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Linear Quadratic Optimal Control Problem for Linear Stochastic Generalized System in Hilbert Spaces

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Abstract: A finite-horizon linear stochastic quadratic optimal control problem is investigated by the GE-evolution operator in the sense of the mild solution in Hilbert spaces. We assume that the coefficient operator of the differential term is a bounded linear operator and that the state and input operators are time-varying in the dynamic equation of the problem. Optimal state feedback along with the well-posedness of the generalized Riccati equation is obtained for the finite-horizon case. The results are also applicable to the linear quadratic optimal control problem of ordinary time-varying linear stochastic systems.

Keywords: linear stochastic quadratic problem; linear stochastic generalized systems; optimal state feedback; GE-evolution operator; Hilbert spaces

MSC: 3E20; 49N10; 49N35; 93C05; 93B52

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1. Introduction

In recent years, there has been increasing interest in the optimal control of stochastic systems (e.g., [1–13]). However, these studies are limited to ordinary stochastic systems with a time-invariant state operator in Hilbert spaces and have not involved a stochastic generalized system in Hilbert spaces. Stochastic generalized systems in Hilbert spaces are inherent in many application fields; among them, we mention input–output economics, evolution of the free surface of seepage liquid, the stochastic generalized wave equation, the heat equation, etc. (e.g., [14–27]). They are essentially different from an ordinary stochastic system in Hilbert spaces. It is necessary to investigate the optimal control problem of such systems in Hilbert spaces. Based on this point, we deal with the linear quadratic optimal control problem of linear stochastic generalized systems in Hilbert spaces, in which the coefficient operator of the differential term is a bounded linear operator and the state and input operators are time-varying. As far as we know, even if the coefficient operator of the differential term is the identical operator, which corresponds to the ordinary time-varying linear stochastic system, there are no research results on this kind of optimal control problem. Therefore, the research results of this paper are also applicable to the linear quadratic optimal control problem of ordinary time-varying linear stochastic systems in Hilbert spaces.

First of all, we formulate the optimal control problem. Let us consider the following linear stochastic generalized system in Hilbert spaces:

$$Ed\xi(t) = (M_1(t)\xi(t) + N_1(t)\eta(t))dt + (M_2(t)\xi(t) + N_2(t)\eta(t))dw(t)$$
$$t \in [s,a], \xi(s) = \xi_0. \tag{1}$$

Here, $E \in L(\mathbb{H},\mathbb{H})$; $M_1(t) : \mathrm{dom}M_1(t) \subseteq \mathbb{H} \to \mathbb{H}$ is a linear operator, where $L(\mathbb{U},\mathbb{H})$ denotes the set of all bounded linear operators from Hilbert space $\mathbb{U}$ to Hilbert space $\mathbb{H}$, and $\mathrm{dom}M_1(t)$ denotes the domain of operator $M_1(t)$. Let $P([s,a], L(\mathbb{H},\mathbb{H})) = \{D(\cdot) \in$

$L(\mathbb{U},\mathbb{H}) : D(\cdot)\eta$ is continuous for every $\eta \in \mathbb{U}$, and $\|D(\cdot)\| = \sup_{t\in[s,a]}\|D(t)\|_{L(\mathbb{U},\mathbb{H})} < +\infty\}$; $\|\cdot\|_{\mathbb{H}}$ (or $\|\cdot\|$) and $\langle\cdot,\cdot\rangle_{\mathbb{H}}$ (or $\langle\cdot,\cdot\rangle$) denote the norm and inner product on Hilbert spaces $\mathbb{H}$, respectively. Let $(\Omega,\mathbb{F},\mathbb{P})$ be a complete probability space, $w(t)$ be a one dimensional standard Wiener process on $(\Omega,\mathbb{F},\mathbb{P})$, and the filtration $\mathbb{F}_t$ be the $\sigma-$algebra generated by $\{w(r) : r \le t\}$. We suppose that all processes are adapted to the filtration $\mathbb{F}_t$. We denote by $L^2_m([s,a],\Omega,\mathbb{F}_t,\mathbb{H})$ the set of all processes $\xi(t) \in \mathbb{H}$ such that

(i) $\int_s^a \|\xi(t)\|^2 dt < +\infty$.
(ii) $\xi(t)$ is $\mathbb{F}_t$-measurable for $\forall t \in [s,a]$.

