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New MCDM Algorithms with Linear Diophantine Fuzzy Soft TOPSIS, VIKOR and Aggregation Operators

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Abstract: In this paper, we focus on several ideas associated with linear Diophantine fuzzy soft sets (LDFSSs) along with its algebraic structure. We provide operations on LDFSSs and their specific features, elaborating them with real-world examples and statistical depictions to construct an inflow of linguistic variables based on linear Diophantine fuzzy soft (LDFSS) information. We offer a study of LDFSSs to the multi-criteria decision-making (MCDM) process of university determination, together with new algorithms and flowcharts. We construct LDFSS-TOPSIS, LDFSS-VIKOR and the LDFSS-AO techniques as robust extensions of TOPSIS (a technique for order preferences through the ideal solution), VIKOR (Vlse Kriterijumska Optimizacija Kompromisno Resenje) and AO (aggregation operator). We use the LDFSS-TOPSIS, LDFSS-VIKOR and LDFSS-AO techniques to solve a real-world agricultural problem. Moreover, we present a small-sized robotic agri-farming to support the proposed technique. A comparison analysis is also performed to examine the symmetry of optimal decision and to analyze the efficiency of the suggested algorithms.

Keywords: linear Diophantine fuzzy soft sets; MCDM; linear Diophantine fuzzy soft topological spaces; symmetry; LDFSS-TOPSIS; LDFSS-VIKOR; LDFSS-AO

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1. Introduction

Many real-world problems have uncertainties, inconsistent information and data are not crisp. Zadeh [1] established the theory known as fuzzy set (FS) to deal with imprecise data. Many generalizations of fuzzy sets can be found which are developed to handle real world problems. Soft sets (SS) are one such extension which are introduced by Molodtsov [2]. This theory can handle uncertain information in a parametric way. Sabir and Naz [3] initiated the concept of soft topological space which presents the parametrized (precomputed) set values of topologies in the primary universe. In addition, Aygunoglu et al. [4] extended soft topological space to fuzzy set theory as fuzzy soft set topology in 2014.

The intuitionistic fuzzy set (IFS) concept was developed by Atanassov [5] in 1986. Like FS theory, IFS can also handle imprecise information with each element in the set having both satisfaction and dis-satisfaction grade values, provided that the addition of these two values should not exceed one. Maji et al. [6] initiated the notion of intuitionistic fuzzy soft sets (IFSSs) by incorporating IFS and SS. Bayramov and Gunduz [7] developed intuitionistic fuzzy soft topological spaces. In their work, they have investigated the properties of continuous mapping. Picture fuzzy set (PFS), [8] introduced by Coung et al. in 2014, is an amplification of Atanassov's IFS theory and Zadeh's FS theory. Picture fuzzy set and its application in decision making [9] is developed to explain when we have the three different answers (yes, avoid, no). Yager [10,11] initiated the concept of Pythagorean fuzzy set (PyFS) and it is introduced to overcome a circumstance when the sum of satisfaction and dis-satisfaction grades exceeds unity. q-rung orthopair fuzzy [12,13] set (q-ROFS) is an extension of PyFS, IFS whose sum of q-power of satisfaction and dis-satisfaction grade values are less than unity. q-rung orthopair picture fuzzy (q-ROPFS) [14] set is an extension of IFS whose sum of q-power of truth, abstinence and false grade values are less than unity. Riaz and Hashmi [15] unravelled the notion of linear Diophantine fuzzy set (LDFS) which is an amplification of fuzzy, intuitionistic fuzzy and picture fuzzy sets provided the addition of $\alpha(x)T(x)$ and $\beta(x)F(x)$ should not exceed unity, where α, β are the reference parameters and $T(x), F(x)$ are the true and false membership grades.

Forging decisions is an essential element of our day to day lives. A highly renowned graphic designer, James Victor, was asked by an interviewer what prompted him to be so versatile. He just stated, "I make decisions." Every day, we make millions of micro-choices, from how to communicate with someone, what to focus our energy on, how to respond to an email, what to consume to meet our health needs. One may easily state that becoming a better and faster decision-maker is the quickest way to increase one's productivity levels. Every individual, whether a layperson or a politician, an employer or an employee, a teacher or a student, a mature man or a child, takes hundreds and thousands, if not millions, of decisions in his or her everyday existence. When a newborn is hungry and unable to communicate, she/he determines to uproar in order to attract the concentration of her/his caregiver and to demonstrate that her/his belly is unfilled through body motions.

We are frequently duped by our tumults into making significant judgments in life, only to have regret afterwards. Assume we are faced with a difficult decision that will have a huge influence on our lives. Every time we believe we've made a decision, the other choice pulls us back. We return to where we began: it's a tie. Should we construct ever-more-detailed lists of advantages and disadvantages and seek advice from increasingly more reliable sources? Should we trust our instincts? Another critical difficulty is deciding how to decide. Mathematics, in addition to its numerous applications, assists us in making scientific judgments. Many researchers in [16–20] presented diverse decision making (DM) techniques utilizing the LDFSs with their applications.

MCDM is designed to make a optimum decision by a single person or group with the help of ranking. The application of MCDM can be seen when shortlisting people for interview, selecting new gadgets, machines, etc. The idea of TOPSIS is that the selected alternant should have a minimum distance positive ideal solution (PIS) and far from negative ideal solution (NIS). The TOPSIS method is used in MCDM because it can choose the optimum alternative among a group of alternants based on MCDM. The VIKOR method is proposed to deal with MCDM. This technique is used to choose an optimum alternative among a group of alternatives by ranking them in the presence of conflicting criteria. Like TOPSIS and VIKOR, aggregation operator is used in MCDM and the main aim of the aggregation operator in MCDM is to aggregate the set of inputs to a single number.

Many authors such as Biswas and Sarkar [21], Boran et al. [22], Kumar and Garg [23], Xu and Zhang [24], Xu [25], Hashmi et al. [26], Eraslan and Karaaslan [27], Peng and Yuan [28], Liu et al. [29], and Garg and Arora [30] applied the concept of VIKOR, TOPSIS and aggregation operator methods for DM problems with the extension of FSs and systems in different disciplines such as graph theory, operations research, etc. Khalid Naeem et al. [31] developed the notion of Pythagorean m-polar fuzzy topological space with the TOPSIS approach. Recently, Gul & Aydogdu [32] introduced and studied TOPSIS in an LDF environment.

Mathematics, in addition to its numerous applications, assists us in making scientific judgments. In this paper, we present an LDFSS decision-making application. Assume we have an aggregate LDFSS; therefore, we must select the optimal alternate form of this set. Using the following approach, we may use an MCDM based on LDFSSs.

The objective of the paper is given below:

(i) In IFS, each element has satisfaction and dis-satisfaction grades. Each element in LDFS has three grades namely, satisfaction, dissatisfaction and refusal with reference parameters provided the sum of product of grades with reference parameters does not exceed unity. Few theories such as IFS, PFS, q-ROFS fail to meet their own conditions in few cases.
(ii) Our goal is to initiate the concept of LDFSS to fill the research gap. In addition, we introduce a notion of linear Diophantine fuzzy soft topological space (LDFSTS) whose members in this LDFSTS are LDFSS.
(iii) LDFSSs, which are the inference of LDFSs and FSSs, are a more valuable medium in DM situations since they are dealing with two parametrized families of LDFS. TOPSIS, VIKOR, and AO techniques are also useful for decision-making challenges. In this work, we created three approaches in the Linear Diophantine fuzzy soft environment by integrating the modelling benefits of LDF flexible sets with the advantages of TOPSIS, VIKOR, and AO methods.
(iv) LDFSS-TOPSIS, LDFSS-VIKOR and the LDFSS-aggregation operators method are designed to apply the proposed notion in MCDM. A real life problem is considered and applied these proposed algorithm.

The structure of the manuscript is as follows: fundamental definitions are bestowed in Section 2. The definition of LDFSTS, neighbourhood, interior, closure, frontier and base are introduced and the properties of LDFSTS are studied in Section 3. We explained the importance of the targeted method for MCDM based on LDFSSs via LDFSS-TOPSIS, LDFSS-VIKOR, LDFSS-AO methods with numerical real life examples in Sections 4–6 respectively. The suggested MCDM approaches are exemplified by numerical examples in the previous sections and are supported by comparative analysis with various current techniques in Section 7. Section 8 detailed this lucubration work with a definite conclusion.

2. Preliminaries

We review and give some fundamental definitions of the LDFSs in this section.

Definition 1 ([15])**.** *An LDFS $\mathfrak{L}_\mathfrak{d}$ is an element on the non-void reference or connecting set $\mathfrak{Q}$ that composes:*

$$\mathfrak{L}_\mathfrak{d} = \{(\xi, \langle \mathfrak{t}_\mathfrak{d}(\xi), \mathfrak{f}_\mathfrak{d}(\xi)\rangle, \langle \alpha_\mathfrak{d}(\xi), \beta_\mathfrak{d}(\xi)\rangle) : \xi \in \mathfrak{Q}\}$$

where, $\mathfrak{t}_\mathfrak{d}(\xi), \mathfrak{f}_\mathfrak{d}(\xi)$, are the satisfaction grade and dis-satisfaction grade, and $\alpha(\xi), \beta(\xi) \in [0,1]$ are the connecting parameters, respectively. These grades gratify the condition $0 \leq \alpha_\mathfrak{d}(\xi)\mathfrak{t}_\mathfrak{d}(\xi) + \beta_\mathfrak{d}(\xi)\mathfrak{f}_\mathfrak{d}(\xi) \leq 1$ for all $\xi \in \mathfrak{Q}$ and with $0 \leq \alpha(\xi) + \beta(\xi) \leq 1$. Comparison parameters aid classifying a specific system. By traversing the tangible meaning of these parameters, we might classify the system. They increase the amount of space available in LDFS for grades and remove restrictions. The rejection (refusal) grade is defined as follows: $\gamma_\mathfrak{d}(\xi)\mathfrak{r}_\mathfrak{d} = (\xi) = 1 - (\alpha_\mathfrak{d}(\xi)\mathfrak{t}_\mathfrak{d}(\xi) + \beta_\mathfrak{d}(\xi)\mathfrak{f}_\mathfrak{d}(\xi))$, where $\gamma_\mathfrak{d}(\xi)$ is the rejection connecting parameter. Linear Diophantine fuzzy number (LDFN) is outlined as $\mathfrak{L}_\mathfrak{d} = (\langle \mathfrak{t}_\mathfrak{d}, \mathfrak{f}_\mathfrak{d}\rangle, \langle \alpha_\mathfrak{d}, \beta_\mathfrak{d}\rangle)$ and with $0 \leq \alpha + \beta \leq 1, 0 \leq \alpha_\mathfrak{d}\mathfrak{t}_\mathfrak{d} + \beta_\mathfrak{d}\mathfrak{f}_\mathfrak{d} \leq 1$.

Definition 2 ([15])**.** *An LDFS on $\mathfrak{Q}$ is called a*

(i) void LDFS, if $\mathfrak{L}^{o}_\mathfrak{d} = \{\xi, (\langle 0,1\rangle, \langle 0,1\rangle) : \xi \in \mathfrak{Q}\}$.
(ii) absolute LDFS, if $\mathfrak{L}^{1}_\mathfrak{d} = \{\xi, (\langle 1,0\rangle, \langle 1,0\rangle) : \xi \in \mathfrak{Q}\}$.

Definition 3 ([15])**.** *Let $\mathfrak{L}_\mathfrak{d} = (\langle \mathfrak{t}_\mathfrak{d}, \mathfrak{f}_\mathfrak{d}\rangle, \langle \alpha_\mathfrak{d}, \beta_\mathfrak{d}\rangle)$ be an LDFN, then*

1. the score function (SF) is displayed by $S_{(\mathfrak{L}_\mathfrak{d})}$ and is depicted as

$$S_{(\mathfrak{L}_\mathfrak{d})} = \frac{1}{2}[(\mathfrak{t}_\mathfrak{d} - \mathfrak{f}_\mathfrak{d}) + (\alpha_\mathfrak{d} - \beta_\mathfrak{d})]$$

where $S : \mathfrak{L}_\mathfrak{d}(\mathfrak{Q}) \longrightarrow [-1,1]$

2. *the accuracy function (AF) is displayed by* $A_{(\mathfrak{L}_\mathfrak{d})}$ *and is depicted as*

$$A_{(\mathfrak{L}_\mathfrak{d})} = \frac{1}{2}[\frac{(\mathfrak{t}_\mathfrak{d} + \mathfrak{f}_\mathfrak{d})}{2} + (\alpha_\mathfrak{d} + \beta_\mathfrak{d})]$$

where $A : \mathfrak{L}_\mathfrak{d}(\mathfrak{Q}) \longrightarrow [0,1]$
where $\mathfrak{L}_\mathfrak{d}(\mathfrak{Q})$ *is the foregathering of every LDFNs on* $\mathfrak{Q}$

Definition 4 ([15]). *Two LDFNs* $\mathfrak{L}_{\mathfrak{d}_1}$ *and* $\mathfrak{L}_{\mathfrak{d}_2}$ *can be comparable using SF and AF. It is defined as follows:*

(i) $\mathfrak{L}_{\mathfrak{d}_1} > \mathfrak{L}_{\mathfrak{d}_2}$ *if* $S(\mathfrak{L}_{\mathfrak{d}_1}) > S(\mathfrak{L}_{\mathfrak{d}_2})$
(ii) $\mathfrak{L}_{\mathfrak{d}_1} < \mathfrak{L}_{\mathfrak{d}_2}$ *if* $S(\mathfrak{L}_{\mathfrak{d}_1}) < S(\mathfrak{L}_{\mathfrak{d}_2})$
(iii) *If* $S(\mathfrak{L}_{\mathfrak{d}_1}) = S(\mathfrak{L}_{\mathfrak{d}_2})$, *then*
   (a) $\mathfrak{L}_{\mathfrak{d}_1} > \mathfrak{L}_{\mathfrak{d}_2}$ *if* $A(\mathfrak{L}_{\mathfrak{d}_1}) > A(\mathfrak{L}_{\mathfrak{d}_2})$
   (b) $\mathfrak{L}_{\mathfrak{d}_1} < \mathfrak{L}_{\mathfrak{d}_2}$ *if* $A(\mathfrak{L}_{\mathfrak{d}_1}) < A(\mathfrak{L}_{\mathfrak{d}_2})$
   (c) $\mathfrak{L}_{\mathfrak{d}_1} = \mathfrak{L}_{\mathfrak{d}_2}$ *if* $A(\mathfrak{L}_{\mathfrak{d}_1}) = A(\mathfrak{L}_{\mathfrak{d}_2})$

Definition 5 ([15]). *Let* $\mathfrak{L}_{\mathfrak{d}_\mathfrak{i}} = (\langle \mathfrak{t}_{\mathfrak{d}_\mathfrak{i}}, \mathfrak{f}_{\mathfrak{d}_\mathfrak{i}} \rangle, \langle \alpha_{\mathfrak{L}_\mathfrak{i}}, \beta_{\mathfrak{L}_\mathfrak{i}} \rangle)$ *for* $\mathfrak{i} \in \Delta$ *be a convene of LDFNs on* $\mathfrak{Q}$ *and* $\mathfrak{X} > 0$ *then*

(i) $\mathfrak{L}^\mathfrak{c}_{\mathfrak{d}_1} = (\langle \mathfrak{f}_{\mathfrak{d}_1}, \mathfrak{t}_{\mathfrak{D}_1} \rangle, \langle \beta_{\mathfrak{L}_1}, \alpha_{\mathfrak{L}_1} \rangle)$
(ii) $\mathfrak{L}_{\mathfrak{d}_1} = \mathfrak{L}_{\mathfrak{d}_2} \Leftrightarrow \mathfrak{t}_{\mathfrak{d}_1} = \mathfrak{t}_{\mathfrak{d}_2}, \mathfrak{f}_{\mathfrak{d}_1} = \mathfrak{f}_{\mathfrak{d}_2}, \alpha_{\mathfrak{L}_1} = \alpha_{\mathfrak{L}_2}, \beta_{\mathfrak{L}_1} = \beta_{\mathfrak{L}_2}$
(iii) $\mathfrak{L}_{\mathfrak{d}_1} \subseteq \mathfrak{L}_{\mathfrak{d}_2} \Leftrightarrow \mathfrak{t}_{\mathfrak{d}_1} \leq \mathfrak{t}_{\mathfrak{d}_2}, \mathfrak{f}_{\mathfrak{d}_1} \geq \mathfrak{f}_{\mathfrak{d}_2}, \alpha_{\mathfrak{L}_1} \leq \alpha_{\mathfrak{L}_2}, \beta_{\mathfrak{L}_1} \geq \beta_{\mathfrak{L}_2}$
(iv) $\mathfrak{L}_{\mathfrak{d}_1} \oplus \mathfrak{L}_{\mathfrak{d}_2} = (\langle \mathfrak{t}_{\mathfrak{d}_1} + \mathfrak{t}_{\mathfrak{d}_2} - \mathfrak{t}_{\mathfrak{d}_1}\mathfrak{t}_{\mathfrak{d}_2}, \mathfrak{f}_{\mathfrak{d}_1}\mathfrak{f}_{\mathfrak{d}_2} \rangle, \langle \alpha_{\mathfrak{L}_1} + \alpha_{\mathfrak{L}_2} - \alpha_{\mathfrak{L}_1}\alpha_{\mathfrak{L}_2}, \beta_{\mathfrak{L}_1}\beta_{\mathfrak{L}_2} \rangle)$
(v) $\mathfrak{L}_{\mathfrak{d}_1} \otimes \mathfrak{L}_{\mathfrak{d}_2} = (\langle \mathfrak{t}_{\mathfrak{d}_1}\mathfrak{t}_{\mathfrak{d}_2}, \mathfrak{f}_{\mathfrak{d}_1} + \mathfrak{f}_{\mathfrak{d}_2} - \mathfrak{f}_{\mathfrak{d}_1}\mathfrak{f}_{\mathfrak{d}_2} \rangle, \langle \alpha_{\mathfrak{L}_1}\alpha_{\mathfrak{L}_2}, \beta_{\mathfrak{L}_1} + \beta_{\mathfrak{L}_2} - \beta_{\mathfrak{L}_1}\beta_{\mathfrak{L}_2} \rangle)$
(vi) $\mathfrak{L}_{\mathfrak{d}_1} \cup \mathfrak{L}_{\mathfrak{d}_2} = (\langle \mathfrak{t}_{\mathfrak{d}_1} \vee \mathfrak{t}_{\mathfrak{d}_2}, \mathfrak{f}_{\mathfrak{d}_1} \wedge \mathfrak{f}_{\mathfrak{d}_2} \rangle, \langle \alpha_{\mathfrak{L}_1} \vee \alpha_{\mathfrak{L}_2}, \beta_{\mathfrak{L}_1} \wedge \beta_{\mathfrak{L}_2} \rangle)$
(vii) $\mathfrak{L}_{\mathfrak{d}_1} \cap \mathfrak{L}_{\mathfrak{d}_2} = (\langle \mathfrak{t}_{\mathfrak{d}_1} \wedge \mathfrak{t}_{\mathfrak{d}_2}, \mathfrak{f}_{\mathfrak{d}_1} \vee \mathfrak{f}_{\mathfrak{d}_2} \rangle, \langle \alpha_{\mathfrak{L}_1} \wedge \alpha_{\mathfrak{L}_2}, \beta_{\mathfrak{L}_1} \vee \beta_{\mathfrak{L}_2} \rangle)$
(viii) $\mathfrak{X}\mathfrak{L}_{\mathfrak{d}_1} = (\langle (1 - (1 - \mathfrak{t}_{\mathfrak{d}_1})^\mathfrak{X}), \mathfrak{f}^\mathfrak{X}_{\mathfrak{d}_1} \rangle, \langle (1 - (1 - {}_{\mathfrak{L}_1})^\mathfrak{X}), {}^\mathfrak{X}_{\mathfrak{L}_1} \rangle)$
(ix) $\mathfrak{L}^\mathfrak{X}_{\mathfrak{d}_1} = (\langle \mathfrak{t}^\mathfrak{X}_{\mathfrak{d}_1}, (1 - (1 - \mathfrak{f}_{\mathfrak{d}_1})^\mathfrak{X}) \rangle, \langle {}^\mathfrak{X}_{\mathfrak{L}_1}, (1 - (1 - {}_{\mathfrak{L}_1})^\mathfrak{X} \rangle)$

