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The *t*-Graphs over Finitely Generated Groups and the Minkowski Metric

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Abstract: In this paper, we introduce *t*-graphs defined on finitely generated groups. We study some general aspects of the *t*-graphs on two-generator groups, emphasizing establishing necessary conditions for their connectedness. In particular, we investigate properties of *t*-graphs defined on finite dihedral groups.

Keywords: finitely generated groups; finite groups; *t*-graph; subgraph; connected components; chromatic number

MSC: 20F05; 20E65; 05C12; 05C15

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1. Introduction

One of the best-known connections between groups and graph theory was presented by A. Cayley [1]. He gave a group G as a directed graph, where the vertices correspond to elements of G and the edges to multiplication by group generators and their inverses. Such a graph is called a Cayley diagram or Cayley graph of G. It is a central tool in combinatorial and geometric group theory.

Recent works reveal many different ways of associating a graph to a given finite group, most of which were inspired by a question posed by P. Erdös [2]. These differences lie in the adjacency criterion used to relate two group elements constituting the set of vertices of such a graph. Some essential authors in this context are A. Abdollahi [3], A. Ballester-Bolinches et al. [4–8], A. Lucchini [9,10], and D. Hai-Reuven [11], among others.

Our notation will be standard, as in [12] and [13] for groups and graphs. Let $G = \langle g_1, \ldots, g_n \rangle$ be a finitely generated group and suppose now that every element $g \in G$ can be uniquely written as follows

$$g = \prod_{i=1}^{n} g_i^{\epsilon_i}, \tag{1}$$

with $0 \leq \epsilon_i < m_i$, and $1 \leq i \leq n$. The numbers m_i can be, for example, the orders of the corresponding elements in the finite case, but they may also differ from these orders.

To determine a measure of the separation between two elements of G, we introduce the following distance map $d_1 : G \times G \longrightarrow \mathbb{N}_0$, defined by

$$d_1(g,h) = d_1\Big(\prod_{i=1}^{n} g_i^{\epsilon_i}, \prod_{i=1}^{n} g_i^{\delta_i}\Big) = \sum_{i=1}^{n} |\epsilon_i - \delta_i|. \tag{2}$$

The set G endowed with this distance d is a metric space. Note that d_1 is just the Minkowski l_p metric for $p = 1$ in $\{(\epsilon_1, \ldots, \epsilon_n) \mid 0 \leq \epsilon_i < m_i\}$. This is also called the taxicab distance, Manhattan distance, or grid distance.

G. Diaz-Porto and A. Torres-Grandisson introduced *t*-graphs using Minkowski's metric in [14,15]. These graphs can be defined by the group G as the underlying set of vertices and the following adjacency criteria: Let t be an integer number with $1 \leq t \leq n$. We say that $g, h \in G$ are adjacent if and only if $d_1(g,h) = t$.

The simplest example is when G is a finite cyclic group. Let $G = \langle g \rangle$ be a cyclic group with the finite order m. That is, $G = \{1, g, \cdots, g^{m-1}\}$. From (2), we have

$$d_1(g^i, g^j) = |i - j|, \text{ for all } 0 \leq i, j \leq m-1. \tag{3}$$

This means that in the t-graph of G there exists an edge between g^i and g^j if and only if $|i - j| = t$. Defining on G the following relation

$$g^i \sim g^j \iff i \equiv j \bmod t, \tag{4}$$

where $\sim$ is an equivalence relation, and then we have a partition of G in t classes given by

$$[g^i] := \{g^j \in G \mid j \equiv i \bmod t\}, \tag{5}$$

where $i \in \{0, 1, \ldots, t-1\}$. Then, the t-graph of a finite cyclic group G can be viewed as the union of t connected components, consisting of path graphs or isolated points. Consequently for $t \geq 2$, the t-graph is non-connected and 2-chromatic. The 1-graph of G is a finite path graph and then connected.

If t is a divisor of the group order m, then it is well known that G has a cyclic subgroup U of order m/t, and the elements of U form a subgraph with m/n vertices, which is a connected component of the t-graph of G.

If $G = \langle g \rangle$ is an infinite cyclic group, then the 1-graph of G is an infinite path graph. This statement follows directly from the definition of the t-graph.

An immediate consequence of the above discussion is that if G is a finite abelian group, say $G = \langle g_1 \rangle \times \cdots \times \langle g_n \rangle$, with $\text{ord}(g_j) = \epsilon_j$. Then, the 1-graph of G is the Cartesian product of n path graphs of lengths ϵ_j, respectively. That is an n-dimensional square grid graph. In general, using the above example, the t-graph of G is the Cartesian product of t components.

In the general case, if G is a direct product of cyclic groups with at least one infinite factor, then the t-graph of G is an infinite rectangular grid graph.