We denote by $L^2_{sm}([s,a],\Omega,\mathbb{F}_t,\mathbb{H})$ the space of all strongly measurable square integral processes $\xi(t) \in \mathbb{H}$ such that $\int_s^a \mathbb{E}\|\xi(t)\|^2 dt < +\infty$, where $\mathbb{E}$ denotes the mathematical expectation. $\eta(\cdot) \in L^2_{sm}([s,a],\Omega,\mathbb{F}_t,\mathbb{U})$. $\xi(\cdot)$ and $\eta(\cdot)$ denote the state and input processes, respectively. $C^2([s,a],\Omega,\mathbb{F}_t,\mathbb{H})$ denotes the set of all strongly measurable and continuous processes from $[s,a]$ to $\mathbb{H}$.

Let $M_1(t)$ be a generator of the strongly continuous GE-evolution operator (i.e., generalized evolution operator). $T(t,s)$ induced by E, i.e.,

(i) $T(t,r)ET(r,s) = T(t,s), 0 \le s \le r \le t$, and $T(r,r) = T_0$ is a definite operator independent of r.
(ii) $T(t,\cdot)$ is strongly continuous on $[0,t]$, and $T(\cdot,s)$ is strongly continuous on $[s,a]$.
(iii) There exist $b \ge 1$ and $\omega > 0$ such that

$$\|T(t,r)\| \le be^{\omega(t-r)}, t \ge r \ge 0,$$

(iv)

$$M_1(t)\xi = \lim_{h\to 0^+} \frac{ET(h+t,t)E - ET(t,t)E}{h}\xi,$$

for every $\xi \in D_0(t)$, where

$$D_0(t) = \{\xi : \xi \in \mathrm{dom}M_1(t) \subseteq \mathbb{H}, T(t,t)E\xi = \xi, \exists \lim_{h\to 0^+} \frac{ET(h+t,t)E - ET(t,t)E}{h}\xi\}.$$

In the following, we assume that

$$D_0(t) = \{\xi \in \mathrm{dom}M_1(t), M_1(t)\xi \in \mathrm{ran}E\} = D_0$$

is independent of $t, \mathbb{D} = \overline{D_0}$, and $(T(t,r)E)|_{\mathbb{D}}$ is unique. Here, $\overline{D_0}$ denotes the closure of D_0, and $(T(t,r)E)|_{\mathbb{D}}$ denotes the limitation of $T(t,r)E$ on $\mathbb{D}$.

See [16,23,26] for the details of the GE-evolution operator.

We introduce the following quadratic cost functional:

$$F(s,\xi_0,\eta(\cdot)) = \mathbb{E}(\int_s^a (\|L(t)\xi(t)\|^2 + \|M(t)\eta(t)\|^2)dt + \|N\xi(a)\|^2), \quad (2)$$

where $L(\cdot), M(\cdot)$, and N satisfy the following Hypothesis 3.

The optimal control problem considered in this paper is as follows:

Problem 1. *For any given initial pair* $(s,\xi_0) \in [0,a] \times \mathbb{D}$, *find a* $\eta_o(\cdot) \in L^2_{sm}([s,a],\Omega,\mathbb{F}_t,\mathbb{U})$, *such that*

$$F_o(s,\xi_0) = F(s,\xi_0,\eta_o(\cdot)) = \min_{\eta(\cdot)\in L^2_{sm}([s,a],\Omega,\mathbb{F}_t,\mathbb{U})} F(s,\xi_0,\eta(\cdot)).$$

Any $\eta_o(\cdot) \in L^2_{sm}([s,a],\Omega,\mathbb{F}_t,\mathbb{U})$ satisfying Problem 1 is called an optimal control of Problem 1 for the initial pair (s,ξ_0), and the corresponding state $\xi_o(\cdot) = \xi(s,\cdot,\xi_0,\eta_o(\cdot))$ is called an optimal state process; the pair $(\xi_o(\cdot),\eta_o(\cdot))$ is called an optimal pair. The function $F_o(\cdot,\cdot)$ is called the value function of Problem 1.

