Article

An Extended Thermodynamics Model for Blood Flow

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Abstract: A model for blood flow is introduced in the context of the Rational Extended Thermodynamics (RET). The balance equations are applied to the two-hierarchy structure recently introduced by Ruggeri and Sugiyama. The constitutive relations are derived with universal physical principles and the remaining constitutive functions are evaluated by use of the kinetic theory. The model herein obtained is a hyperbolic generalization of a classical blood flow m odel. Our equations by construction have the same physical proprieties of the classical system; in addition, owing to its hyperbolic structure, our model avoids the unphysical feature of instantaneous diffusive effects which is typical of parabolic systems. Furthermore we expect that our model, as all RET systems, can describe the physical phenomena better than the classical ones when the fields change rapidly or one has steep gradients.

Keywords: Rational Extended Thermodynamics; hyperbolic system; fluid mixtures; multiphase flow; particle migration

MSC: 80A17; 76T99; 92C17; 92C05

1. Introduction

Gidaspow in 1994 [1] and Gidaspow and Huang in 2009 [2] employed and studied a model for blood flow based on the kinetic theory which assumes blood as a mixture of plasma and red blood cells (RBCs). This model allows to explain the Fahraeus–Lindqvist effect, the decrease of RBCs viscosity when the diameter of blood vessels decreases. The cause of this effect is the random RBCs kinetic energy per unit mass, known as granular temperature. When this temperature rises near the wall of the vessel, RBCs oscillate and they migrate from the wall to the center of the vessel. This mechanism is called the shear induced diffusion and it is a driving force connected to a gradient of shear. It is observed with the microscope that the smaller the diameter of the blood vessel, the stronger the shear diffusion. This creates a layer of free cells near the wall and consequently a collapse of the viscosity in the narrow vessels. The role of kinetic theory is to provide a theoretical expression for the blood viscosity. The results of Gidaspow's paper [2] are in agreement with the experimental data collected for red blood cell concentrations in narrow vessels. Inspired by this work [2], many researchers have developed models on blood flow to study physiological and patalogical cases. Huang, in her PhD dissertation, applies Gidaspow's theory to the transport of LDL and HDL lipoproteins and platelets. In [3], the pulsing flow in a coronary artery is studied using a typical cardiac waveform. The viscosity values of the red blood cells are numerically determined to appreciate the effect and to identify sites where arteriosclerosis could develop. Fedosov et al. [4] deal with the changes in the shape, mechanics and physical properties of red blood cells, determined by infectious diseases, environmental and hereditary factors. They evaluate how the biorheology of blood flow varies in subjects suffering from malaria and sickle cell anemia and they compare with healthy cases. Felicetti et al. [5] numerically treat the detection and the treatment of tumors by recognizing tumor markers on the blood cell surfaces. The information is conveyed

on one side by nanomachines that have similar size to white blood cells and they can flow in vessels. On the other side, smart probes are used to detect information extrabody about the localization of tumors. Wu et al. [6] study benchmark problems including a sudden expansion and various driven slots and crevices through the vessels. Their results turned out to be a predictive model for thrombosis in blood-wetted devices. Lopes et al. [7] treat the flow of blood in the carotid vessels and compare models in which the vessels are idealized with rigid walls and elastic walls. They focus on the properties of the arteries that become more rigid with the appearance of arteriosclerosis. Furthemore, they speculate that the carotid sinus is the area where arteriosclerosis is most likely to occur, given its low values of Time-Averaged Wall Shear Stress. A 3-phase model, consisting of blood, red blood cells and white blood cells, is formulated by Melka et al. [8]. In particular, they analyze the aorta and its thoracic branches of a child of 8 years, affected by a congenital heart defect, the coarctation of the aorta. Another 3-phase model, this time consisting of plasma, platelets and red blood cells, is explained by Gidaspow and Chandra [9]. It is observed numerically that platelets and red blood cells move in opposite directions. The high shear on the walls of the vessel causes an increase in granular pressure and temperature that makes the red blood cells oscillate and push them towards the center. On the contrary, the platelets push towards the walls. From these articles, it is clear how expendable the Gispaspow's theory is in scientific research.

In this paper, we start from this classical model by Gidaspow and Huang [2] in order to derive a hyperbolic system in the context of Rational Extended Thermodynamics [10–12]. Following this non-equilibrium theory, we consider as field variables not only the classical ones in [2], which are mass densities, velocities and RBCs temperature, but also the stress tensors, the RBG dynamic pressure and the heat flux. In order to describe this extended set of field variables, we need to introduce a new system of field equations. Following the idea of RET, we introduce for these fields a new set of balance laws, and, in particular, here we are inspired by a two-hierarchy structure model recently introduced in [11,12] for polyatomic gases. The balance equations needs closure and, following again the main idea of RET, we close them assuming local and instantaneous constitutive relations and using well-known universal physical principles, such as the entropy principle (existence of an entropy inequality and concavity of the entropy density) and the principle of relativity. The exploitation of the principle of relativity helps us in the determination of the velocity-dependence of the constitutive relations, while the principle of entropy implies some useful mathematical relations. The mathematical exploitation of the entropy principle is made by use of the Lagrange multipliers [13] (main field). The constitutive functions are all determined except for the thermal and caloric equation of state and the production terms.

We identify the remaining terms using the expressions of the RBCs pressure, viscosity and heat conductivity determined by use of the kinetic theory in [1]. In this way, we assure that our equations coincide with the classical GH model when we are in presence of slowly changing fields and small gradients. In addition, our model can describe other processes where rapid time changes or when a strong deviation from equilibrium occur. In fact, it has been shown that the field equations of RET can describe a range various non-equilibrium phenomena such as light shattering, sound waves, heat waves, structure of shock waves [10–12]. RET has been applied to monoatomic gases [10] and mixtures [10,11,14,15] with many interesting results. Recently, RET has been generalized to dense and rarefied polyatomic gases both in the classical [11,12,16–18] and in the relativistic framework [12,19,20], for metal electrons [21,22], to quantum systems [23] and also for biological systems [24–27], providing relevant results and good agreement with experimental data.

The model herein introduced, as all ET systems [28,29], is symmetric hyperbolic. Hyperbolicity guarantees finite speeds of propagation, while symmetric hyperbolic systems imply well-posedness of Cauchy problems (i.e., existence, uniqueness and continuous dependence on the data). Hyperbolic models overcome the paradox of infinite propagation speed of disturbances, and they are better suited to describe transient regimes. Moreover,

they offer a richer scenario of dynamics which can be useful to describe biological problems, see for example [24–27] and the references therein. We expect that the model herein introduced can also describe phenomena in blood flow in which a rapid change in the blood pressure occurs, such as the sudden occlusion of a vessel or rapid initial changes in the fields due to external conditions.

Furthermore, from the results obtained for the heat transfer problem in a monoatomic gas or gas-mixtures, see for example [14,15,30,31] and the references therein, we also expect that our model can show more interesting spatial effects also for time-independent phenomena. Indeed, it has been shown that the field equations of RET are able to put in evidence more spatial effects than classical Navier–Stokes–Fourier equations, when a radial symmetry is present or a particular geometry is taken into account. Therefore, we expect that our model can describe more accurately the blood flow in a vessel and in particular in the case of change in the radial symmetry, like stenosis, or other defects of vessels.

Finally, it was recently shown that the field equations of RET are to describe the thermal-diffusion effect, contrary to the classical equations. In consideration of this RET result, we expect that our model is able to show some interesting additional features to the description of the Fahraeus–Lindqvist effect, which is anyway already present in the Gidaspow classical model [2].