The first thing we can observe is that for a group G different generating systems can give different graphs. For instance, the groups $\mathbb{Z}_4 \times \mathbb{Z}_6$ and $\mathbb{Z}_2 \times \mathbb{Z}_{12}$ are isomorphic, but the graphs associated with the natural generating sets corresponding to these ways to present the group G are different.

On the other hand, if two groups admit generating systems such that every element g can be described as in (1), then it is possible that the corresponding t-graphs are the same, even though the groups are not isomorphic. We can see this in the following example. It is well known that the dihedral group D_n and the quaternion group Q_8 have the subsequent group presentation, respectively,

$$D_n = \langle a, b \mid a^2 = b^n = 1,\ aba = b^{-1} \rangle, \tag{6}$$

$$Q_8 = \langle a, b \mid a^4 = 1,\ a^2 = b^2,\ bab^{-1} = a^{-1} \rangle. \tag{7}$$

Furthermore,

$$\mathbb{Z}_2 \times \mathbb{Z}_4 = \langle a, b \mid a^2 = b^4 = 1,\ ab = ba \rangle. \tag{8}$$

Note that, in terms of their generators, the elements of D_4, Q_8, and $\mathbb{Z}_2 \times \mathbb{Z}_4$ can be written as follows

$$\{1, a, b, b^2, b^3, ab, ab^2, ab^3\}. \tag{9}$$

This means that the three groups have the same distance table (see Table 1) and, consequently, the same t-graphs for all t.

Table 1. Table of distances of $\mathbb{Z}_2 \times \mathbb{Z}_4$, D_4, and Q_8.

d_1	1	a	b	b^2	b^3	ab	ab^2	ab^3
1	0	1	1	2	3	2	3	4
a	1	0	2	3	4	1	2	3
b	1	2	0	1	2	1	2	3
b^2	2	3	1	0	1	2	1	2
b^3	3	4	2	1	0	3	2	1
ab	2	1	1	2	3	0	1	2
ab^2	3	2	2	1	2	1	0	1
ab^3	4	3	3	2	1	2	1	0

An illustration of the first four t-graphs of these three groups is presented in the following Figure 1.

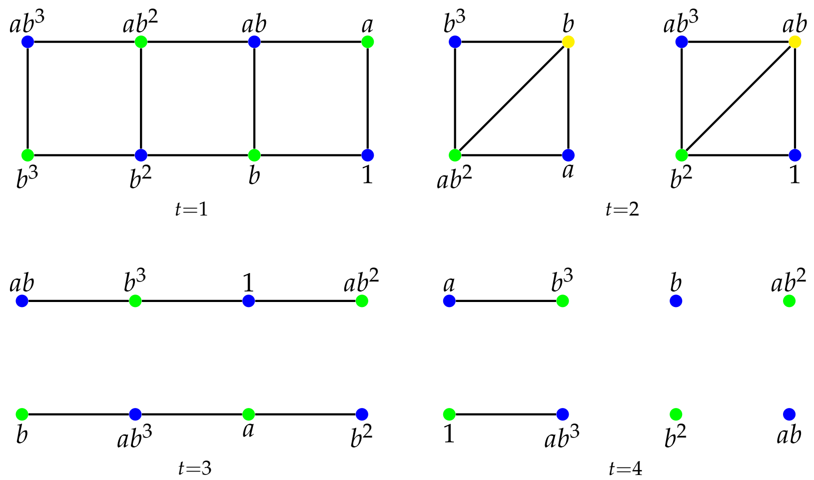

Figure 1. Some t-graphs of $\mathbb{Z}_2 \times \mathbb{Z}_4$, D_4, and Q_8.

Despite being non-isomorphic groups, these groups have precisely the same t-graphs since the metric used to define the adjacency criterion only considers the writing of the group's elements and not how they interact. This leads to the conclusion that any two-generator finite group $G = \langle a, b \rangle$, in which every element can be written in the form $a^i b^j$ with $0 \leq i \leq \text{ord}(a) - 1$ and $0 \leq j \leq \text{ord}(b) - 1$, has the same t-graphs as the group $\mathbb{Z}_{\text{ord}(a)} \times \mathbb{Z}_{\text{ord}(b)}$ since, when considering the form in which its elements are written in terms of the generators, the underlying sets are the same.

Therefore, to study the t-graphs of a finite group G, it is sufficient to consider abelian groups, expressed as products of cyclic groups. Naturally, this implies asking oneself, given an arbitrary group G, how to determine the abelian group with which it will share the same t-graphs. For example, the symmetric group of degree five has the same t-graph as $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_5$. In fact, in general, the group $\text{Sym}(n)$ can be factorized in the form $\text{Sym}(n) = \text{Sym}(n-1)\langle(12\cdots n)\rangle$, and, applying this property inductively, we have that $\text{Sym}(n)$ is generated by the set $\{(12), (123), \cdots, (12\cdots, n)\}$. In particular, the set

$$\{(12)^i(123)^j(1234)^k(12345)^l \mid i = 0,1, j = 0,1,2, k = 0,1,2,3, l = 0,1,2,3,4\}$$

is exactly $\text{Sym}(5)$.

