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Pseudo General Overlap Functions and Weak Inflationary Pseudo BL-Algebras

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Abstract: General overlap functions are generalized on the basis of overlap functions, which have better application effects in classification problems, and the (weak) inflationary BL-algebras as the related algebraic structure were also studied. However, general overlap functions are a class of aggregation operators, and their commutativity puts certain restrictions on them. In this article, we first propose the notion of pseudo general overlap functions as a non-commutative generalization of general overlap functions, so as to extend their application range, then illustrate their relationship with several other commonly used aggregation functions, and characterize some construction methods. Secondly, the residuated implications induced by inflationary pseudo general overlap functions are discussed, and some examples are given. Then, on this basis, we show the definitions of inflationary pseudo general residuated lattices (IPGRLs) and weak inflationary pseudo BL-algebras, and explain that the weak inflationary pseudo BL-algebras can be gained by the inflationary pseudo general overlap functions. Moreover, they are more extensive algebraic structures, thus enriching the content of existing non-classical logical algebra. Finally, their related properties and their relations with some algebraic structures such as non-commutative residuated lattice-ordered groupoids are investigated. The legend reveals IPGRLs include all non-commutative algebraic structures involved in the article.

Keywords: pseudo general overlap functions; fuzzy implications; inflationary pseudo general residuated lattices; weak inflationary pseudo BL-algebras

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1. Introduction

As a special class of aggregation operators [1], overlap functions [2,3] are mainly used in applications involving overlap problems, and also play a significant role on the multi-attribute group decision making and image processing [4–6]. Later, Miguel et al. [7] relaxed two boundary conditions of overlap functions and studied general overlap functions that can be used to handle multiple situations. However, they require commutativity in their definitions, which limits their application in practical problems to a certain extent. On the other hand, following the research of Paiva et al. [8] on non-associative BL-algebras [9] based on overlap functions, they put forward the concept of inflationary BL-algebras in [10], but both of them were gained on commutative residuated lattices. In the paper, we present pseudo general overlap functions as a non-commutative generalization of general overlap functions, and discuss their corresponding non-commutative algebraic structures.

It is necessary to affirm our research motivation. (1) As a kind of extension of overlap functions, general overlap functions are also included in aggregation operators. However, the original aggregation functions, for example, copulas [11], do not require commutativity, which signifies that commutativity of general overlap functions is not necessary. (2) In non-classical mathematical logic [12], the commonly used conjunctions are t-norms [13,14] because they can better express the nature of logical "and". Continuous t-norms are a special class of general overlap functions, pseudo t-norms have been proposed as their non-commutative generalization and some results have been obtained [15–17]. In addition, in recent years, as a non-commutative generalization of overlap functions, pseudo overlap functions [18,19] have also been studied by some scholars, the facts have also proved that they are more effective in some practical applications. Batista et al. have also conducted some related research. Therefore, it is very natural to generalize the general overlap functions. (3) In practical problems, the classes of objects may have different emphasis; that is, they are not symmetrical. At this time, it is natural to think about using non-commutative functions in applications.

In this regard, we detach the commutativity of general overlap functions, give a construction theorem of pseudo general overlap functions, and explore the relationship between them and some other aggregation functions. Then, on the basis of the continuity of the functions, two residuated implications induced by them are discussed. Finally, a kind of non-commutative residuated lattice is defined; the inflationary pseudo general residuated lattices and weak inflationary pseudo BL-algebras are investigated.

The content of the article is arranged as below. In the second part, we recall a few prior pieces of knowledge, including the concepts of general overlap functions, pseudo overlap functions, fuzzy implications, etc. As for Section 3, we introduce the definition of pseudo general overlap functions, elaborate some construction methods, and then analyze their relations with continuous t-norms, continuous copula, pseudo overlap functions, and general overlap functions. After that, we study the residuated implications induced by them, and give some specific examples. In Section 4, inflationary pseudo general residuated lattices and weak inflationary pseudo BL-algebras are described, and the properties satisfied by them are discussed. Finally, the relationship between (weak) inflationary pseudo BL-algebras and other algebras such as non-commutative residuated lattice-ordered groupoids is studied. Conclusions and references are at the end of the paper.

2. Preliminaries

A number of basic notions that will be touched upon are listed.

Definition 1 ([2,8])**.** *Given a binary mapping O on $[0,1]$, if it meets requirements as below for arbitrary $x, y, z \in [0, 1]$, then it is called an overlap function:*

(O1) *O is symmetric;*
(O2) *$xy = 0$ when and only when $O(x,y) = 0$;*
(O3) *$xy = 1$ when and only when $O(x,y) = 1$;*
(O4) *$O(y,x) \geq O(z,x)$ when $y \geq z$;*
(O5) *O satisfies continuity.*

Definition 2 ([7])**.** *Given a binary mapping GO on $[0,1]$, if it meets requirements as below for arbitrary $x, y, z \in [0, 1]$, then it is called a general overlap function (in short GOF):*

(GO1) *GO is symmetric;*
(GO2) *$GO(x,0) = GO(0,y) = 0$;*
(GO3) *$GO(1,1) = 1$;*
(GO4) *$GO(y,x) \geq GO(z,x)$ when $y \geq z$;*
(GO5) *GO is continuous.*

Some examples of two-dimensional general overlap functions are as follows.

Example 1.

(1) *Any overlap function is a* GOF*.*
(2) *The mapping GO defined as $GO(x,y) = max\{x^2 + y^2 - 1, 0\}$ is a* GOF *rather than an overlap function.*

(3) *The function GO defined as* $GO(x,y) = 2xy + x^2y^2 - xy^2 - x^2y$ *is a* GOF *and it is also an overlap function.*

Definition 3 ([18,19])**.** *Given a binary mapping PO on* $[0,1]$*, if it meets requirements as below for arbitrary* $x, y, z \in [0, 1]$*, then it is called a pseudo overlap function:*

(PO1) $xy = 0$ *when and only when* $PO(x,y) = 0$*;*
(PO2) $xy = 1$ *when and only when* $PO(x,y) = 1$*;*
(PO3) *PO satisfies monotonic increasing property;*
(PO4) *PO is countinuous.*

Obviously, every overlap function is a pseudo overlap function. Some other common aggregation operators are defined as below.

Definition 4 ([1,8])**.** *Given a mapping A:* $[0,1]^n \to [0,1]$*, if it meets requirements as below then it is called an aggregation function:*

(A1) *A is non-decreasing about every variable: for arbitrary* $i \in \{1,2,\ldots,n\}$, $A(a_1,\ldots,y,\ldots,a_n) \geq A(a_1,\ldots,a_i,\ldots,a_n)$ *when* $y \geq a_i$*;*
(A2) *A holds two boundary conditions: (i)* $A(a_1,\ldots,a_n) = 0$ *when* $a_i = 0 (i = 1,\ldots,n)$ *and (ii)* $A(a_1,\ldots,a_n) = 1$ *when* $a_i = 1 (i = 1,\ldots,n)$*.*

Definition 5 ([12–14])**.** *Given a binary operation T on* $[0,1]$*, if it meets requirements as below for arbitrary* $x, y, z \in [0, 1]$*, then it is called a t-norm:*

(T1) *T satisfies symmetry;*
(T2) *T satisfies associativity;*
(T3) *T is increasing;*
(T4) *T with 1 as unit element, i.e.,* $T(1,x) = T(x,1) = 1$*.*

Definition 6 ([11])**.** *Given a binary operator C on* $[0,1]$*, if for arbitrary* $s, s', t, t' \in [0,1]$ *satisfying* $s \leq s'$ *and* $t \leq t'$*, it meets requirements as below, then it is called a copula:*

(1) $C(s,t) + C(s',t') \geq C(s,t') + C(s',t)$*;*
(2) $C(s,0) = C(0,t) = 0$*;*
(3) *it has 1 as unit element, i.e.,* $C(s,1) = C(1,s) = s$*.*

Definition 7 ([20])**.** *Given a two-dimensional mapping C on* $[0,1]$*, if it meets requirements as below, then it is called a fuzzy conjunction:*

(1) *C is non-decreasing about each element;*
(2) *boundary conditions:* $C(1,1) = 1, C(0,0) = C(0,1) = C(1,0) = 0$*.*

Fuzzy implication is closely related to operations and is defined as below.