The remainder of this article is the following: In Section 2, we recall the Gidaspow classical model, while in Section 3, we introduce the balance equations for our model. Sections 4–6 are devoted to the applications of the physical principles for the determinations of the constitutive relations. In particular, the Galilean invariance principle helps us to determine the dependence of the constitutive relations on the velocity, while the entropy principle implies some algebraic relations for the constitutive relations. In order to avoid too long calculations, we limit the study near the thermodynamic equilibrium by expanding the constitutive relations in terms of the non-equilibrium fluxes. In Section 7, a summary of the consequences of the principle is made, and in Section 8, the comparison with the Gidaspow model is shown with the identification of the last unknown functions. So all coefficients and functions here introduced can be obtained in terms of those calculated by Gidaspow using the kinetic theory. In Section 9, some simple calculations are presented in order to show with a simple numerical example the applicability of our model. Finally, in Section 10, some conclusions and final remarks are discussed.

2. Gidaspow Model

Gidaspow's model [2] describes blood as a mixture of a fluid and the RBCs and uses as field equations the two conservation laws of mass for the two constituents, the two balance laws of momentum and a balance law for the total energy of RBCs. These equations, typical of classical thermodynamics, can be recast in the following form:

$$
\begin{aligned}
&\frac{\partial \rho_f}{\partial t} + \frac{\partial \left(\rho_f v_k^f \right)}{\partial x_k} = 0, \\[4pt]
&\frac{\partial \rho_s}{\partial t} + \frac{\partial \left(\rho_s v_k^s \right)}{\partial x_k} = 0, \\[4pt]
&\frac{\partial \left(\rho_f v_i^f \right)}{\partial t} + \frac{\partial \left(\rho_{ik}^f + \rho_f v_i^f v_k^f \right)}{\partial x_k} = G_i^f + I_i^f, \\[4pt]
&\frac{\partial \left(\rho_s v_i^s \right)}{\partial t} + \frac{\partial \left(\rho_{ik}^s + \rho_s v_i^s v_k^s \right)}{\partial x_k} = G_i^s + I_i^s, \\[4pt]
&\frac{\partial}{\partial t} \left(\rho_s e_s + \tfrac{1}{2} \rho_s v_l^s v_l^s \right) + \\[4pt]
&\frac{\partial}{\partial x_k} \left(q_k^s + \rho_{kl}^s v_l^s + \rho_s e_s v_k^s + \tfrac{1}{2} \rho_s v_l^s v_l^s v_k^s \right) = v_l^s G_l^s + I^s + v_l^s I_l^s,
\end{aligned}
\tag{1}
$$

where ρ_f and ρ_s are respectively the plasma and the RBCs densities, v_k^f and v_k^s their velocities, ρ_{ik}^f and ρ_{ik}^s the stress tensors, e_s is the RBCs specific interval energy and q_k^s is the

RBCs heat flux. The field Equation (1) are closed by the classical Navier–Stokes and Fourier equations, which assume the stress tensor and the heat flux proportional to the gradient of the velocity or the temperature:

$$\rho_{ij}^s = -2\mu^s \frac{\partial v_{(i}^s}{\partial x_{j)}} + \left(\frac{2}{3}\mu^s - \xi_s\right)\frac{\partial v_k^s}{\partial x_k}\delta_{ij}$$

$$\rho_{ij}^f = -2\mu^f \left(\frac{\partial v_{(i}^f}{\partial x_{j)}} - \frac{2}{3}\frac{\partial v_k^f}{\partial x_k}\delta_{ij}\right),$$

$$q_i^s = -K_s \frac{\partial \theta^s}{\partial x_i}. \tag{2}$$

In the previous equations and through the paper, the round brackets in the indexes denote the symmetric parts of a tensor. The viscosities μ^s, ξ_s and μ^f and the heat conductivity K_s are calculated by Gidaspow by use of the kinetic theory and written explicitly in [2]. The right-hand side terms in Equation (1) are the body forces

$$G_i^f = \rho_f g_i + \varepsilon_s \partial_i p_f,$$

$$G_i^s = \rho_s g_i - \varepsilon_s \partial_i p_f, \tag{3}$$

due to external specific force g_i, the actions that the fluid exerts on the RBCs $-\varepsilon_s \partial_i p_f$ and vice versa (ε_s is the volume fraction of RBCs). Furthermore, the production terms take into account the interaction between plasma and RBCs, that are

$$I_i^f = -I_i^s = \beta\left(v_i^s - v_i^f\right),$$

$$I^s = -\gamma. \tag{4}$$

Moreover, the coefficients β and γ are evaluated in [2].

Equations (1)–(3) form the classical system of field equations, which consists of nine equations in the nine fields ρ_α, v_i^α and θ_α.

3. Balance Equations

Starting from Gidaspow's model, we introduce a system of hyperbolic balance equations. First of all, we consider as field variables not only the densities, the velocities and the temperature but also the stress tensors and the heat flux. The fields equations for all these variables are the two conservation laws of mass for the two constituents, the two balance laws of momentum and the balance law for the total energy of RBCs, as in (1). Furthermore, instead of assuming the classical Navier–Stokes and Fourier expressions (2) for the stress tensors and for the heat flux, we introduce for these new variables supplementary balance equations. In particular, the system of balance equations that we assume has the following form:

$$\frac{\partial \rho^\alpha}{\partial t} + \frac{\partial\left(\rho^\alpha v_k^\alpha\right)}{\partial x_k} = 0,$$

$$\frac{\partial\left(\rho^\alpha v_i^\alpha\right)}{\partial t} + \frac{\partial F_{ik}^\alpha}{\partial x_k} = G_i^\alpha + I_i^\alpha = H_i^\alpha,$$

$$\frac{\partial M_{ll}^\alpha}{\partial t} + \frac{\partial M_{kll}^\alpha}{\partial x_k} = 2v_l^\alpha H_l^\alpha + 2I^\alpha, \tag{5}$$

$$\frac{\partial F_{ij}^\alpha}{\partial t} + \frac{\partial F_{ijk}^\alpha}{\partial x_k} = 2v_{(i}^\alpha H_{j)}^\alpha + S_{ij}^\alpha,$$

$$\frac{\partial M_{ill}^s}{\partial t} + \frac{\partial M_{ikll}^s}{\partial x_k} = 3v_{(i}^s v_l^s H_{l)}^s + 2\frac{\rho_{il}^s}{\rho^s}H_l^s + 2e^s H_i^s + 2v_i^s I^s + S_{ill}^s,$$

with $\alpha = f$ or s.

Equation (5)$_{1,2}$ and (5)$_3$ with $\alpha = s$ coincide exactly with the classical Gidaspow Equation (1) indicating with $2M_{ll}^\alpha$ total energy and $2M_{ill}^\alpha$ the total energy flux. Equation (5)$_3$

with $\alpha = f$ is a supplementary equation here introduced for the total energy for the fluid. The other equations of (5) are introduced assuming that the fluxes in the first equations are the densities in the new equations. This structure was obtained by integration from the Boltzmann equation for a monatomic gasses [10] and recently this structure was modified for polyatomic gasses [11,12,17], where the two hierarchies structure (for the *F*'s and the *M*'s) is introduced by considerations from the kinetic theory.

System (5) represents a set of 24 equations for the 24 field variables that are the two densities ρ^α, the two velocities v_i^α, the two temperatures θ^α, the traceless (here and in the following, square brackets in the indexes denote the traceless part of a symmetric tensor) part stress tensors $\rho^\alpha_{<ij>}$, the RBCs dynamic pressure Δ^s and heat flux q_i^s. For sake of simplicity, we assume that $q_i^f = 0$. The aim of this paper is to close system (5), that is to express all quantities in (5) in terms of these 24 fields.