On the other hand, this situation brings the possibility of studying t-graphs by defining the adjacency criterion in terms of another metric. This change may imply that the group structure plays a more critical role.

The main goal of this paper is to obtain some characterizations of the t-graphs $\mathcal{G}$ associated with the two-generator finite group G that can be expressed in the form

$$G = \langle a, b \rangle = \{a^i b^j \mid 0 \le i \le m,\ \ 0 \le j \le n\}. \tag{10}$$

where $m \le \text{ord}(a)$ y $n \le \text{ord}(b)$; $n, m \in \mathbb{Z}$. These numbers m and n depend exclusively on the structure, namely on the group's presentation and the order of G. We determine the number of connected components of $\mathcal{G}$ depending on whether t is an even or odd number.

2. Preliminaries on t-Graphs

A desirable property of the t-graphs is that every subgroup H of a group G naturally results in a subgraph. However, this is, in general, not true. For example, let G be the Klein four-group, say $G = \{1, a, b, ab\}$ and $H = \{1, ab\}$. Concerning their natural generating systems, ab and one are not adjacent in the 1-graph of G. Nevertheless, in H, they are adjacent.

Lemma 1. *Let $G = \langle g_1, \ldots, g_n \rangle$ be a finitely generated group and $H \le G$, with $H = \langle h_1, \ldots, h_n \rangle$ and $h_j = g_j^{k_j}$ for some natural numbers k_j. Then, the t-graph of H is a subgraph of the t-graph of G.*

Proof. It follows immediately from the definition of the t-graph. □

Lemma 2. *Let $G = \langle g_1, \cdots, g_n \rangle$ be a finitely generated group and suppose now that every element in $g \in G$ can be uniquely written as $g = \prod_{i=1}^n g_i^{\epsilon_i}$ with $0 \le \epsilon_i < m_i$ and $1 \le i \le n$. Further, let $H = \langle h_1, \cdots, h_n \rangle$ be a finitely generated group with the same property. If G and H are isomorphic, then the corresponding t-graphs are isomorphic, for all natural numbers t.*

Proof. Let $f : G \longrightarrow H$ be a group isomorphism with $f(g_i) = h_i$, and let $\mathcal{G} = (G, E_1)$ and $\mathcal{H} = (H, E_2)$ be the corresponding t-graphs of G and H, respectively. Suppose that $\{x, y\} \in E_1$ with $x = \prod_{i=1}^n g_i^{\epsilon_i}$ and $y = \prod_{i=1}^n g_i^{\delta_i}$. Then, $d_1(x, y) = t$, and we have

$$\begin{aligned} d_1(f(x), f(y)) = d_1\Big(\prod_{i=1}^n f(g_i)^{\epsilon_i}, \prod_{i=1}^n f(g_i)^{\delta_i}\Big) = d_1\Big(\prod_{i=1}^n h_i^{\epsilon_i}, \prod_{i=1}^n h_i^{\delta_i}\Big) \\ = \sum_{i=1}^n |\epsilon_i - \delta_i| = d_1(x, y). \end{aligned}$$

It follows that $\{f(x), f(y)\} \in E_2$. □

Remark 1. *Note that the reciprocal of the statement in Lemma 2 is, in general, not true. For example, the t-graphs of the dihedral D_4 and the quaternions group, Q_8 are isomorphic even though $D_4 \ncong Q_8$.*

To study t-graphs in the given context, we can use the spectral theory of graphs, which consists of studying the properties of the Laplacian matrix of a graph, more specifically, its eigenvalues and eigenvectors.

The Laplacian matrix of $\mathcal{G} = (V, E)$ is the $n \times n$ matrix $L = (l_{ij})$ indexed by V, whose (i, j)-entry is defined as follows

$$l_{ij} = \begin{cases} -1 & \text{if } \{v_i, v_j\} \in E \\ \deg(v_i) & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

To analyze the behavior of the number of connected components $k(\mathcal{G})$ of the t-graphs defined on a group G, we use the following theorem, which allows us to realize Tables 2 and 3. A proof of this theorem can be found in [16] (Theorem 7.1).

Theorem 1. *A graph $\mathcal{G}$ has k connected components if and only if the algebraic multiplicity of zero as the Laplacian eigenvalue is k.*

In the following, to study the t-graphs associated with a finite group G, we will consider only finite two-generator groups, which can be expressed in the form (10). These numbers m and n depend exclusively on the structure, namely on the group's presentation and the order of G.

Let G be such a group. To observe the behavior of the number of connected components $k(\mathcal{G})$ of a t-graph $\mathcal{G}$ determined by a group G, we make use of Theorem 1, with which we were able to make the following tables:

Table 2. Number of connected components of the t-graphs on $\mathbb{Z}_n \times \mathbb{Z}_2$.