Example 1. *Let* $\mathfrak{L}_{\mathfrak{d}_1} = (\langle 0.87, 0.63 \rangle, \langle 0.56, 0.21 \rangle)$ *and* $\mathfrak{L}_{\mathfrak{d}_2} = (\langle 0.76, 0.69 \rangle, \langle 0.41, 0.33 \rangle)$ *be two LDFNs, then*

(i) $\mathfrak{L}^\mathfrak{c}_{\mathfrak{d}_1} = (\langle 0.63, 0.87 \rangle, \langle 0.21, 0.56 \rangle)$
(ii) $\mathfrak{L}_{\mathfrak{d}_2} \subseteq \mathfrak{L}_{\mathfrak{d}_1}$ *by the Definition 9 (iii)*
(iii) $\mathfrak{L}_{\mathfrak{d}_1} \oplus \mathfrak{L}_{\mathfrak{d}_2} = (\langle 0.9688, 0.4347 \rangle, \langle 0.7404, 0.0693 \rangle)$
(iv) $\mathfrak{L}_{\mathfrak{d}_1} \otimes \mathfrak{L}_{\mathfrak{d}_2} = (\langle 0.6612, 0.8853 \rangle, \langle 0.2296, 0.4707 \rangle)$
(v) $\mathfrak{L}_{\mathfrak{d}_1} \cup \mathfrak{L}_{\mathfrak{d}_2} = (\langle 0.87, 0.63 \rangle, \langle 0.56, 0.21 \rangle) = \mathfrak{L}_{\mathfrak{d}_1}$
(vi) $\mathfrak{L}_{\mathfrak{d}_1} \cap \mathfrak{L}_{\mathfrak{d}_2} = (\langle 0.76, 0.69 \rangle, \langle 0.41, 0.33 \rangle) = \mathfrak{L}_{\mathfrak{d}_2}$

If $\mathfrak{X} = 0.1$, *then we have the following*
(vii) $\mathfrak{X}\mathfrak{L}_{\mathfrak{d}_1} = (\langle 0.1846, 0.9548 \rangle, \langle 0.0788, 0.8555 \rangle)$
(viii) $\mathfrak{L}^\mathfrak{X}_{\mathfrak{d}_1} = (\langle 0.9862, 0.0946 \rangle, \langle 0.9437, 0.02330 \rangle)$

Definition 6 ([15]). *The euclidean distance within the two LDFSs* $\mathfrak{L}_{\mathfrak{d}_1}$ *and* $\mathfrak{L}_{\mathfrak{d}_2}$ *is determined as* $\mathfrak{d}(\mathfrak{L}_{\mathfrak{d}_1}, \mathfrak{L}_{\mathfrak{d}_2}) = \frac{1}{2}\sqrt{\{(\mathfrak{t}_{\mathfrak{d}_1} - \mathfrak{t}_{\mathfrak{d}_2})^2 + (\mathfrak{f}_{\mathfrak{d}_1} - \mathfrak{f}_{\mathfrak{d}_2})^2 + ({}_{\mathfrak{d}_1} - {}_{\mathfrak{d}_2})^2 + ({}_{\mathfrak{d}_1} - {}_{\mathfrak{d}_2})^2\}}$.

Definition 7 ([2]). *Let* $\mathfrak{E}$ *be the set of attributes and* $\mathfrak{X}$ *be a crisp set. The soft set will be outlined as* $(\psi, \mathfrak{A}) = \{(\mathfrak{e}, \psi(\mathfrak{e})) : \mathfrak{e} \in \mathfrak{A}, \psi(\mathfrak{e}) \in \mathfrak{P}(\mathfrak{X})\}$, *where* $\mathfrak{A} \subseteq \mathfrak{E}$ *and* $\psi : \mathfrak{A} \rightarrow \mathfrak{P}(\mathfrak{X})$ *is the set-valued function.* $\psi_\mathfrak{A}$ *is the shortest method of writing the couplet* $(\psi, \mathfrak{A})$.

Definition 8 ([33]). *Let* $\mathfrak{E}$ *be the set of parameters and* $\mathfrak{X}$ *be the universal set. If we suppose that* $\mathfrak{A} \subseteq \mathfrak{E}$ *and* $LDF^\mathfrak{X}$ *signifies the assembly of all linear Diophantine fuzzy subsets over* $\mathfrak{X}$ *and* $\kappa : \mathfrak{A} \rightarrow LDF^\mathfrak{X}$ *is a mapping. An LDFSS on* $\mathfrak{X}$ *is denoted by* $(\kappa, \mathfrak{A})$ *or* $\kappa_\mathfrak{A}$ *and outlined by* $(\kappa, \mathfrak{A}) = \{\mathfrak{e}, (\zeta, \langle \mathfrak{t}_{\kappa_\mathfrak{A}}(\zeta), \mathfrak{f}_{\kappa_\mathfrak{A}}(\zeta) \rangle, \langle \alpha_{\kappa_\mathfrak{A}}(\zeta), \beta_{\kappa_\mathfrak{A}}(\zeta) \rangle) : \mathfrak{e} \in \mathfrak{A}, \zeta \in \mathfrak{X}\}$.

where $\mathfrak{t}_{\kappa_{\mathfrak{A}}}, \mathfrak{f}_{\kappa_{\mathfrak{A}}}, \alpha_{\kappa_{\mathfrak{A}}}, \beta_{\kappa_{\mathfrak{A}}} : \mathfrak{X} \to [0,1]$ *delineates functions called satisfaction function, dis-satisfaction function, satisfaction parameter function, dis-satisfaction parameter function, respectively. Specifically,* $\mathfrak{t}_{\kappa_{\mathfrak{A}}}(\zeta)$ *denotes the satisfaction grade,* $\mathfrak{f}_{\kappa_{\mathfrak{A}}}(\zeta)$ *represents the dis-satisfaction grade,* $\alpha_{\kappa_{\mathfrak{A}}}(\zeta)$ *denotes the parameter of the satisfaction grade,* $\beta_{\kappa_{\mathfrak{A}}}(\zeta)$ *represents the parameter of the dis-satisfaction grade of the alternative* $\zeta \in \mathfrak{X}$ *to the set* $(\kappa, \mathfrak{A})$ *having the following constraints:*

- $0 \leq \alpha_{\kappa_{\mathfrak{A}}}(\zeta)\mathfrak{t}_{\kappa_{\mathfrak{A}}}(\zeta) + \beta_{\kappa_{\mathfrak{A}}}(\zeta)\mathfrak{f}_{\kappa_{\mathfrak{A}}}(\zeta) \leq 1$ *for all* $\zeta \in \mathfrak{X}$
- $0 \leq \alpha_{\kappa_{\mathfrak{A}}}(\zeta) + \beta_{\kappa_{\mathfrak{A}}}(\zeta) \leq 1$

For each attribute $\mathfrak{e}$, the value $\kappa(\mathfrak{e})$ evinces $\kappa(\mathfrak{e})$-approximate point.

The multitude of all LDFSS over $\mathfrak{X}$ taken from $\mathfrak{E}$ is defined as LDFS class and is represented as LDFS$(\mathfrak{X}, \mathfrak{E})$.

Let us consider $\mathfrak{t}_{ij} = \mathfrak{t}_{\kappa_{\mathfrak{A}}}(\mathfrak{e}_j)(\zeta_i)$, $\mathfrak{f}_{ij} = \mathfrak{f}_{\kappa_{\mathfrak{A}}}(\mathfrak{e}_j)(\zeta_i)$, $\alpha_{ij} = \alpha_{\kappa_{\mathfrak{A}}}(\mathfrak{e}_j)(\zeta_i)$ and $\beta_{ij} = \beta_{\kappa_{\mathfrak{A}}}(\mathfrak{e}_j)(\zeta_i)$ where $\mathfrak{i}$ run from from one to $\mathfrak{m}$ and $\mathfrak{j}$ run from one to $\mathfrak{n}$. Thus the LDFSS $\kappa_{\mathfrak{A}}$ may be written in tabular form as cited in Table 1.

Table 1. Tabular array of LDFSS $\kappa_{\mathfrak{A}}$.

$\kappa_{\mathfrak{A}}$	$\mathfrak{e}_1$	$\mathfrak{e}_2$	$\cdots$	$\mathfrak{e}_n$
ρ_1	$(\langle \mathfrak{t}_{11}, \mathfrak{f}_{11}\rangle, \langle \alpha_{11}, \beta_{11}\rangle)$	$(\langle \mathfrak{t}_{12}, \mathfrak{f}_{12}\rangle, \langle \alpha_{12}, \beta_{12}\rangle)$	$\cdots$	$(\langle \mathfrak{t}_{1n}, \mathfrak{f}_{1n}\rangle, \langle \alpha_{1n}, \beta_{1n}\rangle)$
ρ_2	$(\langle \mathfrak{t}_{21}, \mathfrak{f}_{21}\rangle, \langle \alpha_{21}, \beta_{21}\rangle)$	$(\langle \mathfrak{t}_{22}, \mathfrak{f}_{22}\rangle, \langle \alpha_{22}, \beta_{22}\rangle)$	$\cdots$	$(\langle \mathfrak{t}_{2n}, \mathfrak{f}_{2n}\rangle, \langle \alpha_{2n}, \beta_{2n}\rangle)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
ρ_m	$(\langle \mathfrak{t}_{m1}, \mathfrak{f}_{m1}\rangle, \langle \alpha_{m1}, \beta_{m1}\rangle)$	$(\langle \mathfrak{t}_{m2}, \mathfrak{f}_{m2}\rangle, \langle \alpha_{m2}, \beta_{m2}\rangle)$	$\cdots$	$(\langle \mathfrak{t}_{mn}, \mathfrak{f}_{mn}\rangle, \langle \alpha_{mn}, \beta_{mn}\rangle)$

The corresponding matrix form is

$$(\kappa, \mathfrak{A}) = [\langle \mathfrak{t}_{ij}, \mathfrak{f}_{ij}\rangle, \langle \alpha_{ij}, \beta_{ij}\rangle]_{m\times n} =$$
$$\begin{pmatrix} (\langle \mathfrak{t}_{11}, \mathfrak{f}_{11}\rangle, \langle \alpha_{11}, \beta_{11}\rangle) & (\langle \mathfrak{t}_{12}, \mathfrak{f}_{12}\rangle, \langle \alpha_{12}, \beta_{12}\rangle) & \cdots & (\langle \mathfrak{t}_{1n}, \mathfrak{f}_{1n}\rangle, \langle \alpha_{1n}, \beta_{1n}\rangle) \\ (\langle \mathfrak{t}_{21}, \mathfrak{f}_{21}\rangle, \langle \alpha_{21}, \beta_{21}\rangle) & (\langle \mathfrak{t}_{22}, \mathfrak{f}_{22}\rangle, \langle \alpha_{22}, \beta_{22}\rangle) & \cdots & (\langle \mathfrak{t}_{2n}, \mathfrak{f}_{2n}\rangle, \langle \alpha_{2n}, \beta_{2n}\rangle) \\ \vdots & \vdots & \ddots & \vdots \\ (\langle \mathfrak{t}_{m1}, \mathfrak{f}_{m1}\rangle, \langle \alpha_{m1}, \beta_{m1}\rangle) & (\langle \mathfrak{t}_{m2}, \mathfrak{f}_{m2}\rangle, \langle \alpha_{m2}, \beta_{m2}\rangle) & \cdots & (\langle \mathfrak{t}_{mn}, \mathfrak{f}_{mn}\rangle, \langle \alpha_{mn}, \beta_{mn}\rangle) \end{pmatrix}$$

The matrix displayed above is said to be linear Diophantine fuzzy soft matrix (LDFSM).

Definition 9 ([33])**.** *Let* $(\kappa_1, \mathfrak{A}_1)$ *and* $(\kappa_2, \mathfrak{A}_2)$ *be a convene of LDFSSs on* $\mathfrak{X}$*, then*

(i) $\kappa^c_{\mathfrak{A}_1} = (\langle \mathfrak{f}_{\kappa_1}, \mathfrak{t}_{\kappa_1}\rangle, \langle \beta_{\kappa_1}, \alpha_{\kappa_1}\rangle)$
(ii) $\kappa_{\mathfrak{A}_1} \widetilde{\subseteq} \kappa_{\mathfrak{A}_1}$*, if* $\mathfrak{A}_1 \subseteq \mathfrak{A}_2$ *and* $\kappa_1(\mathfrak{e}) \subseteq \kappa_2(\mathfrak{e})$*, for all* $\mathfrak{e} \in \mathfrak{A}_1$*.*
(iii) $\kappa_{\mathfrak{A}} = \kappa_{\mathfrak{A}_1} \widetilde{\cup} \kappa_{\mathfrak{A}_1}$*, if* $\mathfrak{A}_1 \cup \mathfrak{A}_2$ *and* $\kappa_1(\mathfrak{e}) \cup \kappa_2(\mathfrak{e})$*, for all* $\mathfrak{e} \in \mathfrak{A}$*.*
(iv) $\kappa_{\mathfrak{A}} = \kappa_{\mathfrak{A}_1} \widetilde{\cap} \kappa_{\mathfrak{A}_1} \neq \phi$*, if* $\mathfrak{A}_1 \cap \mathfrak{A}_2$ *and* $\kappa_1(\mathfrak{e}) \cap \kappa_2(\mathfrak{e})$*, for all* $\mathfrak{e} \in \mathfrak{A}$*.*

Definition 10 ([33])**.** *If* τ *is a collection of linear Diophantine fuzzy subsets of a non-void set* $\mathfrak{X}$ *and if*

(i) $1_{\mathfrak{X}}, 0_{\mathfrak{X}} \in \tau$
(ii) $\mathfrak{A}_1 \cap \mathfrak{A}_2 \in \tau$*, for any* $\mathfrak{A}_1, \mathfrak{A}_2 \in \tau$
(iii) $\cup_i \mathfrak{A}_i \in \tau$ *where* $\mathfrak{i} \in \Delta$*, for any* $\mathfrak{A}_i \in \tau$

then the couplet $(\mathfrak{X}, \tau)$ *is known as an LDFTS, where* τ *is known as an LDFTS on* $\mathfrak{X}$*.*

3. Linear Diophantine Fuzzy Soft Topological Spaces

The concept of LDFSTS is constituted and to a greater extent we explored its peculiarities.

Let $\widetilde{\mathfrak{X}}$ be the inception of the universal set and $LDF(\widetilde{\mathfrak{X}}, \widetilde{\mathfrak{E}})$ represents the kindred of LDFSs on $\widetilde{\mathfrak{X}}$.

Definition 11. *An LDFSS* $(\widetilde{\mathfrak{F}}, \widetilde{\mathfrak{E}})$ *aloft* $\widetilde{\mathfrak{X}}$ *is known as*

- *an absolute LDFSS* ($\tilde{\imath}$)*, if and only if for every* $\xi \in \widetilde{\mathfrak{E}}$, $(\widetilde{\mathfrak{F}}, \widetilde{\mathfrak{E}})(\xi) = (\langle \tilde{1}, \tilde{0} \rangle, \langle \tilde{1}, \tilde{0} \rangle)$
- *an empty LDFSS* ($\tilde{o}$)*, if and only if for every* $\xi \in \widetilde{\mathfrak{E}}$, $(\widetilde{\mathfrak{F}}, \widetilde{\mathfrak{E}})(\xi) = (\langle \tilde{0}, \tilde{1} \rangle, \langle \tilde{0}, \tilde{1} \rangle)$

where $\tilde{0}, \tilde{1}, \tilde{0}, \tilde{1}$ *are the value of the grade of satisfaction, grade of dis-satisfaction, the parameter of the satisfaction grade and the parameter of the dis-satisfaction grade, respectively of the absolute and empty LDFSSs over* $\widetilde{\mathfrak{X}}$.

Definition 12. *Let* $\widetilde{\mathcal{T}} \subset LDF(\widetilde{\mathfrak{X}}, \widetilde{\mathfrak{E}})$*, then* $\widetilde{\mathcal{T}}$ *on* $\widetilde{\mathfrak{X}}$ *is said to be an LDFSTS, if the following constraints hold good*

- $\tilde{o}\,, \tilde{\imath} \in \widetilde{\mathcal{T}}$
- $\cap_{i=1}^{n} \widetilde{\mathfrak{L}}_i \in \widetilde{\mathcal{T}} \;\forall\; \widetilde{\mathfrak{L}}_i \in \widetilde{\mathcal{T}}$
- $\cup_{i=1}^{\infty} \widetilde{\mathfrak{L}}_i \in \widetilde{\mathcal{T}} \;\forall\; \widetilde{\mathfrak{L}}_i \in \widetilde{\mathcal{T}}$

The triple $(\widetilde{\mathfrak{X}}, \widetilde{\mathcal{T}}, \widetilde{\mathfrak{E}})$ *over* $\widetilde{\mathfrak{X}}$ *is called an LDFSTS. The objects of* $\widetilde{\mathcal{T}}$ *are known as linear Diophantine fuzzy soft open sets (LDFSOS) and their complements are said to be linear Diophantine fuzzy soft closed sets (LDFSCS).*

Definition 13. *Let* $\widetilde{\mathcal{T}_1}$ *and* $\widetilde{\mathcal{T}_2}$ *be any two LDFSTS. If for every* $\widetilde{\mathcal{L}}_1 \in \widetilde{\mathcal{T}_1}$ *is in* $\widetilde{\mathcal{T}_2}$*, then* $\widetilde{\mathcal{T}_1}$ *is linear Diophantine fuzzy soft coarser (weaker) than* $\widetilde{\mathcal{T}_2}$ *or* $\widetilde{\mathcal{T}_2}$ *is linear Diophantine fuzzy soft finer than* $\widetilde{\mathcal{T}_1}$.