Definition 8 ([3,11,20])**.** *Given a binary function I on* $[0,1]$*, if it meets requirements as below for arbitrary* $a, b, c \in [0, 1]$*, then it is called a fuzzy implication:*

(I1) *decreasing about first element:* $I(b,c) \leq I(a,c)$ *when* $a \leq b$*;*
(I2) *increasing about second variable:* $I(c,a) \leq I(c,b)$ *when* $a \leq b$*;*
(I3) $I(a,b) = 1$ *when* $a = b = 0$ *or* $a = b = 1$*,* $I(1,0) = 0$*.*

After that, some existing residuated lattice structures are given to facilitate the later content.

Definition 9 ([8,21])**.** *An algebra* $L = \langle L, \wedge, \vee, *, \to, 0, 1\rangle$ *is defined as a non-associative residuated lattice (also called "commutative residuated lattice-ordered groupoid") if it meets requirements as below:*

(naR1) $(L, \wedge, \vee, 0, 1)$ *is a lattice with 0 as the lower bound and 1 as the upper bound;*
(naR2) $(L, *, 1)$ *is a commutative groupoid and it has 1 as unit element;*

(naR3) $x * y \leq z$ *when and only when* $x \leq y \to z$ *for arbitrary* $x, y, z \in L$ *(residuation principle).*

Definition 10 ([8,21])**.** *Given a non-associative residuated lattice A, if it meets requirements as below for arbitrary $a, b, x, y \in A$, then it is called a non-associative BL-algebra (in short naBL-algebra):*

(naBL1) $x \wedge y = x * (x \to y)$ *(divisibility)*
(naBL2) $(x \to y) \vee \alpha_b^a(y \to x) = 1$, *where* $\alpha_b^a(y \to x) = (a * b) \to (a * (b * (y \to x)))$ *(α-prelinearity)*
(naBL3) $(x \to y) \vee \beta_b^a(y \to x) = 1$, *where* $\beta_b^a(y \to x) = b \to (a \to ((a * b) * (y \to x)))$ *(β-prelinearity)*

Definition 11 ([10])**.** *Given an algebra $L = \langle L, \wedge, \vee, *, \to, 0, 1\rangle$, if it meets requirements as below, then it is called an inflationary general residuated lattice (in short $IGRL$), where the operator $*$ satisfies $x \leq x * 1$ for arbitrary $x \in L$:*

(L1) $(L, \wedge, \vee, 0, 1)$ *is a lattice with 0 as the lower bound and 1 as the upper bound;*
(L2) $(L, *)$ *is a commutative groupoid, i.e., L is a nonempty set, the operator $*$ is commutative on L;*
(L3) $(*, \to)$ *satisfies the residuation principle.*

3. Pseudo General Overlap Functions and Residuated Implications

In this part, we state the related concepts of pseudo general overlap functions and residuated implications induced by them.

Definition 12. *Given a binary mapping PGO on $[0, 1]$, if it meets the following requirements then it is called a pseudo general overlap function (in short PGOF):*

(PGO1) $PGO(x, 0) = PGO(0, y) = 0$ *for arbitrary* $x, y \in [0, 1]$*;*
(PGO2) $PGO(1, 1) = 1$*;*
(PGO3) *PGO is non-decreasing;*
(PGO4) *PGO is countinuous.*

Example 2.

(1) *Every* GOF *and pseudo overlap function is a* PGOF*.*
(2) *The function PGO defined by $PGO(x, y) = max\{x^p + y^q - 1, 0\}$ where $p, q > 0$ is a* PGOF*, and it is a* GOF *when $p = q$.*
(3) *The function PGO defined by $PGO(x, y) = xy(x - x^2 - xy + x^2y + 1)$ is a* PGOF *and is also a pseudo overlap function, but not a* GOF*.*
(4) *The function PGO defined as $PGO(x, y) = min\{\frac{(x+1)\sqrt{y}}{2}, y\sqrt{x}\}$ is a* PGOF*, but it is not a* GOF*.*

Lemma 1. *Given a conjunctive continuous commutative aggregation function F on $[0, 1]$, F is a* GOF*.*

Proof. This is obvious from [7] (Proposition 2). □

Theorem 1. *Given a t-norm T, if it is continuous, then it meets* (PGO1) $\sim$ (PGO4)*, i.e., it is a* PGOF*.*

Proof. By Lemma 1, we can obtain that T is a general overlap function, and then it is a PGOF. □

Since any continuous pseudo-t-norm is a t-norm (see [15]) and any positive continuous copula is a pseudo overlap function (see [22]), based on the above theorem, we obtain the following relationship diagram between some operators (see Figure 1).

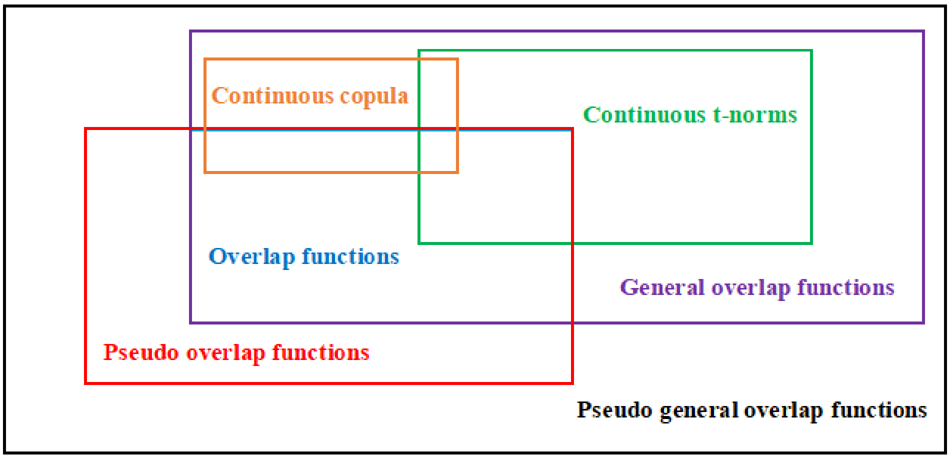

Figure 1. Relationship of the several kinds of functions considered in the paper.

Next, we give a construction theorem of pseudo general overlap function similarly.

Theorem 2. *The mapping PGO on $[0,1]$ is a* PGOF *when and only when there exist binary operators g, h on $[0,1]$ with $PGO(x,y) = \frac{g(x,y)}{g(x,y)+h(x,y)}$ satisfying conditions as below:*

(1) $g(x,z) \geq g(x,y), g(z,x) \geq g(y,x)$ *and* $h(x,z) \leq h(x,y), h(z,x) \leq h(y,x)$ *when* $y \leq z$*;*
(2) $g(x,0) = g(0,x) = 0$ *for arbitrary* $x \in [0,1]$*;*
(3) $h(1,1) = 0$*;*
(4) g *and* h *are continuous;*
(5) $g(x,y) + h(x,y) \neq 0$ *for arbitrary* $x, y \in [0,1]$.

Proof.
($\Rightarrow$) Assume that PGO is a PGOF, and we make $g(x,y) = PGO(x,y), h(x,y) = 1 - PGO(x,y)$; then $PGO(x,y) = \frac{g(x,y)}{g(x,y)+h(x,y)}$, (1) $\sim$ (5) are also obviously established.

($\Leftarrow$) Suppose that there are binary operators g, h on [0, 1] keeping $PGO(x,y) = \frac{g(x,y)}{g(x,y)+h(x,y)}$ and they satisfy the conditions (1)$\sim$(5), we verify that PGO is a PGOF. By (2) and (5), if $xy = 0$, then $g(x,y) = 0$ and $g(x,y) + h(x,y) \neq 0 \Rightarrow PGO(x,y) = 0$, i.e., PGO satisfies (PGO1). If $xy = 1$, by (3) and (5), we obtain $g(x,y) \neq 0$, then $PGO(x,y) = \frac{g(x,y)}{g(x,y)} = 1$, i.e., PGO satisfies (PGO2). Suppose that $x \leq y$, $PGO(z,x) - PGO(z,y) = \frac{g(z,x)}{g(z,x)+h(z,x)} - \frac{g(z,y)}{g(z,y)+h(z,y)} = \frac{g(z,x)h(z,y)-g(z,y)h(z,x)}{[g(z,x)+h(z,x)][g(z,y)+h(z,y)]}$, by (1), we have that $g(z,x) \leq g(z,y)$ and $h(z,y) \leq h(z,x) \Rightarrow g(z,x)h(z,y) - g(z,y)h(z,x) \leq 0$, i.e., $PGO(z,x) - PGO(z,y) \leq 0 \Rightarrow PGO(z,x) \leq PGO(z,y)$, analogously, $PGO(x,z) \leq PGO(y,z)$, so PGO satisfies (PGO3). It is obvious that PGO satisfies (PGO4) by (4). Therefore PGO is a PGOF. □

Several methods for generating new pseudo general overlap functions from existing functions are displayed as below.