$n\backslash t$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
2	1	2	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
3	1	2	4	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
4	1	2	2	6	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
5	1	2	1	4	8	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
6	1	2	1	2	6	10	-	-	-	-	-	-	-	-	-	-	-	-	-	-
7	1	2	1	2	4	8	12	-	-	-	-	-	-	-	-	-	-	-	-	-
8	1	2	1	2	2	6	10	14	-	-	-	-	-	-	-	-	-	-	-	-
9	1	2	1	2	1	4	8	12	16	-	-	-	-	-	-	-	-	-	-	-
10	1	2	1	2	1	2	6	10	14	18	-	-	-	-	-	-	-	-	-	-
11	1	2	1	2	1	2	4	8	12	16	20	-	-	-	-	-	-	-	-	-
12	1	2	1	2	1	2	2	6	10	14	18	22	-	-	-	-	-	-	-	-
13	1	2	1	2	1	2	1	4	8	12	16	20	24	-	-	-	-	-	-	-
14	1	2	1	2	1	2	1	2	6	10	14	18	22	26	-	-	-	-	-	-
15	1	2	1	2	1	2	1	2	4	8	12	16	20	24	28	-	-	-	-	-
16	1	2	1	2	1	2	1	2	2	6	10	14	18	22	26	30	-	-	-	-
17	1	2	1	2	1	2	1	2	1	4	8	12	16	20	24	28	32	-	-	-
18	1	2	1	2	1	2	1	2	1	2	6	10	14	18	22	26	30	34	-	-
19	1	2	1	2	1	2	1	2	1	2	4	8	12	16	20	24	28	32	36	-
20	1	2	1	2	1	2	1	2	1	2	2	6	10	14	18	22	26	30	34	38

Table 3. Number of connected components of the t-graphs on $\mathbb{Z}_n \times \mathbb{Z}_3$.

$n\backslash t$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
2	1	2	4	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
3	1	2	2	7	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
4	1	2	1	4	10	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
5	1	2	1	3	7	13	-	-	-	-	-	-	-	-	-	-	-	-	-	-
6	1	2	1	2	4	10	16	-	-	-	-	-	-	-	-	-	-	-	-	-
7	1	2	1	2	2	7	13	19	-	-	-	-	-	-	-	-	-	-	-	-
8	1	2	1	2	1	4	10	16	22	-	-	-	-	-	-	-	-	-	-	-
9	1	2	1	2	1	3	7	13	19	25	-	-	-	-	-	-	-	-	-	-
10	1	2	1	2	1	2	4	10	16	22	28	-	-	-	-	-	-	-	-	-
11	1	2	1	2	1	2	2	7	13	19	25	31	-	-	-	-	-	-	-	-
12	1	2	1	2	1	2	1	4	10	16	22	28	34	-	-	-	-	-	-	-
13	1	2	1	2	1	2	1	3	7	13	19	25	31	37	-	-	-	-	-	-
14	1	2	1	2	1	2	1	2	4	10	16	22	28	34	40	-	-	-	-	-
15	1	2	1	2	1	2	1	2	2	7	13	19	25	31	37	43	-	-	-	-
16	1	2	1	2	1	2	1	2	1	4	10	16	22	28	34	40	46	-	-	-
17	1	2	1	2	1	2	1	2	1	3	7	13	19	25	31	37	43	49	-	-
18	1	2	1	2	1	2	1	2	1	2	4	10	16	22	28	34	40	46	52	-
19	1	2	1	2	1	2	1	2	1	2	2	7	13	19	25	31	37	43	49	55
20	1	2	1	2	1	2	1	2	1	2	1	4	10	16	22	28	34	40	46	52

Remark 2. *Note in the previous tables that $k(\mathcal{G})$ has the same value up to a certain value of t where, if t is even, $k(\mathcal{G}) = 2$, and, if t is odd, then $k(\mathcal{G}) = 1$ and, when $t > \lceil \frac{m+n-2}{2} \rceil$, then $k(\mathcal{G})$ has a value with the following possible pattern:*

1. *If $m = 2$ (for example by dihedral groups), then the number of connected components of the t-graph increases by four. We conjecture that $K(G) = 2(2t - n) - 2$.*
2. *If $m = 3$, then the number of connected components starts with seven and so progresses from six to six, if n is odd, and starts at four and progresses from six to six when n is even. We conjecture that $K(G) = 3(2t - n - 2) - 2$.*

This fact leads us to state the first theorem in the next section, which allows us to characterize first the t-graphs associated with two-generator groups in the form (10), concerning the number of connected components.

3. The *t*-Graph of Some Two-Generator Groups

This section considers the t-graph of a particular case of two-generator groups. Specifically, we suppose that a is an involution and b has an order n. For example, the group G can be the abelian group $\mathbb{Z}_2 \times \mathbb{Z}_n$ or the dihedral group D_n of order n.