Example 2. *Let* $\widetilde{\mathfrak{X}} = \{\xi_1, \xi_2, \xi_3\}$ *be the reference set (distinct models of bikes) and* $\widetilde{\mathfrak{E}} = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$ *be the attributes or parameters set, where* ζ_1*=affordable,* ζ_2*=caliber,* ζ_3*=comfort,* ζ_4*=recovery service. Let* $\widetilde{\mathfrak{A}} = \{\zeta_1, \zeta_2\} \subset \widetilde{\mathfrak{E}}$ *and* $\widetilde{\mathfrak{B}} = \{\zeta_2\} \subset \widetilde{\mathfrak{E}}$*. Then we contemplate two LDFSS* $(\widetilde{\mathfrak{F}}, \widetilde{\mathfrak{A}})$ *and* $(\widetilde{\mathfrak{G}}, \widetilde{\mathfrak{B}})$ *are given by:*
$(\widetilde{\mathfrak{F}}, \widetilde{\mathfrak{A}}) = \{(\zeta_1, \widetilde{\mathfrak{F}}(\zeta_1)), (\zeta_2, \widetilde{\mathfrak{F}}(\zeta_2))\}$*, and* $(\widetilde{\mathfrak{G}}, \widetilde{\mathfrak{B}}) = \{(\zeta_2, \widetilde{\mathfrak{G}}(\zeta_2))\}$*, where*
$\widetilde{\mathfrak{F}}(\zeta_1) = \{\xi_1 = (\langle 0.7, 0.4 \rangle, \langle 0.4, 0.2 \rangle), \xi_2 = (\langle 0.7, 0.5 \rangle, \langle 0.4, 0.2 \rangle), \xi_3 = (\langle 0.8, 0.3 \rangle, \langle 0.5, 0.2 \rangle)\}$
$\widetilde{\mathfrak{F}}(\zeta_2) = \{\xi_1 = (\langle 0.4, 0.6 \rangle, \langle 0.2, 0.5 \rangle), \xi_2 = (\langle 0.6, 0.7 \rangle, \langle 0.4, 0.3 \rangle), \xi_3 = (\langle 0.6, 0.4 \rangle, \langle 0.6, 0.3 \rangle)\}$
$\widetilde{\mathfrak{G}}(\zeta_2) = \{\xi_1 = (\langle 0.7, 0.5 \rangle, \langle 0.3, 0.5 \rangle), \xi_2 = (\langle 0.4, 0.5 \rangle, \langle 0.2, 0.5 \rangle), \xi_3 = (\langle 0.7, 0.3 \rangle, \langle 0.2, 0.5 \rangle)\}$
Here,

1. $\widetilde{\mathcal{T}} = \{(\widetilde{\mathfrak{F}}, \widetilde{\mathfrak{A}}), (\widetilde{\mathfrak{G}}, \widetilde{\mathfrak{B}}), \tilde{o}, \tilde{\imath}\}$ *is a LDFSTS.*
2. $\widetilde{\mathcal{T}_1} = \{(\widetilde{\mathfrak{F}}, \widetilde{\mathfrak{A}}), \tilde{o}, \tilde{\imath}\}$ *and* $\widetilde{\mathcal{T}_2} = \{(\widetilde{\mathfrak{F}}, \widetilde{\mathfrak{A}}), (\widetilde{\mathfrak{G}}, \widetilde{\mathfrak{B}}), \tilde{o}, \tilde{\imath}\}$ *are two LDFSTSs. It is obvious that* $\widetilde{\mathcal{T}_1} \subseteq \widetilde{\mathcal{T}_2}$*. Thus,* $\widetilde{\mathcal{T}_2}$ *is said to be LDFSS-finer than* $\widetilde{\mathcal{T}_1}$ *and* $\widetilde{\mathcal{T}_1}$ *is said to be LDFS-coarser* $\widetilde{\mathcal{T}_2}$.

Theorem 1. *If* $\widetilde{\mathcal{T}_1} \cap \widetilde{\mathcal{T}_2} = \{\widetilde{\mathfrak{L}} \in LDFSSs(\widetilde{\mathfrak{X}}, \widetilde{\mathfrak{E}}) : \widetilde{\mathfrak{L}} \in \widetilde{\mathcal{T}_1} \cap \widetilde{\mathcal{T}_2}\}$*, where* $(\widetilde{\mathfrak{X}}, \widetilde{\mathcal{T}_1}, \widetilde{\mathfrak{E}})$ *and* $(\widetilde{\mathfrak{X}}, \widetilde{\mathcal{T}_2}, \widetilde{\mathfrak{E}})$ *are two LDFSTSs over* $(\widetilde{\mathfrak{X}}, \widetilde{\mathfrak{E}})$*, then* $\widetilde{\mathcal{T}_1} \cap \widetilde{\mathcal{T}_2}$ *is also an LDFSTS on* $(\widetilde{\mathfrak{X}}, \widetilde{\mathfrak{E}})$.

Proof. (i) It is obvious that $\tilde{\imath}, \tilde{o} \in \widetilde{\mathcal{T}_1}, \widetilde{\mathcal{T}_2}$
(ii) Let $\widetilde{\mathfrak{L}_1}, \widetilde{\mathfrak{L}_2} \in \widetilde{\mathcal{T}_1} \cap \widetilde{\mathcal{T}_2}$. This implies that $\widetilde{\mathfrak{L}_1}, \widetilde{\mathfrak{L}_2} \in \widetilde{\mathcal{T}_1}$ and $\widetilde{\mathfrak{L}_1}, \widetilde{\mathfrak{L}_2} \in \widetilde{\mathcal{T}_2}$, this implies that $\widetilde{\mathfrak{L}_1} \cap \widetilde{\mathfrak{L}_2} \in \widetilde{\mathcal{T}_1}$ and $\widetilde{\mathfrak{L}_1} \cap \widetilde{\mathfrak{L}_2} \in \widetilde{\mathcal{T}_2}$, this implies that $\widetilde{\mathfrak{L}_1} \cap \widetilde{\mathfrak{L}_2} \in \widetilde{\mathcal{T}_1} \cap \widetilde{\mathcal{T}_2}$.
(iii) Let $\{\widetilde{\mathfrak{L}}_i : i \in \Gamma\} \in \widetilde{\mathcal{T}_1} \cap \widetilde{\mathcal{T}_2}$. This implies that $\{\widetilde{\mathfrak{L}}_i\} \in \widetilde{\mathcal{T}_1}$ and $\{\widetilde{\mathfrak{L}}_i\} \in \widetilde{\mathcal{T}_2}$, this implies that $\cup_i \widetilde{\mathfrak{L}}_i \in \widetilde{\mathcal{T}_1}$ and $\cup_i \widetilde{\mathfrak{L}}_i \in \widetilde{\mathcal{T}_2}$, this implies that $\cup_i \widetilde{\mathfrak{L}}_i \in \widetilde{\mathcal{T}_1} \cap \widetilde{\mathcal{T}_2}$.
Therefore, $\widetilde{\mathcal{T}_1} \cap \widetilde{\mathcal{T}_2}$ is an LDFSTS on $(\widetilde{\mathfrak{X}}, \widetilde{\mathfrak{E}})$. □

Remark 1. *The union of two LDFSTSs might not be such.*

Let the reference set be $\widetilde{\mathfrak{X}} = \{\xi_1, \xi_2, \xi_3\}$ *and the attribute set be* $\widetilde{\mathfrak{E}} = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5\}$*. Let* $\widetilde{\mathfrak{A}} = \{\zeta_1, \zeta_2, \zeta_3\} \subset \widetilde{\mathfrak{E}}$ *and* $\widetilde{\mathfrak{B}} = \{\zeta_3, \zeta_4, \zeta_5\} \subset \widetilde{\mathfrak{E}}$*. Now let us take two LDFSSs* $(\widetilde{\mathfrak{F}}, \widetilde{\mathfrak{A}})$ *and* $(\widetilde{\mathfrak{G}}, \widetilde{\mathfrak{B}})$ *such that:*

$(\widetilde{\mathfrak{F}}, \widetilde{\mathfrak{A}}) = \{(\zeta_1, \widetilde{\mathfrak{F}}(\zeta_1)), (\zeta_2, \widetilde{\mathfrak{F}}(\zeta_2)), (\zeta_3, \widetilde{\mathfrak{F}}(\zeta_3))\}$*, and* $(\widetilde{\mathfrak{G}}, \widetilde{\mathfrak{B}}) = \{(\zeta_3, \widetilde{\mathfrak{G}}(\zeta_3)), (\zeta_4, \widetilde{\mathfrak{F}}(\zeta_4)), (\zeta_5, \widetilde{\mathfrak{F}}(\zeta_5))\}$*, where*
$\widetilde{\mathfrak{F}}(\zeta_1) = \{\xi_1 = (\langle 0.6, 0.6 \rangle, \langle 0.3, 0.4 \rangle), \xi_2 = (\langle 0.6, 0.7 \rangle, \langle 0.4, 0.3 \rangle), \xi_3 = (\langle 0.4, 0.4 \rangle, \langle 0.2, 0.3 \rangle\}$
$\widetilde{\mathfrak{F}}(\zeta_2) = \{\xi_1 = (\langle 0.7, 0.5 \rangle, \langle 0.4, 0.2 \rangle), \xi_2 = (\langle 0.5, 0.4 \rangle, \langle 0.3, 0.5 \rangle), \xi_3 = (\langle 0.2, 0.3 \rangle, \langle 0.3, 0.2 \rangle)\}$
$\widetilde{\mathfrak{F}}(\zeta_3) = \{\xi_1 = (\langle 0.5, 0.3 \rangle, \langle 0.3, 0.3 \rangle), \xi_2 = (\langle 0.7, 0.5 \rangle, \langle 0.4, 0.1 \rangle), \xi_3 = (\langle 0.4, 0.3 \rangle, \langle 0.3, 0.1 \rangle)\}$,
$\widetilde{\mathfrak{G}}(\zeta_3) = \{\xi_1 = (\langle 0.3, 0.4 \rangle, \langle 0.6, 0.5 \rangle), \xi_2 = (\langle 0.7, 0.4 \rangle, \langle 0.4, 0.7 \rangle), \xi_3 = (\langle 0.7, 0.5 \rangle, \langle 0.4, 0.3 \rangle)\}$
$\widetilde{\mathfrak{G}}(\zeta_4) = \{\xi_1 = (\langle 0.7, 0.6 \rangle, \langle 0.3, 0.1 \rangle), \xi_2 = (\langle 0.8, 0.3 \rangle, \langle 0.5, 0.4 \rangle), \xi_3 = (\langle 0.5, 0.4 \rangle, \langle 0.2, 0.4 \rangle)\}$
$\widetilde{\mathfrak{G}}(\zeta_5) = \{\xi_1 = (\langle 0.8, 0.4 \rangle, \langle 0.6, 0.3 \rangle), \xi_2 = (\langle 0.6, 0.5 \rangle, \langle 0.3, 0.7 \rangle), \xi_3 = (\langle 0.9, 0.4 \rangle, \langle 0.3, 0.1 \rangle)\}$.

Then, the two LDFSTSs over $\widetilde{\mathfrak{X}}$ are $\widetilde{\mathcal{T}_1} = \{\widetilde{\mathfrak{1}}, \widetilde{\mathfrak{o}}, (\widetilde{\mathfrak{F}}, \widetilde{\mathfrak{A}})\}$ and $\widetilde{\mathcal{T}_2} = \{\widetilde{\mathfrak{1}}, \widetilde{\mathfrak{o}}, (\widetilde{\mathfrak{G}}, \widetilde{\mathfrak{B}})\}$. The opposite hand, since $(\widetilde{\mathfrak{F}}, \widetilde{\mathfrak{A}}), (\widetilde{\mathfrak{G}}, \widetilde{\mathfrak{B}}) \in \widetilde{\mathcal{T}_1} \cup \widetilde{\mathcal{T}_2}$. However, $(\widetilde{\mathfrak{F}}, \widetilde{\mathfrak{A}}) \cup (\widetilde{\mathfrak{G}}, \widetilde{\mathfrak{B}}) \notin \widetilde{\mathcal{T}_1} \cup \widetilde{\mathcal{T}_2}$, $(\widetilde{\mathfrak{F}}, \widetilde{\mathfrak{A}}) \cap (\widetilde{\mathfrak{G}}, \widetilde{\mathfrak{B}}) \notin \widetilde{\mathcal{T}_1} \cup \widetilde{\mathcal{T}_2}$. Thus, $\widetilde{\mathcal{T}_1} \cup \widetilde{\mathcal{T}_2}$ is not an LDFSTS on $\widetilde{\mathfrak{X}}$. But $\widetilde{\mathcal{T}_1} \cap \widetilde{\mathcal{T}_2}$ is an LDFSS on $\widetilde{\mathfrak{X}}$.

Definition 14. *Let $\widetilde{\mathfrak{L}}_1, \widetilde{\mathfrak{L}}_2 \in LDFSS(\widetilde{\mathfrak{X}}, \widetilde{\mathfrak{E}})$ and $\widetilde{\mathcal{T}}$ be an LDFSTS on $(\widetilde{\mathfrak{X}}, \widetilde{\mathfrak{E}})$. Then $\widetilde{\mathfrak{L}}_2$ is called a neighbourhood (nbd) of $\widetilde{\mathfrak{L}}_1$, if $\exists$ an LDFSOS $\widetilde{\gamma}$ (i.e., $\widetilde{\gamma} \in \widetilde{\mathcal{T}}$) $\ni \widetilde{\mathfrak{L}}_1 \subset \widetilde{\gamma} \subset \widetilde{\mathfrak{L}}_1$.*

Theorem 2. *A LDFSS $\widetilde{\gamma} \in LDFSSs(\widetilde{\mathfrak{X}}, \widetilde{\mathfrak{E}})$ is an LDFSOS if and only if $\widetilde{\gamma}$ is a nbd of each LDFSS $\widetilde{\mathfrak{L}}_1 \subset \widetilde{\gamma}$.*

Proof. Let $\widetilde{\mathfrak{L}}_1$ be an LDFSSs in $\widetilde{\gamma}$, where $\widetilde{\gamma}$ is an LDFSOS. As we have $\widetilde{\mathfrak{L}}_1 \subset \widetilde{\gamma} \subset \widetilde{\gamma} \implies \widetilde{\gamma}$ is a nbd of $\widetilde{\mathfrak{L}}_1$. Thereupon, if we suppose $\widetilde{\gamma}$ is an nbd for all LDFSS $\subseteq \widetilde{\gamma}$. Since $\widetilde{\gamma} \subset \widetilde{\gamma}$, $\exists$ an LDFSOS $\widetilde{\mathfrak{L}}_2 \ni \widetilde{\gamma} \subset \widetilde{\mathfrak{L}}_2 \subset \widetilde{\gamma}$. Thus, $\widetilde{\gamma}$ is open and $\widetilde{\gamma} = \widetilde{\mathfrak{L}}_2$. □

Theorem 3. *Let $\widetilde{\gamma} \in (\widetilde{\mathfrak{X}}, \widetilde{\mathfrak{E}})$ and $(\widetilde{\mathfrak{X}}, \widetilde{\mathcal{T}_1}, \widetilde{\mathfrak{E}})$ be an LDFSTS. $\widetilde{\gamma}$ is said to be the nbd system or nbd filter of $\widetilde{\gamma}$, the set of all nbds, upto topology $\widetilde{\mathcal{T}_1}$ (in short, $LDFSSnbd(\widetilde{\gamma})$).*

Theorem 4. *Let the nbd filter of the LDFSS $\widetilde{\gamma}$ be $LDFSSnbd(\widetilde{\gamma})$. Then,*

1. *finite intersections of the members of $LDFSSnbd(\widetilde{\gamma}) \in LDFSSnbd(\widetilde{\gamma})$.*
2. *each LDFSS containing a member of $LDFSSnbd(\widetilde{\gamma}) \in LDFSSnbd(\widetilde{\gamma})$.*

Proof.

1. Let $\widetilde{\mathfrak{L}}_1, \widetilde{\mathfrak{L}}_2 \in LDFSSnbd(\widetilde{\gamma})$. Then $\exists \widetilde{\mathfrak{L}_1'}, \widetilde{\mathfrak{L}_2'} \in \widetilde{\mathcal{T}} \ni \widetilde{\gamma} \subset \widetilde{\mathfrak{L}_1'} \subset \widetilde{\mathfrak{L}}_1$ and $\widetilde{\gamma} \subset \widetilde{\mathfrak{L}_2'} \subset \widetilde{\mathfrak{L}}_2$. Since, $\widetilde{\mathfrak{L}_1'} \cap \widetilde{\mathfrak{L}_2'} \in \widetilde{\mathcal{T}}$, we have, $\widetilde{\gamma} \subset \widetilde{\mathfrak{L}_1'} \cap \widetilde{\mathfrak{L}_2'} \subset \widetilde{\mathfrak{L}}_1 \cap \widetilde{\mathfrak{L}}_2$. Thus, $\widetilde{\mathfrak{L}}_1 \cap \widetilde{\mathfrak{L}}_2 \in LDFSSnbd(\widetilde{\gamma})$.
2. If $\widetilde{\mathfrak{L}}_1 \in LDFSSnbd(\widetilde{\gamma})$ and $\widetilde{\mathfrak{L}}_2$ be an LDFSS containing $\widetilde{\mathfrak{L}}_1$, then $\exists \widetilde{\mathfrak{L}_1'} \in \widetilde{\mathcal{T}} \ni \widetilde{\gamma} \subset \widetilde{\mathfrak{L}_1'} \subset \widetilde{\mathfrak{L}}_1 \subset \widetilde{\mathfrak{L}}_2$. This proves that $\widetilde{\mathfrak{L}}_2 \in LDFSSnbd(\widetilde{\gamma})$

□

Definition 15. *Let $\widetilde{\mathfrak{L}} \in LDFSS(\widetilde{\mathfrak{X}}, \widetilde{\mathfrak{E}})$ be an arbitrary LDFSS and let $(\widetilde{\mathfrak{X}}, \widetilde{\mathcal{T}}, \widetilde{\mathfrak{E}})$ be an LDFSTS over $(\widetilde{\mathfrak{X}}, \widetilde{\mathfrak{E}})$. Then the interior and closure of $\widetilde{\mathfrak{L}}$ are defined as follows:*

1. $\widetilde{\mathfrak{L}}^{\mathfrak{LDFS}o} = \cup\{\widetilde{\mathfrak{G}} : \widetilde{\mathfrak{G}}$ *is LDFSO and* $\widetilde{\mathfrak{G}} \subseteq \widetilde{\mathfrak{L}}\}$,
2. $\widetilde{\mathfrak{L}}^{\mathfrak{LDFS}-} = \cap\{\widetilde{\mathfrak{G}} : \widetilde{\mathfrak{G}}$ *is LDFSC and* $\widetilde{\mathfrak{G}} \supseteq \widetilde{\mathfrak{L}}\}$.

