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Asymptotic Behavior of Solutions of the Cauchy Problem for a Hyperbolic Equation with Periodic Coefficients (Case: $H_0 > 0$)

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Abstract: The main goal of this article is to study the behavior of solutions of non-stationary problems at large timescales, namely, to obtain an asymptotic expansion characterizing the behavior of the solution of the Cauchy problem for a one-dimensional second-order hyperbolic equation with periodic coefficients at large values of the time parameter t. To obtain an asymptotic expansion as $t \to \infty$, the basic methods of the spectral theory of differential operators are used, as well as the properties of the spectrum of the Hill operator with periodic coefficients in the case when the operator is positive: $H_0 > 0$.

Keywords: asymptotic behavior of solutions; second-order hyperbolic equation; periodic coefficients; Cauchy problem; Hill operator

MSC: 35B10; 35B40; 35C20; 35L10; 35Q41

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1. Introduction

We study the behavior of the solution for $|x| < b$ and $t \to \infty$ of the following Cauchy problem:

$$u_{tt}(x,t) - (p(x)\, u_x(x,t))_x + q(x)\, u(x,t) = 0, \quad (x,t) \in \mathbb{R} \times \{t > 0\}, \tag{1}$$

$$u(x,t)|_{t=0} = 0, \quad u_t(x,t)|_{t=0} = f(x), \quad x \in \mathbb{R}, \tag{2}$$

where the functions $p(x)$ and $q(x)$ are periodic with period 1,

$$p(x+1) = p(x) \geq \text{const} > 0, \quad q(x+1) = q(x) \geq 0.$$

We also assume that the functions $p(x)$ and $q(x)$ are continuous or have a finite number of discontinuities of the first kind on the period, $f \in C_0^\infty(\mathbb{R})$, $\operatorname{supp} f \subset [0,1]$, b is an arbitrary fixed constant.

The behavior (as $t \to \infty$) of solutions to problems similar to the problem (1) and (2) with $p(x) = 1$, and of the corresponding multidimensional problems under the condition that the potential differs from a constant by a finite function tends to a constant sufficiently fast at infinity, has been studied in many papers, see, for example, [1], and the bibliography there, as well as other papers.

In this regard, we note the paper [2], in which was received the asymptotic expansion (for $t \to \infty$ and $|x| < a < \infty$) of the solution $u(x,t)$ of the following Cauchy problem:

$$u_{tt}(x,t) - u_{xx}(x,t) + (\alpha_0 + q_0(x))u = 0, \quad (x,t) \in \mathbb{R} \times \{t > 0\},$$

$$u(x,t)|_{t=0} = \varphi(x), \quad u_t(x,t)|_{t=0} = \psi(x), \quad x \in \mathbb{R},$$

where the initial functions are finite, $\varphi(x) \in C^2(\mathbb{R})$, $\psi(x) \in C^1(\mathbb{R})$, and under weaker restrictions on the potential $q(x) = \alpha_0 + q_0(x)$, where $\alpha_0 = \text{const}$ and $q_0(x)$ is a real-valued continuous function for all $x \in \mathbb{R}$, and for some $k \geq 1$, satisfies the condition:

$$\int_{-\infty}^{+\infty} |x|^k |q_0(x)| < \infty.$$

In [3] studied the following Cauchy problem:

$$u_{tt}(x,t) - (a(x)\, u_x(x,t))_x = 0, \quad 0 < a_0 \leq a(x) \leq A_0 < +\infty, \quad (x,t) \in \mathbb{R} \times \{t > 0\},$$

$$u(x,t)|_{t=0} = \varphi(x), \quad u_t(x,t)|_{t=0} = 0, \quad x \in \mathbb{R},$$

for which, under certain assumptions on the tension coefficient $a(x)$, such as:

$$\frac{1}{a_0} \int_{-\infty}^{+\infty} |a'(x)|\, dx < 1,$$

sufficient conditions for the stabilization of the solution $u(x,t)$ as $t \to +\infty$ uniformly in x on any compact set, as well as necessary and sufficient conditions for the stabilization of the solution $u(x,t)$ in the mean where obtained.

In the note [4], it is proved that a perturbed Hill operator with an exponentially decreasing impurity potential has a resonance (or an odd number of resonances) in every sufficiently distant lacuna on the second (“non-physical”) sheet.

In [5], the problem of scattering by a one-dimensional periodic lattice $p(x)$ with impurity potential $q(x)$ is considered. Based on the asymptotics of scattered waves, a stationary scattering matrix is constructed, its properties are studied, and it is shown that it coincides with the non-stationary scattering operator defined in the usual way in the quasi-momentum representation of the unperturbed operator H_0. The inverse scattering problem is also solved, i.e., the problem of recovering $q(x)$ based on $p(x)$ and the scattering data. Here the author follows the scheme proposed in the paper by V. A. Marchenko and L. D. Faddeev. However, to solve the inverse problem in the presence of a periodic lattice, it requires significant modifications of classical arguments. The theory of so-called “global” quasi-momentum serves as analytic basis. In this article, conditions on the scattering data are also found, necessary with a finite second moment, and sufficient for the existence of a unique impurity potential with given scattering characteristics and a finite first moment.

One of the key papers is [6], in which the large-time asymptotic behavior of the Green’s function for the one-dimensional diffusion equation is found in two cases. In the first case, when the potential is a function with compact support, the asymptotic behavior of the Green’s function is expressed in terms of the elements of the scattering matrix of the corresponding Schrödinger operator for negative energies on the spectral plane. In the second case, when the potential is a periodic function of the coordinates, the asymptotic behavior can be expressed in terms of the Floquet–Bloch functions of the corresponding Hill operator for negative energy values on the spectral plane. The results obtained are used to study diffusion in layered media at long times. The case of external force is also considered. In the periodic case, the Seeley coefficients are found.

In [7], the behavior at large time t of the solution of the Cauchy problem for a hyperbolic equation with a periodic potential $q(x)$ was studied.

The main difference of this article from the above papers is that the case of periodic coefficients $p(x)$ and $q(x)$ is considered here. Papers where the coefficients $p(x)$ and $q(x)$ are periodic were not known until our investigations [8,9] appeared, in which the main results are published in the form of short communications. In the same papers, the behavior of the solution of the Cauchy problem for both homogeneous and inhomogeneous hyperbolic equations, as well as the behavior of the solution of a mixed initial-boundary value problem for the same equations, are studied.

Paper [10] deals with the numerical study of the simple one-dimensional Schrödinger operator $-\frac{1}{2}\partial_{xx} + \alpha q(x)$ with $\alpha \in \mathbb{R}$, $q(x) = \cos(x) + \varepsilon \cos(kx)$, $\varepsilon > 0$ and k is irrational. Here the quantum wave function of an independent electron in a crystal lattice perturbed by some impurities is determined, the dissemination of which induces only a long-range order, which is transmitted using a quasi-periodic potential q. Here the author numerically studies what happens for various values of k and ε, and it turns out that for $k > 1$ and $\varepsilon << 1$, that is, when more than one impurity appears inside an elementary cell of the original lattice, "impurity bands" appear, which seem to be k -periodic. When ε grows bigger than one, the opposite case occurs.

We also note paper [11], which investigates a simple one-dimensional model of an incommensurable "harmonic crystal" in terms of the spectrum of the corresponding Schrödinger equation. The paper shows that the lower spectrum of the operator is divided between "Cantor-like bands" and "impurity bands", which correspond to critical and extended eigenstates, respectively. Numerical experiments were also carried out, which are performed both for stationary and non-stationary problems.

The spectral properties of the Hill operator were studied in [4–6,12–16].

In this article, we present the full proofs of the results announced in [8], which were also presented at the international conference in Cyprus [17] (ICMSQUARE 6: International Conference on Mathematical Modeling in Physical Sciences, 25–29 August, 2017, Pafos, Cyprus).

Let us describe the scheme of investigation of the Cauchy problem (1) and (2). Using the Fourier transform, we reduce the Cauchy problem under consideration to a stationary problem, then we write the solution in terms of the resolvent of the Hill operator and do the inverse Fourier transform. In the resulting integral, we shift the integration contour to the lower half-plane, bypassing the branch points of the resolvent of the Hill operator (these points are at the ends of the spectrum zones), and find the asymptotics of the resulting integral.

Notations: $C_0^\infty(\Omega)$ is the space of infinitely differentiable functions in the domain Ω and compactly supported in Ω; $L^2(\Omega)$ is the space of measurable functions in Ω for which

$$||u; L^2(\Omega)|| = \left(\int_\Omega |u|^2 dx\right)^{1/2} < \infty.$$

The Sobolev space $H^1(\Omega)$ in Ω is defined as:

$$H^1(\Omega) = \{u : u \in L^2(\Omega), \nabla u \in L^2(\Omega)\},$$

provided with the norm

$$||u; H^1(\Omega)||^2 = ||u; L^2(\Omega)||^2 + ||\nabla u; L^2(\Omega)||^2.$$

2. Definitions and Auxiliary Statements

Definition 1. *A function $u \in C^2(\mathbb{R} \times \{t \geq 0\})$ is called a periodic (anti-periodic) solution of the Cauchy problem* (1) *and* (2)*, if it satisfies the relation:*

$$u(x+1,t) = (-1)^j u(x,t)$$

for any $x \in \mathbb{R}$ and $t \geq 0$, with $j = 0$ and $j = 1$ in the case of periodic and anti-periodic solutions, respectively.

Spectrum and Green's Function of the Hill Operator

Continuing the function $u(x,t)$ by zero in the region $t < 0$, and applying the Fourier transform with respect to the variable t in the Cauchy problem (1) and (2), for the function

$$y(x,k) = \int_0^\infty u(x,t)\, e^{ikt} dt,$$

we obtain the equation

$$(p(x)\, y'(x,k))' + (k^2 - q(x))\, y(x,k) = -f(x). \tag{3}$$

For any function $g(x)$ from $L^2(-\infty,+\infty)$, we define its norm in the same space

$$||g; L^2|| = ||g; L^2(-\infty,+\infty)||.$$

If the function $g(x)$ is defined on the entire axis $(-\infty,+\infty)$, then by $\hat{g}(x)$ we denote the restriction of this function on the segment $[0,1]$.

Let us present some necessary facts from the spectral theory of differential equations. For any function $g(x,k)$ we denote by g' the derivative with respect to x and by g_k the derivative with respect to k

Let $\{y = \theta(x,k),\, y = \varphi(x,k)\}$ be the fundamental system of solutions of the homogeneous (for $f(x) \equiv 0$) Equation (3) such that:

$$\begin{cases} \theta(0,k) = 1, & \theta'(0,k) = 0, \\ \varphi(0,k) = 0, & \varphi'(0,k) = 1. \end{cases} \tag{4}$$

It is known [16] that $\theta(x,k)$ and $\varphi(x,k)$ are entire functions in k real on the real axis, and for $|k| \to \infty$ have the form:

$$\begin{cases} \theta(x,k) = & \cos kx + O(|k|^{-1} e^{|\tau| x}), \\ \varphi(x,k) = & \frac{1}{k}\sin kx + O(|k|^{-2} e^{|\tau| x}), \quad \tau = Im\, k, \end{cases} \tag{5}$$

uniformly in $x \in [-b,b]$. These expansions can be differentiated with respect to x and with respect to k.

Let us denote $\theta(k) = \theta(1,k)$, $\theta'(k) = \theta'(1,k)$, $\varphi(k) = \varphi(1,k)$, $\varphi'(k) = \varphi'(1,k)$ and $F(k) \equiv \theta(k) + \varphi'(k)$. The functions $\theta(k)$, $\theta'(k)$, $\varphi(k)$, $\varphi'(k)$ and $F(k)$ are even on the real axis of the complex plane of the variable k.

The Hill operator is the minimal closed differential operator,

$$H_0 := -\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x),$$

generated in the Hilbert space $L^2(\mathbb{R})$ by the operation

$$\Lambda_0 y := -(p(x)\, y')' + q(x)\, y,$$

where the functions $p(x)$ and $q(x)$ are periodic with period 1.

The spectrum $\sigma(H_0)$ of the Hill operator H_0 is absolutely continuous and is a finite or infinite sequence of isolated segments (zones) separated by lacunae going to infinity.

Note that the Hill operator has only a continuous spectrum, which lies on the real axis and is left semi-bounded [16]. Let us replace the spectral parameter λ by k^2 so that the spectrum $\sigma(H_0)$ of the operator H_0 on the complex plane of the variable k consists of points for which $H_0 - k^2$ does not have bounded inverse on an everywhere dense set in $L^2(\mathbb{R})$.

For a more detailed characterization of the spectrum $\sigma(H_0)$ of the Hill operator H_0, consider the following periodic (anti-periodic) Sturm–Liouville problems.