Lemma 3. *Let G be a two-generator group in the form (10) with $n, m \geq 2$, and $\mathcal{G}$ is the corresponding t-graph of G. Then, $\mathcal{G}$ has no isolated points if and only if $t \leq \lceil \frac{m+n-2}{2} \rceil$.*

Proof. Let $x = a^i b^j, y = a^k b^l \in G$ with

$$d_1(x, y) = |i - k| + |j - l| = t. \tag{12}$$

Then, $t \in \{0, \ldots, m + n - 2\}$, and suppose $|i - k| = s \in \{0, \ldots, m - 1\}$. This implies that $|j - l| = t - s \in \{0, \ldots, n - 1\}$. Note that if $t - s > n - 1$, the equality (12) is not verified. That is, there is no edge between x and y. Then, in order not to have isolated points, it must be fulfilled that $t - s \leq n - 1$ with $s \in \{0, \ldots, m - 1\}$. Moreover, $t \leq n - 1$. Analogously, it follows that $t \leq m - 1$. Consequently, $2t \leq m + n - 2$, and, therefore, $t \leq \lceil \frac{m+n-2}{2} \rceil$. □

Theorem 2. *Let G be a two-generator group in the form (10) with $n, m \geq 2$, and $\mathcal{G} = (G, E)$ be the corresponding t-graph with $t \leq \lceil \frac{m+n-2}{2} \rceil$.*

1. *If t is an even number, then $k(\mathcal{G}) = 2$.*
2. *If t is an odd number, then $\mathcal{G}$ is connected.*

Proof. From the above lemma, we have that the condition $t \leq \lceil \frac{m+n-2}{2} \rceil$ implies that $\mathcal{G}$ has no isolated points. We now differentiate two possible cases.

1. Let t be an even number. We define $\mathcal{C}_1 = (V_1, E_1)$ and $\mathcal{C}_2 = (V_2, E_2)$, the subgraph of $\mathcal{G}$, as follows:

$$V_1 := \{a^i b^j \mid i + j \equiv 0 \bmod 2\}, \tag{13}$$
$$E_1 := \{\{a^i b^j, a^k b^l\} \mid i + j, k + l \equiv 0 \bmod 2 \wedge |i - k| + |j - l| = t\}, \tag{14}$$

and

$$V_2 := \{a^i b^j \mid i + j \equiv 1 \bmod 2\}, \tag{15}$$
$$E_2 := \{\{a^i b^j, a^k b^l\} \mid i + j, k + l \equiv 1 \bmod 2 \wedge |i - k| + |j - l| = t\}. \tag{16}$$

It is clear that $V_1 \cup V_2 = G$, and then $k(\mathcal{G}) = 2$.

2. Let t be an even number, and $x = a^i b^j \in G$ be arbitrary. If $i + j \equiv 1 \bmod 2$, then we consider the sets

$$\{a^k b^l \mid i, k + l \equiv 0 \bmod 2, j \equiv 1 \bmod 2 \wedge |i - k| + |j - l| = t\} \tag{17}$$
$$\{a^k b^l \mid j, k + l \equiv 0 \bmod 2, i \equiv 1 \bmod 2 \wedge |i - k| + |j - l| = t\}. \tag{18}$$

Since $\mathcal{G}$ has no isolated points, at least one of these sets is non-empty, and then $\{a^i b^j, a^k b^l\} \in E$.
If $i + j \equiv 0 \bmod 2$, then a similar analysis leads to the same conclusion. Then, we have that $\mathcal{G}$ is a connected graph.

□

The next theorem shows that the 1-graph associated with a finite dihedral group D_n has a simple structure. It corresponds to a square $(n \times 2)$-grid, as shown in Figure 2 below. Therefore, this graph is bichromatic or bipartite.

Theorem 3. *The* 1*-graph of* D_n *is bipartite.*

Proof. From (6), we have that

$$D_n = \{1, b, \cdots, b^{n-1}\} \cup \{ab, \ldots, ab^{n-1}\}. \tag{19}$$

Note that

$$d_1(b^i, b^{i+1}) = d_1(ab^i, ab^{i+1}) = 1, \tag{20}$$

then, the sets $\{1, b, \cdots, b^{n-1}\}$ and $\{a, ab, \cdots, ab^{n-1}\}$ form a bipartition of the vertex set D_n. □

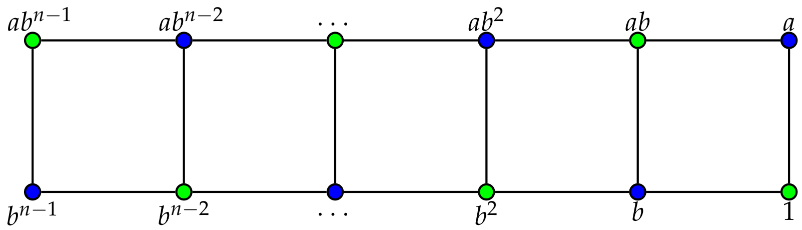

Figure 2. The 1-graph of D_n.