Remark 2. *For any LDFSS $\widetilde{\mathfrak{L}}$ in $(\widetilde{\mathfrak{X}}, \widetilde{\mathcal{T}}, \widetilde{\mathfrak{E}})$, we have*

1. $[\widetilde{\mathfrak{L}}^c]^{\mathfrak{LDFS}-} = [\widetilde{\mathfrak{L}}^{\mathfrak{LDFS}o}]^c$.
2. $[\widetilde{\mathfrak{L}}^c]^{\mathfrak{LDFS}o} = [\widetilde{\mathfrak{L}}^{\mathfrak{LDFS}-}]^c$.
3. $\widetilde{\mathfrak{L}}$ *is an LDFSCS if and only if* $\widetilde{\mathfrak{L}}^{\mathfrak{LDFS}-} = \widetilde{\mathfrak{L}}$.
4. $\widetilde{\mathfrak{L}}$ *is an LDFSOS if and only if* $\widetilde{\mathfrak{L}}^{\mathfrak{LDFS}o} = \widetilde{\mathfrak{L}}$.
5. $\widetilde{\mathfrak{L}}^{LDFS-}$ *is an LDFSCS in* $(\widetilde{\mathfrak{X}}, \widetilde{\mathfrak{E}})$.
6. $\widetilde{\mathfrak{L}}^{\mathfrak{LDFS}o}$ *is an LDFSOS in* $(\widetilde{\mathfrak{X}}, \widetilde{\mathfrak{E}})$.

Theorem 5. *Let $(\widetilde{\mathfrak{X}}, \widetilde{\mathcal{T}}, \widetilde{\mathfrak{E}})$ be an LDFSTS with respect to $(\widetilde{\mathfrak{X}}, \widetilde{\mathfrak{E}})$. Let $\widetilde{\mathfrak{L}}_1$ and $\widetilde{\mathfrak{L}}_2$ be linear Diophantine fuzzy soft subsets of $(\widetilde{\mathfrak{X}}, \widetilde{\mathfrak{E}})$. Then the following holds:*

1. $\widetilde{\mathfrak{L}} \subseteq \widetilde{\mathfrak{L}}^{\mathfrak{LDFS}-}$.
2. $\widetilde{\mathfrak{L}}$ *is an LDFSCS if and only if* $\widetilde{\mathfrak{L}}^{\mathfrak{LDFS}-} = \widetilde{\mathfrak{L}}$.
3. $\widetilde{\mathfrak{o}}^{\mathfrak{LDFS}-} = \widetilde{\mathfrak{o}}$ *and* $\widetilde{\mathfrak{1}}^{\mathfrak{LDFS}-} = \widetilde{\mathfrak{1}}$.
4. $\widetilde{\mathfrak{L}}_1 \subseteq \widetilde{\mathfrak{L}}_2 \Rightarrow \widetilde{\mathfrak{L}}_1^{\mathfrak{LDFS}-} \subseteq \widetilde{\mathfrak{L}}_2^{\mathfrak{LDFS}-}$.
5. $(\widetilde{\mathfrak{L}}_1 \cup \widetilde{\mathfrak{L}}_2)^{\mathfrak{LDFS}-} = \widetilde{\mathfrak{L}}_1^{\mathfrak{LDFS}-} \cup \widetilde{\mathfrak{L}}_2^{\mathfrak{LDFS}-}$.
6. $(\widetilde{\mathfrak{L}}_1 \cap \widetilde{\mathfrak{L}}_2)^{\mathfrak{LDFS}-} = \widetilde{\mathfrak{L}}_1^{\mathfrak{LDFS}-} \cap \widetilde{\mathfrak{L}}_2^{\mathfrak{LDFS}-}$.
7. $(\widetilde{\mathfrak{L}}^{\mathfrak{LDFS}-})^{\mathfrak{LDFS}-} = \widetilde{\mathfrak{L}}^{\mathfrak{LDFS}-}$.

Proof.

1. From Definition 3.5 (ii), $\widetilde{\mathfrak{L}} \subseteq \widetilde{\mathfrak{L}}^{\mathfrak{LDFS}-}$
2. If $\widetilde{\mathfrak{L}}$ is a linear Diophantine fuzzy soft closed set (LDFSCS), then $\widetilde{\mathfrak{L}}$ is the tiniest LDFSCS carrying oneself and therefore $\widetilde{\mathfrak{L}}^{\mathfrak{LDFS}-} = \widetilde{\mathfrak{L}}$. In the reverse way, if $\widetilde{\mathfrak{L}}^{\mathfrak{LDFS}-} = \widetilde{\mathfrak{L}}$, then $\widetilde{\mathfrak{L}}$ is the tiniest LDFSCS containing itself and therefore $\widetilde{\mathfrak{L}}$ is an LDFSCS.
3. Since $\widetilde{0}$ and $\widetilde{1}$ are LDFSCSs in $(\widetilde{\mathfrak{X}}, \widetilde{\mathcal{T}}, \widetilde{\mathfrak{E}})$, $\widetilde{0}^{\mathfrak{LDFS}-} = \widetilde{0}$ and $\widetilde{1}^{\mathfrak{LDFS}-} = \widetilde{1}$.
4. If LDFSS $\widetilde{\mathfrak{L}}_1$ is a subset of LDFSS $\widetilde{\mathfrak{L}}_2$, since LDFSS $\widetilde{\mathfrak{L}}_2$ is a subset of $\widetilde{\mathfrak{L}}_2^{\mathfrak{LDFS}-}$, then LDFSS $\widetilde{\mathfrak{L}}_1$ is a subset of $\widetilde{\mathfrak{L}}_2^{\mathfrak{LDFS}-}$. That is, $\widetilde{\mathfrak{L}}_2^{\mathfrak{LDFS}-}$ is an LDFSCS containing $\widetilde{\mathfrak{L}}_1$. However, $\widetilde{\mathfrak{L}}_1^{\mathfrak{LDFS}-}$ is the littlest LDFSCS containing $\widetilde{\mathfrak{L}}_1$. Therefore, $\widetilde{\mathfrak{L}}_1^{\mathfrak{LDFS}-} \subseteq \widetilde{\mathfrak{L}}_2^{\mathfrak{LDFS}-}$
5. Since the union of two LDFSSs $\widetilde{\mathfrak{L}}_1$ and $\widetilde{\mathfrak{L}}_2$ contains the LDFSS $\widetilde{\mathfrak{L}}_1$ and the union of two LDFSSs $\widetilde{\mathfrak{L}}_1$ and $\widetilde{\mathfrak{L}}_2$ contains the LDFSS $\widetilde{\mathfrak{L}}_2$, $(\widetilde{\mathfrak{L}}_1 \cup \widetilde{\mathfrak{L}}_2)^{\mathfrak{LDFS}-} \supseteq \widetilde{\mathfrak{L}}_1^{\mathfrak{LDFS}-}$. Then the closure of the union of two LDFSSs $\widetilde{\mathfrak{L}}_1$ and $\widetilde{\mathfrak{L}}_2$ contains the closure of LDFSS $\widetilde{\mathfrak{L}}_1$ and the closure of the union of two LDFSSs $\widetilde{\mathfrak{L}}_1$ and $\widetilde{\mathfrak{L}}_2$ contains the closure of LDFSS $\widetilde{\mathfrak{L}}_2$. Hence, the union of closure of LDFSSs $\widetilde{\mathfrak{L}}_1^{\mathfrak{LDFS}-}$, $\widetilde{\mathfrak{L}}_2^{\mathfrak{LDFS}-}$ is a subset of closure of the union of $(\widetilde{\mathfrak{L}}_1^{\mathfrak{LDFS}-}, \widetilde{\mathfrak{L}}_2)^{\mathfrak{LDFS}-}$. By the fact that $\widetilde{\mathfrak{L}}_1 \cup \widetilde{\mathfrak{L}}_2 \subseteq \widetilde{\mathfrak{L}}_1^{\mathfrak{LDFS}-} \cup \widetilde{\mathfrak{L}}_2^{\mathfrak{LDFS}-}$, and since $(\widetilde{\mathfrak{L}}_1 \cup \widetilde{\mathfrak{L}}_2)^{\mathfrak{LDFS}-}$ is the littlest LDFSCS containing $\widetilde{\mathfrak{L}}_1 \cup \widetilde{\mathfrak{L}}_2$, so $(\widetilde{\mathfrak{L}}_1 \cup \widetilde{\mathfrak{L}}_2)^{\mathfrak{LDFS}-} \subseteq \widetilde{\mathfrak{L}}_1^{\mathfrak{LDFS}-} \cup \widetilde{\mathfrak{L}}_2^{\mathfrak{LDFS}-}$. Thus, $(\widetilde{\mathfrak{L}}_1 \cup \widetilde{\mathfrak{L}}_2)^{\mathfrak{LDFS}-} = \widetilde{\mathfrak{L}}_1^{\mathfrak{LDFS}-} \cup \widetilde{\mathfrak{L}}_2^{\mathfrak{LDFS}-}$.
6. Since $\widetilde{\mathfrak{L}}_1 \cap \widetilde{\mathfrak{L}}_2 \subseteq \widetilde{\mathfrak{L}}_1$ and $\widetilde{\mathfrak{L}}_1 \cap \widetilde{\mathfrak{L}}_2 \subseteq \widetilde{\mathfrak{L}}_2$, $(\widetilde{\mathfrak{L}}_1 \cap \widetilde{\mathfrak{L}}_2)^{\mathfrak{LDFS}-} \subseteq \widetilde{\mathfrak{L}}_1^{\mathfrak{LDFS}-} \cap \widetilde{\mathfrak{L}}_2^{\mathfrak{LDFS}-}$.
7. Since $\widetilde{\mathfrak{L}}^{\mathfrak{LDFS}-}$ is a LDFSCS, then $(\widetilde{\mathfrak{L}}^{\mathfrak{LDFS}-})^{\mathfrak{LDFS}-} = \widetilde{\mathfrak{L}}^{\mathfrak{LDFS}-}$.

□

Theorem 6. $(\widetilde{\mathfrak{X}}, \widetilde{\mathcal{T}}, \widetilde{\mathfrak{E}})$ *be a LDFSTS over* $(\widetilde{\mathfrak{X}}, \widetilde{\mathfrak{E}})$*. Let* $\widetilde{\mathfrak{L}}$ *be a linear Diophantine fuzzy soft subset of* $(\widetilde{\mathfrak{X}}, \widetilde{\mathfrak{E}})$*. Then*

1. $\widetilde{1} - \widetilde{\mathfrak{L}}^{\mathfrak{LDFS}\circ} = (\widetilde{1} - \widetilde{\mathfrak{L}})^{\mathfrak{LDFS}-}$.
2. $\widetilde{1} - \widetilde{\mathfrak{L}}^{\mathfrak{LDFS}-} = (\widetilde{1} - \widetilde{\mathfrak{L}})^{\mathfrak{LDFS}\circ}$.

Theorem 7. *Let* $(\widetilde{\mathfrak{X}}, \widetilde{\mathcal{T}}, \widetilde{\mathfrak{E}})$ *be an LDFSTS in relation to* $(\widetilde{\mathfrak{X}}, \widetilde{\mathfrak{E}})$*. Let* $\widetilde{\mathfrak{L}}_1$ *and* $\widetilde{\mathfrak{L}}_2$ *be linear Diophantine fuzzy soft subsets of* $(\widetilde{\mathfrak{X}}, \widetilde{\mathfrak{E}})$*. Then the following claims are true:*

1. $\widetilde{\widetilde{\mathfrak{L}}}$ *is an LDFSOS open if and only if* $\widetilde{\mathfrak{L}}^{\mathfrak{LDFS}\circ} = \widetilde{\mathfrak{L}}$.
2. $\widetilde{0}^{\mathfrak{LDFS}\circ} = \widetilde{0}$ *and* $\widetilde{1}^{\mathfrak{LDFS}\circ} = \widetilde{1}$.
3. $\widetilde{\mathfrak{L}}_1 \subseteq \widetilde{\mathfrak{L}}_2 \Rightarrow \widetilde{\mathfrak{L}}_1^{\mathfrak{LDFS}\circ} \subseteq \widetilde{\mathfrak{L}}_2^{\mathfrak{LDFS}\circ}$.
4. $(\widetilde{\mathfrak{L}}_1 \cup \widetilde{\mathfrak{L}}_2)^{\mathfrak{LDFS}\circ} = \widetilde{\mathfrak{L}}_1^{\mathfrak{LDFS}\circ} \cup \widetilde{\mathfrak{L}}_2^{\mathfrak{LDFS}\circ}$.
5. $(\widetilde{\mathfrak{L}}_1 \cap \widetilde{\mathfrak{L}}_2)^{\mathfrak{LDFS}\circ} = \widetilde{\mathfrak{L}}_1^{\mathfrak{LDFS}\circ} \cap \widetilde{\mathfrak{L}}_2^{\mathfrak{LDFS}\circ}$.
6. $(\widetilde{\mathfrak{L}}^{\mathfrak{LDFS}\circ})^{\mathfrak{LDFS}\circ} = \widetilde{\mathfrak{L}}^{\mathfrak{LDFS}\circ}$.

Proof.

1. $\widetilde{\mathfrak{L}}$ is an LDFSOS if and only if $\widetilde{1} - \widetilde{\mathfrak{L}}$ is an LDFSCS, if and only if $(\widetilde{1} - \widetilde{\mathfrak{L}})^{\mathfrak{LDFS}-} = \widetilde{1} - \widetilde{\mathfrak{L}}$, if and only if $\widetilde{1} - (\widetilde{1} - \widetilde{\mathfrak{L}})^{\mathfrak{LDFS}-} = \widetilde{\mathfrak{L}}$ if and only if $\widetilde{\mathfrak{L}}^{\mathfrak{LDFS}\circ} = \widetilde{\mathfrak{L}}$.
2. As $\widetilde{0}$ and $\widetilde{1}$ are LDFSOSs in $(\widetilde{\mathfrak{X}}, \widetilde{\mathcal{T}}, \widetilde{\mathfrak{E}})$, $\widetilde{0}^{\mathfrak{LDFS}\circ} = \widetilde{0}$ and $\widetilde{1}^{\mathfrak{LDFS}\circ} = \widetilde{1}$.
3. If $\widetilde{\mathfrak{L}}_1 \subseteq \widetilde{\mathfrak{L}}_2$, since $\widetilde{\mathfrak{L}}_2 \supseteq \widetilde{\mathfrak{L}}_2^{\mathfrak{LDFS}\circ}$, then $\widetilde{\mathfrak{L}}_1 \supseteq \widetilde{\mathfrak{L}}_2^{\mathfrak{LDFS}\circ}$. That is, $\widetilde{\mathfrak{L}}_2^{\mathfrak{LDFS}\circ}$ is an LDFSOS containing $\widetilde{\mathfrak{L}}_1$. However, $\widetilde{\mathfrak{L}}_1^{\mathfrak{LDFS}\circ}$ is the largest LDFSOS contained in $\widetilde{\mathfrak{L}}_1$. Therefore, $\widetilde{\mathfrak{L}}_1^{\mathfrak{LDFS}\circ} \subseteq \widetilde{\mathfrak{L}}_2^{\mathfrak{LDFS}\circ}$
4. Since $\widetilde{\mathfrak{L}}_1 \subseteq \widetilde{\mathfrak{L}}_1 \cup \widetilde{\mathfrak{L}}_2$ and $\widetilde{\mathfrak{L}}_2 \subseteq \widetilde{\mathfrak{L}}_1 \cup \widetilde{\mathfrak{L}}_2$, $\widetilde{\mathfrak{L}}_1^{\mathfrak{LDFS}\circ} \subseteq (\widetilde{\mathfrak{L}}_1 \cup \widetilde{\mathfrak{L}}_2)^{\mathfrak{LDFS}\circ}$ and $\widetilde{\mathfrak{L}}_2^{\mathfrak{LDFS}\circ} \subseteq (\widetilde{\mathfrak{L}}_1 \cup \widetilde{\mathfrak{L}}_2)^{\mathfrak{LDFS}\circ}$. Therefore, $\widetilde{\mathfrak{L}}_1^{\mathfrak{LDFS}\circ} \cup \widetilde{\mathfrak{L}}_2^{\mathfrak{LDFS}\circ} \subseteq (\widetilde{\mathfrak{L}}_1 \cup \widetilde{\mathfrak{L}}_2)^{\mathfrak{LDFS}\circ}$. By the fact that $\widetilde{\mathfrak{L}}_1 \cup \widetilde{\mathfrak{L}}_2 \subseteq \widetilde{\mathfrak{L}}_1^{\mathfrak{LDFS}\circ} \cup \widetilde{\mathfrak{L}}_2^{\mathfrak{LDFS}\circ}$, and since $(\widetilde{\mathfrak{L}}_1 \cup \widetilde{\mathfrak{L}}_2)^{\mathfrak{LDFS}\circ}$ is the largest LDFSOS containing $\widetilde{\mathfrak{L}}_1 \cup \widetilde{\mathfrak{L}}_2$, so $(\widetilde{\mathfrak{L}}_1 \cup \widetilde{\mathfrak{L}}_2)^{\mathfrak{LDFS}\circ} \subseteq \widetilde{\mathfrak{L}}_1^{\mathfrak{LDFS}\circ} \cup \widetilde{\mathfrak{L}}_2^{\mathfrak{LDFS}\circ}$. Thus, $(\widetilde{\mathfrak{L}}_1 \cup \widetilde{\mathfrak{L}}_2)^{\mathfrak{LDFS}\circ} = \widetilde{\mathfrak{L}}_1^{\mathfrak{LDFS}\circ} \cup \widetilde{\mathfrak{L}}_2^{\mathfrak{LDFS}\circ}$.
5. Since $\widetilde{\mathfrak{L}}_1 \cap \widetilde{\mathfrak{L}}_2 \subseteq \widetilde{\mathfrak{L}}_1$ and $\widetilde{\mathfrak{L}}_1 \cap \widetilde{\mathfrak{L}}_2 \subseteq \widetilde{\mathfrak{L}}_2$, $(\widetilde{\mathfrak{L}}_1 \cap \widetilde{\mathfrak{L}}_2)^{\mathfrak{LDFS}\circ} \subseteq \widetilde{\mathfrak{L}}_1^{\mathfrak{LDFS}\circ} \cap \widetilde{\mathfrak{L}}_2^{\mathfrak{LDFS}\circ}$.
6. Since $\widetilde{\mathfrak{L}}^{\mathfrak{LDFS}\circ}$ is an LDFSOS, then $(\widetilde{\mathfrak{L}}^{\mathfrak{LDFS}\circ})^{\mathfrak{LDFS}\circ} = \widetilde{\mathfrak{L}}^{\mathfrak{LDFS}\circ}$.