Let $\hat{v}(x,\lambda_n)$ be an eigenfunction of the periodic Sturm–Liouville problem:

$$\begin{gathered} -(p(x)y')' + q(x)y = \lambda_n y, \quad x \in [0,1], \\ y(0) = y(1), \quad y'(0) = y'(1), \end{gathered} \tag{6}$$

normalized by the condition $||\hat{v}; L^2([0,1])|| = 1$, and $\hat{v}(x,\mu_n)$ is the eigenfunction of the anti-periodic Sturm–Liouville problem:

$$\begin{aligned} -(p(x)y')' + q(x)y = \mu_n y, \quad x \in [0,1], \\ y(0) = -y(1), \quad y'(0) = -y'(1), \end{aligned} \tag{7}$$

normalized in $L^2([0,1])$, where λ_n and $\mu_n,\ n = 0,1,2,\ldots$, are eigenvalues of the corresponding problems, which are numbered in ascending order, taking into account the multiplicity.

Continuing the function $\hat{v}(x,\lambda_n)$ (or $\hat{v}(x,\mu_n)$) to the entire real axis, in a periodic (or anti-periodic) way, we get a function, which we denote by $v(x,\lambda_n)$ (or $v(x,\mu_n)$).

It is known ([16], § 21.4) that if the Hill operator H_0 is positive, then all eigenvalues of the periodic (anti-periodic) Sturm–Liouville problem are positive. In addition, between the numbers λ_n and $\mu_n,\ n = 0,1,2,\ldots$ there is a relation,

$$\lambda_0 < \mu_0 \le \mu_1 < \lambda_1 \le \lambda_2 < \mu_2 \le \mu_3 < \lambda_3 \le, \ldots \tag{8}$$

Based on the results of the paper [12], we can state that:

(i) The set M_1 is a union of segments on the real axis, extending in both directions to infinity

$$[-\lambda_{2n+1}, -\mu_{2n+1}], [-\mu_{2n}, -\lambda_{2n}], [\lambda_{2n}, \mu_{2n}], [\mu_{2n+1}, \lambda_{2n+1}], n = 0,1,2.\ldots;$$

(ii) The set M_2 consists of those values λ_n for which the homogeneous (for $f(x) \equiv 0$) Equation (3) has a bounded solution in $-\infty < x < +\infty$;

(iii) The set M_3 consists of those values λ_n (or k) for which $|F(k)| \le 2$.

Hence [12], if the Hill operator H_0 is positive, then the spectrum of $\sigma(H_0)$ coincides with the sets M_1, M_2, M_3, i.e.,

$$\sigma(H_0) = M_1 = M_2 = M_3.$$

The set of points $\pm\lambda_n$ coincides with the set of roots of the equation $F(k) = 2$ (correspondingly, $\pm\mu_n$ coincides with the set of roots of the equation $F(k) = -2$), $n = 0,1,2,\ldots$.

Gaps in the spectrum, that is, intervals not included in the spectrum,

$$(-\mu_{2n+1}, -\mu_{2n}), (-\lambda_{2n+2}, -\lambda_{2n+1}), (\mu_{2n}, \mu_{2n+1}), (\lambda_{2n+1}, \lambda_{2n+2}), n = 0,1,2.\ldots,$$

for which $\mu_{2n} \ne \mu_{2n+1}, \lambda_{2n+1} \ne \lambda_{2n+2}$, are called lacunae.

If λ_n (or μ_n) are ends of a lacunae, then (8) implies that $\pm\lambda_n$ are simple roots of the equation $F(k) = 2$ (or $\pm\mu_n$ are the roots of the equation $F(k) = -2$), $n = 0,1,2,\ldots$, ([12]).

As is known [16], if λ_n (or μ_n) are the ends of a lacuna, then λ_n (or μ_n) are simple proper values of the periodic (or anti-periodic) Sturm–Liouville problem (6) (or (7)).

Note that each lacuna contains exactly one simple zero of the function $F_k(k)$, and the functions $\varphi(k)$ and $\theta'(k)$ have one simple zero in the closure of each lacuna.

If $\lambda = \lambda_{2n} = \lambda_{2n+1}$ (or $\mu = \mu_{2n} = \mu_{2n+1}$), $n \ge 0$, then λ (or μ) is the simple zero of the functions $\varphi(k)$ and $\theta'(k)$ ([16]).

Denote by $\mathbb{C}'$ the complex plane of the variable k with cuts along the vertical rays lying in the lower half-plane and starting at the ends of the lacunae.

Let us put

$$m_1(k) = \frac{\varphi'(k) - \theta(k)}{2\varphi(k)} + \frac{\sqrt{(\theta(k) + \varphi'(k))^2 - 4}}{2\varphi(k)}, \quad k \in \mathbb{C}',$$

$$m_2(k) = \frac{\varphi'(k) - \theta(k)}{2\varphi(k)} - \frac{\sqrt{(\theta(k) + \varphi'(k))^2 - 4}}{2\varphi(k)}, \quad k \in \mathbb{C}',$$

where the branch of the root is determined by the condition $\sqrt{F(k)^2 - 4} > 0$ for $k = 0$.

Note that the function $\sqrt{F(k)^2-4}$ has branching only at the ends of the lacunae [16], so $m_1(k)$ and $m_2(k)$ are single-valued in $\mathbb{C}'$. Then for any $k, Im\, k > 0$

$$\begin{aligned} \psi_1(x,k) \equiv &\quad \theta(x,k) + m_1(k)\,\varphi(x,k) \in L^2(-\infty,0), \\ \psi_2(x,k) \equiv &\quad \theta(x,k) + m_2(k)\,\varphi(x,k) \in L^2(0,+\infty). \end{aligned} \tag{9}$$

Define the Green function of the Equation (3) for k from the upper half-plane,

$$\Gamma(x,\xi,k) = \begin{cases} \frac{\psi_1(\xi,k)\psi_2(x,k)}{m_2(k)-m_1(k)} & \text{for} \quad \xi < x, \\ \frac{\psi_1(x,k)\psi_2(\xi,k)}{m_2(k)-m_1(k)} & \text{for} \quad \xi > x, \end{cases}$$

and, taking into account the identities (9) and the equality,

$$\theta(x,k)\,\varphi'(x,k) - \theta'(x,k)\,\varphi(x,k) = 1,\ x \in \mathbb{R}^1, \tag{10}$$

we get

$$\Gamma(x,\xi,k) = \begin{cases} -\frac{h(x,\xi,k)}{\sqrt{F(k)^2-4}} + \frac{1}{2}(\theta(\xi,k)\,\varphi(x,k) - \theta(x,k)\,\varphi(\xi,k)) & \text{for} \quad \xi < x, \\ -\frac{h(x,\xi,k)}{\sqrt{F(k)^2-4}} + \frac{1}{2}(\theta(x,k)\,\varphi(\xi,k) - \theta(\xi,k)\,\varphi(x,k)) & \text{for} \quad \xi > x, \end{cases} \tag{11}$$

where

$$\begin{aligned} h(x,\xi,k) = \varphi(k)\,\theta(x,k)\,\theta(\xi,k) - \theta'(k)\,\varphi(\xi,k)\varphi(x,k) + \\ + \frac{\varphi'(k)-\theta(k)}{2}(\theta(\xi,k)\,\varphi(x,k) + \theta(x,k)\,\varphi(\xi,k)). \end{aligned} \tag{12}$$

The solution of the Equation (3) for $Im\, k > 0$ using the Green's function is defined as:

$$y(x,t) = -\int_0^1 \Gamma(x,\xi,k)\,f(\xi)\,d\xi,$$

and the solution to the problem (1) and (2) has the form:

$$u(x,t) = -\frac{1}{2\pi}\int_{Im\, k=a}\int_0^1 \Gamma(x,\xi,k)\,f(\xi)\,e^{-ikt}d\xi\,dk, \tag{13}$$

where a is some positive constant.

Note that the Green's function $\Gamma(x,\xi,k)$ for every $x,\xi \in [-b,b]$ continues analytically to $\mathbb{C}'$.

To study the properties of the integral (13), we introduce some notation.

Denote by L_+ (and L_-) the line $Im\, k = a,\ a > 0$ (and $Im\, k = -d,\ d > 0$) , and q_l is the segment $Re\, k = l\pi + \frac{\pi}{3}, -d \le Im\, k \le a$, l is any real number .

Consider the integral,

$$J_1 \equiv -\int_{L+}\int_0^x (\theta(\xi,k)\,\varphi(x,k) - \theta(x,k)\,\varphi(\xi,k))\,f(\xi)\,e^{-ikt}d\xi\,dk, \quad x \in [-b,b]. \tag{14}$$

From the relations (5), it follows that:

$$\int_{q_l}\int_0^x (\theta(\xi,k)\,\varphi(x,k) - \theta(x,k)\,\varphi(\xi,k))\,f(\xi)\,e^{-ikt}d\xi\,dk \to 0 \quad \text{as} \quad |l| \to \infty,$$

moreover, $|l|$ can tend to infinity in any way, so in (14) one can replace the line L_+ by L_-. According to (5), we have:

$$\theta(\xi,k)\,\varphi(x,k) - \theta(x,k)\,\varphi(\xi,k) = S_1(x,\xi,k) + S_2(x,\xi,k),$$

where

$$S_1(x,\xi,k) = \frac{1}{k}\cos k\xi\,\sin kx - \frac{1}{k}\cos kx\,\sin k\xi$$

is an entire function $k \in \mathbb{C}'$ for each $x,\xi \in [-b,b]$, and the function $S_2(x,\xi,k)$ for $k \to \infty$ uniformly in $x,\xi \in [-b,b]$ has the form:

$$S_2(x,\xi,k) = O\Big(|k|^{-2}e^{|\tau|(x+\xi)}\Big).$$

Thus,

$$J_1 = J_1^{(1)} + J_1^{(2)} + J_1^{(3)},$$

where

$$J_1^{(1)} = -\int_{L_-}\int_0^x \frac{1}{k}\cos k\xi\,\sin kx\, f(\xi)\,e^{-ikt}d\xi\,dk, \quad J_1^{(2)} = \int_{L_-}\int_0^x \frac{1}{k}\cos kx\,\sin k\xi\, f(\xi)\,e^{-ikt}d\xi\,dk,$$

and

$$J_1^{(3)} = -\int_{L_-}\int_0^x S_2(x,\xi,k)\,f(\xi)\,e^{-ikt}d\xi\,dk.$$

Let us explore these integrals. Putting $k = \sigma - id$ with $k \in L_-$, we get:

$$J_1^{(1)} = -\int_{-\infty}^{+\infty}\frac{1}{\sigma - id}\sin(\sigma - id)x\,e^{-i\sigma t}e^{-dt}\Phi(\sigma,x)\,d\sigma, \quad x \in [-b,b], \tag{15}$$

where

$$\Phi(\sigma,x) \equiv \int_0^x \cos(\sigma - id)\xi\, f(\xi)\,d\xi = \frac{1}{2}\int_0^x e^{i\sigma\xi}e^{d\xi}f(\xi)\,d\xi + \frac{1}{2}\int_0^x e^{-i\sigma\xi}e^{-d\xi}f(\xi)\,d\xi. \tag{16}$$

Let us examine the first term in (16). Consider the function,

$$w(x,\xi) = \begin{cases} e^{d\xi}f(\xi) & \text{for} \quad \xi < x, \\ 0 & \text{for} \quad \xi > x. \end{cases}$$

For any fixed $x \in [-b,b]$, we have $w \in L^2(-\infty,+\infty)$ and

$$||w;L^2|| = \left(\int_0^x e^{2d\xi}f^2(\xi)\,d\xi\right)^{1/2} \le \left(\int_0^1 e^{2d\xi}f^2(\xi)\,d\xi\right)^{1/2} \le C_1||f;L^2||,$$

where C_1 does not depend on f and x.

For all $x \in [-b,b]$, due to the Parseval equality for the Fourier transform, we have:

$$||\int_0^x e^{i\sigma\xi}e^{d\xi}f(\xi)\,d\xi; L^2(\mathbb{R}^1_\sigma)|| = \sqrt{2\pi}||w;L^2(\mathbb{R}^1_\xi)|| \le C_1\sqrt{2\pi}||f;L^2||.$$

The second term of the equality (16) is studied in a similar way. Therefore, for any fixed $x \in [-b,b]$,

$$||\Phi(\sigma,x);L^2(\mathbb{R}^1_\sigma)|| \le C_2||f;L^2||,$$

where C_2 does not depend on f and x.

By the Cauchy–Bunyakovskii–Schwartz inequality and the last inequality, from (15) we obtain:

$$|J_1^{(1)}| \le C_3e^{-td}||f;L^2||,$$

where C_3 depends only on b.

In the same way we get:

$$|J_1^{(2)}| \le C_4e^{-td}||f;L^2||,$$

where C_4 depends only on b.

To investigate $J_1^{(3)}$, we note that:

$$J_1^{(3)} = -\int_{L_-}\int_0^x S_2(x,\xi,k)\, f(\xi)\, e^{-ikt} d\xi\, dk =$$

$$= -\int_{-\infty}^{+\infty} \frac{1}{\sigma - id}\, e^{-i\sigma\, t} e^{-dt} \left(\int_0^x f(\xi) O\left(\frac{e^{d(x+\xi)}}{|\sigma - id|}\right) d\xi\right) d\sigma.$$

It is easy to show that:

$$\left|\int_0^x f(\xi) O\left(\frac{e^{d(x+\xi)}}{|\sigma - id|}\right) d\xi\right|^2 \le \frac{C}{|\sigma - id|^2} ||f; L^2||.$$

By the Cauchy–Bunyakovskii–Schwartz inequality, we obtain the estimate:

$$|J_1^{(3)}| \le C_5 e^{-td} ||f; L^2||,$$

where C_5 depends only on b.