Proposition 1. *Let $PGO_1, \dots, PGO_n$ be pseudo general overlap functions and $w_1, \dots, w_n$ be positive weights satisfying $\sum\limits_{i=1}^{n} w_i = 1$, then the operation $PGO(x,y) = \sum\limits_{i=1}^{n} w_i \cdot PGO_i(x,y)$ is a pseudo general overlap function.*

Proof. Monotonicity and continuity are obvious. If $xy = 0$, $PGO(x,y) = \sum\limits_{i=1}^{n} w_i PGO_i(x,y) = \sum\limits_{i=1}^{n} w_i \cdot 0 = 0$, and if $xy = 1$, $PGO(x,y) = \sum\limits_{i=1}^{n} w_i \cdot 1 = \sum\limits_{i=1}^{n} w_i = 1$. So PGO satisfies (PGO1) and (PGO2), then PGO is a pseudo general overlap function. □

Proposition 2. *Given two pseudo general overlap function PGO_1, PGO_2, and a continuous t-norm T, then the operator PGO_T formulated as $PGO_T(x,y) = T(PGO_1(x,y), PGO_2(x,y))$ is a* PGOF.

Proof. We can easily obtain $PGO_1(x,y) = PGO_2(x,y) = 0$ when $xy = 0$, since T has 1 as identity element, $PGO_T(x,y) = T(PGO_1(x,y), PGO_2(x,y)) = T(0,0) \leq T(0,1) = 0$, i.e., $T(0,0) = 0$, so PGO_T satisfies (PGO1). If $xy = 1$, $PGO_1(x,y) = PGO_2(x,y) = 1$, then $PGO_T(x,y) = T(PGO_1(x,y), PGO_2(x,y)) = T(1,1) = 1$, so PGO_T satisfies (PGO2). Since T is increasing and continuous, PGO_T is also increasing and continuous. It holds that PGO_T satisfies (PGO3) and (PGO4). □

We know that t-norms are included in aggregation operators, so we can expand to obtain the proposition as below.

Proposition 3. *Given pseudo general overlap functions $PGO_1, \ldots, PGO_n$, and A is a continuous aggregation function, then the mapping PGO_A formulated as $PGO_A(x,y) = A(PGO_1(x,y), \ldots, PGO_n\ (x,y))$ is a* PGOF.

Proof. We can easily obtain $PGO_i(x,y) = 0$ when $xy = 0$, for any $i \in \{1,\ldots,n\}$; then $PGO_A(x,y) = A(0,\ldots,0) = 0$, i.e., PGO_A satisfies (PGO1). Similarly, if $xy = 1$, $PGO_i(x,y) = 1$ for any $i \in \{1,\ldots,n\}$, then $PGO_A(x,y) = A(1,\ldots,1) = 1$, so PGO_A satisfies (PGO2). It is obvious that PGO_A is increasing and continuous. Therefore PGO_A is a PGOF. □

Proposition 4. *Given a pseudo general overlap function PGO, and a continuous t-norm T, then the mapping PGO^T defined as $PGO^T(x,y) = PGO(x,y) \cdot T(x,y)$ is a* PGOF.

Proof. $PGO^T(x,y) = PGO(x,y) \cdot T(x,y) = 0 \cdot T(x,y) = 0$ when $xy = 0$, if $xy = 1$, then $PGO^T(x,y) = PGO(x,y) \cdot T(x,y) = 1 \cdot T(1,1) = 1$, so we have PGO^T satisfies (PGO1) and (PGO2). It is clear that PGO^T satisfies (PGO3) and (PGO4). Therefore PGO^T is a pseudo general overlap function. □

Similarly, we can extend the t-norm in the above proposition to the aggregation function to obtain such a proposition.

Proposition 5. *Given a pseudo general overlap function PGO, and a continuous aggregation operator A, then function PGO^A formulated as $PGO^A(x,y) = PGO(x,y) \cdot A(x,y)$ is also a* PGOF.

Proof. It is obvious that the function PGO^A satisfies (PGO3) and (PGO4). Since $xy = 0 \Rightarrow PGO^A(x,y) = 0 \cdot A(x,y) = 0$ and $PGO^A(x,y) = 1 \cdot A(1,1) = 1$ when $xy = 1$, PGO^A also satisfies (PGO1) and (PGO2). It holds that PGO^A is a PGOF. □

PGOFs also no longer have unit element, so we have the following definition.

Definition 13. *Given a pseudo general overlap function PGO on $[0,1]$, it is called deflationary when it satisfies $\forall x \in [0,1]$, $PGO(x,1) \leq x$, $PGO(1,x) \leq x$; and it is called inflationary when it meets $\forall x \in [0,1]$, $PGO(x,1) \geq x$, $PGO(1,x) \geq x$.*

Example 3.

(1) *The pseudo general overlap function PGO defined by $PGO(x,y) = max\{0, x^p + y^q - 1\}$ where $p, q > 0$; when $p, q > 1$ it is deflationary, since $PGO(x,1) = x^p \leq x$ and $PGO(1,x) = x^q \leq x$. When $0 < p, q < 1$, it is inflationary, since $PGO(x,1) = x^p \geq x$ and $PGO(1,x) = x^q \geq x$.*

(2) *The pseudo general overlap function PGO defined by $PGO(x,y) = min\{\frac{(x+1)\sqrt{y}}{2}, y\sqrt{x}\}$ is inflationary, since $PGO(x,1) \geq x$ and $PGO(1,x) \geq x$.*

Now we discuss the residuated implications induced by pseudo general overlap functions. We first give an existing theorem as follows.

Theorem 3 ([3]). *Given a fuzzy conjunction C satisfying $C(1,y) > 0$ for arbitrary $y \in (0,1]$. For any $x, y \in [0,1]$ consider the fuzzy implication I_C induced by C as $I_C(x,y) = sup\{z \in [0,1] \mid C(x,z) \le y\}$, then statements as below are equivalent:*

(1) *C satisfies left continuity about the second element;*
(2) *C and I_C meet the residuation property, i.e., $C(x,y) \le z \Leftrightarrow I_C(x,z) \ge y$;*
(3) *$I_C(x,y) = max\{z \in [0,1] \mid C(x,z) \le y\}$.*

In fact, we can easily know that not all pseudo general overlap functions satisfy $PGO(1,y) > 0$ for any $y \in [0,1]$; for example, the mapping PGO defined as

$$PGO(x,y) = \begin{cases} 0 & min\{x,y\} \le \frac{1}{4} \\ \frac{16}{9}(x-\frac{1}{4})(y-\frac{1}{4})^2 + \frac{1}{4} & otherwise \end{cases} \tag{1}$$

is a pseudo general overlap function. However, when $y = 0.25 > 0$, $PGO(1,y) = 0$. So we consider the case in the following lemma.

Lemma 2. *Given an inflationary pseudo general overlap function PGO on $[0,1]$, it is a fuzzy conjunction satisfying $PGO(1,y) > 0$ for any $y \in (0,1]$.*

Proof. By definition it is clear that PGO is a fuzzy conjunction. Since PGO is inflationary, $PGO(1,y) \ge y > 0$ for any $y \in (0,1]$. □

Evidently, every PGOF is a fuzzy conjunction satisfying continuity according to definition. When the two arguments of an inflationary PGOF are symmetric, it can induce a residuated implication as an inflationary GOF (see [10]). If the function is no longer commutative, then we have the following proposition.