□

Definition 16. *Let $\widetilde{\mathfrak{L}} \in LDFSSs(\widetilde{\mathfrak{X}}, \widetilde{\mathfrak{E}})$ and $(\widetilde{\mathfrak{X}}, \widetilde{\mathcal{T}}, \widetilde{\mathfrak{E}})$ be a LDFSTS over $(\widetilde{\mathfrak{X}}, \widetilde{\mathfrak{E}})$. Then LDFS frontier of $\widetilde{\mathfrak{L}}$ is represented by $LDFSB(\widetilde{\mathfrak{L}})$ and is outlined as $LDFSB(\widetilde{\mathfrak{L}}) = \widetilde{\mathfrak{L}}^{\mathfrak{LDFS}-} \cap (\widetilde{\mathfrak{L}}^c)^{\mathfrak{LDFS}-}$.*

Theorem 8. *Let $(\widetilde{\mathfrak{X}}, \widetilde{\mathcal{T}}, \widetilde{\mathfrak{E}})$ be an LDFSTS over $(\widetilde{\mathfrak{X}}, \widetilde{\mathfrak{E}})$ and $\widetilde{\mathfrak{L}} \in LDFSSs(\widetilde{\mathfrak{X}}, \widetilde{\mathfrak{E}})$. Then,*

1. $\widetilde{\mathfrak{L}}^{\mathfrak{LDFS}\circ} \cap LDFSB(\widetilde{\mathfrak{L}}) = \widetilde{\mathfrak{o}}$
2. $\widetilde{\mathfrak{L}}^{\mathfrak{LDFS}-} = \widetilde{\mathfrak{L}}^{\mathfrak{LDFS}\circ} \cup LDFSB(\widetilde{\mathfrak{L}})$
3. $LDFSB(\widetilde{\mathfrak{L}}) = \widetilde{\mathfrak{o}}$ *if and only if* $\widetilde{\mathfrak{L}}$ *is both open and closed.*
4. $LDFSB(\widetilde{\mathfrak{L}}) = \widetilde{\mathfrak{L}}^{\mathfrak{LDFS}-} \cap (\widetilde{\mathfrak{L}}^{\mathfrak{LDFS}\circ})^c = \widetilde{\mathfrak{o}}$

Proof.

1. $\widetilde{\mathfrak{L}}^{\mathfrak{LDFS}\circ} \cap LDFSB(\widetilde{\mathfrak{L}}) = \widetilde{\mathfrak{L}}^{\mathfrak{LDFS}\circ} \cap (\widetilde{\mathfrak{L}}^{\mathfrak{LDFS}-} \cap (\widetilde{\mathfrak{L}}^c)^{\mathfrak{LDFS}-}) = \widetilde{\mathfrak{L}}^{\mathfrak{LDFS}\circ} \cap (\widetilde{\mathfrak{L}}^{\mathfrak{LDFS}-} \cap (\widetilde{\mathfrak{L}}^{\mathfrak{LDFS}\circ})^c) = \widetilde{\mathfrak{L}}^{\mathfrak{LDFS}\circ} \cap (\widetilde{\mathfrak{L}}^{\mathfrak{LDFS}\circ})^c \cap \widetilde{\mathfrak{L}}^{\mathfrak{LDFS}-} = \widetilde{\mathfrak{o}} \cap \widetilde{\mathfrak{L}}^{\mathfrak{LDFS}-} = \widetilde{\mathfrak{o}}$.
2. $\widetilde{\mathfrak{L}}^{\mathfrak{LDFS}\circ} \cup LDFSB(\widetilde{\mathfrak{L}}) = \widetilde{\mathfrak{L}}^{\mathfrak{LDFS}\circ} \cup (\widetilde{\mathfrak{L}}^{\mathfrak{LDFS}-} \cap (\widetilde{\mathfrak{L}}^c)^{\mathfrak{LDFS}-}) = \widetilde{\mathfrak{L}}^{\mathfrak{LDFS}\circ} \cup (\widetilde{\mathfrak{L}}^{\mathfrak{LDFS}-} \cap (\widetilde{\mathfrak{L}}^{\mathfrak{LDFS}\circ})^c) = (\widetilde{\mathfrak{L}}^{\mathfrak{LDFS}\circ} \cup \widetilde{\mathfrak{L}}^{\mathfrak{LDFS}-}) \cap (\widetilde{\mathfrak{L}}^{\mathfrak{LDFS}\circ} \cup (\widetilde{\mathfrak{L}}^{\mathfrak{LDFS}\circ})^c) = (\widetilde{\mathfrak{L}}^{\mathfrak{LDFS}\circ} \cup \widetilde{\mathfrak{L}}^{\mathfrak{LDFS}-}) \cap \widetilde{\mathfrak{1}} = (\widetilde{\mathfrak{L}}^{\mathfrak{LDFS}\circ} \cup \widetilde{\mathfrak{L}}^{\mathfrak{LDFS}-}) = \widetilde{\mathfrak{L}}^{\mathfrak{LDFS}-}$. Since $\widetilde{\mathfrak{L}}^{\mathfrak{LDFS}\circ} \subset \widetilde{\mathfrak{L}} \subset \widetilde{\mathfrak{L}}^{\mathfrak{LDFS}-}$.
3. $LDFSB(\widetilde{\mathfrak{L}}) = \widetilde{\mathfrak{L}}^{\mathfrak{LDFS}-} \cap (\widetilde{\mathfrak{L}}^c)^{\mathfrak{LDFS}-} = \widetilde{\mathfrak{o}} \Rightarrow \widetilde{\mathfrak{L}}^{\mathfrak{LDFS}-} \cap (\widetilde{\mathfrak{L}}^{\mathfrak{LDFS}\circ})^c = \widetilde{\mathfrak{o}} \Rightarrow \widetilde{\mathfrak{L}}^{\mathfrak{LDFS}-} \cap ((\widetilde{\mathfrak{L}}^{\mathfrak{LDFS}\circ})^c)^c = \widetilde{\mathfrak{o}} \Rightarrow \widetilde{\mathfrak{L}}^{\mathfrak{LDFS}-} \cap \widetilde{\mathfrak{L}}^{\mathfrak{LDFS}\circ} = \widetilde{\mathfrak{o}} \Rightarrow \widetilde{\mathfrak{L}}^{\mathfrak{LDFS}-} \subset \widetilde{\mathfrak{L}}^{\mathfrak{LDFS}\circ}$ i.e., $\widetilde{\mathfrak{L}} \subset \widetilde{\mathfrak{L}}^{\mathfrak{LDFS}-} \subset \widetilde{\mathfrak{L}}^{\mathfrak{LDFS}\circ} \Rightarrow \widetilde{\mathfrak{L}} \subset \widetilde{\mathfrak{L}}^{\mathfrak{LDFS}\circ}$.
   In addition, we know that $\widetilde{\mathfrak{L}}^{\mathfrak{LDFS}\circ} \subset \widetilde{\mathfrak{L}}$. Thus $\widetilde{\mathfrak{L}}^{\mathfrak{LDFS}\circ} = \widetilde{\mathfrak{L}}$. This shows that $\widetilde{\mathfrak{L}}$ is open.
   Furthermore, $\widetilde{\mathfrak{L}}^{\mathfrak{LDFS}-} \subset \widetilde{\mathfrak{L}}^{\mathfrak{LDFS}\circ} \subset \widetilde{\mathfrak{L}} \Rightarrow \widetilde{\mathfrak{L}}^{\mathfrak{LDFS}-} \subset \widetilde{\mathfrak{L}}$. Moreover, we know that $\widetilde{\mathfrak{L}} \subset \widetilde{\mathfrak{L}}^{\mathfrak{LDFS}-}$. Thus $\widetilde{\mathfrak{L}}^{\mathfrak{LDFS}-} = \widetilde{\mathfrak{L}}$. This shows that $\widetilde{\mathfrak{L}}$ is closed.
   Conversely, if $\widetilde{\mathfrak{L}}$ is open and closed, then $\widetilde{\mathfrak{L}}^{\mathfrak{LDFS}\circ} = \widetilde{\mathfrak{L}}$ and $\widetilde{\mathfrak{L}}^{\mathfrak{LDFS}-} = \widetilde{\mathfrak{L}}$. Now, $LDFSB(\widetilde{\mathfrak{L}}) = \widetilde{\mathfrak{L}}^{\mathfrak{LDFS}-} \cap (\widetilde{\mathfrak{L}}^c)^{\mathfrak{LDFS}-} = \widetilde{\mathfrak{L}}^{\mathfrak{LDFS}-} \cap (\widetilde{\mathfrak{L}}^{\mathfrak{LDFS}\circ})^c = \widetilde{\mathfrak{L}}^{\mathfrak{LDFS}-} \cap \widetilde{\mathfrak{L}}^c = \widetilde{\mathfrak{o}}$.
4. $LDFSB(\widetilde{\mathfrak{L}}) = \widetilde{\mathfrak{L}}^{\mathfrak{LDFS}-} \cap (\widetilde{\mathfrak{L}}^c)^{\mathfrak{LDFS}-} = \widetilde{\mathfrak{L}}^{\mathfrak{LDFS}-} \cap (\widetilde{\mathfrak{L}}^{\mathfrak{LDFS}\circ})^c$

□

Definition 17. *Let $(\widetilde{\mathfrak{X}}, \widetilde{\mathcal{T}}, \widetilde{\mathfrak{E}})$ be an LDFSTS over $(\widetilde{\mathfrak{X}}, \widetilde{\mathfrak{E}})$. The accumulation $\widetilde{\mathfrak{B}} \subset \widetilde{\mathcal{T}}$ is known as a base for $\widetilde{\mathcal{T}}$. If $\forall \mathscr{A} \in \widetilde{\mathcal{T}}$ can be written as the supercilious union of some objects of LDFSS $\widetilde{\mathfrak{B}}$, then $\widetilde{\mathfrak{B}}$ is called as a linear Diophantine fuzzy soft basis (LDFSB) for the LDFST $\widetilde{\mathcal{T}}$. Linear Diophantine fuzzy basic open sets are the elements of $\widetilde{\mathfrak{B}}$.*

Theorem 9. *Let $(\widetilde{\mathfrak{X}}, \widetilde{\mathcal{T}}, \widetilde{\mathfrak{E}})$ be an LDFSTS over $(\widetilde{\mathfrak{X}}, \widetilde{\mathfrak{E}})$ and $\widetilde{\mathfrak{B}}$ an LDFSB for $\widetilde{\mathcal{T}}$. Then, $\widetilde{\mathfrak{T}}$ is the set of linear Diophantine fuzzy soft unions of $\widetilde{\mathfrak{B}}$ components.*

Proof. The evidence is unambiguous. □

Theorem 10. *Let the two LDFSTS over $(\widetilde{\mathfrak{X}}, \widetilde{\mathfrak{E}})$ be $(\widetilde{\mathfrak{X}}, \widetilde{\mathcal{T}}_1, \widetilde{\mathfrak{E}})$ and $(\widetilde{\mathfrak{X}}, \widetilde{\mathcal{T}}_2, \widetilde{\mathfrak{E}})$. Moreover, let $\widetilde{\mathfrak{B}}_1$ be an LDFSB for $\widetilde{\mathcal{T}}_1$ and $\widetilde{\mathfrak{B}}_2$ be an LDFSB for $\widetilde{\mathcal{T}}_2$. If $\widetilde{\mathfrak{B}}_1 \subset \widetilde{\mathfrak{B}}_2$, then $\widetilde{\mathcal{T}}_1 \subset \widetilde{\mathcal{T}}_2$.*

Proof. The proof is straightforward. □

4. MCDM via LDFSS-TOPSIS Approach

TOPSIS is used to select the best choice from a set of venture options. The reasonable compromise is the option that is nearest to the PIS but farthest from the NIS. In this part, we will look at how LDFSSs may be used in MCDM with TOPSIS. Primarily, we will expand TOPSIS to LDFSSs, and then we will look at a stock exchange investing problem. TOPSIS is one of the most powerful strategies available in the literature for dealing with such issues. Every approach has advantages and limitations depending on the nature of the problem at hand.

We start by discussing the targeted approach procedure by procedure. The suggested LDFSS TOPSIS is a generalization of Eraslan and Karaaslan's [27] fuzzy soft TOPSIS.

Step 7: The normalized euclidean distance (NED) of each attribute and its LDFSSV-PIS can be defined as:

$$\mathfrak{d}_{\mathfrak{E}}^{\mathfrak{N}+} = \frac{1}{4n}\sum_{j=1}^{q}[({}^{i}\mathfrak{t}_{ij} -{}^{i} \mathfrak{t}_{j}{}^{+})^{2} + ({}^{i}\mathfrak{f}_{ij} -{}^{i} \mathfrak{f}_{j}{}^{+})^{2} + ({}^{i}\alpha_{ij} -{}^{i} \alpha_{j}{}^{+})^{2} + ({}^{i}\beta_{ij} -{}^{i} \beta_{j}{}^{+})^{2}]$$

The normalized euclidean distance (NED) of each alternative and its LDFSSV-NIS can be defined as:

$$\mathfrak{d}_{\mathfrak{E}}^{\mathfrak{N}-} = \frac{1}{4n}\sum_{j=1}^{q}[({}^{i}\mathfrak{t}_{ij} -{}^{i} \mathfrak{t}_{j}{}^{-})^{2} + ({}^{i}\mathfrak{f}_{ij} -{}^{i} \mathfrak{f}_{j}{}^{-})^{2} + ({}^{i}\alpha_{ij} -{}^{i} \alpha_{j}{}^{-})^{2} + ({}^{i}\beta_{ij} -{}^{i} \beta_{j}{}^{-})^{2}]$$

Step 8: Compute the LDFSS relative closeness with the formula:

$$\mathfrak{C}_{j}^{+} = \frac{\mathfrak{d}_{\mathfrak{E}}^{\mathfrak{N}-}}{\mathfrak{d}_{\mathfrak{E}}^{\mathfrak{N}+} + \mathfrak{d}_{\mathfrak{E}}^{\mathfrak{N}-}}$$

Step 9: Finally, the alternate ranking order is found. The best attribute is the one with the greatest revised coefficient value.

The proposed LDFSS-TOPSIS is portrayed as a flow chart in Figure 1.

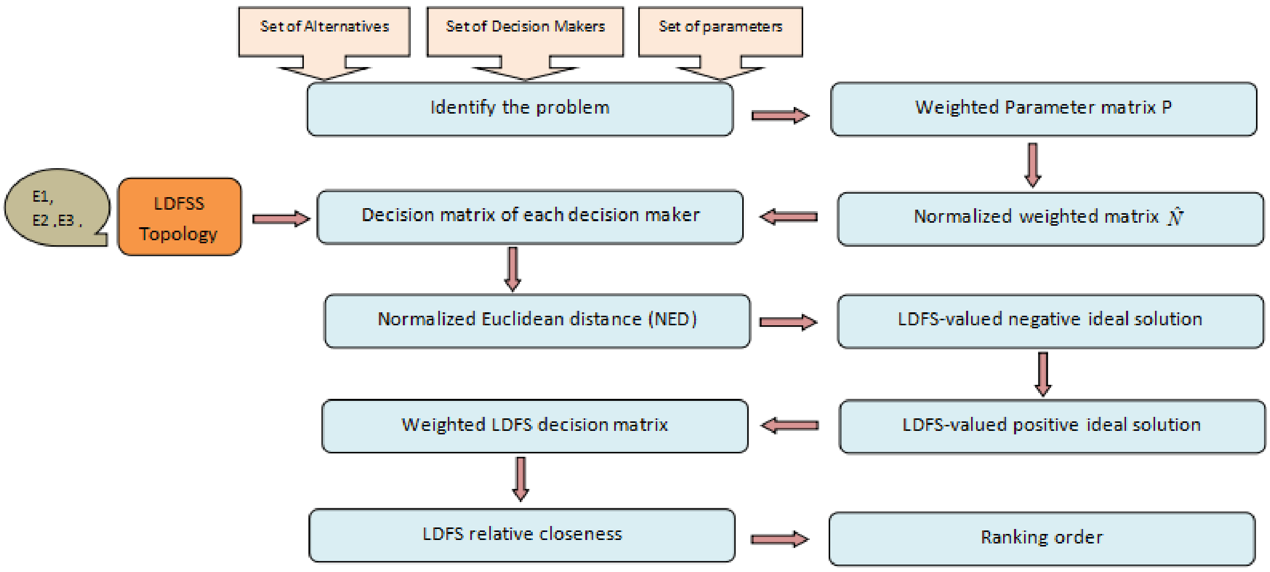

Figure 1. Procedural steps of Algorithm 1.

Algorithm 1: LDFSS-TOPSIS.

Step 1: Identify the issue: $\mathfrak{E} = \{\mathfrak{e}_i\}$ is the set of decision makers/experts, the assemblage of alternatives/attributes is $\mathfrak{A} = \{\mathfrak{a}_j\}$ and $\mathfrak{C} = \{\mathfrak{c}_\mathfrak{k}\}$ is the family of parameters/criteria, where $\mathfrak{i}, \mathfrak{j}, \mathfrak{k} \in N$ and $\mathfrak{i} = \{1, 2, 3, \ldots, \mathfrak{p}\}$, $\mathfrak{j} = \{1, 2, 3, \ldots, \mathfrak{q}\}$,$\mathfrak{k} = \{1, 2, 3, \ldots, \mathfrak{r}\}$.