From the estimates for $J_1^{(1)}$, $J_1^{(2)}$, and $J_1^{(3)}$, it follows that:

$$|J_1| \le C(b) e^{-td} ||f; L^2||. \tag{17}$$

Likewise, for the integral,

$$J_2 \equiv -\int_{L+}\int_x^1 (\theta(x,k)\, \varphi(\xi,k) - \theta(\xi,k)\, \varphi(x,k))\, f(\xi)\, e^{-ikt} d\xi\, dk, \quad x \in [-b, b],$$

we get the estimate

$$|J_2| \le C(b) e^{-td} ||f; L^2||. \tag{18}$$

Thus, we get that the integrals J_1 and J_2 decrease exponentially as $t \to \infty$.

From the point $k = p$ lying on the real axis, let us make a vertical cut into the lower half-plane of the variable k.

Denote by l_p the contour going from the point $p - id$ along the left edge of this cut to the point p, and then from the point p along the right edge of the cut to the point $p - id$, $d > 0$.

On the plane $\mathbb{C}'$, consider the contour L, which can be represented as:

$$L = L_1 \cup L_2 \cup L_3, \tag{19}$$

where

$$L_1 = \left(\bigcup_{n=0}^{\infty} l_{\lambda_n}\right) \bigcup \left(\bigcup_{n=0}^{\infty} l_{-\lambda_n}\right), \quad L_2 = \left(\bigcup_{n=0}^{\infty} l_{\mu_n}\right) \bigcup \left(\bigcup_{n=0}^{\infty} l_{-\mu_n}\right),$$

and

$$L_3 = L_- \cap \mathbb{C}',$$

moreover, if $\lambda_{j+1} = \lambda_j$ (or $\mu_{j+1} = \mu_j$) for some non-negative integer j, then these unions do not include l_{λ_j}, $l_{\lambda_{j+1}}$, $l_{-\lambda_j}$, $l_{-\lambda_{j+1}}$ (respectively, l_{μ_j}, $l_{\mu_{j+1}}$, $l_{-\mu_j}$, $l_{-\mu_{j+1}}$).

Let δ be some finite contour in $\mathbb{C}'$. Denote by J_δ the integral

$$J_\delta = \int_\delta \int_0^1 \frac{h(x,\xi,k)}{\sqrt{(\theta(k) + \varphi'(k))^2 - 4}} f(\xi)\, e^{-ikt} d\xi dk, \quad x \in [-b, b].$$

Now let the contour Δ be unbounded. Let us put

$$J_\Delta = \lim_{j\to\infty} J_{\delta\cap\{k:\,|Rek|\le\pi j+\frac{\pi}{2}\}}, \quad j \in N. \tag{20}$$

Proposition 1. *For the solution of the problem* (1) *and* (2)*, the following representation is valid:*

$$u(x,t) = \frac{1}{2\pi} J_L + v_1(x,t),$$

where the function $v_1(x,t)$ for $x \in [-b,b]$ and $t > 0$ satisfies the estimate:

$$|v_1(x,t)| \le C(b)\, e^{-td}||f; L^2||. \tag{21}$$

Proof. From the Formulas (11) and (13), and the estimates (17) and (18), it follows that

$$u(x,t) = \frac{1}{2\pi} J_{L_+} + v_1(x,t),$$

where the estimate (21) is valid for the function v_1.

To prove the assertion, it remains to show that:

$$J_{q_j} \to 0 \quad \text{as} \quad |j| \to \infty,\ j \in N.$$

From (5) it follows that for $|k| \to \infty$

$$\sqrt{(\theta(k) + \varphi'(k))^2 - 4} = 2\sqrt{-\sin^2 k + O\Big(|k|^{-1} e^{2|\tau|}\Big)} \tag{22}$$

Since $|\sin k| > C_1 > 0$ for $k \in q_j$, $j = 0, \pm1, \pm2, \dots$, then (22) implies that for sufficiently large $|j|$, $|j| \in N$,

$$\left|\sqrt{(\theta(k) + \varphi'(k))^2 - 4}\right| \ge C_2,\ k \in q_j. \tag{23}$$

It follows from (5), (13) and (23) that the modulus of the integrand in the integral J_{q_j} for sufficiently large $|j|$ does not exceed $C\,|k|^{-1} e^{dt}$. Therefore, for any fixed $t > 0$, we get:

$$|J_{q_j}| \le \frac{C}{|\pi j + \frac{\pi}{2}|} \to 0 \quad \text{as} \quad |j| \to \infty,\ |j| \in N.$$

□

Let us pass to the investigation of the integral $J_L = J_{L_1} + J_{L_2} + J_{L_3}$.

Proposition 2. *For any $t > 0$ and $x \in [-b,b]$ we have the estimate:*

$$|J_{L_3}| \le C(b)\, e^{-td}||f; L^2||.$$

Proof. Since there exists $C_1 > 0$ such that $|\sin\ k| \ge C_1 > 0$ for $k \in L_3$, and function $F^2(k) - 4 \equiv (\theta(k) + \varphi'(k))^2 - 4$ has zeros only on the real axis, then (22) implies:

$$\left|\sqrt{(\theta(k) + \varphi'(k))^2 - 4}\right| \ge C_2 > 0 \quad \text{for} \quad k \in L_3.$$

We represent the function $h(x,\xi,k)$ as

$$h(x,\xi,k) = g_1(x,\xi,k) + g_2(x,\xi,k),$$

where $g_1(x,\xi,k)$ is an entire function $k \in \mathbb{C}'$ for every $x,\xi \in [-b,b]$, and

$$g_1(x,\xi,k) = \frac{1}{k}\sin k\,(\cos kx\,\cos k\xi + \sin kx\,\sin k\xi),$$

and the function $g_2(x,\xi,k)$ as $|k| \to \infty$ uniformly with respect to $x,\xi \in [-b,b]$ has the form

$$g_2(x,\xi,k) = O\Big(|k|^{-2}e^{|\tau|(x+\xi+1)}\Big).$$

It is easy to see that for $k = \sigma - i\,d$,

$$|J_{L_3}| \le \int_{-\infty}^{\infty}\left|\int_0^1 \frac{h(x,\xi,\sigma-id)}{\sqrt{(\theta(\sigma-id)+\varphi'(\sigma-id))^2-4}}\,f(\xi)\,e^{-i\sigma t}e^{-dt}d\xi\right|d\sigma.$$

Further, arguing in the same way as when obtaining an estimate for the integral J_1, we can verify the validity of Proposition 2. □

Before turning to the investigation of the integrals J_{L_1} and J_{L_2}, we prove some auxiliary statements.

Denote by $B(a)$ the circle $B(a) = \{k : |k-\pi a| \le \frac{\pi}{4}\}$.

There exists $n_1 \in N$ such that, for $n > n_1$, the following representations

$$\begin{gathered}\lambda_{2n-1} = 2n\pi + O\Big(\tfrac{1}{n}\Big), \quad \lambda_{2n} = 2n\pi + O\Big(\tfrac{1}{n}\Big),\\ \mu_{2n+1} = (2n+1)\pi + O\Big(\tfrac{1}{n}\Big), \quad \mu_{2n+1} = (2n+1)\pi + O\Big(\tfrac{1}{n}\Big),\end{gathered} \tag{24}$$

hold (see, [13,16,18]).

Let us choose a number $d > 0$ involved in the definition of contours L_-, $l_{\pm\lambda_i}$, $l_{\pm\mu_i}$ less than $\frac{\pi}{4}$. Then, (24) implies that there exists $n_2 > n_1, n_2 \in N$ such that for $n > n_2$ the contours $l_{\lambda_{2n}}$, $l_{\lambda_{2n-1}}$, (and $l_{\mu_{2n}}$, $l_{\mu_{2n+1}}$) belong to the circle $B(2n)$ (corresponding to the circle $B(2n+1)$).

It is obvious that the contours $l_{-\lambda_{2n}}$, $l_{-\lambda_{2n-1}}$ (and $l_{-\mu_{2n}}$, $l_{-\mu_{2n+1}}$) belong to the circle $B(-2n)$ (corresponding to the circle $B(-(2n+1))$).

Let us denote $G(k) \equiv (\theta(k)+\varphi'(k))^2 - 4$.

Proposition 3. *The following equalities,*

$$\begin{cases} G(k) = (k-\lambda_{2m-1})(k-\lambda_{2m})\,g_{2m}(k),\ |g_{2m}(k)| \ge C_{2m} > 0 \quad \textit{for} \quad k \in l_{\lambda_{2m-1}} \cup l_{\lambda_{2m}},\\ G(k) = (k+\lambda_{2m+1})(k+\lambda_{2m})\,g_{-2m}(k),\ |g_{-2m}(k)| \ge C_{-2m} > 0 \quad \textit{for} \quad k \in l_{-\lambda_{2m-1}} \cup l_{-\lambda_{2m}},\\ m = 1,2,3,\dots, \end{cases}$$

$$G(k) = (k-\lambda_0)(k+\lambda_0)g_0(k),\ g_0(k) \ge C_0 > 0 \quad \textit{for} \quad k \in l_{\lambda_0} \cup l_{-\lambda_0},$$

$$\begin{cases} G(k) = (k-\mu_{2m})(k-\mu_{2m+1})\,g_{2m+1}(k),\ |g_{2m+1}(k)| \ge C_{2m+1} > 0 \quad \textit{for} \quad k \in l_{\mu_{2m}} \cup l_{\mu_{2m+1}},\\ G(k) = (k+\mu_{2m})(k+\mu_{2m+1})\,g_{-(2m+1)}(k),\ |g_{-(2m+1)}(k)| \ge C_{-(2m+1)} > 0 \quad \textit{for} \quad k \in l_{-\mu_{2m}} \cup l_{-\mu_{2m+1}},\\ m = 0,1,2,\dots. \end{cases}$$

are satisfied, where the constants $C_{\pm 2m}$ and $C_{\pm(2m+1)}$ depend only on m.

Proof. The validity of the first of the equalities follows from the fact that the entire function $G(k)$ on the contours $l_{\lambda_{2m}}$ and $l_{\lambda_{2m-1}}$ has no other zeros, except for λ_{2m} and λ_{2m-1}.

The validity of the remaining equalities is proved similarly. □

Proposition 4. *For sufficiently large $n > n_2$, the following equalities,*

$$\begin{cases} G(k) = (k-\lambda_{2n-1})(k-\lambda_{2n})\, g_{2n}(k),\ |g_{2n}(k)| \geq C > 0 \quad \text{for} \quad k \in B(2n), \\ G(k) = (k+\lambda_{2n-1})(k+\lambda_{2n})\, g_{-2n}(k),\ |g_{-2n}(k)| \geq C > 0 \quad \text{for} \quad k \in B(-2n), \end{cases}$$

$$\begin{cases} G(k) = (k-\mu_{2n})(k-\mu_{2n+1})\, g_{2n+1}(k),\ |g_{2n+1}(k)| \geq C > 0 \quad \text{for} \quad k \in B(2n+1), \\ G(k) = (k+\mu_{2n})(k+\mu_{2n+1})\, g_{-(2n+1)}(k),\ |g_{-(2n+1)}(k)| \geq C > 0 \quad \text{for} \quad k \in B(-(2n+1)), \end{cases}$$

are satisfied, where the constant C does not depend on n.

Proof. Let us prove the first equality. The rest of the equalities are proved in a similar way.

By the definition of the number n_2, for $n > n_2$ the numbers λ_{2n} and λ_{2n-1} belong to the circle $B(2n)$, and the function $G(k)$ has no other zeros in this circle [16].

Hence it follows that the function $G(k)$ for $k \in B(2n)$ can be written in the form

$$G(k) = (k-\lambda_{2n-1})(k-\lambda_{2n})\, g_{2n}(k),$$

where $g_{2n}(k) \neq 0$ for $k \in B(2n)$.

In the circle $B(0) = \{k : |k| \leq \frac{\pi}{4}\}$ the function $h_0(k) = -\frac{4\sin^2 k}{k^2}$ has no zeros, and therefore there exists $C_1 > 0$ such that $|h_0(k)| > C_1$ for $k \in B(0)$.

After the change of variable $k = k' + 2n\pi$, $k' \in B(0)$, the functions $G(k)$ and $g_{2n}(k)$, become $G_{2n}(k') = G(k' + 2n\pi)$ and $\tilde{g}_{2n}(k') = g_{2n}(k' + 2n\pi)$.