In order to study the optimal control problem, we need the following hypotheses:

Hypothesis 1. *$M_1(t)$ is a generator of the strongly continuous GE-evolution operator $T(t,r)$ induced by E and $\|T(t,r)\| \le c_T, 0 \le r \le t \le a$, where c_T is a constant.*

Hypothesis 2. *$M_2(\cdot) \in P([s,a], L(\mathbb{D}, E(\mathbb{D})))$; $N_1(\cdot), N_2(\cdot) \in P([s,a], L(\mathbb{U}, E(\mathbb{D})))$; $\|M_2(\cdot)\| \le c_{MN}$, and $\|N_1(\cdot)\| \le c_{MN}, \|N_2(\cdot)\| \le c_{MN}$, where c_{MN} is a constant number.*

Hypothesis 3. *$L(\cdot) \in P([s,a], L(\mathbb{D}, \mathbb{L}))$; $M(\cdot) \in P([s,a], L(\mathbb{U}, \mathbb{U}))$ is a strongly positive operator on $[s,a]$; $N \in L(\mathbb{D}, \mathbb{N})$, where $\mathbb{L}, \mathbb{N}$ are Hilbert spaces.*

Hypothesis 4. *Stochastic GE-evolution operator $V(t,r)$ induced by E is related to the linear homogeneous equation*

$$Ed\xi(t) = M_1(t)\xi(t)dt + M_2(t)\xi(t)dw(t), t \in [s,a], \xi(s) = \xi_0, \tag{3}$$

i.e., $\xi(t) = V(t,s)E\xi_0$ is the mild solution of System (3) and satisfies

$$\sup_{r\in[s,a]} \|V(t,r)\xi_0\|^2_{\mathbb{D}} \le c_V \|\xi_0\|^2.$$

See [19,23] for the details of the stochastic GE-evolution operator.

Remark 1. *Hypotheses 1 and 2 can guarantee the existence and uniqueness of the mild solution of System (1); Hypotheses 1–4 can guarantee that the optimal control solution satisfying Problem 1 is unique.*

The organization of this paper is as follows: In Section 2, we discuss the mild solution to the linear stochastic generalized System (1). In Section 3, we consider the existence of the solution to the generalized integral Riccati equation and deal with the properties of the Riccati operator. In Section 4, we investigate the generalized differential Riccati equation from the generalized integral Riccati equation. In Section 5, we deal with the relation between the solution to the generalized Riccati equation and the optimal control, extend the result globally in time, and study the uniqueness of the solution to the generalized Reccati equation. In Section 6, the main results are proved. In Section 7, the linear quadratic optimal control problem for a class of linear stochastic generalized systems is discussed. In Section 8, we give three examples to illustrate the theory. The conclusions are given in Section 9.

It should be noted that paper [16], published by the author, mainly studies the controllability problem, while this paper mainly studies the optimal control problem. The contents of the two studies are completely different.

2. Mild Solution of System (1)

Definition 1. *A function $\xi : [s,a] \to \mathbb{D}$ is called to be a mild solution of the linear stochastic generalized System (1) if $\xi(\cdot) \in C^2([s,a], \Omega, \mathbb{F}_t, \mathbb{D})$ and satisfies*

$$\xi(t) = T(t,s)E\xi_0 + \int_s^a T(t,r)N_1(r)\eta(r)dr$$
$$+ \int_s^a T(t,r)M_2(r)\xi(r)dw(r)$$
$$+ \int_s^a T(t,r)N_2(r)\eta(r)dw(r).$$

Theorem 1. *Suppose that Hypotheses 1–2 are true. For a given process $\eta(\cdot) \in L^2_{sm}([s,a], \Omega, \mathbb{F}_t, \mathbb{U})$ and an initial value $\xi(s) = \xi_0 \in \mathbb{D}$, there is a unique mild solution $\xi(t) \in C^2([s,a], \Omega, \mathbb{F}_t, \mathbb{D})$ to the linear stochastic generalized System (1).*