First of all, in the spirit of Rational Extended Thermodynamics, we assume that the unknown quantities in (5) are local and instantaneous. This means that is at one time and one point they depend on the field variables at the same time and at the same point. So, indicating a generic unknown quantity with φ, it must hold

$$\varphi = \varphi\left(\rho^\alpha, v_i^\alpha, \theta^\alpha, \rho^\alpha_{<ij>}, \Delta^s, q_i^s\right). \tag{6}$$

Then, if the functions φ are known, we can insert them into the balance Equation (5) in order to obtain a closed system of 24 field equations in the 24 fields. The obtained system, for the assumption in (6), will be a set of quasi-linear first order differential equations, whose solution is called thermodynamic process.

4. Decomposition in Convective and Non Convective Terms

We start now to decompose the densities, fluxes and productions in (5) into convective and non-convective parts, according to the single component case. Indeed, assuming that the balance equations for each constituent of the mixture must be invariant under a Galilean transformation, it is possible to determine the velocity dependence of the quantities in (5) on the two velocities, so we have:

$$
\begin{aligned}
F_{ik}^\alpha &= \rho_{ik}^\alpha + \rho^\alpha v_i^\alpha v_k^\alpha, \\
F_{ijk}^\alpha &= \rho_{ijk}^\alpha + 3\rho_{(ij}^\alpha v_{k)}^\alpha + \rho^\alpha v_i^\alpha v_j^\alpha v_k^\alpha, \\
M_{ll}^\alpha &= 2\rho^\alpha e^\alpha + \rho^\alpha v_l^\alpha v_l^\alpha, \\
M_{ill}^s &= 2q_i^s + 2\rho_{il}^s v_l^s + 2\rho^s e^s v_i^s + \rho^s v_l^s v_l^s v_i^s, \\
M_{ikll}^s &= m_{ikll}^s + 2\rho_{ikl}^s v_l^s + 4q_{(i}^s v_{k)}^s + 4\rho_{l(i}^s v_{k)}^s v_l^s \\
&\quad + \rho_{ik}^s v_l^s v_l^s + 2\rho^s e^s v_i^s v_k^s + \rho^s v_i^s v_k^s v_l^s v_l^s, \\
S_{ij}^\alpha &= s_{ij}^\alpha, \\
S_{ill}^s &= s_{ill}^s + 2v_l^s s_{ij}^s.
\end{aligned}
\tag{7}
$$

The explicit calculations for expressions (7) are shown in [10] pp. 53, 54.

It is easy to recognize the decomposition (7)$_1$ of the moment fluxes in the stress tensors and velocity dependent parts, the decomposition (7)$_3$ of the total energies in internal and kinetic energies and the decomposition (7)$_4$ for the total energy flux M_{ill}^s in the sum of the heat flux and the velocity dependent parts. These three decompositions are already present in (1). Then, we may say that ρ_{ik}^α in (7) are the stress tensors which, according to Gidaspow's model, are expressed by

$$
\begin{aligned}
\rho_{ik}^s &= \rho_{<ik>}^s + (p^s + \Delta^s)\delta_{ik}, \\
\rho_{ik}^f &= \rho_{<ik>}^f + p^f \delta_{ik},
\end{aligned}
\tag{8}
$$

while q_i^s is the RBCs heat flux.

Then, by insertion of the decomposition (7) into the balance Equation (5), one obtains

$$
\frac{d_\alpha \rho^\alpha}{dt} + \rho^\alpha \frac{\partial v_k^\alpha}{\partial x_k} = 0,
$$

$$
\rho^\alpha \frac{d_\alpha v_i^\alpha}{dt} + \frac{\partial \rho_{ik}^\alpha}{\partial x_k} = H_i^\alpha,
$$

$$
\frac{d_\alpha}{dt}\left(\rho^\alpha e^\alpha\right) + \rho^\alpha e^\alpha \frac{\partial v_k^\alpha}{\partial x_k} + \rho_{kl}^\alpha \frac{\partial v_l^\alpha}{\partial x_k} + \frac{\partial q_k^\alpha}{\partial x_k} = I^\alpha, \tag{9}
$$

$$
\frac{d_\alpha \rho_{ij}^\alpha}{dt} + \rho_{ij}^\alpha \frac{\partial v_k^\alpha}{\partial x_k} + 2\rho_{k(i}^\alpha \frac{\partial v_{j)}^\alpha}{\partial x_k} + \frac{\partial \rho_{kij}^\alpha}{\partial x_k} = s_{ij}^\alpha,
$$

$$
2\frac{d_s q_i^s}{dt} + \frac{\partial m_{ikll}^s}{\partial x_k} + 2\rho_{ikl}^s \frac{\partial v_l^s}{\partial x_k} + 4q_{(i}^s \frac{\partial v_{k)}^s}{\partial x_k} - 2\frac{\rho_{il}^s}{\rho^s}\frac{\partial \rho_{lk}^s}{\partial x_k} - 2e^s \frac{\partial \rho_{ik}^s}{\partial x_k} = s_{ill}^s,
$$

where the two material derivatives $d_\alpha/dt = \partial/\partial t + v_k^\alpha \partial/\partial x_k$ have been introduced. It can be clearly seen that these equations do not depend on the velocities but only on their derivatives and this is in agreement with the Galileian invariance applied to each constituent of the mixture.

With these decompositions, the unknown functions become ρ_{ijk}^α, m_{ikll}^s, s_{ij}^α and s_{ill}^s. These quantities must be determined in terms of the set ρ^β, θ^β, $\rho_{<ij>}^\beta$, Δ^s, q_i^s and, in virtue of the principle of material frame indifference applied to the whole mixture ([10] p. 86, [11] p. 313, [12], [32] pp. 6, 177, [33]), they cannot depend on the velocity of the two components v_i^β but only on their relative velocity $V_i^s = v_i^s - v_i^f$. Equivalently, they can be considered as functions of the diffusion velocity

$$
u_i^s = v_i^s - v_i \quad \text{where} \quad v_i = \frac{\rho^s v_i^s + \rho^f v_i^f}{\rho^s + \rho^f} \tag{10}
$$

is the mixture velocity. Summarizing, the constitutive functions to be determined are

$$
\rho_{ijk}^\alpha = \rho_{ijk}^\alpha\left(\rho^\beta, \theta^\beta, u_i^s, \rho_{<ij>}^\beta, \Delta^s, q_i^s\right),
$$

$$
m_{ikll}^s = m_{ikll}^s\left(\rho^\beta, \theta^\beta, u_i^s, \rho_{<ij>}^\beta, \Delta^s, q_i^s\right),
$$

$$
s_{ij}^\alpha = s_{ij}^\alpha\left(\rho^\beta, \theta^\beta, u_i^s, \rho_{<ij>}^\beta, \Delta^s, q_i^s\right), \tag{11}
$$

$$
s_{ill}^s = s_{ill}^s\left(\rho^\beta, \theta^\beta, u_i^s, \rho_{<ij>}^\beta, \Delta^s, q_i^s\right).
$$

5. Entropy Principle

The entropy principle asserts that it exists a quantity h, called entropy, which satisfies a balance equation of the form

$$
\frac{\partial h}{\partial t} + \frac{\partial h_k}{\partial x_k} = \Phi \geq 0 \tag{12}
$$

for all thermodynamic processes, that is for all solutions of the field equations. In this paper, in line with the methods of Rational Extended Thermodynamics [10–12], we use this physical principle in order to determine the constitutive Equation (11).