From the Formula (5) it follows that:

$$G(k) = -4\sin^2 k + O\Big(|k|^{-1} e^{2|\tau|}\Big) \quad \text{as} \quad |k| \to \infty,$$

from here

$$G_{2n}(k') = -4\sin^2 k' + O\Big(n^{-1} e^{2|\tau|}\Big) \quad \text{as} \quad n \to \infty. \tag{25}$$

Further,

$$\tilde{g}_{2n}(k') = g_{2n}(k' + 2n\pi) = \frac{G(k' + 2n\pi)}{(k' + 2n\pi - \lambda_{2n-1})(k' + 2n\pi - \lambda_{2n})}.$$

From the Formulas (25) and (24), it follows that on the circle $|k'| = \frac{\pi}{4}$ the sequence $\tilde{g}_{2n}(k')$ tends uniformly to $h_0(k')$ as $n \to \infty$. □

Remark 1. *The functions $k\varphi(k)$ and $\frac{1}{k}\theta'(k)$ each have one simple zero in the segments $[\lambda_{2n-1}, \lambda_{2n}]$, $n \geq 1$ and $[\mu_{2n}, \mu_{2n+1}]$, $n \geq 0$. Therefore, just as in Statement 3, we can prove that for sufficiently large $n \in N$ in the circle $B(n)$, the equalities*

$$k\,\varphi(k) = (k - k'_n)\,\varphi_n(k) \quad \text{and} \quad \frac{1}{k}\,\theta'(k) = (k - k''_n)\,\theta'_n(k),$$

are satisfied, where k'_n and k''_n are the zeros of the functions $\varphi(k)$ and $\theta'(k)$, respectively, and $|\varphi_n(k)| \leq C$, $|\theta'_n(k)| \leq C$ for $k \in B(n)$.

As is known [16], if λ_n and μ_n are the ends of a lacuna, then $\lambda = \lambda_n$ is a simple eigenvalue of the periodic Sturm–Liouville problem:

$$\begin{gathered} -(p(x)\,y')' + q(x)\,y = \lambda y, \\ y(0) = y(1), \quad y'(0) = y'(1), \end{gathered} \tag{26}$$

and $\lambda = \mu_n$ is a simple eigenvalue of the anti-periodic Sturm–Liouville problem:

$$\begin{gathered} -(p(x)\,y')' + q(x)\,y = \lambda y, \\ y(0) = -y(1), \quad y'(0) = -y'(1). \end{gathered} \tag{27}$$

An eigenfunction corresponding to the eigenvalue λ_n, we will search in the form

$$\hat{v}(x,\lambda_n) = A\,\hat{\theta}(x,\lambda_n) + B\,\hat{\varphi}(x,\lambda_n).$$

Therefore, we get the following system:

$$\begin{cases} A\,(\theta(\lambda_n)-1) + B\,\varphi(\lambda_n) = 0, \\ A\,\theta'(\lambda_n) + B\,(\varphi'(\lambda_n)-1) = 0. \end{cases} \tag{28}$$

Since λ_n are simple eigenvalues of the problem, (26), then the determinant of the system (28) is equal to zero and all coefficients of the system do not vanish simultaneously. Together with the equality $F(k) \equiv \theta(k) + \varphi'(k) = 2$ for $k = \lambda_n$, which served as the definition of the numbers λ_n, this leads to the fact that at the points λ_n satisfies one of the following relations:

$$A_1)\ \theta(\lambda_n) \neq 1,\ \theta'(\lambda_n) \neq 0,\ \varphi(\lambda_n) \neq 0,\ \varphi'(\lambda_n) \neq 1;$$

$$A_2)\ \theta(\lambda_n) = 1,\ \theta'(\lambda_n) \neq 0,\ \varphi(\lambda_n) = 0,\ \varphi'(\lambda_n) = 1;$$

$$A_3)\ \theta(\lambda_n) = 1,\ \theta'(\lambda_n) = 0,\ \varphi(\lambda_n) \neq 0,\ \varphi'(\lambda_n) = 1.$$

Note that for any $x, \xi \in \mathbb{R}^1$, the functions $\theta(x,k)$, $\varphi(x,k)$, $h(x,\xi,k)$ and $F^2(k) - 4$ are even on the real axis of the complex plane of variable k.

Lemma 1. *For points $\pm\lambda_n$, $n = 0, 1, 2, \dots$, if λ_n are the ends of lacunae (that is, simple zeros of the function $F(k) - 2$), then the equalities,*

$$h(x,\xi,\pm\lambda_n) = C_{\lambda_n} v(x,\lambda_n)\, v(\xi,\lambda_n), \quad -b \leq x,\ \xi \leq b, \tag{29}$$

are satisfied, where the function $v(x,\lambda_n) = v(x,-\lambda_n)$ is the eigenfunction of the periodic Sturm–Liouville problem, and the numbers C_{λ_n} depending on the cases $A_1)$, $A_2)$, $A_3)$ have the form

$$\begin{array}{rl} A_1 : C_{\lambda_n} = & \varphi(\lambda_n)\int_0^1 \left(\theta(x,\lambda_n) + \frac{1-\theta(\lambda_n)}{\varphi(\lambda_n)}\,\varphi(x,\lambda_n)\right)^2 dx; \\ A_2 : C_{\lambda_n} = & -\theta'(\lambda_n)\int_0^1 (\varphi(x,\lambda_n))^2 dx; \\ A_3 : C_{\lambda_n} = & \varphi(\lambda_n)\int_0^1 (\theta(x,\lambda_n))^2 dx. \end{array}$$

Proof. If a function h depends on x and on ξ, then by $\hat{h}$ we denote its restriction on the square $0 \leq x,\ \xi \leq 1$.

Consider the case $A_1)$; the reasoning for the other cases is similar. From the system (28) we get:

$$\hat{v}(x,\lambda_n) = A\left(\hat{\theta}(x,\lambda_n) + \frac{1-\theta(\lambda_n)}{\varphi(\lambda_n)}\,\hat{\varphi}(x,\lambda_n)\right). \tag{30}$$

Because $||v; L^2([0,1]) = 1$, then (30) implies:

$$A = \frac{1}{\sqrt{\int_0^1 \left(\theta(x,\lambda_n) + \frac{1-\theta(\lambda_n)}{\varphi(\lambda_n)}\varphi(x,\lambda_n)\right)^2 dx}}. \tag{31}$$

Since $F(k) \equiv \varphi'(k) + \theta(k) = 2$ for $k = \lambda_n$, then we get:

$$1 - \theta(\lambda_n) = \frac{\varphi'(\lambda_n) + \theta(\lambda_n)}{2} - \theta(\lambda_n) = \frac{\varphi'(\lambda_n) - \theta(\lambda_n)}{2} \tag{32}$$

and

$$(1-\theta(\lambda_n))^2 = (1-\theta(\lambda_n))(\varphi'(\lambda_n)-1) = \varphi'(\lambda_n)+\theta(\lambda_n)-1-\theta(\lambda_n)\varphi'(\lambda_n) = \\ = 1-\theta(\lambda_n)\varphi'(\lambda_n) = -\varphi(\lambda_n)\theta'(\lambda_n). \tag{33}$$

The last equality follows from the fact that the Wronskii determinant of the functions θ and φ is equal to one.

From (30) and (31), it follows that the right-hand side of the equality (29) for $0 \le x, \xi \le 1$ is equal to

$$\varphi(\lambda_n)\left(\hat{\theta}(x,\lambda_n) + \frac{1-\theta(\lambda_n)}{\varphi(\lambda_n)}\,\hat{\varphi}(x,\lambda_n)\right)\left(\hat{\theta}(\xi,\lambda_n) + \frac{1-\theta(\lambda_n)}{\varphi(\lambda_n)}\,\hat{\varphi}(\xi,\lambda_n)\right).$$

Further, expanding the brackets and replacing $1-\theta(\lambda_n)$ and $(1-\theta(\lambda_n))^2$ according to the Formulas (32) and (33), respectively, we obtain that the right-hand side of the equality (29) coincides with the right-hand side of the equality (12) for $0 \le x, \xi \le 1$, i.e.,

$$\hat{h}(x,\xi,\pm\lambda_n) = C_{\lambda_n}\hat{v}(x,\lambda_n)\,\hat{v}(\xi,\lambda_n).$$

To complete the proof of the lemma, we show that the function $h(x,\xi,\pm\lambda_n)$ is a function periodic in x and ξ with period 1.

We fix $x \in \mathbb{R}^1$ and consider

$$h(x,\xi,\lambda_n) = \varphi(\lambda_n)\,\theta(x,\lambda_n)\,\theta(\xi,\lambda_n) - \theta'(\lambda_n)\,\varphi(x,\lambda_n)\,\varphi(\xi,\lambda_n) + \\ + \tfrac{\varphi'(\lambda_n)-\theta(\lambda_n)}{2}\,(\theta(\xi,\lambda_n)\,\varphi(x,\lambda_n) + \theta(x,\lambda_n)\,\varphi(\xi,\lambda_n)).$$

Taking into account the relations (4), and since the Wronskii determinant of the functions θ and φ does not depend on x, then the identity (10) holds.

By definition of the number λ_n, we have $F(\lambda_n) \equiv \varphi'(\lambda_n) + \theta(\lambda_n) = 2$, and after elementary transformations we get that $h(x,\xi,\pm\lambda_n)$ is a periodic function in x and ξ. The lemma is proven. □

In the same way as for the relations A_1), A_2), A_3), it is proved that at the ends of lacunae μ_n one of the relations holds:

$$B_1)\ \theta(\mu_n) \neq -1,\ \theta'(\mu_n) \neq 0,\ \varphi(\mu_n) \neq 0,\ \varphi'(\mu_n) \neq -1;$$

$$B_2)\ \theta(\mu_n) = -1,\ \theta'(\mu_n) \neq 0,\ \varphi(\mu_n) = 0,\ \varphi'(\mu_n) = -1;$$

$$B_3)\ \theta(\mu_n) = -1,\ \theta'(\mu_n) = 0,\ \varphi(\mu_n) \neq 0,\ \varphi'(\mu_n) = -1.$$

Lemma 2. *For points $\pm\mu_n, n = 0,1,2,\ldots$, if μ_n are ends of lacunae (that is, simple zeros of the function $F(k)+2$), then the equalities*

$$h(x,\xi,\pm\mu_n) = C_{\mu_n} v(x,\mu_n)\,v(\xi,\mu_n), \quad -b \le x,\xi \le b,$$

are satisfied, where the function $v(x,-\mu_n) = v(x,\mu_n)$ is the eigenfunction of the anti-periodic Sturm–Liouville problem, and the numbers C_{μ_n} depending on the cases B_1), B_2), B_3) have the form

$$\begin{aligned} B_1 : C_{\mu_n} &= \varphi(\mu_n)\int_0^1\left(\theta(x,\mu_n) + \tfrac{1+\theta(\mu_n)}{\varphi(\mu_n)}\,\varphi(x,\mu_n)\right)^2 dx;\\ B_2 : C_{\mu_n} &= -\theta'(\mu_n)\int_0^1(\varphi(x,\mu_n))^2 dx;\\ B_3 : C_{\mu_n} &= \varphi(\mu_n)\int_0^1(\theta(x,\mu_n))^2 dx. \end{aligned}$$

The proof of Lemma 2 is the same as for Lemma 1.

Lemma 3. *If $n > n_2$ and $\lambda_{2n} \neq \lambda_{2n-1}$ (i.e., λ_{2n} and λ_{2n-1} are the ends of lacunae), then the following estimates,*

$$|C_{\lambda_{2n}}| \leq C\,\frac{\lambda_{2n}-\lambda_{2n-1}}{\lambda_{2n-1}}, \quad |C_{\lambda_{2n-1}}| \leq C\,\frac{\lambda_{2n}-\lambda_{2n-1}}{\lambda_{2n-1}},$$

hold, where C does not depend on f and b, and the numbers $C_{\lambda_{2n}}$, $C_{\lambda_{2n-1}}$ are defined in Lemma 1.

Proof. Since λ_{2n} and λ_{2n-1} are the ends of a lacunae, then at each of these points one of the conditions $A_1)$, $A_2)$, $A_3)$ is satisfied.

Let us prove the lemma for the case $A_1)$; in other cases it is proved similarly.