Theorem 2 leads to a complete characterization of the t-graphs associated with D_n. However, before characterizing the t-graphs on dihedral groups, let us first look at some useful lemmas.

Lemma 4. *Let* $\mathcal{G} = (D_n, E)$ *be the t-graph of* D_n*. Then,*

$$|E| = \begin{cases} 4(n-t) + 2 & \text{If } t > 1 \\ 3n - 2 & \text{If } t = 1. \end{cases} \tag{21}$$

Proof. Let $x = a^i b^j, y = a^k b^l \in D_n$, then, $0 \le i, k \le 1$ and $0 \le j, l \le n - 1$. If $d_1(x, y) = |i - k| + |j - l| = t$, then, for $|i - k|$, we have the following cases:

1. If $i = k$, then $|j - l| = t$. Note that there are $n - t$ ways to choose $j, l \in \{0, \ldots, n - 1\}$ such that the absolute value of their difference is t.
2. If $i \neq k$, then $|j - l| = t - 1$. In this case, there are $n - t + 1$ forms to choose $j, l \in \{0, \ldots, n - 1\}$ such that the absolute value of their difference is $t - 1$.

If $t > 1$, then there are $2(n - t) + 2(n - t + 1)$ ways of constructing an edge between two elements of D_n. Therefore, we have that $|E| = 4(n - t) + 2$.

If $t = 1$ then we the same argument we have that $|E| = 3n - 2$. □

Lemma 5. *Let $f : D_n \longrightarrow D_n$ be defined as follows*

$$f(a^i b^j) = \begin{cases} b^j & \text{If } i = 1 \\ ab^j & \text{If } i = 0. \end{cases} \tag{22}$$

Then, f is an isometry under the Minkowski metric (2). Further, if we restrict f to $U \subset D_n$, we have that U and $f(U)$ are also isometric under the Minkowski metric.

Proof. It is immediate that f is an injective function and $(f \circ f)(x) = x$, for all $x \in D_n$. That is, f is bijective. To prove that f is an isometry, let $a^i b^j, a^k b^l \in D_n$. Then,

1. If $i, k = 1$, then $d_1(f(a^i b^j), f(a^k b^l)) = d_1(b^j, b^l) = d_1(a^i b^j, a^k b^l)$.
2. If $i, k = 0$, then it is similar to the previous case.
3. If $i = 0$ and $k = 1$, then $d_1(f(a^i b^j), f(a^k b^l)) = d_1(ab^j, b^l) = d_1(a^i b^j, a^k b^l)$.
4. If $i = 1$ and $k = 0$, then it is similar to the previous case.

Therefore, f is an isometry on D_n. The other statement is clear. □

Theorem 4. *(**Characterization of t-graphs on** D_n)*
Let $\mathcal{G} = (D_n, E)$ the t-graph of D_n with $n \geq 2$. We define $r := \lceil \frac{n}{2} \rceil$.

1. *If $t \leq r$ and t is an even number, then $k(\mathcal{G}) = 2$, and these connected components are isomorphic.*
2. *If $t \leq r$ and t is an odd number, then $\mathcal{G}$ is an connected graph.*
3. *If $t = r + s$, with $1 \leq s \leq n - r$, then the number $K(\mathcal{G})$ of connected components of $\mathcal{G}$ is given by*

$$k(\mathcal{G}) = \begin{cases} 4(s-1) + 2 & \text{If } n \text{ is even} \\ 4s & \text{If } n \text{ is odd,} \end{cases} \tag{23}$$

where two of the connected components of $\mathcal{G}$ are an isomorphic path graph.

Proof.

1. It follows from Theorem 2 that $k(\mathcal{G}) = 2$. The connected components of $\mathcal{G}$ are $\mathcal{C}_1 = (V_1, E_1)$ and $\mathcal{C}_2 = (V_2, E_2)$, as in the proof of Theorem 2 (1). It is then sufficient to show that $\mathcal{C}_1 \cong \mathcal{C}_2$. Using the function f defined in Lemma 5, we have for $a^i b^j, a^k b^l \in V_1$ that

$$\{a^i b^j, a^k b^l\} \in E_1 \Longleftrightarrow \{f(a^i b^j), f(a^k b^l)\} \in E_2, \tag{24}$$

which leads to $\mathcal{C}_1 \cong \mathcal{C}_2$.
2. This follows immediately from Theorem 2.
3. We differentiate two cases:
   (a) Suppose t is an even number. The condition $t > r$ implies that $\mathcal{G}$ has isolated points, and then, using Theorem 2, we have that $\mathcal{G}$ has at least two connected components. Let $\mathcal{C}_1 = (V_1, E_1)$ and $\mathcal{C}_2 = (V_2, E_2)$ for the connected components constructed in the proof of Theorem 2 (1).
   We prove first that $|V_1| = |V_2|$. In fact, we have that $|j - l| = t$ or $|j - l| = t - 1$, which implies that