First of all, as described before for the fields, the concave entropy density h, the entropy flux, $h_k = hv_k + \phi_k$, and the entropy production Φ must depend on the field variables through constitutive functions of the form

$$
h = h\left(\rho^\beta, \theta^s, u_i^s, \rho_{<ij>}^\beta, \Delta^s, q_i^s\right),
$$

$$
\phi_k = \phi_k\left(\rho^\beta, \theta^\beta, u_i^\beta, \rho_{<ij>}^\beta, \Delta^s, q_i^s\right), \tag{13}
$$

$$
\Phi = \Phi\left(\rho^\beta, \theta^\beta, u_i^\beta, \rho_{<ij>}^\beta, \Delta^s, q_i^s\right).
$$

Then, since the entropy principle must hold for all thermodynamic processes that is for all solutions of the field Equation (9), we use the field Equation (9) as constraints for the validity of the entropy principle, This can be mathematically done, by introduction [13] of the so-called Liu–Lagrange multipliers λ's, so the entropy inequality (12) becomes

$$
\begin{aligned}
&\frac{\partial h}{\partial t} + \frac{\partial (h v_k + \phi_k)}{\partial x_k} \\
&- \sum_{\alpha=f,s} \left\{ \lambda^\alpha \left[\frac{d_\alpha \rho^\alpha}{dt} + \rho^\alpha \frac{\partial v_k^\alpha}{\partial x_k} \right] \right. \\
&+ \frac{1}{\rho^\alpha} \left(\rho^\alpha \lambda_i^\alpha - 2\rho_{ik}^s \lambda_{knn}^s - 3\rho^s \theta^s \lambda_{inn}^s \right) \left[\rho^\alpha \frac{d_\alpha v_i^\alpha}{dt} + \frac{\partial \rho_{ik}^\alpha}{\partial x_k} - H_i^\alpha \right] \\
&+ \hat{\lambda}_{ll}^\alpha \left[\frac{d_\alpha}{dt}(\rho^\alpha e^\alpha) + \rho^\alpha e^\alpha \frac{\partial v_k^\alpha}{\partial x_k} + \rho_{kl}^\alpha \frac{\partial v_l^\alpha}{\partial x_k} + \frac{\partial q_k^\alpha}{\partial x_k} - I^\alpha \right] \\
&+ \lambda_{ij}^\alpha \left[\frac{d_\alpha \rho_{ij}^\alpha}{dt} + \rho_{ij}^\alpha \frac{\partial v_k^\alpha}{\partial x_k} + 2\rho_{k(i}^\alpha \frac{\partial v_{j)}^\alpha}{\partial x_k} + \frac{\partial \rho_{kij}^\alpha}{\partial x_k} - s_{ij}^\alpha \right] \right\} \\
&- \lambda_{inn}^s \left[2\frac{d_s q_i^s}{dt} + \frac{\partial m_{ikll}^s}{\partial x_k} + 2\rho_{ikl}^s \frac{\partial v_l^s}{\partial x_k} + 4q_{(i}^s \frac{\partial v_{k)}^s}{\partial x_k} - 2\frac{\rho_{il}^s}{\rho^s}\frac{\partial \rho_{lk}^s}{\partial x_k} - 2e^s \frac{\partial \rho_{ik}^s}{\partial x_k} - s_{ill}^s \right] = \Psi \geq 0,
\end{aligned}
\tag{14}
$$

which must be valid for all values of ρ^β, θ^s, u_i^s, $\rho_{<ij>}^\beta$, Δ^s and q_i^s.

We start now the mathematical evaluations of (14). We have to say that the introduction of the Lagrange multipliers adds new unknown quantities. In agreements with the other quantities, we assume that also the λ's must depend on the fields through constitutive relations of the form (11) and (13).

It is easy to see that in (14) there is only one term which depends on the external force H_i^α so, since the constitutive relations must be valid independently of the forces applied to the system, we can conclude that

$$
\rho^\alpha \lambda_i^\alpha - 2\rho_{ik}^s \lambda_{knn}^s - 3\rho^s \theta^s \lambda_{inn}^s = 0
\tag{15}
$$

must hold.

Then, in equality (14) there are only few terms that depend only on the fields and not on their derivatives. These terms are the production terms in the balance equations and the entropy production, so we recover the so-called residual inequality, that yields the entropy production in terms of the other quantities

$$
\Phi = \hat{\lambda}_{ll}^s I^s + \sum_\alpha \lambda_{ij}^\alpha s_{ij}^\alpha + \lambda_{inn}^s s_{ill}^s \geq 0
\tag{16}
$$

and whose sign must be verified when the other quantities are determined.

Another relation, that can be easily determined form (14), is the following:

$$
\begin{aligned}
&\sum_\alpha \left[\hat{\lambda}_{vv}^s \rho_{ij}^s + 2\lambda_{iv}^\alpha \rho_{jv}^\alpha + 2\lambda_{vnn}^s \left(\rho_{ijv}^s + q_j^s \delta_{iv} \right) \right] \\
&= \left[h - \sum_\alpha \left(\rho^\alpha \lambda^\alpha + \frac{3}{2}\rho^s \theta^s \lambda_{ll}^s + \lambda_{kv}^\alpha \rho_{kv}^\alpha + 2q_v^s \lambda_{vll}^s \right) \right] \delta_{ij}.
\end{aligned}
\tag{17}
$$

This relation comes out eliminating all coefficients of the velocity v_k from equality (14).

We can now insert the first relations (15)–(17) into the balance law (14) in order to obtain a simpler equation. A very accurate evaluation of the complete entropy principle requires a lot of complex calculations that we prefer not to present here. In the next section, we start by limiting our analysis to processes near the thermodynamic equilibrium. A more general case will be presented in [34].

6. Evaluation of the Quantities Near Equilibrium

As explained before, from now on we limit our attention to processes close to thermodynamic equilibrium, characterized by vanishing non-equilibrium fluxes $\rho_{<ij>}^\alpha$, Δ^s, u_i^α and

q_i^α. This can be done by mathematically expanding the constitutive relations around the thermodynamic equilibrium and considering the terms which are at least quadratic in the non-equilibrium fluxes. In this way the entropy quantities become

$$h = \sum_\alpha \left[h_0^\alpha + h_1^\alpha \rho_{<ij>}^\alpha \rho_{<ij>}^\alpha + h_2^\alpha q_i^\alpha q_i^\alpha + h_3^\alpha \Delta^\alpha \Delta^\alpha + h_4^\alpha u_i^\alpha u_i^\alpha + h_5^\alpha q_i^\alpha u_i^\alpha \right] + O(3),$$

$$\phi_i = \sum_\alpha \left[\phi_1^\alpha q_i^\alpha + \phi_2^\alpha u_i^\alpha + \phi_3^\alpha \rho_{<ij>}^\alpha q_j^\alpha + \phi_4^\alpha \rho_{<ij>}^\alpha u_j^\alpha + \phi_5^\alpha q_i^\alpha \Delta^\alpha + \phi_6^\alpha u_i^\alpha \Delta^\alpha \right] + O(3),$$

(18)

the constitutive relations are given by

$$\rho_{ijk}^\alpha = 3 \left[\beta_1^\alpha q_{(i}^\alpha \delta_{jk)} + \beta_2^\alpha u_{(i}^\alpha \delta_{jk)} \right] + O(2),$$

$$m_{ijll}^s = \gamma_1^s \rho_{<ij>}^s + \frac{1}{3} \left[\gamma_0^s + \gamma_2^s \Delta^s \right] \delta_{ij} + O(2)$$

(19)

and the Lagrange multipliers take the form

$$\lambda^\alpha = \lambda_0^\alpha + \lambda_2^\alpha \Delta^\alpha + O(2),$$

$$\hat{\lambda}_{ll}^\alpha = \nu_0^\alpha + \nu_2^\alpha \Delta^\alpha + O(2),$$

$$\lambda_{ij}^\alpha = r_1^\alpha \rho_{<ij>}^\alpha + \frac{1}{3} \left[r_0^\alpha + r_2^\alpha \Delta^\alpha \right] \delta_{ij} + O(2),$$

$$\lambda_{ill}^s = s_1 q_i^s + s_2 u_i^s + O(2).$$

(20)

So the determination of the constitutive functions reduces to the calculations of all coefficients in (18)–(20), which depends only the equilibrium variables $(\rho^\alpha, \theta^\alpha)$.