Proof. Let S be the following operator:

$$S(\xi)(t) = T(t,s)E\xi_0 + \int_s^t T(t,r)N_1(r)\eta(r)ds$$

$$+ \int_s^t T(t,r)M_2(r)\xi(r))dw(r) + \int_s^t T(t,r)N_2(r)\eta(r))dw(r)]$$

$$= T(t,s)E\xi_0 + S_0(\xi)(t) + S_1(\xi)(t) + S_2(\xi)(t),$$

$$t \in [s,a], \eta \in L^2_{sm}([s,a], \Omega, \mathbb{F}_t, \mathbb{U}).$$

We show that S maps $C^2([s,a], \Omega, \mathbb{F}_t, \mathbb{D})$ into $C^2([s,a], \Omega, \mathbb{F}_t, \mathbb{D})$.
Since $\|S_0(\xi)\|^2 \leq c_T^2 c_{MN}^2 a\mathbb{E}\int_s^a \|\eta(r)\|^2 dr$,

$$\|S_1(\xi)\|^2 \leq c_T^2 c_{MN}^2 \mathbb{E}\int_s^a \|\xi(r)\|^2 dr \leq c_T^2 c_{MN}^2 a\|\xi\|^2,$$

$$\|S_2(\xi)\|^2 \leq c_T^2 c_{MN}^2 \mathbb{E}\int_s^a \|\eta(r)\|^2 dr;$$

thus S_0, S_1, and S_2 map $C^2([s,a], \Omega, \mathbb{F}_t, \mathbb{D})$ into $C^2([s,a], \Omega, \mathbb{F}_t, \mathbb{D})$.

In the following, let ξ_1, ξ_2 be arbitrary processes from $C^2([0,a], \Omega, \mathbb{F}_t, \mathbb{D})$, then

$$\|S(\xi_2) - S(\xi_1)\| \leq \|S_1(\xi_2) - S_1(\xi_1)\| = J_1$$

and

$$J_1^2 \leq c_T^2 c_{MN}^2 \mathbb{E}\int_s^a \|(\xi_2(r) - \xi_1(r)\|^2 dr$$

$$\leq c_T^2 c_{MN}^2 a\|\xi_2 - \xi_1\|^2$$

for all $\xi_1, \xi_2 \in C^2([s,a], \Omega, \mathbb{F}_t, \mathbb{D})$. Therefore, if

$$c_T^2 c_{MN}^2 a < 1 \tag{4}$$

then the operator S has a fixed point ξ in $C^2([s,a], \Omega, \mathbb{F}_t, \mathbb{D})$, which, as is easy to see, is a mild solution of the linear stochastic generalized System (1). The extra condition (4) on a can be easily removed by considering the equation on $[0,a_1], [a_1, 2a_1], \cdots$ with a_1 satisfy (4). □

Remark 2. *The linear homogeneous Equation (3) is the linear homogeneous system of System (1). Under Hypotheses 1, 2, and 4, the mild solution $\xi(t)$ of System (3) satisfying*

$$\xi(t) = T(t,s)E\xi_0 + \int_s^a T(t,r)M_2(r)\xi(r)dw(r)$$

can be expressed as

$$\xi(t) = V(t,s)E\xi_0.$$

3. The Generalized Integral Riccati Equation

In this section, we consider the existence of a solution to a generalized integral Riccati equation. The relevant generalized integral form of the generalized differential Riccati equation is

$$R(t) = \int_t^a E^*T^*(r,t)L^*(r)L(r)U(r,t)Edr + \int_t^a E^*T^*(r,t)M_2^*T_0^*R(r)T_0M_2(r)U(r,t)Edr$$

$$+ \int_t^a [E^*T^*(r,t)M_2^*T_0^*R(r)T_0N_2(r)(M^2(r) + N_2^*(r)T_0^*R(r)T_0N_2(r))^{-1}\cdot$$

$$(N_1^*(r)T_0^*R(r) + N_2^*(r)T_0^*R(r)T_0M_2(r))U(r,t)E]dr$$

$$+ E^*T^*(a,t)N^*NU(a,t)E. \tag{5}$$

and

$$\langle (M^2(r) + N_2^*(r)T_0^*R(r)T_0N_2(r))\eta, \eta\rangle > 0, \forall \eta \neq 0, \eta \in \mathbb{U}.$$