Proposition 6. *Given the function PGO is an inflationary* PGOF, *the mappings $I^{(1)}_{PGO}$, $I^{(2)}_{PGO}$ defined by $I^{(1)}_{PGO}(x,y) = sup\{z \in [0,1] \mid PGO(z,x) \le y\}$, $I^{(2)}_{PGO}(x,y) = sup\{z \in [0,1] \mid PGO(x,z) \le y\}$, respectively, are fuzzy implication, PGO and they satisfy the residuation principle. In addition, $I^{(1)}_{PGO}$, $I^{(2)}_{PGO}$ can be marked as follows: $I^{(1)}_{PGO}(x,y) = max\{z \in [0,1] \mid PGO(z,x) \le y\}$, $I^{(2)}_{PGO}(x,y) = max\{z \in [0,1] \mid PGO(x,z) \le y\}$.*

Proof. For arbitrary $x,y,z \in [0,1]$, $I^{(1)}_{PGO}(x,z) = \sup\{z' \in [0,1] \mid PGO(z',x) \le z\}$, $I^{(1)}_{PGO}(y,z) = \sup\{z' \in [0,1] \mid PGO(z',y) \le z\}$. Since PGO is increasing, it holds that $PGO(z',x) \le PGO(z',y)$ when $x \le y$. Then $PGO(z',y) \le z \Rightarrow PGO(z',x) \le z$, so $\{z' \in [0,1] \mid PGO(z',y) \le z\} \subseteq \{z' \in [0,1] \mid PGO(z',x) \le z\} \Rightarrow \sup\{z' \in [0,1] \mid PGO(z',y) \le z\} \le \sup\{z' \in [0,1] \mid PGO(z',x) \le z\}$, i.e., $I^{(1)}_{PGO}(y,z) \le I^{(1)}_{PGO}(x,z)$. Therefore, the function $I^{(1)}_{PGO}$ is decreasing about the first element. Similarly, we can obtain that the function $I^{(2)}_{PGO}$ is also decreasing about the first element. On the other hand, $I^{(1)}_{PGO}(z,x) = \sup\{z' \in [0,1] \mid PGO(z',z) \le x\}$ and $I^{(1)}_{PGO}(z,y) = \sup\{z' \in [0,1] \mid PGO(z',z) \le y\}$. Since $PGO(z',z) \le x \le y$, we have that $\sup\{z' \in [0,1] \mid PGO(z',z) \le x\} \le \sup\{z' \in [0,1] \mid PGO(z',z) \le y\}$, i.e., $I^{(1)}_{PGO}(z,x) \le I^{(1)}_{PGO}(z,y)$, so the function $I^{(1)}_{PGO}$ is increasing about the second element. Similarly, we also have that the function $I^{(2)}_{PGO}$ is increasing about the second element. Moreover, it holds that $I^{(1)}_{PGO}(0,0) = \sup\{z \in [0,1] \mid PGO(z,0) \le 0\} = \sup\{z \in [0,1] \mid 0 \le 0\} = 1$, $I^{(1)}_{PGO}(1,1) = \sup\{z \in [0,1] \mid PGO(z,1) \le 1\} = 1$. $I^{(1)}_{PGO}(1,0) = \sup\{z \in [0,1] \mid PGO(z,1) \le 0\}$, since PGO is inflationary, $PGO(z,1) \ge z$. If $z \in (0,1]$, $PGO(z,1) > 0$. Therefore, $I^{(1)}_{PGO}(1,0) = 0$. Similarly, we have that $I^{(2)}_{PGO}(0,0) = 1$, $I^{(2)}_{PGO}(1,1) = 1$, $I^{(2)}_{PGO}(1,0) = 0$. Therefore the function $I^{(1)}_{PGO}$ and $I^{(2)}_{PGO}$ are fuzzy implication. By the above theorem and lemma, since PGO is continuous, it is clear that PGO and $I^{(1)}_{PGO}$,

$I_{PGO}^{(2)}$ satisfy $PGO(x,y) \le z$ iff $I_{PGO}^{(1)}(y,z) \ge x$ iff $I_{PGO}^{(2)}(x,z) \ge y$. Moreover, $I_{PGO}^{(1)}(x,y) = \max\{z \in [0,1] \mid PGO(z,x) \le y\}$, $I_{PGO}^{(2)}(x,y) = \max\{z \in [0,1] \mid PGO(x,z) \le y\}$. □

Two fuzzy implications $I_{PGO}^{(1)}$, $I_{PGO}^{(2)}$ are also called residuated implications (R_{PGO}-implication) induced by the pseudo general overlap function PGO, and they form adjoint pairs $(PGO, I_{PGO}^{(1)})$ and $(PGO, I_{PGO}^{(2)})$. We give some examples as shown in the table below (see Table 1).

Table 1. Pseudo general overlap functions and their residuated implications.

	Pseudo General Overlap Function	Residuated Implications
(1)	$PGO(x,y) = max\{0, x^p + y^q - 1\}, 0 < p,q < 1$	$I_{PGO}^{(1)}(x,y) = \begin{cases} 1, & x^q \le y \\ \sqrt[p]{y - x^q + 1}, & otherwise \end{cases}$, $I_{PGO}^{(2)}(x,y) = \begin{cases} 1, & x^p \le y \\ \sqrt[q]{y - x^p + 1}, & otherwise \end{cases}$
(2)	$PGO(x,y) = xy(x - x^2 - xy + x^2y + 1)$	$I_{PGO}^{(1)}(x,y) = \begin{cases} \phi(x,y), & x > y \\ 1, & otherwise \end{cases}$, $I_{PGO}^{(2)}(x,y) = \begin{cases} \frac{-1-x+x^2+\sqrt{(1+x-x^2)^2+4(x-1)y}}{2(x^2-x)}, & x > y \\ 1, & otherwise \end{cases}$
(3)	$PGO(x,y) = min\{\frac{(x+1)\sqrt{y}}{2}, y\sqrt{x}\}$	$I_{PGO}^{(1)}(x,y) = \begin{cases} 1, & x \le y \\ \frac{y^2}{x^2}, & x > y \end{cases}$, $I_{PGO}^{(2)}(x,y) = \begin{cases} 1, & x \le y^2 \\ \frac{y}{\sqrt{x}}, & x > y^2 \end{cases}$
(4)	$PGO(x,y) = max\{min\{x, \frac{y}{2}\}, x + y - 1\}$	$I_{PGO}^{(1)}(x,y) = \begin{cases} max\{min\{\frac{x}{2}, y\}, y - x + 1\}, & x > y \\ 1, & otherwise \end{cases}$, $I_{PGO}^{(2)}(x,y) = \begin{cases} min\{max\{y - x + 1, 2 - 2x\}, 2y, 2x\}, & x > y \\ 1, & otherwise \end{cases}$
(5)	$PGO(x,y) = max\{xy^2, \frac{xy}{2}\}$	$I_{PGO}^{(1)}(x,y) = \begin{cases} 1, & max\{x^2, \frac{x}{2}\} \le y \\ min\{\frac{2y}{x}, \frac{y}{x^2}\}, & otherwise \end{cases}$, $I_{PGO}^{(2)}(x,y) = \begin{cases} 1, & x \le y \\ min\{\frac{2y}{x}, max\{\frac{1}{2}, \sqrt{\frac{y}{x}}\}\}, & otherwise \end{cases}$
(6)	$PGO(x,y) = max\{x^2y, x + y - 1\}$	$I_{PGO}^{(1)}(x,y) = \begin{cases} min\{max\{\frac{1-x}{x}, y - x + 1\}, \sqrt{\frac{y}{x}}\}, & x > y \\ 1, & otherwise \end{cases}$, $I_{PGO}^{(2)}(x,y) = \begin{cases} min\{max\{y - x + 1, \frac{1}{x+1}\}, \frac{y}{x^2}\}, & x > y \\ 1, & otherwise \end{cases}$

Where $\phi(x,y) = \frac{1}{3} + \sqrt[3]{-\alpha + \sqrt{\alpha^2 + \beta^3}} + \sqrt[3]{-\alpha - \sqrt{\alpha^2 + \beta^3}}$, and $\alpha = \frac{-27y + 9x(x^2-x) - 2(x-x^2)^2}{54(x^2-x)^2}$, $\beta = \frac{3x(x^2-x) - (x-x^2)^2}{9(x^2-x)^2}$.