We then proceed by insertion of relations (15)–(20) into the entropy inequality (14) which becomes a very long but linear equation in the derivatives of the field variables ρ^β, θ^s, u_i^s, $\rho_{<ij>}^\beta$, Δ^s and q_i^s. As already said, Equation (14) must be valid for all field variables; therefore, it must be valid also for all gradients of these variables. Then, setting the coefficients of the derivatives of ρ^β, θ^s, u_i^s, $\rho_{<ij>}^\beta$, Δ^s and q_i^s equal to zero, we obtain a big set of algebraic equations for the determination of the functions in (18)–(20). We prefer not to present all these equations here, but an interested reader can find similar calculations in [10–12,34]. We present here only some equilibrium relations and the results.

6.1. Gibbs Equation

We evaluate the coefficients in expansions (18)–(20), starting with the equilibrium terms. Indeed, from the derivatives of ρ^α and θ^α in (18)–(20) we obtain

$$\frac{\partial h_0}{\partial \rho^\alpha} = \frac{\partial h_0^\alpha}{\partial \rho^\alpha} = \lambda_0^\alpha + \nu_0^\alpha \frac{\partial}{\partial \rho^\alpha} (\rho^\alpha e^\alpha),$$

$$\frac{\partial h_0}{\partial \theta^\alpha} = \frac{\partial h_0^\alpha}{\partial \theta^\alpha} = \qquad \nu_0^\alpha \frac{\partial}{\partial \theta^\alpha} (\rho^\alpha e^\alpha).$$

(21)

which can be recasted in the differential form

$$d\left(\frac{h_0^\alpha}{\rho^\alpha} \right) = \left[\nu_0^\alpha \frac{\partial e^\alpha}{\partial \theta^\alpha} d\theta^\alpha + \left(\frac{\lambda_0^\alpha}{\rho_\alpha} + \nu_0^\alpha \frac{\partial e^\alpha}{\partial \rho^\alpha} + \frac{\nu_0^\alpha e^\alpha}{\rho^\alpha} - \frac{h^\alpha}{\rho_\alpha^2} \right) d\rho^\alpha \right].$$

(22)

This last equation can be compared with the well-known Gibbs relation, that expresses the variation of the equilibrium entropy density in terms of the thermodynamic variables. The expression of the Gibbs relation appropriate to each component of a mixture can be found for example in [10] p. 90 and it assumes the form

$$d\left(\frac{h_0^\alpha}{\rho^\alpha} \right) = \frac{1}{\theta^\alpha} \left(de^\alpha - \frac{p^\alpha}{\rho_\alpha^2} d\rho^\alpha \right).$$

(23)

Then, by comparison of (22) and (23), taking also into account that $e^\alpha = e^\alpha(\rho^\alpha, \theta^\alpha)$, it is possible to determine the equilibrium part of the Lagrange multipliers, i.e.,

$$v_0^\alpha = \tfrac{1}{\theta^\alpha}, \quad \lambda_0^\alpha = \tfrac{h^\alpha}{\rho_\alpha} - \tfrac{p^\alpha}{\rho_\alpha \theta^\alpha} - \tfrac{e^\alpha}{\theta^\alpha} = \tfrac{\mu^\alpha}{\theta^\alpha}, \quad r_0^\alpha = 0, \tag{24}$$

where μ^α represents the chemical potential.

6.2. Evaluation of the Functions Outside Equilibrium

By use of the Lagrange multipliers at equilibrium (24) it is possible to evaluate from the entropy inequality all coefficients in the constitutive relations (14) and (18)–(20) as explained before. We have the entropy quantities

$$h = \sum_{\alpha = f,s} \rho_\alpha \left[\tfrac{\mu^\alpha}{\theta^\alpha} + \tfrac{1}{\theta^\alpha}\left(e^\alpha + \tfrac{p^\alpha}{\rho_\alpha}\right)\right] - \tfrac{\rho_{<ij>}^\alpha \rho_{<ij>}^\alpha}{4p^\alpha \theta^\alpha} + \tfrac{q_i^s q_i^s}{\Psi^s} + \tfrac{\Delta^\alpha \Delta^\alpha}{2\theta^\alpha \Gamma^\alpha},$$

$$\phi_i = \tfrac{q_i^s}{\theta^s} + \sum_{\alpha = f,s} \rho_\alpha \left[\tfrac{\mu^\alpha}{\theta^\alpha} + \tfrac{1}{\theta^\alpha}\left(e^\alpha + \tfrac{p^\alpha}{\rho_\alpha}\right)\right] u_i^\alpha - \tfrac{\left[\gamma_1^s - 2\left(e^s + \tfrac{p^s}{\rho_s}\right)\right]}{\Psi^s}\rho_{<ij>}^s q_j^s \tag{25}$$

$$+ \tfrac{1}{\theta^s \Gamma^s}\left[-\tfrac{p_\theta^s}{\rho^s c_V^s} + \tfrac{5}{3}\tfrac{p^\alpha \theta^s \left[\gamma_1^s - 2\left(e^s + \tfrac{p^s}{\rho_s}\right)\right]}{\Psi^s}\right] q_i^s \Delta^s,$$

the first order terms in the constitutive relations

$$\rho_{ijk}^\alpha = 3\beta_1^\alpha q_{(i}^\alpha \delta_{jk)},$$

$$m_{ijll}^s = \gamma_1^s \rho_{<ij>}^s + \tfrac{1}{3}\left[\gamma_0^s + \gamma_2^s \Delta^s\right]\delta_{ij} \tag{26}$$

and in the Lagrange multipliers

$$\lambda^\alpha = \tfrac{\mu^\alpha}{\theta^\alpha} + \tfrac{1}{\theta^\alpha \Gamma^\alpha}\left[-p_\rho^\alpha + \tfrac{p_\theta^\alpha}{\rho^\alpha c_V^\alpha}\left(e^\alpha + \rho^\alpha e_\rho^\alpha\right)\right]\Delta^\alpha,$$

$$\lambda_i^\alpha = \tfrac{2}{\Psi^\alpha}\left(\tfrac{p^\alpha}{\rho^\alpha} + e^\alpha\right)q_i^\alpha,$$

$$\hat\lambda_{ll}^\alpha = \tfrac{1}{\theta^\alpha} - \tfrac{1}{\theta^\alpha \Gamma^\alpha}\tfrac{p_\theta^\alpha}{\rho^\alpha c_V^\alpha}\Delta^\alpha, \tag{27}$$

$$\lambda_{ij}^\alpha = -\tfrac{1}{2\theta^\alpha p^\alpha}\rho_{<ij>}^\alpha + \tfrac{1}{3\theta^\alpha \Gamma^\alpha}\Delta^\alpha \delta_{ij},$$

$$\lambda_{ill}^s = \tfrac{1}{\Psi^s}q_i^s,$$

with

$$\tfrac{\partial \gamma_0^\alpha}{\partial \rho_\alpha} = 6\left(e^\alpha + \tfrac{p^\alpha}{\rho_\alpha}\right)p_\rho^\alpha,$$

$$\tfrac{\partial}{\partial \rho_\alpha}\left[p^\alpha \gamma_1^\alpha\right] = 2\left[2\tfrac{p^\alpha}{\rho_\alpha} + e^\alpha\right]p_\rho^\alpha,$$

