estimates of the model parameters $(a, b, \sigma^2, \omega^2)$ are given by $\widetilde{a} = 2.1744 \times 10^{-4}, \widetilde{b} = 1.6 \times 10^{-3}, \widetilde{\sigma^2} = 6.5151 \times 10^{-4}$, $\widetilde{\omega^2} = 7.2874 \times 10^{-8}$, respectively. The estimates of $\mu_i, i = 1, 2, 3$, are given by $\widetilde{\mu_1} = 2.0 \times 10^{-4}, \widetilde{\mu_2} = 3.0 \times 10^{-4}$ and $\widetilde{\mu_3} = 1.9 \times 10^{-3}$, respectively. Utilize the proposed method in Section 4, based on the real degradation data, the GCIs/GPIs, LCLs and UCLs of the model parameters and some quantities are given in Table 7.

Table 7. The GCIs (GPIs), GLCL and GUCL of model parameters and some quantities.

Parameter	Level	GCI/GPI	GLCL	GUCL
a	90%	$(0.0022, 0.4338) \times 10^{-3}$	0.0514×10^{-3}	0.3833×10^{-3}
	95%	$(-0.0418, 0.4790) \times 10^{-3}$	0.0022×10^{-3}	0.4338×10^{-3}
b	90%	$(1.2000, 2.1000) \times 10^{-3}$	1.3000×10^{-3}	2.0000×10^{-3}
	95%	$(1.1000, 2.2000) \times 10^{-3}$	1.2000×10^{-3}	2.1000×10^{-3}
σ^2	90%	$(0.5173, 0.8519) \times 10^{-3}$	0.5445×10^{-3}	0.8036×10^{-3}
	95%	$(0.4943, 0.9004) \times 10^{-3}$	0.5173×10^{-3}	0.8519×10^{-3}
ω^2	90%	$(0.0197, 0.1891) \times 10^{-6}$	0.0313×10^{-6}	0.1590×10^{-6}
	95%	$(0.0101, 0.2188) \times 10^{-6}$	0.0197×10^{-6}	0.1891×10^{-6}
$R(5000)$	90%	(0.8393, 0.9797)	0.8673	0.9734
	95%	(0.8117, 0.9839)	0.8393	0.9797
MTTF	90%	$(1.8784, 6.0114) \times 10^4$	2.1982×10^4	5.4981×10^4
	95%	$(1.6418, 6.4548) \times 10^4$	1.8784×10^4	6.0114×10^4
$X_0(10,000)$	90%	$(-4.6179, 9.0402)$	-3.0454	7.4440
	95%	$(-5.9605, 10.5122)$	-4.6179	9.0402

Based on the proposed degradation model, the real data are analyzed completely, using the proposed GPQ method. The point estimations and generalized interval estimations of the model parameters are given. In addition, the interval estimations of the reliability function and MTTF of the carbon-film resistor are obtained, and the generalized prediction interval of the resistance degradation of the carbon-film resistor under normal operating conditions is also obtained. It can be seen from the calculation results that these interval estimations match well with the corresponding point estimations.

5. Conclusions

In this paper, we consider a CSADT model of the Wiener process with random effects. In most cases, we mainly focus on the interval estimations of the model parameters and some quantities of interest. As is known to all, it is difficult to obtain the exact confidence intervals for them. Thus, we develop the GCIs of the model parameters and some quantities, such as the reliability function, the MTTF and the degradation rate at normal using condition, by constructing a GPQ. In addition, the GPIs of the degradation characteristic at the normal using condition are derived. The simulations are conducted to investigate the performances of the proposed GCIs (GPIs) and GCLs in terms of the CP and AIL. The results reveal that the proposed GCIs (GPIs), GLCLs and GUCLs perform well in terms of the CP. The proposed interval estimation methods are successfully applied to the degradation data of carbon-film resistors.

In this study, we mainly focus on the Wiener CSADT model with a random drift parameter and fixed diffusion parameter. Although this model works well for many degradation data, in the literature, the Wiener process with two random effects (i.e., both the drift and diffusion parameters are considered as random variables) also appears. It will be a meaningful research direction for the future to develop interval estimation methods for the Wiener CSADT model with two random effects under a small sample size. Other stochastic processes, such as the gamma process and inverse Gaussian process, are also widely used in a degradation analysis. It will be a meaningful research direction for the future to develop interval estimation methods for the gamma and inverse Gaussian processes.

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