$$j \in \{t-1, \ldots, n-1\} \cup \{0, \ldots, n-t\} =: A, \tag{25}$$

since $l \in \{0, \ldots, n-1\}$.
It is clear that $\{t-1, \ldots, n-1\} \cap \{0, \ldots, n-t\} = \emptyset$, therefore

$$|A| = 2(n-t) + 2. \tag{26}$$

On the other hand, it follows immediately that $j \in A$ and $i + j$ are even numbers if and only if $a^i b^j \in V_1$, and then

$$|V_1| = |A| = 2(n-t) + 2. \quad (27)$$

Analogously, $|V_2| = |A|$, and we have $|V_1| = |V_2|$.
To demonstrate that $\mathcal{C}_1 \cong \mathcal{C}_2$, we consider again the function f defined in Lemma 5. Note that $f(V_1) = V_2$, and, since f is an isometry, we have the statement.
Finally, using Lemma 4, we have that $|E| = 4(n-t) + 2$, and the isomorphy between $\mathcal{C}_1$ and $\mathcal{C}_2$ implies that $|E_1| = |E_2|$. Further, note that the minimum value for $|E_1|$ and $|E_2|$ is $2(n-t) + 1$. This proves that $\mathcal{C}_1$ and $\mathcal{C}_2$ are the unique connected components of $\mathcal{G}$, which are not isolated points, and these are actually isomorphic path graphs.
The number of isolated points of $\mathcal{G}$ is $|D_n| - |V_1| - |V_2| = 2n - 4(n-t) - 4 = -2n + 4t - 4$, and, consequently, $k(\mathcal{G}) = -2n + 4t - 2 = -2n + 4r + 4s - 2$. That is,

- If n is even, then $k(\mathcal{G}) = -2n + 4(\frac{n}{2}) + 4s - 2 = 4(s-1) + 2$.
- If n is odd, then $k(\mathcal{G}) = -2n + 4(\frac{n+1}{2}) + 4s - 2 = 4s$.

(b) Suppose now that t is an odd number. Similar to before, the graph $\mathcal{G}$ has isolated points, and the set

$$\{\{a^i b^j, a^k b^l\} \mid i + j \equiv 0 \bmod 2, k + l \equiv 1 \bmod 2 \wedge |i-k| + |j-l| = t\}, \quad (28)$$

is a subset of E. Let V' be the set consisting of the non-isolated points of $\mathcal{G}$. Using the same argument as in (a), we obtain

$$|V'| = 2|A| = 4(n-t) + 4. \quad (29)$$

By Lemma 4, we have that $|E| = 4(n-t) + 2$, then, comparing $|V'|$ and $|E|$ excluding the isolated points, it follows that $\mathcal{G}$ cannot be connected.
Let m be an even number such that

$$\begin{cases} 0 \leq m \leq n-t-1 & \text{if } n-t-1 \text{ is even, and} \\ 0 \leq m \leq n-t & \text{if } n-t-1 \text{ is odd,} \end{cases} \quad (30)$$

and consider the subgraph $\mathcal{C}_1 = (V_1, E_1)$ of $\mathcal{G}$ with the following edges:

$$\{ab^{t+m-1}, b^m\},\ \{b^m, b^{t+m}\},\ \{b^{t+m}, ab^{m+1}\},\ \{ab^{m+1}, ab^{t+m+1}\}.$$

Then, $\mathcal{C}_1$ is a connected component of $\mathcal{G}$, and, furthermore,

$$|V_1| = 2(n-t) + 2 \ \wedge\ |E_1| = 2(n-t) + 1, \quad (31)$$

whence it is concluded that $\mathcal{C}_1$ is a path graph.
As before, using the function f from Lemma 5, we have that there exists another connected component $\mathcal{C}_2 = (f(V_1), E_2)$, isomorphic to $\mathcal{C}_1$. Thus,

$$|E_1| + |E_2| = |E| \ \wedge\ |V_1| + |V_2| = |V'|. \quad (32)$$

This means that $\mathcal{C}_1$ and $\mathcal{C}_2$ are the unique connected components of $\mathcal{G}$, and, analogously to the previous case, we have the same values for $k(\mathcal{G})$.

□

The following corollary is a generalization of Theorem 3.

Corollary 1. *Let G be a two-generator group in the form (10) with $n, m \geq 2$, and t be an odd number. Let further r be defined as in Theorem 2. If $t \leq r$, then $\mathcal{G} = (G, E)$ is a bipartite graph.*

Proof. From Theorem 4, we have that $\mathcal{G}$ is connected. Now, we define the sets V_1 and V_2 as follows

$$V_1 := \{a^i b^j \mid i + j \equiv 0 \bmod 2\} \quad (33)$$
$$V_2 := \{a^i b^j \mid i + j \equiv 1 \bmod 2\} \quad (34)$$

It is immediate to verify that V_1 and V_2 form a bipartition of G, and $\mathcal{G}$ is a bipartite graph. □

An illustration of the previous Corollary is presented in Figure 3.