Taking into account the form $C_{\lambda_{2n}}$ in case $A_1)$, we have,

$$\begin{aligned} |C_{\lambda_{2n}}| &\leq 2|\varphi(\lambda_{2n})| \int_0^1 (\theta(x,\lambda_{2n}))^2 dx + 2\,\tfrac{(1-\theta(\lambda_{2n}))^2}{|\varphi(\lambda_{2n})|} \int_0^1 (\varphi(x,\lambda_{2n}))^2 dx = \\ &= \tfrac{2}{\lambda_{2n}} |\lambda_{2n}\,\varphi(\lambda_{2n})| \int_0^1 (\theta(x,\lambda_{2n}))^2 dx - \lambda_{2n}\,\tfrac{\varphi(\lambda_{2n})\,\theta'(\lambda_{2n})}{\lambda_{2n}|\varphi(\lambda_{2n})|} \int_0^1 (\varphi(x,\lambda_{2n}))^2 dx. \end{aligned} \tag{34}$$

Note that the last equality in (34) follows from (33). By (5), the first integral $\int_0^1 (\theta(x,\lambda_{2n}))^2 dx$ is uniformly bounded, and the integral

$$\int_0^1 (\varphi(x,\lambda_{2n}))^2 dx \leq \frac{C}{\lambda_{2n}^2}.$$

Taking into account Remark 1, we get:

$$|C_{\lambda_{2n}}| \leq \frac{C_1}{\lambda_{2n}}|\lambda_{2n}-k'_{2n}| + \frac{C_2}{\lambda_{2n}}|\lambda_{2n}-k''_{2n}| \leq \frac{C}{\lambda_{2n}}\,(\lambda_{2n}-\lambda_{2n-1}) \leq \frac{C}{\lambda_{2n-1}}\,(\lambda_{2n}-\lambda_{2n-1}).$$

When obtaining the second inequality, it was taken into account that $k'_{2n}, k''_{2n} \in [\lambda_{2n-1}, \lambda_{2n}]$.

In a similar way, we obtain an estimate for $C_{\lambda_{2n-1}}$. □

Lemma 4. *If $n \in N$ and $\mu_{2n} \neq \mu_{2n+1}$ (i.e., μ_{2n} and μ_{2n+1} are the ends of lacunae), then the following estimate:*

$$|C_{\mu_{2n}}| \leq C\,\frac{\mu_{2n+1}-\mu_{2n}}{\mu_{2n}}, \quad |C_{\mu_{2n+1}}| \leq C\,\frac{\mu_{2n+1}-\mu_{2n}}{\mu_{2n}},$$

hold, where C does not depend on f and b, and the numbers $C_{\mu_{2n}}$, $C_{\mu_{2n+1}}$ are defined in Lemma 2.

Let us choose $d > 0$ so that $l_{\lambda_{2n}}$ and $l_{\lambda_{2n-1}}$ (or $l_{-\lambda_{2n}}$ and $l_{-\lambda_{2n}-1}$) belonged to the circle $B(2n)$ (respectively, $B(-2n)$) for sufficiently large $n \in N$, and this choice is possible due to (24).

The proof of Lemma 4 is carried out in the same way as Lemma 3.

Lemma 5. *For any $n = 1, 2, \ldots$, such that $\lambda_{2n} \neq \lambda_{2n-1}$, and for $n = 0$ the equalities,*

$$J_{l_{\pm\lambda_{2n}}} = -\frac{\pi}{i\sqrt{t}}\, b_{\pm\lambda_{2n}} a_{\lambda_{2n}} v(x,\lambda_{2n})\, e^{\mp i\lambda_{2n}t - i\frac{\pi}{4}} + b^{(1)}_{\pm\lambda_{2n}}(t)\, a_{\lambda_{2n}} v(x,\lambda_{2n}) + R_{\pm\lambda_{2n}}(x,t), \tag{35}$$

hold, where

$$a_{\lambda_{2n}} = \int_0^1 f(x)\, v(x,\lambda_{2n}) dx, \tag{36}$$

$$b_{\lambda_{2n}} = -\frac{2iC_{\lambda_{2n}}\Gamma(\frac{1}{2})}{\pi} \cdot \lim_{k\to\lambda_{2n}} \frac{\sqrt{k-\lambda_{2n}}}{\sqrt{G(k)}}, \quad b_{-\lambda_{2n}} = -ib_{\lambda_{2n}}, \tag{37}$$

$$R_{\pm\lambda_{2n}}(x,t)=\int_0^1\int_{l_{\pm\lambda_{2n}}}\left(\frac{h(x,\xi,k)}{r_{\pm 2n}(k)}-\frac{h(x,\xi,\pm\lambda_{2n})}{r_{\pm 2n}(\pm\lambda_{2n})}\right)\frac{f(\xi)\,e^{-ikt}}{\sqrt{k\mp\lambda_{2n}}\sqrt{k\mp\lambda_{2n-1}}}\,dkd\xi;$$

here the function $h(x,\xi,k)$ is defined by (12), and the function $r_{\pm 2n}(k)$ will be defined below (see, (40)). Moreover, there exists n_3 such that for $n>n_3$, and $t>0$, $x\in[-b,b]$, the following estimates are true:

$$|b_{\pm\lambda_{2n}}|\le\frac{C\sqrt{\lambda_{2n}-\lambda_{2n-1}}}{\lambda_{2n-1}},\quad |b^{(1)}_{\pm\lambda_{2n}}(t)|\le\frac{1}{t}\frac{C}{\lambda_{2n-1}},\tag{38}$$

the constant C depends only on the segment $[-b,b]$, and the numbers $C_{\lambda_{2n}}$ are defined in Lemma 1.

Proof. Let us prove the lemma for the case of the following integral,

$$J_{l_{\lambda_{2n}}}=\int_0^1\int_{l_{\lambda_{2n}}}\frac{h(x,\xi,k)}{\sqrt{G(k)}}\,f(\xi)e^{-ikt}dkd\xi,\quad x\in[-b,b].$$

The integral $J_{l_{-\lambda_{2n}}}$ is investigated in a similar way.

Let $n>n_2$, and by Proposition 4 in the circle $B(2n)$ the function G can be represented as

$$G(k)=(k-\lambda_{2n})\,(k-\lambda_{2n-1})\,g_{2n}(k).\tag{39}$$

Let us choose single-valued branches of the roots of each factor in (39). Denote by $\sqrt{k-\lambda_{2n}}$ and $\sqrt{k-\lambda_{2n-1}}$ the single-valued branches of these roots in $\mathbb{C}'$, defined by the condition of their positivity for positive values of $k-\lambda_{2n}$ and $k-\lambda_{2n-1}$.

Since the single-valued branch of the function $\sqrt{G(k)}$ for $k\in\mathbb{C}'$ has been chosen earlier, then:

$$r_{2n}(k)=\sqrt{g_{2n}(k)},\quad \text{for}\quad k\in B(2n)\cap\mathbb{C}'\tag{40}$$

is uniquely defined. Then we have:

$$J_{l_{\lambda_{2n}}}=J^{(1)}_{\lambda_{2n}}+R_{\lambda_{2n}}(x,t),\tag{41}$$

where

$$J^{(1)}_{\lambda_{2n}}=\int_0^1\int_{l_{\lambda_{2n}}}\frac{h(x,\xi,\lambda_{2n})}{r_{2n}(\lambda_{2n})}\cdot\frac{1}{\sqrt{k-\lambda_{2n}}\sqrt{k-\lambda_{2n-1}}}\,f(\xi)\,e^{-ikt}dkd\xi,$$

$$R_{\lambda_{2n}}(x,t)=\int_0^1\int_{l_{\lambda_{2n}}}\left(\frac{h(x,\xi,k)}{r_{2n}(k)}-\frac{h(x,\xi,\lambda_{2n})}{r_{2n}(\lambda_{2n})}\right)\cdot\frac{1}{\sqrt{k-\lambda_{2n}}\sqrt{k-\lambda_{2n-1}}}\,f(\xi)\,e^{-ikt}dkd\xi.$$

We will take into account that $k=\lambda_{2n}+i\tau,\ -d\le\tau\le 0$ for $k\in l_{\lambda_{2n}}$.

It is clear that if k belongs to the left side of the contour $l_{\lambda_{2n}}$, then $\sqrt{k-\lambda_{2n}}=\sqrt{i\tau}=e^{i\frac{3\pi}{4}}\sqrt{|\tau|}$; and if k belongs to the right side of the contour $l_{\lambda_{2n}}$, then the modulo root has the same sign.

The values of the roots of the remaining factors on the right side of the Formula (39) coincide at the corresponding points of the left and right sides of the contour, $l_{\lambda_{2n}}$.

Taking into account the Lemma 1, and the fact that τ changes to $(-d,0)$ on the left side of the contour $l_{\lambda_{2n}}$, and τ changes to $(0,-d)$ on the right side of the contour $l_{\lambda_{2n}}$, we get

$$\begin{aligned}J^{(1)}_{\lambda_{2n}}&=2ie^{-\frac{3}{4}\pi i}e^{-i\lambda_{2n}t}\int_0^1\tfrac{h(x,\xi,\lambda_{2n})}{r_{2n}(\lambda_{2n})}\,f(\xi)\left(\int_{-d}^0\tfrac{e^{t\tau}}{\sqrt{|\tau|}\sqrt{(\lambda_{2n}-\lambda_{2n-1})+i\tau}}d\tau\right)d\xi=\\&=\tfrac{2ie^{-\frac{3}{4}\pi i}e^{-i\lambda_{2n}t}C_{\lambda_{2n}}a_{\lambda_{2n}}v(x,\lambda_{2n})}{r_{2n}(\lambda_{2n})}\int_0^d\tfrac{e^{-t\tau}}{\sqrt{\tau}\sqrt{(\lambda_{2n}-\lambda_{2n-1})-i\tau}}\,d\tau,\end{aligned}\tag{42}$$

where the Fourier coefficients $a_{\lambda_{2n}}$ are defined by the Formula (36).

Let us write the last integral from (42) in the form:

$$\int_0^d \frac{e^{-t\tau}}{\sqrt{\tau}\sqrt{(\lambda_{2n}-\lambda_{2n-1})-i\tau}}\,d\tau = J^{(3)}_{\lambda_{2n}} + J^{(4)}_{\lambda_{2n}}, \tag{43}$$

where

$$J^{(3)}_{\lambda_{2n}} = \int_0^d \frac{e^{-t\tau}}{\sqrt{\tau}}\cdot\left(\frac{1}{\sqrt{(\lambda_{2n}-\lambda_{2n-1})-i\tau}} - \frac{1}{\sqrt{(\lambda_{2n}-\lambda_{2n-1})}}\right)d\tau,$$

$$J^{(4)}_{\lambda_{2n}} = \frac{1}{\sqrt{\lambda_{2n}-\lambda_{2n-1}}}\int_0^d \frac{e^{-t\tau}}{\sqrt{\tau}}\,d\tau.$$

We investigate the integrals $J^{(3)}_{\lambda 2n}$ and $J^{(4)}_{\lambda_{2n}}$ separately. For the integral $J^{(3)}_{\lambda_{2n}}$, we have

$$J^{(3)}_{\lambda 2n} = \int_0^d \frac{e^{-t\tau}}{\sqrt{\tau}}\cdot\frac{\sqrt{\lambda_{2n}-\lambda_{2n-1}}-\sqrt{(\lambda_{2n}-\lambda_{2n-1})-i\tau}}{\sqrt{(\lambda_{2n}-\lambda_{2n-1})-i\tau}\sqrt{\lambda_{2n}-\lambda_{2n-1}}}\,d\tau =$$

$$= i\int_0^d \frac{1}{\sqrt{(\lambda_{2n}-\lambda_{2n-1})-i\tau}\sqrt{\lambda_{2n}-\lambda_{2n-1}}}\cdot\frac{\sqrt{\tau}e^{-t\tau}}{\sqrt{(\lambda_{2n}-\lambda_{2n-1})-i\tau}+\sqrt{\lambda_{2n}-\lambda_{2n-1}}}\,d\tau.$$

It is easy to see that for $\tau > 0$,

$$|\sqrt{(\lambda_{2n}-\lambda_{2n-1})-i\tau}| \geq \sqrt{\tau},$$

$$|\sqrt{(\lambda_{2n}-\lambda_{2n-1})-i\tau}+\sqrt{\lambda_{2n}-\lambda_{2n-1}}| \geq \sqrt{\lambda_{2n}-\lambda_{2n-1}}.$$

Hence,

$$|J^{(3)}_{\lambda_{2n}}| \leq \frac{1}{t}\frac{1}{\lambda_{2n}-\lambda_{2n-1}}. \tag{44}$$

Now we investigate the integral $J^{(4)}_{\lambda_{2n}}$. We have:

$$J^{(4)}_{\lambda_{2n}} = \frac{1}{\sqrt{\lambda_{2n}-\lambda_{2n-1}}}\left(\int_0^\infty - \int_d^\infty\right)\frac{e^{-t\tau}}{\sqrt{\tau}}\,d\tau = \frac{1}{\sqrt{\lambda_{2n}-\lambda_{2n-1}}}\left(\frac{1}{\sqrt{t}}\cdot\Gamma\left(\frac{1}{2}\right) - \int_d^\infty\frac{e^{-t\tau}}{\sqrt{\tau}}\,d\tau\right). \tag{45}$$

Since $\lambda_{2n}-\lambda_{2n-1}\to 0$ for $n\to 0$ (see, (24)), then there exists a number $n_3 > n_2$ such that:

$$\lambda_{2n}-\lambda_{2n-1} < 1 \quad \text{for} \quad n > n_3.$$

From the obvious estimate,

$$\int_d^\infty \frac{e^{-t\tau}}{\sqrt{\tau}}\,d\tau \leq \frac{C}{t}$$

and from the Formulas (44) and (45), it follows that the integral (43) has the form:

$$\frac{1}{\sqrt{\lambda_{2n}-\lambda_{2n-1}}}\frac{1}{\sqrt{t}}\cdot\Gamma\left(\frac{1}{2}\right) + k_{\lambda_{2n}},$$

where

$$|k_{\lambda_{2n}}| \leq \frac{C}{t}\cdot\frac{1}{\lambda_{2n}-\lambda_{2n-1}} \quad \text{for} \quad n > n_3, \tag{46}$$

C does not depend on the function f.