Here, $U(t,s)$ satisfies

$$U(t,s)E\xi_0 = T(t,s)E\xi_0 - \int_s^t [T(t,r)N_1(r)(M^2(r) + N_2^*(r)T_0^*R(r)T_0N_2(r))^{-1} \cdot$$

$$(N_1^*(r)T_0^*R(r) + N_2^*(r)T_0^*R(r)T_0M_2(r))U(r,s)E\xi_0]dr. \tag{6}$$

The main result of this section is the following theorem:

Theorem 2. *The generalized integral Equations (5) and (6) have unique solutions $R(t) \in P([s,a], L(\mathbb{D},\mathbb{D}))$, and $U(\cdot,s) \in P([s,a], L(E(\mathbb{D}),\mathbb{D}))$ for $s = a_{max} < a$ chosen such that $a - a_{max}$ is sufficiently small. Moreover, $R(t)$ is a positive self-adjoint operator on $\mathbb{D}$.*

First, we introduce the following marks: $C([s,a],\mathbb{B})$ denotes the Banach space of all continuous functions $g(\cdot)$ on $[s,a]$ into a Banach space $\mathbb{B}$ with the norm $\|g\| = \sup_{t\in[s,a]}\|g(t\|_{\mathbb{B}}$. Suppose $\Delta_s = \{(t,r) \in \mathbb{R}^2 : s \le r \le t \le a\}.C(\Delta_s, L(\mathbb{D},\mathbb{D}))$ denotes the Banach space with the norm

$$\|g\|_{C(\Delta_s,L(\mathbb{D},\mathbb{D}))} = \sup_{(t,r)\in\Delta_s}\|g(t,r)\|_{L(\mathbb{D},\mathbb{D})}.$$

Let $C_s = \int_s^t T(t,r)N_1(r)dr, C_s^* = \int_s^t N_1^*(r)T^*(t,r)dr$. Then, C_s is continuous from $C([s,a],\mathbb{U})$ to $C([s,a],\mathbb{D})$; C_s^* is continuous from $C([s,a],\mathbb{D})$ to $C([s,a],\mathbb{U})$.

3.1. Linear Generalized Integral Equation

Now we investigate the linear generalized integral equation

$$R_1(t) = \int_t^a E^*T^*(r,t)L^*(r)L(r)U_1(r,t)Edr + \int_t^a E^*T^*(r,t)S^*(r)S(r)U_1(r,t)Edr$$

$$+ \int_t^a E^*T^*(r,t)M_3^*(r)R_1(r)M_3(r)U_1(r,t)Edr$$

$$- \int_t^a E^*T^*(r,t)\phi^*(r)N_1^*(r)T_0^*R_1(r)U_1(r,s)Edr$$

$$+ E^*T^*(a,t)N^*NU_1(a,t)E. \tag{7}$$

and

$$U_1(t,s)E\xi_0 = T(t,s)E\xi_0 - \int_s^t T(t,r)N_1(r)\phi(r)U_1(r,t)E\xi_0dr. \tag{8}$$

In the following, we prove the existence of the solutions $R_1(t)$ and $U_1(t,s)E$ to the linear generalized integral Equations (7) and (8).

Lemma 1. *Let $S(t), M_3(t), \phi(t)$ be given bounded operators for every $t \in [s,a]$ satisfying*

$$\|S(t)\xi\|_{\mathbb{H}}, \|M_3(t)\xi\|_{\mathbb{H}}, \|\phi(t)\xi\|_{\mathbb{H}} \le c_r\|\xi\|_{\mathbb{H}}, \forall \xi \in \mathbb{D}, t \in [s,a] \tag{9}$$

for some suitable chosen $c_r > 0$. Then, there exist $R_1(t) \in P([a_0,a], L(\mathbb{D},\mathbb{D}))$ and $U_1(\cdot,\cdot)E \in C(\Delta_{a_0}, L(\mathbb{D},\mathbb{D}))$ to the set of linear generalized integral Equations (7) and (8).