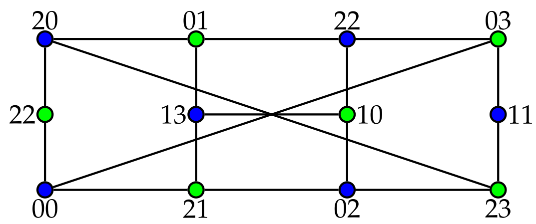

Figure 3. The 3-graph of $G = \mathbb{Z}_3 \times \mathbb{Z}_4$.

Corollary 2. *Let n be an odd number, $n \geq 5$ and $t = \frac{n+1}{2}$.*

1. *If t is odd, then $\mathcal{G} = (D_n, E)$ is a cycle of even length.*
2. *If t is even, then $\mathcal{G} = (D_n, E)$ is non-connected, and it has two isomorphic components, which are two cycles. Furthermore, $\chi(\mathcal{G}) = 3$.*

Proof. These statements follow directly from Theorem 4. Note that $t = r$.

1. From Lemma 4, it follows that

$$|E| = 4(n - (\tfrac{n+1}{2})) + 2 = 2n = |D_n|, \quad (35)$$

and then $\mathcal{G}$ is a cycle of even length.

2. $\mathcal{G}$ has two isomorphic connected components, say $\mathcal{C}_1 = (V_1, E_1)$ and $\mathcal{C}_2 = (V_2, E_2)$. Lemma 4 implies that

$$|E| = 4(n - (\tfrac{n+1}{2})) + 2 = 2n, \quad (36)$$

and it follows that $|E_1| = |E_2| = n$, so $\mathcal{G}$ is constituted by two isomorphic cycles. Finally, note that each component has an odd number of vertices. Then, $\chi(\mathcal{G}) = 3$.

□

Corollary 3. *Let n be an even number, $n \geq 2$ and $t = \frac{n}{2} + 1$. Then, the t-graph of D_n consists of two isomorphic paths graphs.*

Proof. Using Theorem 4, and since n is an even number, we have that $r = \frac{n}{2}$, and then $t = r + 1$, and $k(\mathcal{G}) = 2$. The rest is clear. □

Corollary 4. *Let $n \geq 2$ and r be as in Theorem 4. Then, the t-graph of D_n is 2-chromatic if $t \leq r$ and t is an odd number or $t > r$.*

Proof. It follows immediately from Theorem 4 and Corollary 1. □

Corollary 5. *The n-graph of D_n has $2(n-1)$ connected components, and two of these are path graphs with two vertices.*

Proof. Let $\mathcal{G} = (D_n, E)$ be the n-graph of D_n. From Lemma 4, it follows that $|E| = 2$. Note that

$$\{a, b^{n-1}\},\ \{ab^{n-1}, 1\} \in E. \quad (37)$$

The other $2n - 4$ elements of D_n are isolated points, and the proof is complete. □

4. Some Questions and Conjectures

Some open questions and conjectures are presented below.

Question 1. *Is it possible to characterize the t-graphs on two-generator groups, when $t > r$ and r is as in Theorem 2?*

Question 2. *Is it possible to generalize a version of Theorem 2 for an n-generator group for n, an arbitrary natural number?*

Question 3. *It is possible to determine in a finite group the existence (or not) of a generating system with the conditions stated for the definition of the t-graphs?*

Conjecture 1. *With respect to Theorem 2, if m is an even number and $t \leq r$, it follows that the two connected components of the t-graph $\mathcal{G}$ are isomorphic.*

Conjecture 2. *Let $n \geq 2$ and r be as in Theorem 4. Then, the t-graph of D_n is 3-chromatic, if $t \leq r$ and t is an even number.*

Conjecture 3. *If $G = \mathbb{Z}_n \times \mathbb{Z}_2$, then $K(G) = 2(2t - n) - 2$.*

Conjecture 4. *If $G = \mathbb{Z}_n \times \mathbb{Z}_3$, then $K(G) = 3(2t - n - 2) - 2$.*

5. Discussion

In the present research, we introduce and investigate the t-graph on a finitely generated group G. It leads to an interesting combinatorial problem. We establish conditions for t to guarantee the existence of isolated points in the t-graph when G is a two-generator group. We also propose an expression to determine the number of the connected components of the t-graph. Other results have to do with the conditions that must be fulfilled for the t-graphs of the dihedral groups to be a path graph or a cycle. Consequently, we can characterize the chromatic number of the t-graph depending exclusively on the parity of t.

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