4. Inflationary Pseudo General Residuated Lattices and Weak Inflationary Pseudo BL-Algebras

In this part, we extend the inflationary general residuated lattices to the inflationary pseudo general residuated lattices, defined as below.

Definition 14. *Given an algebra $A = \langle L, \wedge, \vee, *, \rightarrow, \rightsquigarrow, 0, 1\rangle$, where $*$ is a non-commutative inflationary binary operator, i.e., $1 * x \ge x$ and $x * 1 \ge x$, if it meets requirements as below then it is an inflationary pseudo general residuated lattice (in short $IPGRL$):*

(A1) *$\langle L, \wedge, \vee, 0, 1\rangle$ is a lattice with 0 as the lower bound and 1 as the upper bound;*
(A2) *$\langle L, *\rangle$ is a groupoid;*
(A3) *$x * z \le y$ when and only when $z \le x \rightarrow y$, $z * x \le y$ when and only when $z \le x \rightsquigarrow y$, for arbitrary $x, y, z \in L$ (two-residuation principle).*

Remark 1. *Given an IPGRL L,* (A3) *can also be marked as* $x * y \leq z$ *when and only when* $y \leq x \to z$ *when and only when* $x \leq y \rightsquigarrow z$ *for arbitrary* $x, y, z \in L$.

Example 4. *Given an inflationary* PGOF $*$, $\to$, *and* $\rightsquigarrow$ *are residuated implications induced from* $*$, *i.e.,* $x \to y = max\{z \in [0,1] \mid x * z \leq y\}$, $x \rightsquigarrow y = max\{z \in [0,1] \mid z * x \leq y\}$, *then* $L = \langle [0,1], \wedge, \vee, *, \to, \rightsquigarrow, 0, 1\rangle$ *is an IPGRL.*

Next, we discuss some properties satisfied by IPGRLs.

Proposition 7. *Given an IPGRL* $L = \langle L, \wedge, \vee, *, \to, \rightsquigarrow, 0, 1\rangle$, *for arbitrary* $a, b, c \in L$, *it satisfies:*

(A4) $a * (a \to b) \leq b$, $(a \rightsquigarrow b) * a \leq b$;
(A5) $a * b \leq a * c$ *and* $b * a \leq c * a$ *when* $b \leq c$;
(A6) $a \to b \leq a \to c$ *and* $a \rightsquigarrow b \leq a \rightsquigarrow c$ *when* $b \leq c$;
(A7) $c \to a \leq b \to a$ *and* $c \rightsquigarrow a \leq b \rightsquigarrow a$ *when* $b \leq c$;
(A8) $(a \vee b) * c = (a * c) \vee (b * c)$, $c * (a \vee b) = (c * a) \vee (c * b)$;
(A9) $(a \vee b) \to c = (a \to c) \wedge (b \to c)$, $(a \vee b) \rightsquigarrow c = (a \rightsquigarrow c) \wedge (b \rightsquigarrow c)$;
(A10) $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$, $a \rightsquigarrow (b \wedge c) = (a \rightsquigarrow b) \wedge (a \rightsquigarrow c)$;
(A11) $a \leq b \rightsquigarrow (a * b)$, $a \leq b \to (b * a)$;
(A12) $a * 0 = 0 * a = 0$;
(A13) $a \to 1 = a \rightsquigarrow 1 = 1$;
(A14) $a \leq b$ *when* $a \to b = 1$ *or* $a \rightsquigarrow b = 1$;
(A15) $a \leq b$ *iff a and b are comparable and* $a \to (b * 1) = 1$ *iff a and b are comparable and* $a \rightsquigarrow (1 * b) = 1$;
(A16) $1 * a = a * 1 = a$ *when and only when* $a \wedge b \geq a * b$ *and* $a \wedge b \geq b * a$;
(A17) *If* $*$ *is associative, then* $(a \to b) * (b \to c) \leq a \to c$ *and* $(b \rightsquigarrow c) * (a \rightsquigarrow b) \leq a \rightsquigarrow c$.

Proof.

(A4) By (A3), $a \to b \leq a \to b \Rightarrow a * (a \to b) \leq b$; similarly, $a \rightsquigarrow b \leq a \rightsquigarrow b$, then $(a \rightsquigarrow b) * a \leq b$;
(A5) Since $a * c \leq a * c$, by (A3), it is clear that $b \leq c \to (c * b)$; then $a \leq c \to (c * b)$ when $a \leq b$, so $c * a \leq c * b$. Similarly, since $b * c \leq b * c$, it holds that $a \leq b \leq c \rightsquigarrow (b * c)$; then $a * c \leq b * c$;
(A6) If $a \leq b$, by (A4) it is clear that $c * (c \to a) \leq a \leq b$, by (A3) it holds that $c \to a \leq c \to b$; similarly, $(c \rightsquigarrow a) * c \leq a \leq b$; then $c \rightsquigarrow a \leq c \rightsquigarrow b$;
(A7) If $a \leq b$, by (A4) and (A5), $a * (b \to c) \leq b * (b \to c) \leq c$; then by (A3) it is clear that $b \to c \leq a \to c$. Similarly, $(b \rightsquigarrow c) * a \leq (b \rightsquigarrow c) * b \leq c$, so $b \rightsquigarrow c \leq a \rightsquigarrow c$;
(A8) The former has been proved in detail in [10], here we only prove the latter. Because $a \leq a \vee b$ and $b \leq a \vee b$, by (A5), $c * a \leq c * (a \vee b)$ and $c * b \leq c * (a \vee b)$, so $c * (a \vee b) \geq (c * a) \vee (c * b)$. On the other hand, $c * a \leq (c * a) \vee (c * b)$ and $c * b \leq (c * a) \vee (c * b)$, so by (A3) it holds that $a \leq c \to [(c * a) \vee (c * b)]$ and $b \leq c \to [(c * a) \vee (c * b)]$, so $a \vee b \leq c \to [(c * a) \vee (c * b)]$. Therefore $c * (a \vee b) \leq [(c * a) \vee (c * b)]$. Then $c * (a \vee b) = (c * a) \vee (c * b)$;
(A9) Since $a \leq a \vee b$ and $b \leq a \vee b$, by (A7) it holds that $(a \vee b) \to c \leq a \to c$ and $(a \vee b) \to c \leq b \to c$, so $(a \vee b) \to c \leq (a \to c) \wedge (b \to c)$. Analogously, it holds that $(a \vee b) \rightsquigarrow c \leq a \rightsquigarrow c$ and $(a \vee b) \rightsquigarrow c \leq b \rightsquigarrow c$; then $(a \vee b) \rightsquigarrow c \leq (a \rightsquigarrow c) \wedge (b \rightsquigarrow c)$. On the other hand, assume that $p \leq a \to c$ and $p \leq b \to c$; then by (A3), $a * p \leq c$ and $b * p \leq c$. By (A8) it is clear that $(a \vee b) * p = (a * p) \vee (b * p) \leq c \Rightarrow p \leq (a \vee b) \to c$ according to (A3). So $(a \to c) \wedge (b \to c) \leq (a \vee b) \to c$. Similarly, $(a \rightsquigarrow c) \wedge (b \rightsquigarrow c) \leq (a \vee b) \rightsquigarrow c$.
(A10) Since $b \wedge c \leq b$ and $b \wedge c \leq c$, by (A6) we have that $a \to (b \wedge c) \leq a \to b$ and $a \to (b \wedge c) \leq a \to c$, so $a \to (b \wedge c) \leq (a \to b) \wedge (a \to c)$. Similarly, $a \rightsquigarrow (b \wedge c) \leq a \rightsquigarrow b$ and $a \rightsquigarrow (b \wedge c) \leq a \rightsquigarrow c$; then $a \rightsquigarrow (b \wedge c) \leq (a \rightsquigarrow b) \wedge (a \rightsquigarrow c)$. On the other hand, the certificate is the same as above;