In order to prove the existence of solutions $R_1(t)$ and $U_1(t,s)E$, we can use the fixed point theory on the map O defined by

$$O\begin{bmatrix} l \\ m \end{bmatrix}(t) = \begin{bmatrix} O_{11}(m)(t) + O_{12}(m)(t) + O_{13}(l,m) + O_{14}(l,m) + O_{15}(m) \\ O_2(m) \end{bmatrix}(t)$$

for $t \in [s,a]$ on the space

$$\mathbb{Y} = P([s,a], L(\mathbb{D},\mathbb{D})) \times C(\Delta_s, L(\mathbb{D},\mathbb{D})),$$

where

$$O_{11}(m)(t) = \int_t^a E^*T^*(r,t)L^*(r)L(r)m(r,t)dr$$

$$O_{12}(m)(t) = \int_t^a E^*T^*(r,t)S^*(r)S(r)m(r,t)dr$$

$$O_{13}(l,m)(t) = \int_t^a E^*T^*(r,t)l(r)M_3^*(r)m(r,t)dr$$

$$O_{14}(l,m)(t) = -\int_t^a E^*T^*(r,t)\phi^*(r)N_1^*(r)T_0^*l(r)m(r,t)dr$$

$$O_{15}(m)(t) = -E^*T^*(a,t)N^*N\int_t^a T(a,r)N_1(r)\phi(r)m(r,t)dr + E^*T^*(a,t)N^*NT(a,t)E$$

and

$$O_2(m)(t) = T(t,s)E - C_sT_0N_1(t)\phi(\cdot)m(\cdot,\cdot)(t).$$

Both of these two quantities are defined on $\mathbb{Y}$. The fixed point l, m represents the operators $R_1(t)$ and $U_1(t,s)E$, respectively.

Lemma 2. *The operator O maps the ball $S_{c_r}(0) \subset \mathbb{Y}$ into itself continuously, and is a contraction on $S_{c_r}(0)$ for suitably chosen $c_r > 0$ and $s = a_0$ such that $a - a_0$ is sufficiently small.*

Proof. Let $\begin{bmatrix} l \\ m \end{bmatrix}$ be an element in the ball $S_{c_r}(0)$. According to each component, we estimate the norm of $O\begin{bmatrix} l \\ m \end{bmatrix}$ in $\mathbb{Y}$. Based on these estimates, we can obtain that there exists $b > 1$ such that when $c_r = 2b$ and

$$a - s < \frac{1}{2b + 8b^2 + 24b^3},$$

O acts from $S_{c_r}(0)$ into $S_{c_r}(0)$ in $\mathbb{Y}$. The property of contraction of map O can be estimated by the norm of the difference of $O\begin{bmatrix} l_1 \\ m_1 \end{bmatrix}$ and $O\begin{bmatrix} l_2 \\ m_2 \end{bmatrix}$. Taking $s = a_0$ such that $a - a_0$ is sufficiently small, we can obtain that O is a contraction on $S_{c_r}(\mathbb{Y})$. Hence, map O has a unique fixed point $\begin{bmatrix} l \\ m \end{bmatrix} \in \mathbb{Y}$. □

According to Lemma 2, the fixed point $\begin{bmatrix} l \\ m \end{bmatrix}$ represents solution $\begin{bmatrix} R_1(t) \\ U_1(t,s)E \end{bmatrix} \in \mathbb{Y}$ to (7) and (8). Therefore, Lemma 1 is proved.

3.2. Property of Operator $R_1(t)$

In this subsection, we consider the positivity and self-adjointness of operator $R_1(t)$, which is the solution to (7), and the evolution property of $U_1(t,s)$ on $C(\Delta_s, L(E(\mathbb{D}),\mathbb{D}))$.

Theorem 3. *(i) $U_1(t,s)$, defined by (8), is a GE-evolution operator on $C(\Delta_s, L(E(\mathbb{D}),\mathbb{D}))$.*
(ii) $R_1(t)$, defined by (7), is self-adjoint and is positive on $\mathbb{D}$.

Proof. (i) This can be derived by a standard method using the property of the GE-evolution operator.