Step 2: If w_{ij} denotes the weight allocated by $\mathfrak{E}_\mathfrak{k}$ to $\mathfrak{C}_j$ keeping in view the linguistic variables (LVs) Table 2, build a weighted criteria matrix

$$\mathcal{P} = [w_{ij}]_{\mathfrak{p}\times\mathfrak{q}} = \begin{pmatrix} w_{11} & w_{12} & \ldots & w_{1q} \\ w_{21} & w_{22} & \ldots & w_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ w_{p1} & w_{p2} & \ldots & w_{pq} \end{pmatrix}.$$

Step 3: Normalize the weighted parameter matrix $\mathcal{P}$ that was created in Step 2 above. There is no need to split the criteria as cost and benefits. As a result, we apply the normalized approach described below to convert the cost criteria to the benefit parameter. The normalized values are represented as a matrix indicated by

$$\widehat{\mathfrak{N}} = [\widehat{\mathfrak{n}}_{ij}]_{\mathfrak{p}\times\mathfrak{q}} = \begin{pmatrix} \widehat{\mathfrak{n}}_{11} & \widehat{\mathfrak{n}}_{12} & \ldots & \widehat{\mathfrak{n}}_{1q} \\ \widehat{\mathfrak{n}}_{21} & \widehat{\mathfrak{n}}_{22} & \ldots & \widehat{\mathfrak{n}}_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{\mathfrak{n}}_{p1} & \widehat{\mathfrak{n}}_{p2} & \ldots & \widehat{\mathfrak{n}}_{pq} \end{pmatrix}$$, where $\widehat{\mathfrak{n}}_{ij} = \frac{w_{ij}}{\sqrt{\sum_{i=1}^{p} w_{ij}^2}}$ and acquire the weight vector $\mathfrak{W} = (\eta_j : j = 1, 2, ..., \mathfrak{q})$, where $\eta_j = \frac{\sum_{i=1}^{q} \widehat{\mathfrak{n}}_{ij}}{\mathfrak{n}\sum_{\mathfrak{k}=1}^{m} \widehat{\mathfrak{n}}_{i\mathfrak{k}}}$

$\omega_\mathfrak{k} = \frac{1-\sqrt{[(1-({}^\mathfrak{k}\mathfrak{t}(\xi)^2+{}^\mathfrak{k}\mathfrak{f}(\xi))^2)+(1-({}^\mathfrak{k}\alpha(\xi)^2+({}^\mathfrak{k}\beta(\xi))^2)]/2}}{\sum_{\mathfrak{k}=1}^{\mathfrak{r}}[1-\sqrt{[(1-({}^\mathfrak{k}\mathfrak{t}(\xi)^2+{}^\mathfrak{k}\mathfrak{f}(\xi)^2))+(1-({}^\mathfrak{k}\alpha(\xi)^2+{}^\mathfrak{k}\beta(\xi)^2)]/2}]}$ where $\mathfrak{k} = 1, 2, 3, \ldots, \mathfrak{r}$ and $\sum_{\mathfrak{k}=1}^{\mathfrak{r}} \omega_\mathfrak{k} = 1$,

Step 4: Construct the LSFS-decision matrix, where $\mathfrak{a}_{ij}$ is a LDFSS element, for the ith decision maker makes LDFSS topology for each $\mathfrak{i}$. The decision matrix is represented as $\mathfrak{D}_i = [a_{ij}]_{\mathfrak{p}\times\mathfrak{q}} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1q} \\ a_{21} & a_{22} & \ldots & a_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1} & a_{p2} & \ldots & a_{pq} \end{pmatrix}$ where $a_{\mathfrak{r}\mathfrak{q}}$ is a LDFSS-element, for $\mathfrak{k}$ expert/decision maker so that $\mathcal{D}$ makes LDFSS-topology for each $\mathfrak{i}$. Then bring out the aggregated matrix $\mathfrak{A} = \frac{\mathfrak{L}_1+\mathfrak{L}_2+\ldots+\mathfrak{L}_\mathfrak{n}}{\mathfrak{n}} = [\dot{\mathfrak{z}}_{\mathfrak{k}j}]_{\mathfrak{r}\times\mathfrak{q}}$.

Step 5: Acquire the weighted LDFSS decision matrix $\mathfrak{I} = [\breve{\mathfrak{z}}_{j\mathfrak{k}}]_{\mathfrak{l}\times\mathfrak{q}} = \begin{pmatrix} \breve{\mathfrak{z}}_{11} & \breve{\mathfrak{z}}_{12} & \cdots & \breve{\mathfrak{z}}_{1q} \\ \breve{\mathfrak{z}}_{21} & \breve{\mathfrak{z}}_{22} & \cdots & \breve{\mathfrak{z}}_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ \breve{\mathfrak{z}}_{p1} & \breve{\mathfrak{z}}_{p2} & \cdots & \breve{\mathfrak{z}}_{pq} \end{pmatrix}$,

where $\breve{\mathfrak{z}}_\mathfrak{k} = \mathfrak{w}_\mathfrak{k} \times \dot{\mathfrak{z}}_{j\mathfrak{k}}$

$$\begin{aligned} \breve{\mathfrak{z}}_\mathfrak{k} &= LDFWG(\mathfrak{z}_{ij}{}^1, \mathfrak{z}_{ij}{}^2, \ldots, \mathfrak{z}_{ij}^\mathfrak{k}) \\ &= \xi_1\mathfrak{z}_{ij}{}^1 \otimes \xi_2\mathfrak{z}_{ij}{}^2 \otimes \cdots \otimes \xi_\mathfrak{k}\mathfrak{z}_{ij}^\mathfrak{k} \\ &= (\langle \Pi_{\mathfrak{k}=1}^{\mathfrak{r}}(\mathfrak{t}_{ij}{}^\mathfrak{k})^{\xi_\mathfrak{k}}, 1 - \Pi_{\mathfrak{k}=1}^{\mathfrak{r}}(1 - \mathfrak{f}_{ij}{}^\mathfrak{k})^{\xi_\mathfrak{k}}\rangle, \\ &\langle \Pi_{\mathfrak{k}=1}^{\mathfrak{r}}(\alpha_{ij}{}^\mathfrak{k})^{\xi_\mathfrak{k}}, 1 - \Pi_{\mathfrak{k}=1}^{\mathfrak{r}}(1 - \beta_{ij}{}^\mathfrak{k})^{\xi_\mathfrak{k}}\rangle). \end{aligned}$$

Step 6: Locate LDFSS-valued PIS (LDFSSV-PIS) and LDFSS-valued NIS (LDFSSV-NIS), employing in order

$$\mathfrak{s}_j^+ = \{\ddot{\rho}_1^+, \ddot{\rho}_2^+, ..., \ddot{\rho}_q^+\} = \{\langle \vee^i \mathfrak{t}_{ij}, \wedge^i \mathfrak{f}_{ij}\rangle, \langle \vee^i \alpha_{ij}, \wedge^i \beta_{ij}\rangle\}$$

$$\mathfrak{s}_j^- = \{\ddot{\rho}_1^-, \ddot{\rho}_2^-, ..., \ddot{\rho}_q^-\} = \{\langle \wedge^i \mathfrak{t}_{ij}, \vee^i \mathfrak{f}_{ij}\rangle, \langle \wedge^i \alpha_{ij}, \vee^i \beta_{ij}\rangle\}$$

where, $\wedge$ represents LDFSS intersection and $\vee$ represents LDFSS union and.

Table 2. Lingual phrases for importance weights of criteria.

Linguistic Variables	Fuzzy Weights
Less crop production (LCP)	0.10
Ordinary crop production (OCP)	0.30
Good crop production (GCP)	0.50
More crop production (MCP)	0.70
Exceptional crop production (ECP)	0.90

4.1. Numerical Example: MCDM for Robotic Agri-Farming

This section outlines MCDM, which is used to rank alternatives from high to low relevance. In MCDM, DMs must choose the best option from a set of appropriate attributes in a specific scenario. Although there are several aggregation approaches, we suggest extensions of TOPSIS, VIKOR, and aggregation operators to LDFSSs and topologies for MCDM in this context. As an example, the application we are describing here is connected to farming. Alternatives are compared against the chosen criteria to get the best response. As a result, we may conclude that MCDM is a collection of options, various criteria, and subsequent comparability. With the aid of MCDM, we must select those choices that are ideal in every manner.

4.1.1. An Empirical Case Study

Farming is the practice of cultivating food and rearing livestock. Farming includes raising animals and cultivating crops, both of which provide humans with food and raw resources. Farming originated nearly millions of years ago, but we do not know when or where it started. Farming is a way of life, not simply a profession. This also lends credence to modern civilisation, and without it, our survival on Earth would be impossible. Agriculture was once described as “the most beneficial, most valuable, and most honorable occupation of men” by former American President George Washington. Actually, we are all farmers since we all like gardening, whether at home or at fields. We cultivate plants in little mud pots at home, but we are free to grow crops, plants, or trees in the field. This passion of horticulture must be a lifelong habit, whether you are young or elderly. We now are dismantling our homelands and reducing cultivable areas in the name of industrialization, reinforcement, and habitation communities. Food costs will skyrocket as a result of the land destruction process, and we will have to pay considerably more for our daily food requirements. Agriculture is the science and practice of raising plants and livestock. Overall, there are about ten types of farming practiced across the world such as arable farming, commercial farming, extensive farming, fish farming, intense farming, mixed farming, nomadic farming, pastoral farming, poultry farming, sedentary farming, and subsistence farming.

People require more food to survive as the world’s population grows rapidly. Because of the strong demand for food, farmers are under pressure to increase crop production. To address this dilemma, farmers must focus on increasing crop output through the use of agricultural robots. The employment of robots in agriculture is an example of creativity that goes beyond innovation. Agriculture is run like an industry, and it is on its way to becoming a high-tech enterprise in the future. Farmers’ agricultural capacities are rising at a rapid pace as technology advances. Robotics and automation technologies are now increasing manufacturing yields. Agriculture robotic uses include harvesting, weeding, trimming, sowing, spraying, sorting, and packing etc. Agriculture robots are also referred to as “agri-bots” or “agri-robots.” Agribots will play an important part in agriculture in the future. We are just examining one application here, the usage of robots in horticulture. Horticulture is the cultivation of comfort plants, material plants, food plants, and beautiful plants. A next generation robot called “Terra Sentia” (the smallest robot with a width of 12.5 inches and a height of 12.5 inches and a weight of 30 pounds) appears like a lawn mower and navigates a field by producing laser pulses to scan it. It is used to find the plant health and size, plant counting, portrait of the field, stem diameter, and fruit producing plants. This robot has been demonstrated to be useful in a variety of areas, including almond farms, apple orchards, citrus crops, wheat, maize, soybean, tomatoes, cotton, strawberries, sorghum, and vineyards.

We are investigating the effectiveness of farming robots. The characteristics of robotic agri-farming are listed below.

(i) Accuracy and perfection in placement: Plant placement is critical in the field. The precision will result in excellence. Automation of nursing operations completes grafting, propagation, and spacing.
(ii) Automating manual chores: Farmers enhance their productivity by spending little time on duties and more time on amelioration by adopting automation.
(iii) Completion of a difficult work: Scientists, technicians, researchers, and farmers are all in agreement that the utilization of automation will accomplish the difficult duty in a easy and simple manner.
(iv) High quality production: Quality goods are influenced by certain farming aspects such as (soil, time of ripeness, climate, fertilizer etc). Cereal yield is affected by maturity level and degree of dryness (barley, oats, wheat, rice etc.)
(v) Lowering production costs: There is an innovative method for reducing production costs in agriculture by employing robots. We must handle some uncontrolled aspects that reduce profit margins, such as weather stipulations, acquiring various brands of seeds, and employing an adequate amount of pesticides.
(vi) Minimizing necessity physical labour: Because labour costs are substantially higher in agriculture, i.e., (paying to manual labor and skilled workers).
(vii) Persistent function to complete a task: To perform an agreeable role, the farm must be operated using artificial intelligence (automate the entire agricultural process from sowing to harvesting).

4.1.2. Problem Description

Exemplification: A farmer running a large agriculture farm; it may be an expensive endeavor, but he wants to gain a lot of money from it. He comes from a farming family and inherited the skills and enthusiasm for comprehensive sustainable agri-farming. He aspires to live a happy life and provide outstanding education for his children. He wants to update his vision using robots in order to fulfill his thoughts, ambitions, and worries by decreasing available resources and making this career a high-tech vocation. To turn it into a profitable business, the farmer delegated this responsibility to his sons in order to reach a consensus conclusion based upon that technically controlled method.

We apply Algorithm 1 (LDF-TOPSIS) in this example as follows:

Step 1: Let $\mathfrak{E} = \{\mathfrak{e}_i : i = 1,2,3,4\}$ be the family of experts, $\mathfrak{A} = \{\mathfrak{a}_j : j = 1,2,3,\ldots,5\}$ the set of alternatives for robotic agri-farming under study and we determine the possible set of qualities or criterion for robotic agri-farming $\mathfrak{C} = \{\mathfrak{c}_{\mathfrak{k}} : \mathfrak{k} = 1,2,\ldots,7\}$, where, $\mathfrak{c}_1$ = Accuracy and perfection in placement, $\mathfrak{c}_2$ = Automating manual chores, $\mathfrak{c}_3$ = Completion of a difficult work, $\mathfrak{c}_4$ = High quality production, $\mathfrak{c}_5$ = Lowering production costs, $\mathfrak{c}_6$ = Minimizing necessary physical labour, and $\mathfrak{c}_7$ = Persistent function to complete a task.

Step 2: The board of family specialists generates linear Diophantine fuzzy soft information as a weighted parameter matrix, shown in Table 2, by reviewing the track record of the list of agri-farming robots and their performance.

$$\mathcal{P} = [w_{ij}]_{4\times 7} = \begin{pmatrix} ECP & GCP & MCP & LCP & LCP & OCP & GCP \\ MCP & GCP & ECP & OCP & GCP & ECP & LCP \\ GCP & MCP & OCP & ECP & LCP & MCP & ECP \\ ECP & MCP & ECP & MCP & OCP & GCP & LCP \end{pmatrix}$$

$$= \begin{pmatrix} 0.9 & 0.5 & 0.7 & 0.1 & 0.1 & 0.3 & 0.5 \\ 0.7 & 0.5 & 0.9 & 0.3 & 0.5 & 0.9 & 0.1 \\ 0.5 & 0.7 & 0.3 & 0.9 & 0.1 & 0.7 & 0.9 \\ 0.9 & 0.7 & 0.9 & 0.7 & 0.3 & 0.5 & 0.1 \end{pmatrix}.$$

where w_{ij} is the weight provided by the experts $\mathfrak{e}_i$ (row-wise) to each quality or criterion $\mathfrak{c}_j$ (column-wise).

Step 3: The normalized weighted matrix is

$$\widehat{\mathfrak{N}} = [\widehat{\mathfrak{n}}_{ij}]_{4\times7} = \begin{pmatrix} 0.5859 & 0.4110 & 0.4719 & 0.0845 & 0.1667 & 0.2343 & 0.4811 \\ 0.4557 & 0.4110 & 0.6068 & 0.2535 & 0.8333 & 0.7028 & 0.0962 \\ 0.3255 & 0.5754 & 0.2023 & 0.7606 & 0.1667 & 0.5466 & 0.8660 \\ 0.5859 & 0.5754 & 0.6068 & 0.5916 & 0.5000 & 0.3904 & 0.0962 \end{pmatrix}$$

Hence the weight vectors are $\omega = (0.16\ 0.21\ 0.17\ 0.11\ 0.12\ 0.10\ 0.13)$

Step 4: Taking into account the historical track record of the agri-bots, the LDFSS decision matrix $\mathfrak{D}$ of each specialist is provided, with choices indicated row-wise and parameters expressed column-wise. $\mathfrak{D} = \frac{\mathfrak{L}_1+\mathfrak{L}_2+\mathfrak{L}_3+\mathfrak{L}_4}{4} = [\dot{\mathfrak{z}}_{\mathfrak{k}j}]_{5\times7}$
The aggregated decision matrix is now stated as

$$\begin{pmatrix} (\langle 0.91,0.18\rangle,\langle 0.62,0.12\rangle) & (\langle 0.90,0.17\rangle,\langle 0.36,0.64\rangle) & (\langle 0.49,0.56\rangle,\langle 0.55,0.26\rangle) & (\langle 0.73,0.28\rangle,\langle 0.62,0.19\rangle) & (\langle 0.94,0.28\rangle,\langle 0.76,0.23\rangle) & (\langle 0.55,0.44\rangle,\langle 0.27,0.32\rangle) & (\langle 0.65,0.62\rangle,\langle 0.29,0.34\rangle) \\ (\langle 0.57,0.52\rangle,\langle 0.22,0.31\rangle) & (\langle 0.95,0.31\rangle,\langle 0.88,0.11\rangle) & (\langle 0.52,0.38\rangle,\langle 0.40,0.36\rangle) & (\langle 0.61,0.37\rangle,\langle 0.52,0.22\rangle) & (\langle 0.56,0.76\rangle,\langle 0.67,0.28\rangle) & (\langle 0.67,0.55\rangle,\langle 0.25,0.36\rangle) & (\langle 0.92,0.12\rangle,\langle 0.74,0.26\rangle) \\ (\langle 0.69,0.41\rangle,\langle 0.28,0.41\rangle) & (\langle 0.56,0.66\rangle,\langle 0.10,0.70\rangle) & (\langle 0.63,0.27\rangle,\langle 0.49,0.16\rangle) & (\langle 0.35,0.70\rangle,\langle 0.30,0.50\rangle) & (\langle 0.87,0.41\rangle,\langle 0.81,0.17\rangle) & (\langle 0.89,0.15\rangle,\langle 0.56,0.36\rangle) & (\langle 0.74,0.27\rangle,\langle 0.55,0.31\rangle) \\ (\langle 0.71,0.46\rangle,\langle 0.26,0.38\rangle) & (\langle 0.58,0.49\rangle,\langle 0.47,0.32\rangle) & (\langle 0.56,0.76\rangle,\langle 0.58,0.39\rangle) & (\langle 0.50,0.45\rangle,\langle 0.45,0.35\rangle) & (\langle 0.97,0.32\rangle,\langle 0.67,0.25\rangle) & (\langle 0.83,0.29\rangle,\langle 0.33,0.61\rangle) & (\langle 0.83,0.29\rangle,\langle 0.23,0.67\rangle) \\ (\langle 0.87,0.37\rangle,\langle 0.24,0.45\rangle) & (\langle 0.63,0.49\rangle,\langle 0.27,0.46\rangle) & (\langle 0.66,0.69\rangle,\langle 0.36,0.35\rangle) & (\langle 0.50,0.55\rangle,\langle 0.50,0.40\rangle) & (\langle 0.83,0.29\rangle,\langle 0.23,0.67\rangle) & (\langle 0.87,0.41\rangle,\langle 0.91,0.02\rangle) & (\langle 0.36,0.16\rangle,\langle 0.37,0.22\rangle) \end{pmatrix}$$