(A11) Since $a * b \leq a * b$, then $a \leq b \rightsquigarrow (a * b)$ by (A3); similarly, because $b * a \leq b * a$, we can easily obtain $a \leq b \rightarrow (b * a)$;
(A12) Since $a \rightarrow 0 \geq 0$ and $a \rightsquigarrow 0 \geq 0$, by (A3), it is clear that $a * 0 \leq 0$ and $0 * a \leq 0$. Since $a * 0 \geq 0$ and $0 * a \geq 0$, it is clear that $a * 0 = 0 * a = 0$;
(A13) Since $a * 1 \leq 1$, by (A3) we have that $1 \leq a \rightarrow 1$, but $a \rightarrow 1 \leq 1$, so $a \rightarrow 1 = 1$. Similarly, $1 * a \leq 1 \Rightarrow 1 \leq a \rightsquigarrow 1 \Rightarrow a \rightsquigarrow 1 = 1$. Therefore, $a \rightarrow 1 = a \rightsquigarrow 1 = 1$.
(A14) If $a \rightarrow b = 1$, then $1 \leq a \rightarrow b$, by (A3), $a * 1 \leq b$, since $*$ is inflationary, $a \leq a * 1 \leq b$, so $a \leq b$. In the same way if $a \rightsquigarrow b = 1$, then $1 \leq a \rightsquigarrow b \Rightarrow 1 * a \leq b \Rightarrow a \leq 1 * a \leq b$; thus $a \leq b$.
(A15) We can easily obtain $a \leq b \Rightarrow a * 1 \leq b * 1$ and $1 * a \leq 1 * b$ by (A5); then $1 \leq a \rightarrow (b * 1)$ and $1 \leq a \rightsquigarrow (1 * b)$ according to (A3), since $a \rightarrow (b * 1) \leq 1$ and $a \rightsquigarrow (1 * b) \leq 1$, so $a \rightarrow (b * 1) = a \rightsquigarrow (1 * b) = 1$. On the other hand, if $a \rightsquigarrow (1 * b) = 1$ then $1 * a \leq 1 * b$, since a and b are comparable, $a \leq b$. Similarly, $a \rightarrow (b * 1) = 1 \Rightarrow a * 1 \leq b * 1$, according to a and b being comparable, it also holds that $a \leq b$.
(A16) For arbitrary $a, b \in L$, if $1 * a = a * 1 = a$, by (A5), $a * b \leq a * 1 = a$ and $a * b \leq 1 * b = b$, $b * a \leq 1 * a = a$ and $b * a \leq b * 1 = b$, so $a * b \leq a \wedge b$ and $b * a \leq a \wedge b$. On the other hand, if $a * b \leq a \wedge b$ and $b * a \leq a \wedge b$, it is clear that $a * 1 \leq a \wedge 1 = a$ and $1 * a \leq a \wedge 1 = a$. Because $*$ is inflationary, we can obtain that $a * 1 \geq a$ and $1 * a \geq a$. Thus, $a * 1 = a = 1 * a$.
(A17) By (A4) and (A5), $a * (a \rightarrow b) * (b \rightarrow c) \leq b * (b \rightarrow c) \leq c$. According to the residuation property, $a * [(a \rightarrow b) * (b \rightarrow c)] \leq c \Leftrightarrow (a \rightarrow b) * (b \rightarrow c) \leq a \rightarrow c$. Similarly, when $[(b \rightsquigarrow c) * (a \rightsquigarrow b)] * a = (b \rightsquigarrow c) * [(a \rightsquigarrow b) * a]$, we have that $(b \rightsquigarrow c) * (a \rightsquigarrow b) \leq a \rightsquigarrow c$.

□

Theorem 4. *Given an IPGRL $L = \langle L, \wedge, \vee, *, \rightarrow, \rightsquigarrow, 0, 1\rangle$, for arbitrary $x, y \in L$, it holds that $x \rightarrow y = \vee\{z \in L \mid x * z \leq y\}$ and $x \rightsquigarrow y = \vee\{z \in L \mid z * x \leq y\}$ when $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice. Moreover, $1 \rightarrow x \leq x$ and $1 \rightsquigarrow x \leq x$.*

Proof. By (A4), $x * (x \rightarrow y) \leq y$, so $(x \rightarrow y) \in \{z \in L \mid x * z \leq y\}$. Since $x * z \leq y$ when and only when $z \leq x \rightarrow y$, i.e., $x \rightarrow y = \max\{z \in L \mid x * z \leq y\}$, so $x \rightarrow y = \vee\{z \in L \mid x * z \leq y\}$. Similarly, since $(x \rightsquigarrow y) * x \leq y \Rightarrow (x \rightsquigarrow y) \in \{z \in L \mid z * x \leq y\}$, $z * x \leq y$ when and only when $z \leq x \rightsquigarrow y$, then $x \rightsquigarrow y = \max\{z \in L \mid z * x \leq y\} = \vee\{z \in L \mid z * x \leq y\}$. Secondly, according to the above, we can obtain $1 \rightarrow x = \vee\{z \in L \mid 1 * z \leq x\}$ and $1 \rightsquigarrow x = \vee\{z \in L \mid z * 1 \leq x\}$. Since $1 * x \geq x$ and $x * 1 \geq x$ for arbitrary $x \in L$, it holds that $z \leq 1 * z$ $(z * 1) \leq x$, i.e., $\vee\{z \in L \mid 1 * z \leq x\} \leq x$ and $\vee\{z \in L \mid z * 1 \leq x\} \leq x$. So $1 \rightarrow x \leq x$ and $1 \rightsquigarrow x \leq x$. □

Then the notion and some properties of weak inflationary pseudo BL-algebras are presented. We first give the definition of weak inflationary BL-algebra.

Definition 15. *Given an algebra $L = \langle L, \wedge, \vee, *, \rightarrow, 0, 1\rangle$, if it is an* IGRL *and satisfies the following requirements then it is called a weak inflationary BL-algebra (in short weak IBL-algebra):*
(1) *If $y \leq x$, $y = x * (x \rightarrow y)$ (general divisibility);*
(2) $(x \rightarrow (y * 1)) \vee (y \rightarrow (x * 1)) = 1$ *(general prelinearity).*

In fact, weak inflationary BL-algebras are a non-associative algebraic structure including BL-algebras (see [9]), which can be gained by the inflationary general overlap functions with unit element 1. (If an inflationary GOF satisfies the divisibility, it is equivalent to it having 1 as unit element.) Some scholars have also conducted generalization research on the basis of pseudo BL-algebras [23]. Then, based on inflationary pseudo general residuated lattices and considering the two residuated implications induced by inflationary PGOFs, we introduce a new kind of algebraic structure in which operator is non-associative, non-commutative, and does not have unit element.

Definition 16. *Given an $IPGRL$ $L = \langle L, \wedge, \vee, *, \rightarrow, \rightsquigarrow, 0, 1\rangle$, if it meets the following requirements then it is a weak inflationary pseudo BL-algebra (in short $WIPBL$):*
(WIPBL1) $y = x * (x \rightarrow y) = (x \rightsquigarrow y) * x$ *when* $y \leq x$ *(general two-divisibility);*
(WIPBL2) $(x \rightarrow (y * 1)) \vee (y \rightarrow (x * 1)) = (x \rightsquigarrow (1 * y)) \vee (y \rightsquigarrow (1 * x)) = 1$ *(general two-prelinearity).*

After that, we certify that there is a one-to-one correspondence between the inflationary PGOFs and the WIPBLs.

Lemma 3. *Given the function PGO is an inflationary* PGOF *on* $[0, 1]$*, PGO satisfies general two-divisibility and general two-prelinearity.*

Proof. We might as well write the function PGO as $*$, and $\rightarrow$ and $\rightsquigarrow$ represent two residuated implications induced by PGO.

(*i*) Since $*$ is inflationary, $y \leq x * 1$ when $y \leq x$. Moreover, $y \geq 0 = x * 0$, so there is $t \in [0, 1]$ satisfying $y = x * t$. Then we gain $y \geq x * t$, and $t \leq x \rightarrow y$ according to residuation property (A3). Thus by (A5), $y = x * t \leq x * (x \rightarrow y)$. On the other hand, by (A4), $x * (x \rightarrow y) \leq y$. So $x * (x \rightarrow y) = y$. Analogously, $(x \rightsquigarrow y) * x = y$ when $y \leq x$. Therefore PGO meets (WIPBL1).