(ii) From (8), we have

$$T(r,t)E\xi = U_1(r,t)E\xi + \int_t^r T(r,v)N_1(v)\phi(v)U_1(v,t)E\xi dv.$$

Substituting the above expression into (7) and taking the inner product of (7) with $x \in \mathbb{D}$, we can obtain

$$\begin{aligned}
\langle R_1(t)\xi, x\rangle = & \int_t^a \langle L(r)U_1(r,t)E\xi, L(r)U_1(r,t)Ex\rangle dr \\
& + \int_t^a \langle S^*(r)S(r)U_1(r,t)E\xi, \int_t^r T(r,v)N_1(v)\phi(v)U_1(v,t)Exdv\rangle dr \\
& + \int_t^a \langle S(r)U_1(r,t)E\xi, S(r)U_1(r,t)Ex\rangle dr \\
& + \int_t^a \langle S^*(r)S(r)U_1(r,t)E\xi, \int_t^a T(r,v)N_1(v)\phi(v)U_1(v,t)Exdv\rangle dr \\
& + \int_t^a \langle M_3^*(r)R_1(r)M_3(r)U_1(r,t)E\xi, U_1(r,t)E\xi\rangle dr \\
& + \int_t^a \langle M_3^*(r)R_1(r)M_3(r)U_1(r,t)E\xi, \int_t^r T(r,v)N_1(v)\phi(v)U_1(v,t)Exdv\rangle dr \\
& - \int_t^a \langle \phi^*(r)N_1^*(r)T_0^*R_1(r)U_1(r,t)E\xi, U_1(r,t)E\xi\rangle dr \\
& - \int_t^a \langle \phi^*(r)N_1^*(r)T_0^*R_1(r)U_1(r,t)E\xi, \int_t^r T(r,v)N_1(v)\phi(v)U_1(v,t)Exdv\rangle dr \\
& + \langle NU_1(a,t)E\xi, NU_1(a,t)Ex\rangle \\
& + \langle N^*NU_1(a,t)Ex, \int_t^a T(a,v)N_1(v)\phi(v)U_1(v,t)Exdv\rangle. \\
= & \int_t^a \langle L(r)U_1(r,t)E\xi, L(r)U_1(r,t)Ex\rangle dr \\
& + \int_t^a \langle S(r)U_1(r,t)E\xi, S(r)U_1(r,t)Ex\rangle dr \\
& + \int_t^a \langle M_3^*(r)R_1(r)M_3(r)U_1(r,t)E\xi, U_1(r,t)Ex\rangle dr \\
& + \langle NU_1(a,t)E\xi, NU_1(a,t)Ex\rangle,
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
\langle R_1^*(t)\xi, x\rangle = & \int_t^a \langle L(r)U_1(r,t)E\xi, L(r)U_1(r,t)Ex\rangle dr \\
& + \int_t^a \langle S(r)U_1(r,t)E\xi, S(r)U_1(r,t)Ex\rangle dr \\
& + \int_t^a \langle M_3^*(r)R_1^*(r)M_3(r)U_1(r,t)E\xi, U_1(r,t)Ex\rangle dr \\
& + \langle NU_1(a,t)E\xi, NU_1(a,t)Ex\rangle.
\end{aligned}$$

Therefore, we can obtain

$$\langle (R_1 - R_1^*)(t)\xi, x\rangle = \int_t^a \langle M_3^*(r)(R_1 - R_1^*)(r)M_3(r)U_1(r,t)E\xi, U_1(r,t)Ex\rangle dr.$$

This implies that there exists a constant $c_{R_1} > 0$ such that

$$\|R_1 - R_1^*\| \le c_{R_1} \int_t^a \|R_1(r) - R_1^*(r)\| dr.$$

According to Gronwall's inequality, we obtain $R_1(r) = R_1^*(r)$ for all $r \in [s,a]$.
In order to prove positivity, we define the operator O_1 on $P([s,a], L(\mathbb{D},\mathbb{D}))$ by

$$\langle O_1(R_1)(t)\xi, x\rangle = \int_t^a \langle L(r)U_1(r,t)E\xi, L(r)U_1(r,t)Ex\rangle dr$$
$$+ \int_t^a \langle S(r)U_1(r,t)E\xi, S(r)U_1(r,t)Ex\rangle dr$$
$$+ \int_t^a \langle M_3^*(r)R_1(r)M_3(r)U_1(r,t)E\xi, U_1(r,t)Ex\rangle dr$$
$$+\langle NU_1(a,t)E\xi, NU_1(a,t)Ex\rangle.$$