$$\gamma_2^\alpha = -\tfrac{3}{\Gamma^\alpha}\left[\tfrac{5}{3}p^\alpha \gamma_1^\alpha - 2\left(e^\alpha + \tfrac{p^\alpha}{\rho_\alpha}\right)\left(\Gamma^\alpha + \tfrac{5}{3}p^\alpha\right) + \tfrac{\Psi^\alpha p_\theta^\alpha}{\rho^\alpha c_V^\alpha \theta^\alpha}\right], \tag{28}$$

$$\Psi^\alpha = -\tfrac{T_\alpha^2}{3}\tfrac{\partial \gamma_0^\alpha}{\partial \theta^\alpha} + 2\theta_\alpha^2\left(e^\alpha + \tfrac{p^\alpha}{\rho_\alpha}\right)p_\theta^\alpha,$$

$$\Gamma^\alpha = -\tfrac{5}{3}p^\alpha + \rho^\alpha p_\rho^\alpha + \tfrac{\theta^\alpha \left(p_\theta^\alpha\right)^2}{\rho^\alpha c_V^\alpha},$$

$$\beta_1^\alpha = \tfrac{p^\alpha \theta^\alpha}{\Psi^\alpha}\left[\gamma_1^\alpha - 2\left(\tfrac{p^\alpha}{\rho^\alpha} + e^\alpha\right)\right].$$

The symbol c_V^α represents the specific heat at constant volume, defined as $c_V^\alpha = \partial e^\alpha/\partial \theta^\alpha$. We underline that these constitutive relations are in agreement with relations (5.58) p. 123 of Ruggeri and Sugiyama [11] since the structure for RBCs equations is equivalent to their two hierarchies system. We conclude that all functions introduced in the present analysis are determined except for the pressure p^α, the internal energy e^α and the production terms.

6.3. Production Terms

In this subsection, we will limit the generality of the production terms indeed we assume also for the production terms linear constitutive functions and in particular we apply the BGK assumption [35]. Therefore, the production term in each equation is proportional to the density of the same equation minus its equilibrium value. Explicitly we have:

$$s^\alpha_{<ij>} = -\frac{\rho^\alpha_{<ij>}}{\tau^\alpha_\sigma}, \quad s^s_{ll} = -3\frac{\Delta^s}{\tau^s_\Delta} \quad s^s_{ill} = -2\frac{q^s_i}{\tau^s_q}, \tag{29}$$

where τ^α_σ, τ^s_Δ and τ^s_q are the well-known relaxation times that are functions of ρ^α and θ^α. In the next section we identify these relaxation times by comparison with the Gidaspow's model [2].

7. Summary of the Consequences of the Physical Principles

We summarize here all consequences of the Galilein and entropy principles for the determination of the field equations. Our field equations are given by system (9) with the stress tensors in (8). The constitutive functions, which we calculated here, explicitly (limiting our analysis to linear terms) are written in (26) with the coefficients (28). The production terms are instead summarized in (29). The terms due to the external forces H^α_i coincide with those introduced by Gidaspow (3); The interaction terms in H^α_i and I^α are given by (4) in terms of the coefficients β and γ.

Therefore, all is explicit in terms of the fields, when the functions p^α and e^α, the relaxation terms τ^α_σ, τ^s_Δ and τ^s_q and the coefficients β and γ in the interaction terms are known in terms of the fields. In the next section, we evaluate these last coefficients by comparison of our model with that of Gidaspow, where he obtained these function with the help of the kinetic theory [1].

8. Comparison with Gidaspow's Model

In this section, we finally determine the remaining functions by comparison with the Gidaspow–Huang model [2]. First of all, we identify the RBCs pressure, that by calculations [1] in kinetic theory is given by

$$p_s = (1 + 2\chi)\rho_s\theta_s \quad \text{with} \quad \chi = (1 + e)\epsilon_s g_0, \tag{30}$$

where ϵ_s is the RBCs volume fraction, g_0 their radial distribution function, that can be chosen according to the Bagnold's equations, and e is the restitution coefficient, a measure of the elasticity of the particle collisions (see [1,2,36] for more details). We inserted in Table 1 some values for these quantities taken in [2]. Furthermore, we assume that the RBCs energy density is given by $e_s = 3\theta_s$, so that $c^s_V = 3$.

By use of the expression for the RBCs pressure (30), it is possible to evaluate all functions in (28) in terms of the RBCs density and temperature, which are

$$\gamma^s_0 = 12(2 + \chi)(1 + 2\chi)\rho_s\theta^2_s,$$
$$\gamma^s_1 = 2(5 + 4\chi)\theta_s,$$
$$\Psi^s = -4(2 + \chi)(1 + 2\chi)\rho_s\theta^3_s,$$
$$\Gamma^\alpha = -\frac{1}{3}(1 - 4\chi^2)\rho_s\theta_s, \tag{31}$$
$$\gamma^s_2 = 6(5 + 4\chi)\theta_s,$$
$$\beta^s_1 = -\frac{1}{2}\frac{1 + 2\chi}{2 + \chi}.$$

In order to evaluate the relaxation times of the model, we substitute the constitutive relations (26) and the production terms (29) into the balance Equation $(9)_{4,5}$ and we proceed with the first Maxwellian iterations [10]. This method is really simple to be used, but it is very useful: In practice, we neglect from the left-hand side of Equation $(9)_{4,5}$ the non-

equilibrium terms, that is, we consider the case in which no-rapid changes in the fields occur. The obtained equations become

$$2p^\alpha \frac{\partial v^\alpha_{<i}}{\partial x_{j>}} = -\frac{\rho^\alpha_{<ij>}}{\tau^\alpha_\sigma},$$

$$\frac{5}{3} p^s \frac{\partial v^s_k}{\partial x_k} = -\frac{\Delta^s}{\tau^\alpha_\Delta},$$ (32)

$$\left[\frac{1}{3} \frac{\partial \gamma^s_0}{\partial \theta_s} - 2\left(\frac{p_s}{\rho_s} + e_s\right) \frac{\partial p_s}{\partial \theta_s}\right] \frac{\partial \theta_s}{\partial x_k} = -2 \frac{q^s_k}{\tau^\alpha_q},$$

which have exactly the form of the stress tensor, the dynamic pressure and the heat flux predicted by the Navier–Stokes and the Fourier laws (2). This method is particularly useful; indeed, we can firstly evaluate, by comparison of (2) with (32), our relaxation times in terms of the Gidaspow viscosities and heat conductivity. Furthermore, we make sure that our equations predict the same results of the classical model when no-rapid changes in the fields occur. From the comparison and taking into account the expression of the viscosities and heat conductivity obtained in [2], we easily obtain

$$\tau^s_\sigma = \frac{1}{1+2\chi}\left[\frac{4}{5} + \frac{10}{96} \frac{\pi \epsilon_s}{\chi^2}\left(1 + \frac{4}{5}\chi\right)^2\right] \frac{\chi d_p}{\sqrt{\pi \theta_s}},$$

$$\tau^f_\sigma = \epsilon_f \frac{\mu_f}{p_f},$$

$$\tau^s_\Delta = \frac{4}{5} \frac{1}{1+2\chi} \frac{\chi d_p}{\sqrt{\pi \theta_s}},$$ (33)

$$\tau^s_q = \frac{1}{(1+2\chi)(2+\chi)}\left[1 + \frac{75}{384} \frac{\pi \epsilon_s}{\chi^2}\left(1 + \frac{6}{5}\chi\right)^2\right] \frac{\chi d_p}{\sqrt{\pi \theta_s}},$$

where d_p represents the RBCs diameter, while ϵ_f represents the fluid volume fraction with $\epsilon_s + \epsilon_f = 1$.