So, according to (42)

$$J^{(1)}_{\lambda_{2n}} = \frac{2ie^{-\frac{3}{4}\pi i}e^{-i\lambda_{2n}t}C_{\lambda_{2n}}a_{\lambda_{2n}}v(x,\lambda_{2n})\Gamma\left(\frac{1}{2}\right)}{r_{2n}(\lambda_{2n})\sqrt{\lambda_{2n}-\lambda_{2n-1}}}\cdot\frac{1}{\sqrt{t}} + \frac{2ie^{-\frac{3}{4}\pi i}e^{-i\lambda_{2n}t}C_{\lambda_{2n}}a_{\lambda_{2n}}\,v(x,\lambda_{2n})}{r_{2n}(\lambda_{2n})}\,k_{\lambda_{2n}}. \tag{47}$$

Let us denote

$$
\begin{gathered}
b_{\lambda_{2n}} = -\frac{2ie^{-\frac{3}{4}\pi i}C_{\lambda_{2n}}\Gamma\left(\frac{1}{2}\right)e^{i\frac{\pi}{4}}}{\pi r_{2n}(\lambda_{2n})\sqrt{\lambda_{2n}-\lambda_{2n-1}}} = -\frac{2iC_{\lambda_{2n}}\Gamma\left(\frac{1}{2}\right)}{\pi}\cdot \lim_{k\to\lambda_{2n}}\frac{\sqrt{k-\lambda_{2n}}}{\sqrt{G(k)}}, \\
b^{(1)}_{\lambda_{2n}}(t) = \frac{2ie^{-\frac{3}{4}\pi i}e^{-i\lambda_{2n}t}C_{\lambda_{2n}}k_{\lambda_{2n}}}{r_{2n}(\lambda_{2n})}.
\end{gathered} \tag{48}
$$

Therefore, from (41) and (47), the validity of (35) follows. The correctness of the estimate (38) follows from the estimates (46) and (48), the Lemma 3, and the Proposition 4. Thus, for $n > n_3$, for the integral $J_{l_{\lambda_{2n}}}$, the Lemma 5 is proved.

Applying the Proposition 3 and reasoning similarly, we obtain that the equality (35) is also valid for $0 \le n \le n_3$. In this case, the estimates (38) are replaced by the estimates,

$$
|b_{\pm\lambda_{2n}}| \le C_{\pm\lambda_{2n}}, \quad |b^{(1)}_{\pm\lambda_{2n}}| \le \frac{C_{\pm\lambda_{2n}}}{t}, \quad 0 \le n \le n_3.
$$

Performing similar calculations for the integral $J_{l_{-\lambda_{2n}}}$ and denoting

$$
b_{-\lambda_{2n}} = -i\frac{2ie^{-\frac{3}{4}\pi i}C_{\lambda_{2n}}\Gamma\left(\frac{1}{2}\right)e^{i\frac{\pi}{4}}}{\pi r_{2n}(\lambda_{2n})\sqrt{\lambda_{2n}-\lambda_{2n-1}}} = -i\frac{2iC_{\lambda_{2n}}\Gamma\left(\frac{1}{2}\right)}{\pi r_{2n}(\lambda_{2n})\sqrt{\lambda_{2n}-\lambda_{2n-1}}} = -ib_{\lambda_{2n}},
$$

we are convinced of the validity of the Lemma 5. □

Lemma 6. *For any $n = 1, 2, \ldots$, such that $\lambda_{2n-1} \neq \lambda_{2n}$ the following equalities,*

$$
\begin{aligned}
J_{l_{\pm\lambda_{2n-1}}} = &-\frac{\pi}{i\sqrt{t}}\, b_{\pm\lambda_{2n-1}}a_{\lambda_{2n-1}}v(x,\lambda_{2n-1})e^{\mp i\lambda_{2n-1}t+\frac{\pi}{4}i}+ \\
&+b^{(1)}_{\pm\lambda_{2n-1}}a_{\lambda_{2n-1}}v(x,\lambda_{2n-1}) + R_{\pm\lambda_{2n-1}}(x,t),
\end{aligned}
$$

hold, where

$$
a_{\lambda_{2n-1}} = \int_0^1 f(x)\, v(x,\lambda_{2n-1})dx,
$$

$$
b_{\lambda_{2n-1}} = \frac{2iC_{\lambda_{2n-1}}}{\pi}\lim_{k\to\lambda_{2n-1}}\frac{\sqrt{k-\lambda_{2n-1}}}{\sqrt{G(k)}}, \quad b_{-\lambda_{2n-1}} = ib_{\lambda_{2n-1}},
$$

and

$$
R_{\pm\lambda_{2n-1}}(x,t) = \int_0^1\int_{l_{\pm\lambda_{2n-1}}}\left(\frac{h(x,\xi,k)}{r_{\pm 2n}(k)} - \frac{h(x,\xi,\pm\lambda_{2n-1})}{r_{\pm 2n}(\lambda_{2n-1})}\right)\frac{f(\xi)\, e^{-ikt}dk\, d\xi}{\sqrt{k\mp\lambda_{2n}}\sqrt{k\mp\lambda_{2n-1}}}.
$$

Moreover, there exists n_3 such that for $n > n_3$, and $t > 0$, $x \in [-b, b]$, the following estimates are true:

$$
|b_{\pm\lambda_{2n-1}}| \le \frac{C\sqrt{\lambda_{2n}-\lambda_{2n-1}}}{\lambda_{2n-1}}, \quad |b^{(1)}_{\pm\lambda_{2n-1}}(t)| \le \frac{C}{t}\cdot\frac{1}{\lambda_{2n-1}};
$$

the constant C depends only on the segment $[-b, b]$, and the numbers $C_{\lambda_{2n-1}}$ are defined in Lemma 1.

The proof of Lemma 6 is similar to Lemma 5.

Remark 2. *Lemmas 5 and 6 remain valid for all $n = 1, 2, \ldots$ if we replace λ_j with μ_{j-1} in them.*

3. Main Results

Theorem 1. *If the Hill operator H_0 is positive, $p(x) \geq \text{const} > 0$ and $q(x) \geq 0$, then there exist compact operators*

$$M_1, M_3 : L^2[0,1] \longmapsto H^1[0,1],$$
$$M_2, M_4 : L^2[0,1] \longmapsto L^2[0,1],$$

such that for $|x| < b$ and $t > 0$, the solution of the Cauchy problem (1) and (2), has the form

$$u(x,t) = \frac{1}{\sqrt{t}}\{u_1(x,t) + u_2(x,t)\} + v(x,t),$$

where $u_1(x,t)$ is a periodic solution of the Cauchy problem for which

$$\hat{u}(x,t)|_{t=0} = M_1 f(x), \quad \hat{u}_t(x,t)|_{t=0} = M_3 f(x),$$

$u_2(x,t)$ is an anti-periodic solution of the Cauchy problem for which

$$\hat{u}(x,t)|_{t=0} = M_2 f(x), \quad \hat{u}_t(x,t)|_{t=0} = M_4 f(x),$$

and for the function $v(x,t)$ for $|x| < b$ and $t > 0$, the following estimate is valid

$$|v(x,t)| \leq \frac{C(b)}{t} ||f; L^2(\mathbb{R}^1)||;$$

the functions $u_1(x,t)$ and $u_2(x,t)$ have the form:

$$u_1(x,t) = \sum_{n=0}^{\infty} b_{\lambda_n} a_{\lambda_n} v(x,\lambda_n) \sin(\lambda_n t + (-1)^n \frac{\pi}{4}),$$
$$u_2(x,t) = \sum_{n=0}^{\infty} b_{\mu_n} a_{\mu_n} v(x,\mu_n) \sin(\mu_n t + (-1)^{n+1} \frac{\pi}{4}),$$

where a_{λ_n} (a_{μ_n}) are the coefficients of the expansion of the function $f(x)$ in a Fourier series in the system $\{\hat{v}(x,\lambda_n)\}$ $(\{\hat{v}(x,\mu_n)\})$, b_{λ_n} (b_{μ_n}) are some constants of order $o(\frac{1}{n})$ as $n \to \infty$, and they are given by the Formula (37).

Here the summation is carried out only over those n for which λ_n (or μ_n) are simple eigenvalues of the periodic (or anti-periodic) Sturm–Liouville problem.

Proof. From the Proposition 1, it follows that for $x \in [-b,b]$ and $t > 0$ the solution of the Cauchy problem (1) and (2), can be represented as:

$$u(x,t) = \frac{1}{2\pi} J_L + v_1(x,t),$$

where for $t > 0$ the function $v_1(x,t)$ satisfies the estimate:

$$|v_1(x,t)| \leq C(b)\, e^{-td} ||f; L^2||, \quad x \in [-b,b],$$

and the function J_L is defined by the Formula (20).

According to the Formula (19)

$$J_L = J_{L_3} + \sum_{n=0}^{\infty} (J_{l_{\lambda_n}} + J_{l_{-\lambda_n}}) + \sum_{n=0}^{\infty} (J_{l_{\mu_n}} + J_{l_{-\mu_n}}),$$

moreover, in the second (third) term on the right-hand side, the summation is over those n for which l_{λ_n} (respectively, l_{μ_n}) is included in L_1 (in L_2).

From here and from the Proposition 2, it follows that:

$$u(x,t) = \frac{1}{2\pi}\sum_{n=0}^{\infty}\Big(J_{l_{\lambda_n}} + J_{l_{-\lambda_n}}\Big) + \sum_{n=0}^{\infty}\Big(J_{l_{\mu_n}} + J_{l_{-\mu_n}}\Big) + v_2(x,t), \tag{49}$$

where for $t > 0$ the function $v_2(x,t)$ satisfies the following estimate

$$|v_2(x,t)| \leq C(b)\, e^{-td}||f; L^2||, \quad x \in [-b,b].$$

Let us investigate the first term in the Formula (49). Since

$$b_{-\lambda_{2n}} = -ib_{\lambda_{2n}}, \quad b_{-\lambda_{2n-1}} = ib_{\lambda_{2n-1}},$$

then

$$-\pi b_{\lambda_{2n}} a_{\lambda_{2n}} v(x,\lambda_{2n}) e^{-i\lambda_{2n}t - \frac{\pi}{4}i} - \pi b_{-\lambda_{2n}} a_{\lambda_{2n}} v(x,\lambda_{2n}) e^{i\lambda_{2n}t - \frac{\pi}{4}i} =$$
$$= 2\pi i b_{\lambda_{2n}} a_{\lambda_{2n}} v(x,\lambda_{2n}) \sin\Big(\lambda_{2n}t + \frac{\pi}{4}\Big) -$$
$$-\pi b_{\lambda_{2n-1}} a_{\lambda_{2n-1}} v(x,\lambda_{2n-1}) e^{-i\lambda_{2n-1}t + \frac{\pi}{4}i} - \pi b_{-\lambda_{2n-1}} a_{\lambda_{2n-1}} v(x,\lambda_{2n-1}) e^{i\lambda_{2n-1}t + \frac{\pi}{4}i} =$$
$$= 2\pi i b_{\lambda_{2n-1}} a_{\lambda_{2n-1}} v(x,\lambda_{2n-1}) \sin\Big(\lambda_{2n-1}t - \frac{\pi}{4}\Big).$$

These equalities, together with Lemmas 5 and 6, show that the following equality is true:

$$\begin{aligned} \tfrac{1}{2\pi}\sum_{n=0}^{\infty}\Big(J_{l_{\lambda_n}} + J_{l_{-\lambda_n}}\Big) &= \tfrac{1}{\sqrt{t}}\sum_{n=0}^{\infty} b_{\lambda_n} a_{\lambda_n} v(x,\lambda_n)\sin\big(\lambda_n t + (-1)^n \tfrac{\pi}{4}\big) + \\ &+ \tfrac{1}{2\pi}\sum_{n=0}^{\infty}\Big(b^{(1)}_{\lambda_n}(t) + b^{(1)}_{-\lambda_n}(t)\Big) a_{\lambda_n} v(x,\lambda_n) + \tfrac{1}{2\pi}\big(R_{\lambda_0} + R_{-\lambda_0}\big) + \\ &+ \tfrac{1}{2\pi}\sum_{n=1}^{\infty}\big(R_{\lambda_{2n}}(x,t) + R_{\lambda_{2n-1}}(x,t)\big) + \tfrac{1}{2\pi}\sum_{n=1}^{\infty}\big(R_{-\lambda_{2n}}(x,t) + R_{-\lambda_{2n-1}}(x,t)\big). \end{aligned} \tag{50}$$

Consider the second term in (50). From Steklov's theorem on the expansion of a twice continuously differentiable function in terms of eigenfunctions of the problems (26) and (27) (see, [18,19]), it follows that:

$$\sum_{n=0}^{\infty} |a_{\lambda_n}|^2 = ||f; L^2([0,1])||^2 = ||f; L^2||^2. \tag{51}$$

From the estimates for the coefficients $b^{(1)}_{\pm\lambda_n}(t)$ and (24), it follows that:

$$\begin{aligned} \sum_{n=0}^{\infty}\Big|b^{(1)}_{\lambda_n}(t) + b^{(1)}_{-\lambda_n}(t)\Big|^2 &= \sum_{n=0}^{n_3}\Big|b^{(1)}_{\lambda_n}(t) + b^{(1)}_{-\lambda_n}(t)\Big|^2 + \sum_{n=n_3+1}^{\infty}\Big|b^{(1)}_{\lambda_n}(t) + b^{(1)}_{-\lambda_n}(t)\Big|^2 \leq \\ &\leq \tfrac{C_1}{t^2} + \tfrac{C_2}{t^2}\sum_{n=n_3+1}^{\infty}\tfrac{1}{(2n)^2} \leq \tfrac{C_3}{t^2}. \end{aligned} \tag{52}$$

Taking into account now that the functions $v(x,\lambda_n)$ are functions uniformly bounded with respect to n (see [18]), from (51) and (52) for $|x| < b$ and $t > 0$, we get:

$$\left|\frac{1}{2\pi}\sum_{n=0}^{\infty}\Big(b^{(1)}_{\lambda_n}(t) + b^{(1)}_{-\lambda_n}(t)\Big) a_{\lambda_n} v(x,\lambda_n)\right| \leq \frac{C}{t}||f; L^2||, \tag{53}$$

where C does not depend on a.