Step 5: The weighted LDFSS decision matrix is $\mathfrak{B} = [\breve{\mathfrak{z}}_{\mathfrak{k}j}]_{\mathfrak{r}\times\mathfrak{q}} = \mathfrak{w}_j \times \dot{\mathfrak{z}}_{\mathfrak{k}j}$

$$\begin{pmatrix} (\langle 0.320,0.967\rangle,\langle 0.143,0.712\rangle) & (\langle 0.383,0.689\rangle,\langle 0.089,0.911\rangle) & (\langle 0.108,0.906\rangle,\langle 0.127,0.795\rangle) & (\langle 0.134,0.869\rangle,\langle 0.101,0.833\rangle) & (\langle 0.287,0.858\rangle,\langle 0.157,0.838\rangle) & (\langle 0.077,0.921\rangle,\langle 0.031,0.038\rangle) & (\langle 0.128,0.940\rangle,\langle 0.044,0.869\rangle) \\ (\langle 0.126,0.901\rangle,\langle 0.039,0.829\rangle) & (\langle 0.467,0.782\rangle,\langle 0.359,0.629\rangle) & (\langle 0.117,0.848\rangle,\langle 0.083,0.841\rangle) & (\langle 0.098,0.896\rangle,\langle 0.078,0.847\rangle) & (\langle 0.094,0.968\rangle,\langle 0.125,0.858\rangle) & (\langle 0.105,0.942\rangle,\langle 0.028,0.044\rangle) & (\langle 0.280,0.759\rangle,\langle 0.161,0.839\rangle) \\ (\langle 0.171,0.867\rangle,\langle 0.051,0.867\rangle) & (\langle 0.158,0.916\rangle,\langle 0.022,0.928\rangle) & (\langle 0.156,0.800\rangle,\langle 0.108,0.732\rangle) & (\langle 0.046,0.962\rangle,\langle 0.038,0.927\rangle) & (\langle 0.217,0.899\rangle,\langle 0.181,0.808\rangle) & (\langle 0.198,0.827\rangle,\langle 0.079,0.044\rangle) & (\langle 0.161,0.843\rangle,\langle 0.099,0.859\rangle) \\ (\langle 0.180,0.883\rangle,\langle 0.047,0.857\rangle) & (\langle 0.167,0.861\rangle,\langle 0.125,0.787\rangle) & (\langle 0.130,0.954\rangle,\langle 0.137,0.852\rangle) & (\langle 0.073,0.916\rangle,\langle 0.064,0.891\rangle) & (\langle 0.343,0.872\rangle,\langle 0.125,0.847\rangle) & (\langle 0.162,0.884\rangle,\langle 0.039,0.090\rangle) & (\langle 0.206,0.851\rangle,\langle 0.033,0.949\rangle) \\ (\langle 0.279,0.853\rangle,\langle 0.043,0.880\rangle) & (\langle 0.188,0.861\rangle,\langle 0.064,0.850\rangle) & (\langle 0.168,0.939\rangle,\langle 0.073,0.837\rangle) & (\langle 0.073,0.936\rangle,\langle 0.073,0.904\rangle) & (\langle 0.192,0.862\rangle,\langle 0.031,0.953\rangle) & (\langle 0.185,0.915\rangle,\langle 0.214,0.002\rangle) & (\langle 0.056,0.788\rangle,\langle 0.058,0.821\rangle) \end{pmatrix}$$

Step 6: Find a positive ideal solution (LDFSSV-PIS) with an LDFSS value, as well as the LDFSS-valued negative ideal solution (LDFSSV-NIS) using in order and are listed, respectively, as

LDFSSV-PIS= $\mathfrak{s}_j^+ = \{\ddot{\rho}_1^+, \ddot{\rho}_2^+, ..., \ddot{\rho}_q^+\} =$
$\{(\langle 0.320, 0.853\rangle, \langle 0.143, 0.712\rangle), (\langle 0.467, 0.689\rangle, \langle 0.359, 0.629\rangle), (\langle 0.168, 0.800\rangle, \langle 0.137, 0.732\rangle),$
$(\langle 0.134, 0.869\rangle, \langle 0.101, 0.833\rangle), (\langle 0.343, 0.858\rangle, \langle 0.181, 0.808\rangle), (\langle 0.198, 0.827\rangle, \langle 0.214, 0.002\rangle),$
$(\langle 0.280, 0.759\rangle, \langle 0.161, 0.821\rangle)\}$
LDFSSV-NIS= $\mathfrak{s}_j^- = \{\ddot{\rho}_1^-, \ddot{\rho}_2^-, ..., \ddot{\rho}_q^-\} =$
$\{(\langle 0.126, 0.967\rangle, \langle 0.039, 0.880\rangle), (\langle 0.158, 0.916\rangle, \langle 0.022, 0.928\rangle), (\langle 0.108, 0.954\rangle, \langle 0.073, 0.852\rangle),$
$(\langle 0.046, 0.962\rangle, \langle 0.038, 0.927\rangle), (\langle 0.094, 0.968\rangle, \langle 0.031, 0.953\rangle), (\langle 0.077, 0.942\rangle, \langle 0.028, 0.090\rangle),$
$(\langle 0.056, 0.940\rangle, \langle 0.033, 0.949\rangle)\}$

Step 7: We determine the Table 3 LDFSS relative PIS and NIS for the calculated aggregated weighted.

Table 3. Distance Measure of Each Alternative.

Alternatives $\mathfrak{a}_j$	$\mathfrak{d}_{\mathfrak{E}}^{\mathfrak{N}+}$	$\mathfrak{d}_{\mathfrak{E}}^{\mathfrak{N}-}$
$\mathfrak{a}_1$	0.0116	0.0112
$\mathfrak{a}_2$	0.0083	0.0172
$\mathfrak{a}_3$	0.0178	0.0065
$\mathfrak{a}_4$	0.0141	0.0066
$\mathfrak{a}_5$	0.0167	0.0061

Step 8: Table 4 shows the proximity coefficients calculated via LDFSS-Euclidean distances of each alternative from LDFSSV-PIS and LDFSSV-NIS:

Table 4. LDF Closseness Coefficient of Each Alternative.

Alternatives $\mathfrak{a}_j$	$\mathfrak{C}_j^+$	Rank
$\mathfrak{a}_1$	0.49083	2
$\mathfrak{a}_2$	0.67360	1
$\mathfrak{a}_3$	0.26764	4
$\mathfrak{a}_4$	0.32023	3
$\mathfrak{a}_5$	0.26758	5

Step 9: The priority order of the robots as seen in Table 4 is $\mathfrak{a}_2 > \mathfrak{a}_1 > \mathfrak{a}_4 > \mathfrak{a}_3 > \mathfrak{a}_5$: thus, $\mathfrak{a}_2$ is the best robot for the concerned problem of agriculture.

5. MCDM Using LDFSS VIKOR Method

VIKOR (Vlse Kriterijumska Optimizacija Kompromisno Resenje) is a Serbian direct reference to various efficiency and compromise parameters. Serafim Opricovic developed it to alleviate decision-making problems with contrasting and non-commensurable (different units) demands, assuming that compromise is suitable for conflict management, the decision-maker appears to have a workable alternative that is the closest to the ideal, and the alternatives are analysed using all indicators. VIKOR rates the options and determines the workable compromise that is closest to the ideal.

We will start by demonstrating the proposed approach step by step:

We begin by breaking down the proposed approach piece by piece. We omit the very first six stages since they are the same as in Algorithm 1 for the LDFSS TOPSIS technique. The remaining stages are as follows:

Figure 2 depicts a flow chart of the planned LDFSS-VIKOR (Algorithm 2).

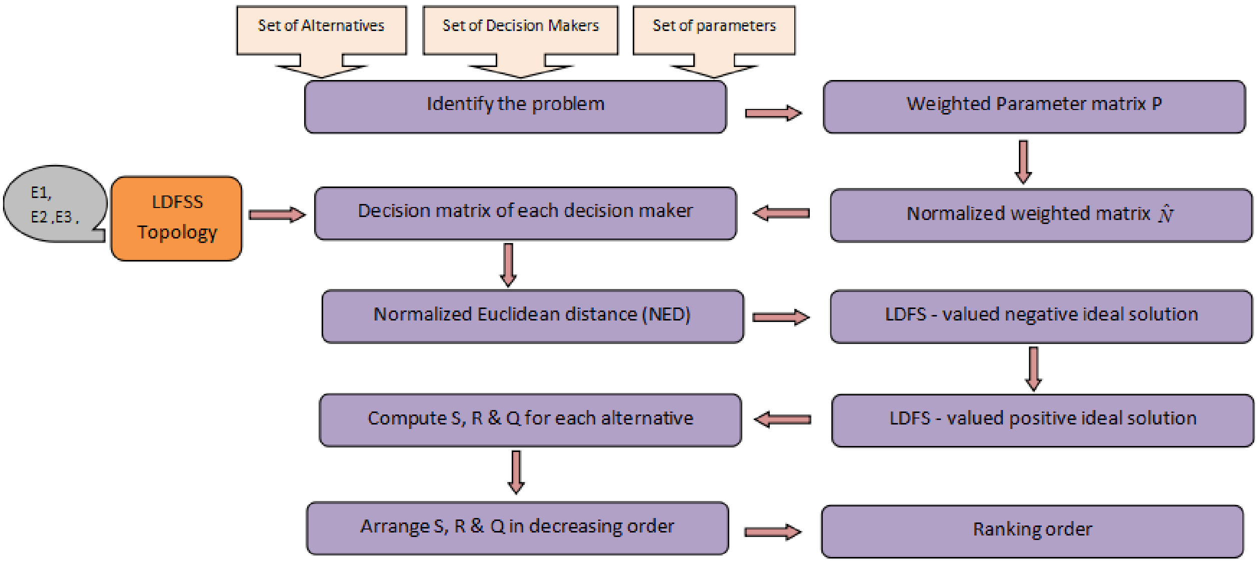

Figure 2. Procedural steps of Algorithm 2.

Algorithm 2: LDFSS-VIKOR.

Step 1 to 6: See Algorithm 1

Step 7: Generate the VIKOR method's core characteristics for each alternative, namely the group utility value $\mathfrak{S}_j$, individual regret value $\mathfrak{R}_j$, and compromise value $\mathfrak{Q}_j$, using

$$\mathfrak{S}_j = \sum_{\mathfrak{k}=1}^{\mathfrak{r}} \mathfrak{W}_{\mathfrak{k}} \left(\frac{d(\ddot{\rho}_{\mathfrak{k}}^{+}, \ddot{\rho}_{j\mathfrak{k}})}{d(\ddot{\rho}_{\mathfrak{k}}^{+}, \ddot{\rho}_{\mathfrak{k}}^{-})} \right)$$

$$\mathfrak{R}_j = \max_{\mathfrak{k}=1}^{\mathfrak{r}} \mathfrak{W}_{\mathfrak{k}} \left(\frac{d(\ddot{\rho}_{\mathfrak{k}}^{+}, \ddot{\rho}_{j\mathfrak{k}})}{d(\ddot{\rho}_{\mathfrak{k}}^{+}, \ddot{\rho}_{\mathfrak{k}}^{-})} \right)$$

$$\mathfrak{Q}_j = \kappa \left(\frac{\mathfrak{S}_j - \mathfrak{S}^{-}}{\mathfrak{S}^{+} - \mathfrak{S}^{-}} \right) + (1-\kappa) \left(\frac{\mathfrak{R}_j - \mathfrak{R}^{-}}{\mathfrak{R}^{+} - \mathfrak{R}^{-}} \right)$$

where $\mathfrak{S}^{+} = max_j \mathfrak{S}_j$, $\mathfrak{S}^{-} = min_j \mathfrak{S}_j$, $\mathfrak{R}^{+} = max_j \mathfrak{R}_j$, $\mathfrak{R}^{-} = min_j \mathfrak{R}_j$. The real value κ is referred to as the decision mechanism coefficient. The purpose of the coefficient κ is that if the compromise option is to be chosen by majority vote, we use $\kappa > 0.5$; for concurrence, we use $\kappa = 0.5$; and $\kappa < 0.5$ symbolises veto. The weight of the $\mathfrak{k}$ criteria is represented by $\mathfrak{W}_{\mathfrak{k}}$, which reflects its relative relevance.

Step 8: Sort the options and come up with a reasonable solution. Organize $\mathfrak{S}_i$, $\mathfrak{R}_i$, and $\mathfrak{Q}_i$ in ascending order to create three rating lists, $\mathfrak{S}_{[.]}$, $\mathfrak{R}_{[.]}$, and $\mathfrak{Q}_{[.]}$. The alternative $\ddot{\rho}_{\eta}$ will be designated the compromise solution if it ranks first in $\mathfrak{Q}_{[.]}$ (with the least value) and concurrently meets the accompanying main specifications:

C1 Acceptable advantage:
If $\ddot{\rho}_{\eta}$ and $\ddot{\rho}_{\zeta}$ represent top two alternatives in $\mathfrak{Q}_j$, then

$$\mathfrak{Q}(\ddot{\rho}_{\eta}) - \mathfrak{Q}(\ddot{\rho}_{\zeta}) \geq \frac{1}{n-1}$$

where n is the number of parameters.

C2 Acceptable stability:
The alternative $\ddot{\rho}_{\eta}$ should be best ranked by $\mathfrak{S}_j$ and/or $\mathfrak{R}_j$.

If the aforementioned two requirements are not satisfied simultaneously, there are several compromise solutions:

(i) If only criterion [C1] is met, then both possibilities $\ddot{\rho}_{\eta}$ and $\ddot{\rho}_{\zeta}$ are the compromise solutions.

(ii) If condition [C1] is not met, the options $\ddot{\rho}_{\eta}, \ddot{\rho}_{\zeta}, \ldots, \ddot{\rho}_{\gamma}$ would be the acceptable compromise solutions, where $\ddot{\rho}_{\gamma}$ may be calculated via

$$\mathfrak{Q}(\ddot{\rho}_{\zeta}) - \mathfrak{Q}(\ddot{\rho}_{\gamma}) \geq \frac{1}{n-1}$$

for the maximum.

Example

We re-solve Example Section 4.1.2 using the VIKOR approach and the strategy described in Algorithm 2. The first six stages are identical to those in Example Section 4.1.2. So we will start with step 7.

Step 1 to 6: Refer Algorithm 1

Step 7: By taking $\kappa = 0.5$, we determine the important components of the VIKOR approach for each choice, namely the group utility value $\mathfrak{S}_i$, the individual regret value

$\mathfrak{R}_i$, and the conciliation value $\mathfrak{Q}_i$, using the following formulas the values are calculated and displayed in Table 5 and Figure 3:

$$\mathfrak{S}_j = \sum_{\mathfrak{k}=1}^{\mathfrak{r}} \mathfrak{W}_{\mathfrak{k}} \left(\frac{d(\ddot{\rho}_{\mathfrak{k}}^{+}, \ddot{\rho}_{j\mathfrak{k}})}{d(\ddot{\rho}_{\mathfrak{k}}^{+}, \ddot{\rho}_{\mathfrak{k}}^{-})} \right)$$

$$\mathfrak{R}_j = \max_{\mathfrak{k}=1}^{\mathfrak{r}} \mathfrak{W}_{\mathfrak{k}} \left(\frac{d(\ddot{\rho}_{\mathfrak{k}}^{+}, \ddot{\rho}_{j\mathfrak{k}})}{d(\ddot{\rho}_{\mathfrak{k}}^{+}, \ddot{\rho}_{\mathfrak{k}}^{-})} \right)$$

$$\mathfrak{Q}_j = \kappa \left(\frac{\mathfrak{S}_j - \mathfrak{S}^{-}}{\mathfrak{S}^{+} - \mathfrak{S}^{-}} \right) + (1-\kappa) \left(\frac{\mathfrak{R}_j - \mathfrak{R}^{-}}{\mathfrak{R}^{+} - \mathfrak{R}^{-}} \right)$$

Table 5. The values of $\mathfrak{S}_j$, $\mathfrak{R}_j$ and $\mathfrak{Q}_j$ of Each Alternative.

Alternatives $\mathfrak{a}_j$	$\mathfrak{S}_j$	$\mathfrak{R}_j$	$\mathfrak{Q}_j$
$\mathfrak{a}_1$	0.6708	0.1416	0.1898
$\mathfrak{a}_2$	0.6456	0.1356	0.0000
$\mathfrak{a}_3$	0.8424	0.2100	0.8888
$\mathfrak{a}_4$	0.8471	0.1586	0.5529
$\mathfrak{a}_5$	0.8986	0.1749	0.7644

Step 8: The following are the options in order of preference:
By $\mathfrak{S}_i : \mathfrak{a}_2 \prec \mathfrak{a}_1 \prec \mathfrak{a}_3 \prec \mathfrak{a}_4 \prec \mathfrak{a}_5$
By $\mathfrak{R}_i : \mathfrak{a}_2 \prec \mathfrak{a}_1 \prec \mathfrak{a}_4 \prec \mathfrak{a}_5 \prec \mathfrak{a}_3$
By $\mathfrak{Q}_i : \mathfrak{a}_2 \prec \mathfrak{a}_1 \prec \mathfrak{a}_4 \prec \mathfrak{a}_5 \prec \mathfrak{a}_3$
We have $\mathfrak{Q}\{(\mathfrak{a}_1) - (\mathfrak{a}_2)\} = 0.1898 - 0.0000 = 0.189 \geq \frac{1}{6}$, condition C1 is met. As a result, we conclude that $\mathfrak{a}_2$ is an acceptable advantage solution. Therefore, $\mathfrak{a}_2$ is the best robot for the concerned problem of agriculture.

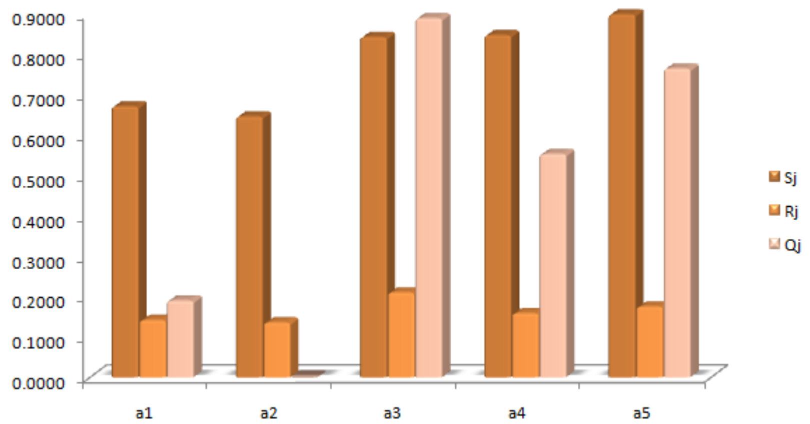

Figure 3. Bar chart of rankings.

6. Multiple Criteria Decision Making Using LDFSS-AO Method

To begin, we generalize the LDFSS aggregation operators to meet our case. The very first five phases are identical to those in Algorithm 1. As a consequence, we bypass them and proceed to step 6.

The proposed LDFSS-VIKOR is represented as a flow chart in the Figure 4.