Finally, the coefficients β and γ in the expression of the interaction terms I^f_i and I^s in (9), are taken from the analogous terms in [2] so we set

$$\beta = \begin{cases} \frac{150\epsilon^2_s \mu_f}{(1-\epsilon_s)d^2_p} & \text{if} \quad \epsilon_f \le 0.8, \\ 0 & \text{if} \quad \epsilon_f > 0.8, \end{cases} \quad \text{and} \quad \gamma = \frac{12(1-e^2)g_0}{d_p\sqrt{\pi}} \rho_s \epsilon_s \theta^{\frac{3}{2}}_s.$$ (34)

The first relation takes into account only the linear term of [2] that is more appropriate for the linear constitutive equations presented in this paper.

All coefficients of our model are evaluated in terms of the field variables; furthermore, with the obtained functions, $-h$ is a convex function and the residual inequality (16) is verified.

Table 1. Values of the parameters of the problems [2].

Tube diameter	0.19 mm
Plasma density	1020 kg m^{-3}
Plasma viscosity	0.0012 kg m^{-1} s^{-1}
RBC size	8 μm
Plasma density	1092 kg m^{-3}
Restitution coefficient	0.95

Summarizing, system (9) with the relations (8), (26), (30) and (31), the productions terms (29) and (33), the terms due to the external forces in (3) and the interaction terms (4) and (34) is our system of field equations. It is a closed system of 24 field equations for the 24 field variables ρ^α, v^α_i, θ^α, $\rho^\alpha_{<ij>}$, Δ^s and q^s_i.

9. A Simple Numerical Example

In order to show clearly the applicability of our model, we present here a simple numerical example. We stress that these results are only a first simple application of our equations and they are not exhaustive of all the applicability of our model. They are also part of another paper [37], so an interested reader can found there more details.

We use the field equations herein obtained in order to describe the stationary blood flow in a vessel, which is idealized as the gap between two infinite parallel plates. We can assume that the flow is stationary and the fields depend only on the *x*-coordinate orthogonal to the two walls, so the complete set of field equations becomes a more simple set of ODE. We also assume that the RBCs and fluid velocities have the direction of the flow and they are driven by an external pressure, considered as external force.

It order to recover easily the solution, we linearize the ODE around an equilibrium state characterized by constant ρ_0^α, θ_0^α and vanishing fluxes. We illustrate in Figures 1 and 2 the solution obtained for $\epsilon_s = 0.24$, with the parameters in Table 1, and in terms of the dimentionless variables

$$\hat{x} = \frac{x}{L}, \quad \hat{\theta}^\alpha = \frac{\theta^s}{\theta_0^\alpha}, \quad \hat{\rho}^\alpha_{<ij>} = \frac{\rho^\alpha_{<ij>}}{\rho_0^s \theta_0^s}, \quad \hat{v}_3^\alpha = \frac{v_3^\alpha}{\sqrt{\theta_0^s}}. \tag{35}$$

The solutions show that the RBCs temperature decreases in the center of the vessel, where instead the RBCs density increases. This result is in agreement with the Fahraeus–Linqvist effect, as already shown in [2]. The two velocities are illustrated in Figure 2, showing that the fluid anticipates RBCs, as they are carried from the fluid self.
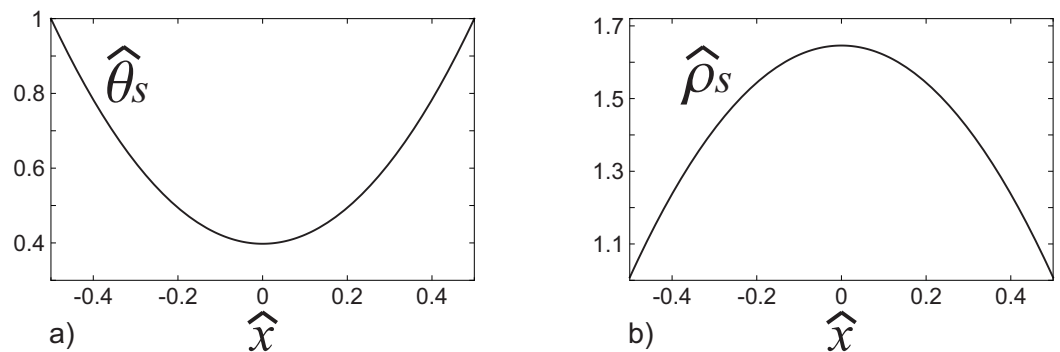

Figure 1. Dimensionless RBCs temperature (**a**) and dimensionless RBCs density (**b**) obtained in terms of $\hat{x}$ with $\epsilon_s = 0.24$.
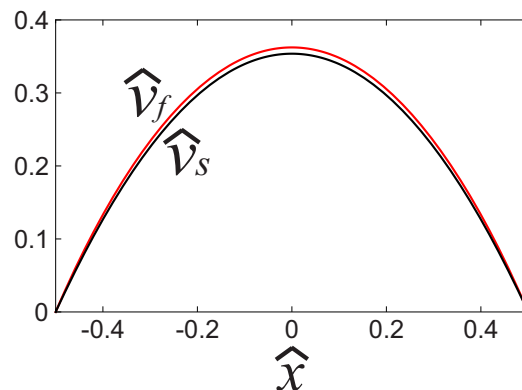

Figure 2. Dimensionless RBCs and fluid velocities in function of $\hat{x}$ obtained for $\epsilon_s = 0.24$.

The solutions in Figures 1 and 2 are somehow less general than those presented in [2], since here only the 1D plane case is considered, no-radial or 2D geometry, nor slip at the boundary as in [2] and furthermore the equations are linearized around the constant state.

Anyway, integration of this very simple and linearized system yields (see for details [37]) already the presence of some non-vanishing stress components. These non-vanishing components do not affect visibly the classical fields in Figures 1 and 2, due to the strong assumptions in this chapter, but they will play surely an important role in all fields when all simplifying assumptions will be removed. This will be shown clearly in further research.

10. Conclusions and Final Remarks

In this paper, starting from a model for blood flow obtained by Gidaspow and Huang [2] in Classical Thermodynamics, we derived an hyperbolic model based on the methods proper of Rational Extended Thermodynamics [10–12]. The stress tensors and the heat flux are considered as new field variables and a new set of field equations is assumed. The introduction of new equations involves new quantities that are here determined as constitutive functions in terms of the fields. This was done using the Galileian and entropy principles. We determine the remaining functions with Gidaspow's coefficients obtained in [1]. All parameters herein introduced are determined explicitly in terms of the fields and a simple numerical solution is shown in order to illustrate the applicability of our model.

The model herein obtained is hyperbolic, so it overcomes the paradox of infinite propagation speed of disturbances typical of parabolic ones. Our model can describe transient regimes better than the classical one and it offers a richer scenario of solutions as many other hyperbolic systems. Furthermore, by construction our model converges to the classical one [2] when the field gradients are negligible. On the contrary, we expect a better description of the phenomena in blood flow when strong changes in the blood occur. As seen for gases and mixture flow, we also expect that our Extended Thermodynamics model can describe more properly situations in with the domain presents more curvature of changes in the radial symmetry.

It must be said that, unfortunately, the integration of our field equations is more complicate than the classical ones, since our equations contain in practice the differential Equation (9)$_{4,5}$ instead of the classical Navier–Stokes and Fourier laws (2). This will imply a more complicate (normally numeric) integration of the whole set of equations and the introduction of initial values for stress tensors and heat flux.

Finally, our model has the same range of applicability of the classical model [2], so it can describe a great range of blood phenomena and cases. Some applications of our model are already under investigation and they will be part of further works.