(*ii*) Since $[0, 1]$ is linearly ordered, it holds that $x \leq y$ or $y \leq x$ for any $x, y \in [0, 1]$. When $x \leq y$, by (A15), $x \rightarrow (y * 1) = 1$ and $x \rightsquigarrow (1 * y) = 1$, that is, $(x \rightarrow (y * 1)) \vee (y \rightarrow (x * 1)) = (x \rightsquigarrow (1 * y)) \vee (y \rightsquigarrow (1 * x)) = 1$. Otherwise, $y \leq x \Rightarrow y \rightarrow (x * 1) = 1$ as well as $y \rightsquigarrow (1 * x) = 1$, i.e., PGO meets (WIPBL2).

Therefore PGO satisfies general two-divisibility and general two-prelinearity. □

Proposition 8. *Given the algebra $A = \langle [0, 1], \wedge, \vee, *, \rightarrow, \rightsquigarrow, 0, 1\rangle$, A is a* WIPBL *when $\wedge$ is the minimization operator, $\vee$ is the maximization operator, $*$ is an inflationary* PGOF*, $\rightarrow$ and $\rightsquigarrow$ are residuated implications induced by $*$.*

Proof. We can easily obtain $A = \langle [0, 1], \wedge, \vee, *, \rightarrow, \rightsquigarrow, 0, 1\rangle$ is a IPGRL; then by Lemma 3 it is clear that A is a WIPBL. □

Example 5. *Given the algebra $L = \langle [0, 1], \wedge, \vee, *, \rightarrow, \rightsquigarrow, 0, 1\rangle$, where $x \wedge y = min\{x, y\}$, $x \vee y = max\{x, y\}$, $x * y = min\{\frac{(x+1)\sqrt{y}}{2}, y\sqrt{x}\}$, $x \rightsquigarrow y = \begin{cases} 1, & x \leq y \\ \frac{y^2}{x^2}, & x > y \end{cases}$ and $x \rightarrow y = \begin{cases} 1, & x \leq y^2 \\ \frac{y}{\sqrt{x}}, & x > y^2 \end{cases}$ as* (3) *as shown in* Table 1*. It is clear that L satisfies* (A1) $\sim$ (A3)*, i.e., it is an $IPGRL$. If $y \leq x$, $x \rightarrow y = \frac{y}{\sqrt{x}}$, and $x \rightsquigarrow y = \frac{y^2}{x^2}$; then $x * (x \rightarrow y) = min\{\frac{(x+1)\sqrt{\frac{y}{\sqrt{x}}}}{2}, y\} = y$ (since $y \leq x \Rightarrow (y + \sqrt{x})^2 \leq (x + 1)^2 \Rightarrow (y + \sqrt{x})^2 - (y - \sqrt{x})^2 < (x + 1)^2 \Rightarrow 4y\sqrt{x} < (x + 1)^2 \Rightarrow 2\sqrt{y\sqrt{x}} < x + 1 \Rightarrow y < \frac{(x+1)\sqrt{\frac{y}{\sqrt{x}}}}{2}$), and $(x \rightsquigarrow y) * x = min\{\frac{(\frac{y^2}{x^2}+1)\sqrt{x}}{2}, y\} = y$ (since $(x - \sqrt{x}y)^2 + y^2(1 - x) = x^2 + y^2 - 2xy\sqrt{x} \geq 0 \Rightarrow \sqrt{x}y^2 + x^2\sqrt{x} \geq 2x^2y \Rightarrow y \leq \frac{(\frac{y^2}{x^2}+1)\sqrt{x}}{2}$), so L satisfies general two-divisibility. Moreover, since $(x \rightarrow (y * 1)) \vee (y \rightarrow (x * 1)) = (x \rightarrow \sqrt{y}) \vee (y \rightarrow \sqrt{x}) = 1$, $(x \rightsquigarrow (1 * y)) \vee (y \rightsquigarrow (1 * x)) = (x \rightsquigarrow y) \vee (y \rightsquigarrow x) = 1$, L satisfies general two-prelinearity. Thus the algebra L is a* WIPBL*.*

A few natures met by WIPBLs are shown as follows.

Proposition 9. *Given* WIPBL $L = \langle L, \wedge, \vee, *, \rightarrow, \rightsquigarrow, 0, 1\rangle$*, it satisfies requirements as below for arbitrary $a, b, c \in L$:*

(WIPBL3) $(a \rightsquigarrow b) * a \leq a$ *and* $a * (a \rightarrow b) \leq a$ *if* $b \leq a$*;*
(WIPBL4) $b \leq a \rightarrow b$ *and* $b \leq a \rightsquigarrow b$ *if* $1 * a = a * 1 = a$ *and* $b \leq a$*;*

(WIPBL5) $a \to b \le (b \to c) \rightsquigarrow (a \to c)$ *and* $a \rightsquigarrow b \le (b \rightsquigarrow c) \to (a \rightsquigarrow c)$ *if* $a * (b * c) = (a * b) * c$ *and* $c \le b \le a$;

(WIPBL6) $a \to b \le (c \to a) \to (c \to b)$ *and* $a \rightsquigarrow b \le (c \rightsquigarrow a) \rightsquigarrow (c \rightsquigarrow b)$ *if* $a * (b * c) = (a * b) * c$ *and* $b \le a \le c$;

(WIPBL7) $a \to b \le (c * a) \to (c * b)$ *and* $a \rightsquigarrow b \le (a * c) \rightsquigarrow (b * c)$ *if* $a * (b * c) = (a * b) * c$ *and* $b \le a$;

(WIPBL8) $(b \to c) * a \le b \to (c * a)$ *and* $a * (b \rightsquigarrow c) \le b \rightsquigarrow (a * c)$ *if* $a * (b * c) = (a * b) * c$ *and* $c \le b$.

Proof.

(WIPBL3) Since weak inflationary pseudo BL-algebra satisfies general two-divisibility, then $(a \rightsquigarrow b) * a = b \le a$ when $b \le a$, similarly, $a * (a \to b) \le a$.

(WIPBL4) If $b \le a$, $b = a * (a \to b) = (a \rightsquigarrow b) * a$. By (A16), $1 * a = a * 1 = a \Rightarrow a * (a \to b) \le a \wedge (a \to b) \le a \to b$, so $b \le a \to b$. Analogously, we can obtain $b \le a \rightsquigarrow b$.

(WIPBL5) According to general two-divisibility and (A5), if $c \le b \le a$, then $a * (a \to b) * (b \to c) = b * (b \to c) = c \le c$. Then by (A3) and associativity, it holds that $(a \to b) * (b \to c) \le a \to c \Rightarrow a \to b \le (b \to c) \rightsquigarrow (a \to c)$. Analogously, when $a * (b * c) = (a * b) * c$, we have that $a \rightsquigarrow b \le (b \rightsquigarrow c) \to (a \rightsquigarrow c)$.

(WIPBL6) According to two-divisibility and (A5), if $b \le a \le c$, then $(a \wedge c) * (a \to b) \le a * (a \to b) = b \le b$, and $a \wedge c = a = c) * (c \to a)$, so $[c * (c \to a)] * (a \to b) \le b$. If $a * (b * c) = (a * b) * c$, then $c * [(c \to a) * (a \to b)] = [c * (c \to a)] * (a \to b) \le b$, by (A3) $(c \to a) * (a \to b) \le c \to b \Rightarrow a \to b \le (c \to a) \to (c \to b)$. Analogously, $a \rightsquigarrow b \le (c \rightsquigarrow a) \rightsquigarrow (c \rightsquigarrow b)$.

(WIPBL7) Since weak inflationary pseudo BL-algebra satisfies two-divisibility, $a * (a \to b) = b \le b$ if $b \le a$. By (A5), $c * [a * (a \to b)] \le c * b$, thus $(c * a) * (a \to b) \le c * b$ according to the associativity. By (A3), it holds that $a \to b \le (c * a) \to (c * b)$. Similarly, $(a \rightsquigarrow b) * a = b \le b \Rightarrow (a \rightsquigarrow b) * a * c \le b * c \Rightarrow a \rightsquigarrow b \le (a * c) \rightsquigarrow (b * c)$.

(WIPBL8) By (A5), $(b \wedge c) * a \le c * a$. Since $c = b * (b \to c)$ when $c \le b$, $[b * (b \to c)] * a = b * [(b \to c) * a] \le c * a$, by (A3), $(b \to c) * a \le b \to (c * a)$. Similarly, if $a * (b * c) = (a * b) * c$, we obtain $a * (b \rightsquigarrow c) \le b \rightsquigarrow (a * c)$.