Now, from the expansion of (50), we investigate the following term:

$$\frac{1}{2\pi}\sum_{n=1}^{\infty}\big(R_{\lambda_{2n}}(x,t) + R_{\lambda_{2n-1}}(x,t)\big),$$

writing the first term in the form:

$$\sum_{n=1}^{\infty} R_{\lambda_{2n}}(x,t) = I_1 + I_2,$$

where

$$I_1 = \sum_{n=1}^{n_3} R_{\lambda_{2n}}(x,t), \quad I_2 = \sum_{n=n_3+1}^{\infty} R_{\lambda_{2n}}(x,t).$$

Applying the same reasoning as in the study of the integral $J^{(1)}_{\lambda_{2n}}$, we rewrite I_2 as:

$$\begin{aligned} I_2 := \sum_{n=n_3+1}^{\infty} R_{\lambda_{2n}}(x,t) = \sum_{n=n_3+1}^{\infty} \int_0^1 \int_{l_{\lambda_n}} \left(\tfrac{h(x,\xi,k)}{r_{2n}(k)} - \tfrac{h(x,\xi,\lambda_{2n})}{r_{2n}(\lambda_{2n})} \right) \cdot \tfrac{f(\xi)\, e^{-ikt} dk\, d\xi}{\sqrt{k-\lambda_{2n}}\sqrt{k-\lambda_{2n-1}}} = \\ = \sum_{n=n_3+1}^{\infty} 2ie^{-\frac{3\pi}{4}i} e^{-i\lambda_{2n}t} \Big\{ \int_0^d \int_0^1 \left(\tfrac{1}{r_{2n}(\lambda_{2n}-i\tau)} - \tfrac{1}{r_{2n}(\lambda_{2n})} \right) \cdot \tfrac{h(x,\xi,\lambda_{2n}-i\tau)\, e^{-t\tau} d\xi\, d\tau}{\sqrt{\tau}\sqrt{(\lambda_{2n}-\lambda_{2n-1})-i\tau}} + \\ + (h(x,\xi,\lambda_{2n}-i\tau) - h(x,\xi,\lambda_{2n})) \cdot \tfrac{f(\xi) e^{-it} d\xi\, d\tau}{r_{2n}(\lambda_{2n})\sqrt{\tau}\sqrt{(\lambda_{2n}-\lambda_{2n-1})-i\tau}} \Big\} = \\ = 2ie^{-\frac{3\pi}{4}i} \sum_{n=n_3+1}^{\infty} e^{-i\lambda_{2n}t} \Big\{ R^{(1)}_{\lambda_{2n}}(x,t) + R^{(2)}_{\lambda_{2n}}(x,t) \Big\}. \end{aligned} \tag{54}$$

We will investigate the first term on the right side of the equality (54)

$$\sum_{n=n_3+1}^{\infty} e^{-i\lambda_{2n}t} R^{(1)}_{\lambda_{2n}}(x,t).$$

Note that in the Proposition 4 the circle $B(2n)$ could be replaced by the circle $B_\varepsilon(2n)$ with the same center and radius $\frac{\pi}{4} + \varepsilon$, where $\varepsilon > 0$ is sufficiently small. In addition, when proving Proposition 4, it was possible to obtain, without any additional reasoning, that the function $|g_{2n}(k)|$ is bounded uniformly in n not only from below, but also from above.

Thus, $|g_{2n}(k)| \le C$ for $k \in B_\varepsilon(2n)$. Since the function $r_{2n}(k) \equiv \sqrt{g_{2n}(k)}$ is holomorphic in the circle $B_\varepsilon(2n)$ and the derivatives of the holomorphic function in the circle $B_\varepsilon(2n)$ are estimated in terms of the maximum of the modulus of the function in the circle $B_\varepsilon(2n)$, then $|(r_{2n}(k))_k| \le C$ for $k \in B_\varepsilon(2n)$, where C does not depend on n. Since

$$\left| \frac{\tau}{\sqrt{\tau} \cdot \sqrt{(\lambda_{2n} - \lambda_{2n-1}) - i\tau}} \right| \le 1 \quad \text{for} \quad 0 \le \tau \le d, \tag{55}$$

then, together with the Proposition 4, this argument leads to the inequality,

$$\left| \frac{(r_{2n}(k))_k}{r_{2n}(\lambda_{2n} - i\tau)\, r_{2n}(\lambda_{2n})} \cdot \frac{i\tau}{\sqrt{\tau} \cdot \sqrt{(\lambda_{2n} - \lambda_{2n-1}) - i\tau}} \right| \le C, \tag{56}$$

where C does not depend on n, and $k = \lambda_{2n} - i\tau$, $0 \le \tau \le d$.

Just as in the proof of the Preposition 2, we represent the function $h(x,\xi,k)$ in the form:

$$h(x,\xi,k) = \frac{1}{k} \sin k \, \cos kx \, \cos k\xi + \frac{1}{k} \sin k \, \sin kx \, \sin k\xi + O\Big(|k|^2 e^{|\tau|(\xi+x)} \Big) \tag{57}$$

Now, after the above remarks, we have:

$$\sum_{n=n_3+1}^{\infty} e^{-i\lambda_{2n}t} R^{(1)}_{\lambda_{2n}}(x,t) = I_3 + I_4 + I_5, \tag{58}$$

where

$$I_3=\sum_{n=n_3+1}^{\infty}e^{-i\lambda_{2n}t}\int_0^d\frac{i\tau e^{-t\tau}}{\lambda_{2n}-i\tau}\cdot\frac{(Re\, r_{2n}(k))_k|_{k=k'_{2n}}+(Im\, r_{2n}(k))_k|_{k=k''_{2n}}}{r_{2n}(\lambda_{2n}-i\tau)r_{2n}(\lambda_{2n})\sqrt{\tau}\sqrt{(\lambda_{2n}-\lambda_{2n-1})-i\tau}}\times$$

$$\sin(\lambda_{2n}-i\tau)\,\cos(\lambda_{2n}-i\tau)x\cdot\left(\int_0^1 f(\xi)\,\cos(\lambda_{2n}-i\tau)\xi\,d\xi\right)d\tau,$$

$$I_4=\sum_{n=n_3+1}^{\infty}e^{-i\lambda_{2n}t}\int_0^d\frac{-i\tau e^{-t\tau}}{\lambda_{2n}-i\tau}\cdot\frac{(Re\, r_{2n}(k))_k|_{k=k'_{2n}}+(Im\, r_{2n}(k))_k|_{k=k''_{2n}}}{r_{2n}(\lambda_{2n}-i\tau)r_{2n}(\lambda_{2n})\sqrt{\tau}\sqrt{(\lambda_{2n}-\lambda_{2n-1})-i\tau}}\times$$

$$\sin(\lambda_{2n}-i\tau)\,\sin(\lambda_{2n}-i\tau)x\cdot\left(\int_0^1 f(\xi)\,\sin(\lambda_{2n}-i\tau)\xi\,d\xi\right)d\tau,$$

$$I_5=\sum_{n=n_3+1}^{\infty}e^{-i\lambda_{2n}t}\int_0^d\frac{(Re\, r_{2n}(k))_k|_{k=k'_{2n}}+(Im\, r_{2n}(k))_k|_{k=k''_{2n}}}{r_{2n}(\lambda_{2n}-i\tau)r_{2n}(\lambda_{2n})\sqrt{\tau}\sqrt{(\lambda_{2n}-\lambda_{2n-1})-i\tau}}\times$$

$$i\tau e^{-t\tau}\left(\int_0^1 f(\xi)\,\mathrm{O}\left(|\lambda_{2n}-i\tau|^{-2}\right)d\xi\right)d\tau;$$

here

$$k'_{2n}=\lambda_{2n}-i\tau'_{2n},\quad 0\le\tau'_{2n}\le\tau\le d,$$
$$k''_{2n}=\lambda_{2n}-i\tau''_{2n},\quad 0\le\tau''_{2n}\le\tau\le d.$$

To investigate I_3 in the Formula (58), note that $n_3>n_2>n_1$, and so

$$\lambda_{2n}=2n\pi+\mathrm{O}\left(\frac{1}{n}\right).$$

Let us make elementary transformations

$$\int_0^1 f(\xi)\,\cos(\lambda_{2n}-i\tau)\xi\,d\xi=\frac{1}{2}\int_0^1 f(\xi)\,(e^{i\lambda_{2n}\xi}e^{\tau\xi}+e^{-i\lambda_{2n}\xi}e^{-\tau\xi})d\xi=$$

$$=\frac{1}{2}\int_0^1 f(\xi)\,e^{\tau\xi}e^{i2n\pi\xi}d\xi+\frac{1}{2}\int_0^1 f(\xi)\,e^{-\tau\xi}e^{-i2n\pi\xi}d\xi+\int_0^1 f(\xi)\,\mathrm{O}\left(\frac{1}{n}\right)d\xi,$$

and note that the number,

$$d_{2n}=\int_0^1 f(\xi)\,e^{\tau\xi}e^{i2n\pi\xi}d\xi\quad\left(d_{-2n}=\int_0^1 f(\xi)\,e^{-\tau\xi}e^{-i2n\pi\xi}d\xi\right),$$

is the coefficient of the expansion of the function $f(\xi)\,e^{\tau\xi}$ (or the function $f(\xi)\,e^{-\tau\xi}$) into a Fourier series with respect to the system $\{e^{in\pi}\}_{n=-\infty}^{\infty}$.

It is obvious that:

$$\left|\int_0^1 f(\xi)\,\mathrm{O}\left(\frac{1}{n}\right)d\xi\right|\le\frac{C}{n}||f;L^2||,$$

where C does not depend on n.

Note also that the functions $\sin(\lambda_{2n}-i\tau)\,\cos(\lambda_{2n}-i\tau)x$ are uniformly bounded by n for $|x|<b$ and $0\le\tau\le d$.

After these remarks, from the Cauchy–Bunyakovsky–Schwartz inequality for an infinite sum and (56), for $|x|<b$ and $t>0$ we obtain

$$|I_3|\le C_4\int_0^d e^{-t\tau}\sqrt{\sum_{n=n_3+1}^{\infty}\frac{1}{\lambda_{2n}^2}}\cdot\sqrt{\sum_{n=n_3+1}^{\infty}\left(d_{2n}^2+d_{-2n}^2+\frac{1}{n^2}||f;L^2||^2\right)}\,d\tau\le$$

$$\leq \frac{C_5}{t}\sqrt{||f(\xi)e^{-\tau\xi} : L^2([0,1])||^2 + ||f(\xi)e^{\tau\xi}; L^2([0,1])||^2 + C_6||f; L^2||} \leq \frac{C}{t}||f; L^2||.$$

Similarly, for $|x| < b$ and $t > 0$ we get:

$$|I_4| \leq \frac{C_7}{t}||f; L^2||.$$

From the inequalities (56) and (24), it follows that for $|x| < b$ and $t > 0$

$$|I_5| \leq \frac{C_7}{t}||f; L^2|| \sum_{n=n_3+1}^{\infty} \frac{1}{\lambda_{2n}^2} \leq \frac{C}{t}||f; L^2||.$$

From (58) and estimates for I_3, I_4, I_5 for $|x| < b$ and $t > 0$ we get

$$\left| \sum_{n=n_3+1}^{\infty} e^{-i\lambda_{2n}t} R^{(1)}_{\lambda_{2n}}(x,t) \right| \leq \frac{C}{t}. \tag{59}$$

Note that the residual function in the Formula (57)

$$Q(x,\xi,k) = O\left(|k|^{-2}e^{|\tau|(\xi+x+1)}\right)$$

is a differentiable function for $-b \leq x, \xi \leq b$ and $Q_k(x,\xi,k)$ has the same descending order as $|k| \to \infty$ the function $Q(x,\xi,k)$.