Author Contributions: These authors contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Gidaspow, D. *Multiphase Flow and Fluidization Continuum and Kinetic Theory Descriptions*; Academic Press: Cambridge, MA, USA, 1994; ISBN 0-12-282470-9.
2. Gidaspow, D.; Huang, J. Kinetic theory based model for blood flow and its viscosity. *Ann. Biomed. Eng.* **2009**, *37*, 1534–1545. [CrossRef] [PubMed]
3. Huang, J.; Gidaspow, R.L.D. Pulsatile flow in a coronary artery using multiphase kinetic theory. *J. Biomech.* **2009**, *42*, 743–754. [CrossRef] [PubMed]
4. Fedosov, D.; Dao, M.; Karniadakis, G.; Suresh, S. Computational Biorheology of Human Blood Flow in Health and Disease. *Ann. Biomed. Eng.* **2014**, *42*, 368–387. [CrossRef] [PubMed]

5. Felicetti, L.; Femminella, M.; Reali, G.; Liò, P. A Molecular Communication System in Blood Vessels for Tumor Detection. In Proceedings of the NANOCOM' 14: ACM The First Annual International Conference on Nanoscale Computing and Communication, Atlanta, GA, USA, 6–9 May 2014.

6. Wu, W.; Yang, F.; Antaki, J.; Aubry, N.; Massoudi, M. Study of blood flow in several benchmark micro-channels using a two-fluid approach. *Int. J. Eng. Sci.* **2015**, *95*, 49–59. [CrossRef]

7. Lopes, D.; Puga, H.; Teixeira, J.C.; Teixeira, S.F. Influence of arterial mechanical properties on carotid blood flow: Comparison of CFD and FSI, studies. *Int. J. Mech. Sci.* **2019**, *160*, 209–218. [CrossRef]

8. Melka, B.; Gracka, M.; Adamczyk, W.; Rojczyk, M.; Golda, A.; Nowak, A.; Białecki, R.; Ostrowski, Z. Multiphase simulation of blood flow within main thoracic arteries of 8-year-old child with coarctation of the aorta. *Heat Mass Transfer* **2018**, *54*, 2405–2413. [CrossRef]

9. Gidaspow, D.; Chandra, V. Unequal granular temperature model for motion of platelets to the wall and red blood cells to the center. *Chem. Eng. Sci.* **2014**, *117*, 107–113. [CrossRef]

10. Müller, I.; Ruggeri, T. *Rational Extended Thermodynamics*; Springer: New York, NY, USA, 1998.

11. Ruggeri, T.; Sugiyama, M. *Rational Extended Thermodynamics Beyond the Monatomic Gas*; Springer: New York, NY, USA, 2015.

12. Ruggeri, T.; Sugiyama, M. *Classical and Relativistic Rational Extended Thermodynamics of Gases*; Springer: Cham, Switzerland, 2021.

13. Liu, I.S. Method of Lagrange multipliers for exploitation of the entropy principle. *Arch. Ration. Mech. Anal.* **1972**, *46*, 131. [CrossRef]

14. Barbera, E.; Brini, F. Heat transfer in gas mixtures: advantages of an extended thermodynamics approach. *Phys Lett. A* **2011**, *375*, 827–831. [CrossRef]

15. Barbera, E.; Brini, F. Frame dependence of stationary heat transfer in an inert mixture of ideal gases. *Acta Mech.* **2014**, *225*, 3285–3307. [CrossRef]

16. Carrisi, M.C.; Montisci, S.; Pennisi, S. Entropy Principle and Galilean Relativity for Dense Gases, the General Solution without Approximations. *Entropy* **2013**, *15*, 1035–1056. [CrossRef]

17. Arima, T.; Taniguchi, S.; Ruggeri, T.; Sugiyama, M. Extended thermodynamics of dense gases. *Contin. Mech. Thermodyn.* **2011**, *24*, 271. [CrossRef]

18. Arima, T.; Carrisi, M.C.; Pennisi, S.; Ruggeri, T. Which moments are appropriate to describe gases with internal structure in Rational Extended Thermodynamics? *Int. J.-Non-Linear Mech.* **2021**, *137*, 103820. [CrossRef]

19. Pennisi, S.; Ruggeri, T. Relativistic extended thermodynamics of rarefied polyatomic gases. *Ann. Phys.* **2017**, *377*, 414–445. [CrossRef]

20. Arima, T.; Carrisi, M.C.; Pennisi, S.; Ruggeri, T. Relativistic Kinetic Theory of Polyatomic Gases: Classical Limit of a New Hierarchy of Moments and Qualitative Analysis. *Partial. Differ. Equ. Appl.* **2022**, *3*, 1–21. [CrossRef]

21. Barbera, E.; Brini, F. A rational extended thermodynamics model for metal electrons in bounded domains. *Ric. Mat.* **2019**, *68*, 37–56. [CrossRef]

22. Barbera, E.; Brini, F. New extended thermodynamics balance equations for an electron gas confined in a metallic body. *Ric. Mat.* **2021**, *70*, 181–194. [CrossRef]

23. Trovato, M. Quantum maximum entropy principle and quantum statistics in extended thermodynamics. *Acta Appl. Math.* **2014**, *132*, 605–619. [CrossRef]

24. Barbera, E.; Curró, C.; Valenti, G. On discontinuous travelling wave solutions for a class of hyperbolic reaction-diffusion models. *Physical D* **2015**, *308*, 116. [CrossRef]

25. Consolo, G.; Curró, C.; Grifó, G.; Valenti, G. Oscillatory periodic pattern dynamics in hyperbolic reaction-advection-diffusion models. *Phys. Rev. E* **2022**, *105*, 034206. [CrossRef]

26. Consolo, G.; Curró, C.; Valenti, G. Turing vegetation patterns in a generalized hyperbolic Klausmeier model. *Math. Methods Appl. Sci.* **2020**, *43*, 10474–10489. [CrossRef]

27. Consolo, G.; Grifó, G. Eckhaus instability of stationary patterns in hyperbolic reaction–diffusion models on large finite domains. *Partial. Differ. Equations Appl.* **2022**, *57*. [CrossRef]

28. Boillat, G. Sur l'existence et la recherche d'équations de conservation supplémentaires pour les systèmes hyperboliques. *Compt. Rend. Hebd. Seances Acad. Sci. A* **1974**, *278*, 909.

29. Ruggeri, T.; Strumia, A. Main field and convex covariant density for quasi-linear hyperbolic systems. Relativistic fluid dynamics. *Ann. Inst. Henri Poincaré* **1981**, *34*, 65.

30. Barbera, E.; Brini, F. Non-isothermal axial flow of a rarefied gas between two coaxial cylinders. *AAPP* **2017**, *95*, A1.

31. Barbera, E.; Brini, F. Stationary heat transfer in helicoidal flows of a rarefied gas. *EPL* **2017**, *120*, 34001. [CrossRef]

32. Müller, I. *Thermodynamics*; Pitman: London, UK, 1985; ISBN 0-273-08577-8.

33. Kremer, G.M. Extended thermodynamics of mixtures of ideal gases. *Int. J. Eng. Sci.* **1987**, *25*, 95–115. [CrossRef]

34. Pollino, A. (University of Messina, Messina, Italy). Dissertation, 2024. *in preparation*.

35. Bhatnagar, P.L.; Gross, E.P.; Krook, M. A model for collision processes in gases. I. Small amplitude processes in charged and neutral one-component systems. *Phys. Rev.* **1954**, *94*, 511–525. [CrossRef]

36. Savage, S.B. Granular flow at high shear rates. In *Theory of Dispersed Multiphase Flow*; Meyer, R.E., Ed.; Academic Press: New York, NY, USA, 1983; pp. 339–358.

37. Barbera, E.; Pollino, A. (University of Messina, Messina, Italy). A Linear Simulation of Blood Flow in a Vessel, 2022. *in preparation*.