□

Proposition 10. *Each commutative* WIPBL *is a weak inflationary BL-algebra.*

Proof. Assume that $A = \langle L, \wedge, \vee, *, \to, \rightsquigarrow, 0, 1 \rangle$ is a WIPBL, we only need to confirm that A is commutative when and only when $x \to y = x \rightsquigarrow y$, for arbitrary $x, y \in A$. For any $x, y, z \in A$, if A is commutative, $x * z = z * x$, then by (A3), $x \le z \to y$ iff $z * x \le y$, that is, $x * z \le y \Leftrightarrow x \le z \rightsquigarrow y$, so $z \to y = z \rightsquigarrow y$. Moreover, if $z \to y = z \rightsquigarrow y$, we have that $x \le z \to y$ iff $x \le z \rightsquigarrow y$, by (A3), $z * x \le y$ iff $x * z \le y$, i.e., $z * x = x * z$; therefore A is commutative. □

In addition, inflationary pseudo BL-algebras are introduced by us, and they can be regarded as a noncommutative generalization of inflationary BL-algebras; they are also a subclass of WIPBLs, defined as below.

Definition 17. *Given an* $IPGRL$ $L = \langle L, \wedge, \vee, *, \to, \rightsquigarrow, 0, 1 \rangle$, *if for arbitrary* $x, y \in L$ *the following statements hold then it is called an inflationary pseudo BL-algebra (in short* $IPBL$*):*

(IPBL1) $x \wedge y = x * (x \to y) = (x \rightsquigarrow y) * x$ *(two-divisibility);*

(IPBL2) $(x \to (y * 1)) \vee (y \to (x * 1)) = (x \rightsquigarrow (1 * y)) \vee (y \rightsquigarrow (1 * x)) = 1$.

Example 6. *Given the function PGO* $(*)$ *formulated as* $PGO(x, y) = max\{x + y - 1, min\{\frac{y}{2}, x\}\}$, *the residuated implications* $I^{(1)}_{PGO}(\rightsquigarrow)$, $I^{(2)}_{PGO}(\to)$ *induced by it as* (4) *as shown in* Table 1. *It is* *clear that the algebra $L = \langle [0,1], \wedge, \vee, *, \rightarrow, \rightsquigarrow, 0, 1\rangle$ satisfies* (IPBL1) *and* (IPBL2), *i.e., L is an inflationary pseudo BL-algebra.*

We can easily know that an inflationary pseudo BL-algebra must be a weak inflationary pseudo BL-algebra, and vice versa. For example, the weak inflationary pseudo BL-algebra L given in Example 5 above is not an IPBL, since $x * (x \rightarrow y) = \frac{1}{4} \neq x \wedge y = \frac{1}{9}$ when we take $x = \frac{1}{9}$ and $y = \frac{1}{4}$.

In fact, we can obtain that inflationary pseudo BL-algebras can be obtained from the inflationary PGOFs with unit element 1; the details are as follows.

Proposition 11. *Given the algebra $L = \langle [0,1], \wedge, \vee, *, \rightarrow, \rightsquigarrow, 0, 1\rangle$, where $\wedge$ is the minimization operator, $\vee$ is the maximum operator, $*$ is an inflationary* PGOF *with unit element 1, $\rightarrow$ and $\rightsquigarrow$ are residuated implications induced by $*$. Then L is an inflationary pseudo BL-algebra.*

Proof. By Proposition 7 we obtain that the algebra L is a weak inflationary pseudo BL-algebra, i.e., $y = x * (x \rightarrow y) = (x \rightsquigarrow y) * x$ when $y \leq x$ and $(x \rightarrow (y * 1)) \vee (y \rightarrow (x * 1)) = (x \rightsquigarrow (1 * y)) \vee (y \rightsquigarrow (1 * x)) = 1$. If the operator $*$ has 1 as unit element, then by (A15) $x \leq y \Rightarrow x \rightarrow (y * 1) = 1$ and $x \rightsquigarrow (1 * y) = 1 \Rightarrow x \rightarrow y = 1$ and $x \rightsquigarrow y = 1$, so $x * (x \rightarrow y) = (x \rightsquigarrow y) * x = x = x \wedge y$; that is, L satisfies two-divisibility. Thus L is an inflationary pseudo BL-algebra. □

Finally, the definition of non-commutative residuated lattice-ordered groupoids is given as below so that we can analyze the relationship between several classes of noncommutative algebraic structures.

Definition 18. *Given a lattice $L = \langle L, \wedge, \vee, *, \rightarrow, \rightsquigarrow, 0, 1\rangle$, if it meets requirements as below for arbitrary $x, y, z \in L$, then it is called a non-commutative residuated lattice-ordered groupoid (in short RLG):*

(RLG1) $(L, \wedge, \vee, 0, 1)$ *has 0 as the lower bound and 1 as the upper bound;*
(RLG2) $(L, *)$ *is a groupoid and has 1 as unit element;*
(RLG3) *L satisfies the two-residuation principle.*

It is not difficult to find that WIPBLs and IPBLs are included in class IPGRL, and the intersection of WIPBLs and RLGs is the inflationary pseudo BL-algebras. Then the relation diagram between several types of algebras is as follows (see Figure 2).

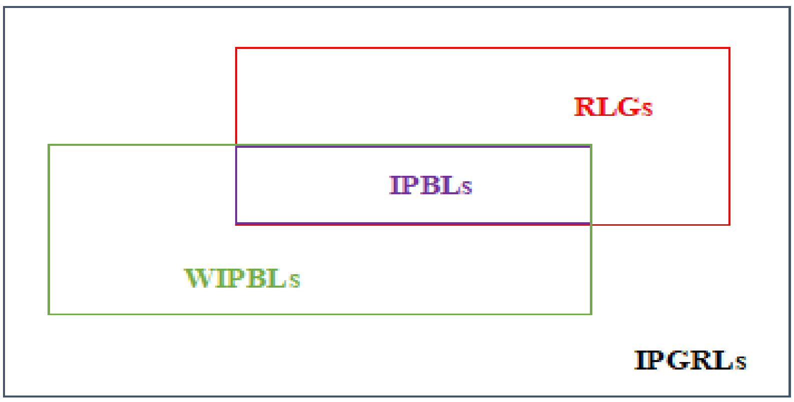

Figure 2. Relationship of the several kinds of non-commutative algebras considered in the paper.

5. Conclusions

In the article we first introduce a class of pseudo general overlap functions as a non-commutative generalization of general overlap functions, including pseudo overlap functions, continuous t-norms, and GOFs. Thus, the limitation of commutativity is cancelled and the application range of functions is broadened. Moreover, their construction theorem and the methods of constituting pseudo general overlap functions through continuous aggregation functions are given. Then we prove that the inflationary pseudo general overlap functions and their induced fuzzy implications satisfy the residuation property. Further, we propose the concept of IPGRL, generalize the IGRL to a non-commutative case, and discuss their properties. In addition, on the basis of inflationary pseudo general residuated lattices, we also extend the weak inflationary BL-algebras to be non-commutative, present weak inflationary pseudo BL-algebras satisfying general two-divisibility and general two-prelinearity, and verify that they can be obtained from pseudo general overlap functions. Moreover, inflationary pseudo BL-algebras are also studied as a non-commutative generalization of inflationary BL-algebras. Finally, a diagram revealing the relationship between several classes of lattice structures is shown in Figure 2; it is convenient to research other related algebraic structures later.

As the next research content, the properties of pseudo general overlap functions and their application in practical problems are worth discussing. In fact, from another point of view, the pseudo general overlap functions can also be regarded as the pseudo overlap functions with relaxed boundary conditions. Moreover, filters of IPGRLs and non-commutative residuated lattice-ordered groupoids can also be studied. It is worth mentioning that the operators corresponding to the (weak) inflationary pseudo BL-algebras introduced in this paper require complete continuity; that is, they are continuous for each variable, but , the operators can induce residuated implication as long as they meet the left continuity, so we can make further generalization about this. Besides, there is a class of MTL-algebras [24] on bounded lattices with a wider range than BL-algebras. Similarly, we can also investigate its corresponding non-commutative generalization. Furthermore, the fusions of pseudo general overlap functions, fuzzy rough sets, and related non-classical logic algebras (see [25–27]) are also interesting topics worthy of research.

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