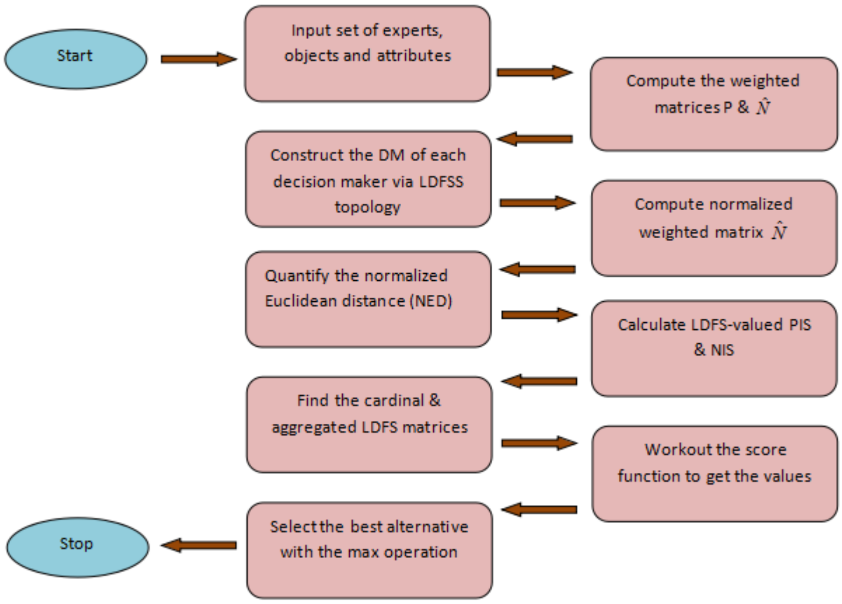

Figure 4. Procedural steps of Algorithm 3.

Example

We repeat Example Section 4.1.2 using the extended LDFSS aggregation operators as described in Algorithm 3.

Algorithm 3: LDFSS-aggregation operator(LDFSS-AO).

Step 1 to 5: Refer Algorithm 1

Step 6: Compute the cardinal matrix

$$\mathfrak{M}_{\mathfrak{C}(\mathfrak{B})} = \left[\frac{1}{\mathfrak{p}}\sum_{\mathfrak{i}=1}^{\mathfrak{p}} a_{\mathfrak{pq}} : \mathfrak{j} = 1, 2, \ldots, \mathfrak{q}\right]_{\mathfrak{p}\times\mathfrak{q}}$$

Step 7: Calculate the aggregated LDFSS matrix $\mathfrak{M}^*$ using $\mathfrak{M}^* = \frac{\mathfrak{B}\times\mathfrak{M}^{\mathfrak{T}}_{\mathfrak{C}(\mathfrak{B})}}{|\mathfrak{E}|}$.

Step 8: The score function values are calculated with the formula $S(\mathfrak{L}_\mathfrak{d}) = \frac{1}{2}[(\mathfrak{t}_\mathfrak{d} - \mathfrak{f}_\mathfrak{d}) + (\alpha_\mathfrak{d} - \beta_\mathfrak{d})]$. The best option is the one with the largest $S(\mathfrak{L}_\mathfrak{d})$ value.

Step 6: The cardinal matrix

$$\mathfrak{M}_{\mathfrak{C}(\mathfrak{B})} = \left[\frac{1}{5}\sum_{\mathfrak{i}=1}^{5} a_{\mathfrak{ij}} : \mathfrak{j} = 1, 2, \ldots, 7\right]_{5\times 7}$$

$\mathfrak{M}_{\mathfrak{C}(\mathfrak{J})} = [(\langle 0.215, 0.894\rangle, \langle 0.065, 0.829\rangle), (\langle 0.273, 0.822\rangle, \langle 0.132, 0.821\rangle),$
$(\langle 0.136, 0.890\rangle, \langle 0.106, 0.811\rangle), (\langle 0.085, 0.916\rangle, \langle 0.071, 0.880\rangle),$
$(\langle 0.227, 0.892\rangle, \langle 0.124, 0.861\rangle), (\langle 0.145, 0.898\rangle, \langle 0.078, 0.043\rangle),$
$(\langle 0.166, 0.836\rangle, \langle 0.079, 0.868\rangle)]$

Step 7: Gauge the aggregated LDFSS matrix $\mathfrak{M}^*$ with the formula $\mathfrak{M}^* = \frac{\mathfrak{B} \times \mathfrak{M}^{\mathfrak{T}}_{\mathfrak{C}(\mathfrak{B})}}{|\mathfrak{E}|}$.

$$= \begin{pmatrix} (\langle 0.0424, 0.9838\rangle, \langle 0.0096, 0.8451\rangle) \\ (\langle 0.0374, 0.9823\rangle, \langle 0.0135, 0.8429\rangle) \\ (\langle 0.0299, 0.9845\rangle, \langle 0.0081, 0.8493\rangle) \\ (\langle 0.0348, 0.9858\rangle, \langle 0.0085, 0.8568\rangle) \\ (\langle 0.0314, 0.9842\rangle, \langle 0.0070, 0.8465\rangle) \end{pmatrix}$$

Step 8: The score function for the alternatives found in step 7 is calculated using the formula $S(\mathfrak{L}_{\mathfrak{d}}) = \frac{1}{2}[(\mathfrak{t}_{\mathfrak{d}} - \mathfrak{f}_{\mathfrak{d}}) + (\alpha_{\mathfrak{d}} - \beta_{\mathfrak{d}})]$.
$S(\mathfrak{a}_1) = -0.8401, S(\mathfrak{a}_2) = -0.8369, S(\mathfrak{a}_3) = -0.8448, S(\mathfrak{a}_4) = -0.8523, S(\mathfrak{a}_5) = -0.8420$. Thus, the archetypal of the robots is $\mathfrak{a}_2 \succ \mathfrak{a}_1 \succ \mathfrak{a}_5 \succ \mathfrak{a}_3 \succ \mathfrak{a}_4$. The optimal choice is the one with the greatest score function value. i.e., $S(\mathfrak{a}_2)$.

7. Comparison and Advantages

7.1. Three Techniques Are Compared: Commentary

Figure 5 depicts the agri-robot ranks obtained using the TOPSIS, VIKOR, and generalised LDFSS aggregation operator techniques. To make the comparison possible, we used the values $1 - \mathfrak{Q}$ instead of $\mathfrak{Q}$ in VIKOR. In addition, to render the columns representing score values legible, we normalized the scores by multiplying them by 1000. TOPSIS is the first series on the left, VIKOR is the second, and scaled score values are the third.

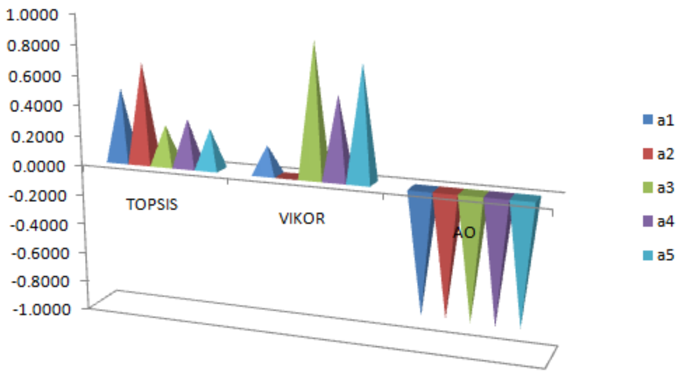

Figure 5. LDFSSS-TOPSIS, VIKOR, and generalized AO approaches were used to compare rankings.

We can see that the best option for all three tactics is the same, which is $\mathfrak{a}2$. TOPSIS simply has one check: the optimal solution must be closest to the positive ideal solution and the furthest away from the negative ideal solution. VIKOR has a number of checkpoints. For example, we choose $\mathfrak{Q}_j$, $\mathfrak{R}_j$, and $\mathfrak{S}_j$ values to ensure appropriate advantage and stability. As a result, if a poor solution passes one check, it will be rejected at the next. VIKOR offers a variety of compromise choices.

TOPSIS uses the grade metric, which takes into account distances between PIS and NIS. Without consideration for their virtual importance, the predicted distances are simply added. The distance may naturally represent some equilibrium between overall and individual happiness, but in VIKOR, it does so in a different way. In VIKOR, the weight κ is quite well. Both methods establish a ranking grade. The top-ranked answer obtained by VIKOR is nearly perfect. Nonetheless, TOPSIS's top-ranked choice takes priority in the ranking table. This does not imply that it is always close to the ultimate solution. Apart from ranking, VIKOR offers a compromise alternative with an improvement (advantage) level.

In comparison to the other two ways, the method of generalized LDFSS aggregation operators is easier to handle and gives greater computational ease. Based on this debate, we may infer that the VIKOR model outperforms TOPSIS and produces more dependable results. However, in terms of computing convenience, the approach of generalized PFS aggregation operators is preferred. How much precision we want depends on the problem under consideration. We select the procedure based on the amount of precision necessary.

7.2. Analysis of Comparisons and the Superiority of Suggested Work

We see that using any of the three algorithms in this article yields the same best answer. Furthermore, the techniques provided in this article are simple to use and produce clear results. Table 6 shows a comparison of final ranks with several known techniques.

Table 6. The proposed approaches are compared to certain existing procedures.

Methodology	The Best Option
Prioritized weighted AOs (Liu et al. [29])	$\mathfrak{a}_2$
IF AOs (Xu [25])	$\mathfrak{a}_2$
Generalized IF soft power AOs (Garg and Arora [30])	$\mathfrak{a}_2$
Pythagorean fuzzy AOs (Peng and Yuan [28])	$\mathfrak{a}_2$
Algorithms 1–3 (Proposed)	$\mathfrak{a}_2$

Furthermore, in this section, we compared and analyzed the existing soft topological space under different environments with the defined notion. Each FST, IFST, PyFST, SFST, LDFT and LDFST is superior to the other but also has its own in-build limitation given in Table 7.

Table 7. Comparison of different fuzzy soft extensions.

Set	Advantages	Limitations
FST [4]	It can handle imprecise parametrized element	It cannot handle dis-satisfaction grade values of parametrized element
IFST [7]	It can handle both satisfaction and dis-satisfaction grade of parametrized element	This theory could not support for some cases when sum of satisfaction and dis-satisfaction grade of parametrized element exceeds unity. This concepts failed to address grades such as abstinence
PyFST [34]	This notion can support when satisfaction and dis-satisfaction grade of parametrized element exceeds 1	It has inherent limitations like sum of square of satisfaction and dis-satisfaction grade of parametrized element exceeds 1. This concepts failed to address grades like abstinence
SFST [35]	This concept can handle each parametrized elements positive, neural and negative membership grade	It is not developed with reference parameters. It cannot handle when the sum of squares of parametrized elements positive, neural and negative membership grade exceeds 1.
LDFST (Proposed)	This concept is initiated to deal the parametrized elements with reference parameter	We cannot use for some case which do not have reference parameters.

8. Conclusions

We proposed the concept of linear Diophantine fuzzy soft set topological spaces and analyzed their features. We also suggested three strategies for modeling uncertainties in the MCDM problem from agri robot selection using LDFSSs: LDFSS-TOPSIS, LDFSS-VIKOR, and the extended LDFSS-AO approach. The suggested algorithms have been successfully used to rate various robots. A brief but detailed description of the various types of robots, as well as their job efficiency, was provided. We used statistical graphics to help us understand the final ranks. A comparison of three ranks, as well as a good argument for the more viable technique, was also discussed. With the help of a statistical chart, we compared the final gradings provided by the three models. The suggested model has enormous theoretical and application potential, and it may be conveniently utilized in different hybrid architectures of fuzzy sets with little modifications. The notion may be utilized to deal with uncertainty successfully in a variety of real-world situations, including artificial intelligence, business, chemical engineering, coding theory, electoral system, energy management, environment management, forecasting, game theory, image processing, logistics, machine learning, manufacturing, marketing, medical diagnosis, pattern recognition, recruitment, robotics, and trade analysis problems.

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References

1. Zadeh, L.A. Fuzzy sets. *Inf. Control* **1965**, *8*, 338–353. [CrossRef]
2. Molodtsov, D. Soft Set Theory-First Results. *Comput. Math. Appl.* **1999**, *37*, 19–31. [CrossRef]
3. Shabir, M.; Naz, M. On Soft Topological Spaces. *Comput. Math. Appl.* **2011**, *61*, 1786–1799. [CrossRef]
4. Aygunoglu, A.; Cetkin, V.; Aygun, H. An Introduction to Fuzzy Soft Topological Spaces. *Hacet. J. Math. Stat.* **2014**, *43*, 197–208.
5. Atanassov, K.T. Intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1986**, *20*, 87–96. [CrossRef]
6. Maji, P.K.; Biswas, R.; Roy, A.R. Intuitionistic Fuzzy Soft Sets. *J. Fuzzy Math.* **2001**, *9*, 677–691.
7. Bayramov, S.; Gunduz, G. On intuitionistic fuzzy soft topological spaces. *Int. J. Pure Appl. Math.* **2014**, *5*, 66–79.
8. Cuong, B.C. Picture fuzzy sets. *J. Comput. Sci. Cybern.* **2014**, *30*, 409–420.
9. Wang, L.; Zhang, H.Y.; Wang, J.Q.; Wu, G.F. Picture fuzzy multi-criteria group decision-making method to hotel building energy efficiency retrofit project selection. *RAIRO-Oper. Res.* **2020**, *54*, 211–229. [CrossRef]
10. Yager, R.R.; Abbasov, A.M. Pythagorean membership grades, complex numbers, and decision making. *Int. J. Intell. Syst.* **2013**, *28*, 436–452. [CrossRef]
11. Yager, R.R. Pythagorean membership grades in multi-criteria decision making. *IEEE Trans. Fuzzy Syst.* **2013**, *22*, 958–965. [CrossRef]
12. Ali, M.I. Another view on q-rung orthopair fuzzy sets. *Int. J. Intell. Syst.* **2018**, *33*, 2139–2153. [CrossRef]
13. Akram, M. Multi-criteria decision-making methods based on q-rung picture fuzzy information. *J. Intell. Fuzzy Syst.* **2021**, *40*, 10017–10042. [CrossRef]
14. Pinar, A.; Boran, F.E. A novel distance measure on q-rung picture fuzzy sets and its application to decision making and classification problems. *Artif. Intell. Rev.* **2022**, *55*, 1317–1350. [CrossRef]
15. Riaz, M.; Hashmi, M.R. Linear Diophantine fuzzy set and its applications towards multi-attribute decision-making problems. *J. Intell. Fuzzy Syst.* **2019**, *37*, 5417–5439. [CrossRef]
16. Ayub, S.; Shabir, M.; Riaz, M.; Aslam, M.; Chinram, R. Linear Diophantine fuzzy relations and their algebraic properties with decision making. *Symmetry* **2021**, *13*, 945. [CrossRef]
17. Iampan, A.; Garcia, G.S.; Riaz, M.; Athar Farid, H.M.; Chinram, R. Linear Diophantine fuzzy Einstein aggregation operators for multi-criteria decision-making problems. *J. Math.* **2021**, *2021*, 5548033. [CrossRef]
18. Parimala, M.; Jafari, S.; Riaz, M.; Aslam, M. Applying the Dijkstra algorithm to solve a linear Diophantine fuzzy environment. *Symmetry* **2021**, *13*, 1616. [CrossRef]
19. Riaz, M.; Farid, H.M.A.; Aslam, M.; Pamucar, D.; Bozanic, D. Novel approach for third-party reverse logistic provider selection process under linear Diophantine fuzzy prioritized aggregation operators. *Symmetry* **2021**, *13*, 1152. [CrossRef]
20. Riaz, M.; Farid, H.M.A.; Wang, W.; Pamucar, D. Interval-Valued Linear Diophantine Fuzzy Frank Aggregation Operators with Multi-Criteria Decision-Making. *Mathematics* **2022**, *10*, 1811. [CrossRef]
21. Biswas, A.; Sarkar, B. Pythagorean fuzzy TOPSIS for multi-criteria group decision-making with unknown weight information through entropy measure. *Int. J. Intell. Syst.* **2019**, *34*, 1108–1128. [CrossRef]
22. Boran, F.E.; Genc, S.; Kurt, M.; Akay, D. A multi-criteria intuitionistic fuzzy group decision making for supplier selection with TOPSIS method. *Expert Syst. Appl.* **2009**, *36*, 11363–11368. [CrossRef]
23. Kumar, K.; Garg, H. TOPSIS method based on the connection number of set pair analysis under interval-valued intuitionistic fuzzy set environment. *Comput. Appl. Math.* **2018**, *37*, 1319–1329. [CrossRef]
24. Xu, Z.; Zhang, X. Hesitant fuzzy multi-attribute decision making based on TOPSIS with incomplete weight information. *Knowl.-Based Syst.* **2013**, *52*, 53–64. [CrossRef]
25. Xu, Z. Intuitionistic fuzzy aggregation operators. *IEEE Trans. Fuzzy Syst.* **2007**, *15*, 1179–1187.
26. Hashmi, M.R.; Tehrim, S.T.; Riaz, M.; Pamucar, D.; Cirovic, G. Spherical Linear Diophantine Fuzzy Soft Rough Sets with Multi-Criteria Decision Making. *Axioms* **2021**, *10*, 185. [CrossRef]

27. Eraslan, S.; Karaaslan, F. A group decision making method based on TOPSIS under fuzzy soft environment. *J. New Theory* **2015**, *3*, 30–40.
28. Peng, X.D.; Yuan, H.Y. Fundamental properties of Pythagorean fuzzy aggregation operators. *Fundam. Inform.* **2016**, *147*, 415–446. [CrossRef]
29. Liu, P.; Akram, M.; Sattar, A. Extensions of prioritized weighted aggregation operators for decision-making under complex q-rung orthopair fuzzy information. *J. Intell. Fuzzy Syst.* **2020**, *39*, 7469–7493. [CrossRef]
30. Garg, H.; Arora, R. Generalized intuitionistic fuzzy soft power aggregation operator based on t-norm and their application in multicriteria decision-making. *Int. J. Intell. Syst.* **2019**, *34*, 215–246. [CrossRef]
31. Naeem, K.; Riaz, M.; Peng, X.; Afzal, D. Pythagorean m-polar fuzzy topology with TOPSIS approach in exploring most effectual method for curing from COVID-19. *Int. J. Biomath.* **2020**, *13*, 2050075. [CrossRef]
32. Gül, S.; Aydoğdu, A. Novel distance and entropy definitions for linear Diophantine fuzzy sets and an extension of TOPSIS (LDF-TOPSIS). *Expert Syst.* **2022**. [CrossRef]
33. Riaz, M.; Hashmi, M.R.; Kalsoom, H.; Pamucar, D.; Chu, Y.M. Linear Diophantine fuzzy soft rough sets for the selection of sustainable material handling equipment. *Symmetry* **2020**, *12*, 1215. [CrossRef]
34. Riaz, M.; Naeem, K.; Aslam, M.; Afzal, D.; Almahdi, F.A.A.; Jamal, S.S. Multi-criteria group decision making with Pythagorean fuzzy soft topology. *J. Intell. Fuzzy Syst.* **2020**, *39*, 6703–6720. [CrossRef]
35. Garg, H.; Perveen P.A., F.; John, S.J.; Perez-Dominguez, L. Spherical Fuzzy Soft Topology and Its Application in Group Decision-Making Problems. *Math. Probl. Eng.* **2022**, *2022*, 1007133. [CrossRef]