Consider now the second term on the right side of the equality (54), i.e.,

$$\sum_{n=n_3+1}^{\infty} e^{-i\lambda_{2n}t} R^{(2)}_{\lambda_{2n}}(x,t).$$

Let us write this sum as:

$$\sum_{n=n_3+1}^{\infty} e^{-i\lambda_{2n}t} R^{(2)}_{\lambda_{2n}}(x,t) = F_1 + F_2 + F_3,$$

where

$$F_1 = \sum_{n=n_3+1}^{\infty} e^{-i\lambda_{2n}t} \int_0^d \int_0^1 \left(\frac{\sin(\lambda_{2n} - i\tau)\, \cos(\lambda_{2n} - i\tau)x\, \cos(\lambda_{2n} - i\tau)\xi}{\lambda_{2n} - i\tau} - \right.$$

$$\left. - \frac{\sin\lambda_{2n}\, \cos\lambda_{2n}x\, \cos\lambda_{2n}\xi}{\lambda_{2n}} \right) \cdot \frac{f(\xi)\, e^{-t\tau} d\xi\, d\tau}{r_{2n}(\lambda_{2n})\sqrt{\tau}\sqrt{(\lambda_{2n} - \lambda_{2n-1}) - i\tau}},$$

$$F_2 = \sum_{n=n_3+1}^{\infty} e^{-i\lambda_{2n}t} \int_0^d \int_0^1 \left(\frac{\sin(\lambda_{2n} - i\tau)\, \sin(\lambda_{2n} - i\tau)x\, \sin(\lambda_{2n} - i\tau)\xi}{\lambda_{2n} - i\tau} - \right.$$

$$\left. - \frac{\sin\lambda_{2n}\, \sin\lambda_{2n}x\, \sin\lambda_{2n} - i\tau\xi}{\lambda_{2n}} \right) \cdot \frac{f(\xi)\, e^{-t\tau} d\xi d\tau}{r_{2n}(\lambda_{2n})\sqrt{\tau}\sqrt{(\lambda_{2n} - \lambda_{2n-1}) - i\tau}},$$

and

$$F_3 = \sum_{n=n_3+1}^{\infty} e^{-i\lambda_{2n}t} \int_0^d \int_0^1 (Q(x,\xi,\lambda_{2n} - i\tau) - Q(x,\xi,\lambda_{2n})) \cdot \frac{f(\xi)\, e^{-t\tau} d\xi\, d\tau}{r_{2n}(\lambda_{2n})\sqrt{\tau}\sqrt{(\lambda_{2n} - \lambda_{2n-1}) - i\tau}}.$$

We first investigate the integral F_1. From the obvious equality,

$$\begin{aligned}
&\tfrac{1}{\lambda_{2n}-i\tau}\cdot\sin(\lambda_{2n}-i\tau)\,\cos(\lambda_{2n}-i\tau)\,\cos(\lambda_{2n}-i\tau)\xi-\tfrac{1}{\lambda_{2n}}\sin\lambda_{2n}\,\cos\lambda_{2n}x\,\cos\lambda_{2n}\xi=\\
&\quad=\tfrac{1}{\lambda_{2n}-i\tau}\cdot\sin(\lambda_{2n}-i\tau)\,\cos(\lambda_{2n}-i\tau)x\,(\cos(\lambda_{2n}-i\tau)\xi-\cos\lambda_{2n}\xi)+\\
&\qquad+\tfrac{1}{\lambda_{2n}-i\tau}\cdot\cos\lambda_{2n}\xi\,\sin(\lambda_{2n}-i\tau)\,(\cos(\lambda_{2n}-i\tau)x-\cos\lambda_{2n}x)+\\
&\qquad\quad+\tfrac{1}{\lambda_{2n}-i\tau}\cdot\cos\lambda_{2n}\xi\,\cos\lambda_{2n}x\,(\sin(\lambda_{2n}-i\tau)-\sin\lambda_{2n})+\\
&\qquad\qquad+\left(\tfrac{1}{\lambda_{2n}-i\tau}-\tfrac{1}{\lambda_{2n}}\right)\cdot\cos\lambda_{2n}\xi\,\cos\lambda_{2n}x\,\sin\lambda_{2n},
\end{aligned}\tag{60}$$

it follows that the left side of the equality (60) can be represented as:

$$\frac{\tau}{\lambda_{2n}-i\tau}\cdot\left(\cos\lambda_{2n}\xi\,h^{(1)}_{2n}(x,\xi,\tau)+\sin\lambda_{2n}\xi\,h^{(2)}_{2n}(x,\xi,\tau)\right),$$

and besides $|h^{(i)}_{2n}(x,\xi,\tau)|\le C$, $i=1,2$, for $|x|<b$, $|\xi|<b$, and $0\le\tau\le d$, and C does not depend on n.

Then, taking into account (55) and Proposition 4, and reasoning in the same way as in the I_3 estimate, we obtain:

$$|F_1|\le\frac{C}{t}||f;L^2||\quad\text{for}\quad |x|<b,\,t>0.\tag{61}$$

In a similar way we get:

$$|F_2|\le\frac{C}{t}||f;L^2||\quad\text{for}\quad |x|<b,\,t>0.\tag{62}$$

To estimate F_3, note that:

$$Q(x,\xi,\lambda_{2n}-i\tau)-Q(x,\xi,\lambda_{2n})=$$

$$=i\tau\left((Re\,Q(x,\xi,k))_k\big|_{k=\lambda_{2n}-i\tau^{(1)}_{2n}}+(Im\,Q(x,\xi,k))_k\big|_{k=\lambda_{2n}-i\tau^{(2)}_{2n}}\right),$$

besides $0\le\tau^{(1)}_{2n}\le\tau\le d$, $0\le\tau^{(2)}_{2n}\le\tau\le d$.

Further, taking into account the fact that:

$$Q_k(x,\xi,k)=O\left(|k|^{-2}e^{|\tau|(x+\xi+1)}\right)\quad\text{as}\quad|k|\to\infty$$

just as for F_1 and F_2, we get:

$$|F_3|\le\frac{C}{t}||f;L^2||\quad\text{for}\quad |x|<b,\,t>0.\tag{63}$$

From (61)–(63) it follows that:

$$\left|\sum_{n=n_3+1}^{\infty}e^{-i\lambda_{2n}t}R^{(2)}_{\lambda_{2n}}(x,t)\right|\le\frac{C}{t}||f;L^2||,\tag{64}$$

where C does not depend on the function f.

From (54), (59) and (64) it follows that:

$$\left|\sum_{n=n_3+1}^{\infty}R_{\lambda_{2n}}(x,t)\right|\le\frac{C}{t}||f;L^2||\quad\text{for}\quad|x|<b,\,t>0,\tag{65}$$

where C does not depend on the function f.

Applying Proposition 3 and reasoning similarly, it is easy to show that:

$$\left| R_{\lambda_0}(x,t) + \sum_{n=1}^{n_3} R_{\lambda_{2n}}(x,t) \right| \leq \frac{C}{t} ||f; L^2|| \quad \text{for} \quad |x| < b,\ t > 0, \tag{66}$$

As a result, from (65) and (66), it follows that:

$$\left| R_{\lambda_0}(x,t) + \sum_{n=1}^{\infty} R_{\lambda_{2n}}(x,t) \right| \leq \frac{C}{t} ||f; L^2|| \quad \text{for} \quad |x| < b,\ t > 0, \tag{67}$$

In the same way, one can show that:

$$\left| \sum_{n=1}^{\infty} R_{\lambda_{2n-1}}(x,t) \right| \leq \frac{C}{t} ||f; L^2|| \quad \text{for} \quad |x| < b,\ t > 0, \tag{68}$$

and

$$\left| R_{-\lambda_0}(x,t) + \sum_{n=1}^{\infty} \left(R_{-\lambda_{2n}}(x,t) + R_{-\lambda_{2n-1}}(x,t) \right) \right| \leq \frac{C}{t} ||f; L^2|| \quad \text{for} \quad |x| < b,\ t > 0, \tag{69}$$

where C does not depend on the function f.

From (50), (53), (67), (68) and (69) it follows that:

$$\frac{1}{2\pi} \sum_{n=0}^{\infty} \left(J_{l_{\lambda_{2n}}} + J_{l_{-\lambda_{2n}}} \right) = \frac{1}{\sqrt{t}} u_1(x,t) + v_3(x,t), \tag{70}$$

where

$$u_1(x,t) = \sum_{n=1}^{\infty} b_{\lambda_n} a_{\lambda_n} v(x, \lambda_n) \sin(\lambda_n t + (-1)^n \frac{\pi}{4}),$$

and for the function $v_3(x,t)$ for $t > 0$, the following estimate is valid:

$$|v_3(x,t)| \leq \frac{C}{t} ||f; L^2||, \quad |x| < b;$$

here C does not depend on the function f.

It is proved similarly that:

$$\frac{1}{2\pi} \sum_{n=0}^{\infty} \left(J_{l_{\mu_{2n}}} + J_{l_{-\mu_{2n}}} \right) = \frac{1}{\sqrt{t}} u_2(x,t) + v_4(x,t), \tag{71}$$

where

$$u_2(x,t) = \sum_{n=1}^{\infty} b_{\mu_n} a_{\mu_n} v(x, \mu_n) \sin(\mu_n t + (-1)^{n+1} \frac{\pi}{4}),$$

and for the function $v_4(x,t)$ for $t > 0$, the following estimate is valid

$$|v_4(x,t)| \leq \frac{C}{t} ||f; L^2||, \quad |x| < b;$$

here C does not depend on the function f.

Finally, from (49), (70) and (71), we get

$$u(x,t) = \frac{1}{\sqrt{t}} (u_1(x,t) + u_2(x,t)) + v(x,t)$$

where for the function $v(x,t)$ for $t > 0$, the following estimate is valid

$$|v(x,t)| \leq \frac{C(b)}{t}||f;L^2||, \quad |x| < b.$$

To complete the proof, it remains to note that:

$$M_1 f = \sum_{n=0}^{\infty}(-1)^n \frac{\sqrt{2}}{2} b_{\lambda_n} a_{\lambda_n} \hat{v}(x,\lambda_n), \quad M_2 f = \sum_{n=0}^{\infty} \frac{\sqrt{2}}{2} \lambda_n b_{\lambda_n} a_{\lambda_n} \hat{v}(x,\lambda_n),$$

$$M_3 f = \sum_{n=0}^{\infty}(-1)^{n+1} \frac{\sqrt{2}}{2} b_{\mu_n} a_{\mu_n} \hat{v}(x,\mu_n), \quad M_4 f = \sum_{n=0}^{\infty} \frac{\sqrt{2}}{2} \mu_n b_{\mu_n} a_{\mu_n} \hat{v}(x,\mu_n),$$

and the compactness of these operators in these spaces is an obvious consequence of the estimates obtained in Lemmas 5 and 6 for coefficients b_{λ_j} and b_{μ_j}. □

Remark 3. *In the case of a finite-gap potential, the functions $u_1(x,t)$ and $u_2(x,t)$ are represented as a finite sum of terms oscillating with respect to t, since the spectrum of the operator H_0 has a band structure, and the ends of the bands coincide with the simple eigenvalues λ_n and μ_n [16].*

4. Applications

The need to solve the equations of mathematical physics with variable coefficients is due to the large number of applied problems leading to them. In particular, problems of this kind are led by current issues of studying the non-stationary interaction of fields of different nature, in which one-dimensional problems of non-stationary interaction of mechanical and electromagnetic fields are solved (see, for example, [20,21]). A subtle study of narrower classes of equations is conditioned by the need to study the behavior of solutions of such problems during the transition from a non-stationary regime to a steady one.

5. Conclusions

The main difference of this paper from the papers cited in the Introduction and included in the bibliography is that the case of periodic coefficients $p(x)$ and $q(x)$ is considered here. In this paper, the periodic coefficients $p(x)$ and $q(x)$ are considered for the first time. The main results, including the results of this paper, have been published in well-known scientific journals in the form of short reports, as well as presented at International Conferences.

As a conclusion, we would like to announce some developments of the problem under consideration in the following vein:

(1) Study of the asymptotic behavior of the solution of the Cauchy problem (1) and (2), in the case when the left end of the spectrum of the Hill operator is non-positive;

(2) Obtaining the principle of limiting amplitude for the Cauchy problem (1) and (2);

(3) Study of the asymptotic behavior of the solution of the mixed problem on the half-axis, that is, the following condition is added to the Cauchy problem (1) and (2): $u(x,t)|_{x=0} = 0$.

We also note papers [22,23], in which the construction of uniform asymptotics is proposed by the method of resurgent analysis based on the Laplace–Borel transform [24].

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