It is obvious that O_1 maps a positive operator to a positive operator. The set of positive operators denoted by O_+ in $L(\mathbb{D},\mathbb{D})$ is a convex set, and the existence of a unique fixed point for O_1 on $P([a_0,a], O_+)$ follows by the contraction mapping theorem, for a_0 chosen such that $a - a_0$ is sufficiently small. The unique fixed point of map O_1 is $R_1(t)$. □

3.3. *Proof of Theorem 2*

Proof. In order to prove Theorem 2, we use an iteration scheme.

$$R_{1,k+1}(t) = \int_t^a E^*T^*(r,t)L^*(r)L(r)U_{1,k}(r,t)Edr$$
$$+ \int_t^a E^*T^*(r,t)S_k^*(r)S_k(r)U_{1,k}(r,t)Edr$$
$$+ \int_t^a E^*T^*(r,t)M_{3,k}^*(r)R_{1,k+1}(r)M_{3,k}(r)U_{1,k}(r,t)Edr$$
$$- \int_t^a E^*T^*(r,t)\phi_k^*(r)N_1^*(r)T_0^*R_{1,k+1}(r)U_{1,k}(r,t)Edr$$
$$+ E^*T^*(a,t)N^*NU_{1,k}(a,t)E. \tag{11}$$

Here,

$$S_k(r) = (M^2(r) + N_2^*(r)T_0^*R_{1,k}(r)T_0N_2(r))^{-1}\cdot$$
$$(N_1^*(r)T_0^*R_{1,k}(r) + N_2^*(r)T_0^*R_{1,k}(r)T_0M_2(r)),$$
$$M_{3,k} = T_0M_2(r) - T_0N_2(r)(M^2(r) + N_2^*(r)T_0^*R_{1,k}(r)T_0N_2(r))^{-1}\cdot$$
$$(N_1^*(r)T_0^*R_{1,k}(r) + N_2^*(r)T_0^*R_{1,k}(r)T_0M_2(r)),$$
$$\phi_k(r) = (M^2(r) + N_2^*(r)T_0^*R_{1,k}(r)T_0N_2(r))^{-1}\cdot$$
$$(N_1^*(r)T_0^*R_{1,k}(r) + N_2^*(r)T_0^*R_{1,k}(r)T_0M_2(r)),$$
$$R_{1,0}(t) = E^*T^*(a,t)N^*NT(a,t)E,$$

and $U_{1,k}$ is the solution of

$$U_{1,k}(t,s)E\xi = T(t,s)E\xi - \int_s^t T(t,v)N_1(v)\phi(v)U_{1,k}(v,s)E\xi dv. \tag{12}$$

According to Lemma 1 and Theorem 3, each $R_{1,k}$ is well defined, positively self-adjoint, and bounded with $\|R_{1,k}\| \le c_r$, $\forall k \in \mathbb{N}$, and $U_{1,k} \in C(\Delta_s, L(E(\mathbb{D}),\mathbb{D}))$ satisfies

$$\|U_{1,k}\| \le c_r,$$

and this implies that $(M^2(r) + N_2^*(r)T_0^*R_{1,k}(r)T_0N_2(r))^{-1}$ is well defined and bounded on $\mathbb{D}$ at each step. We can prove that the sequence $\{R_{1,k}, U_{1,k}\}$ is Cauchy in $\mathbb{Y}$ for $s = a_{max} \ge a_0$, chosen such that $a - a_{max}$ is sufficiently small and thus converges to $\begin{bmatrix} R(t) \\ U(t,s) \end{bmatrix}$. From the limit in (11) and (12), we can obtain (5) and (6). □

4. The Generalized Differential Riccati Equation

In this section, we investigate the solution of the generalized differential Riccati equation from the generalized integral Riccati Equation (5). The main result of this section is the following theorem.

Theorem 4. *The operator $R(t)$, solving the generalized integral Riccati Equation (5), is a solution to the generalized differential Riccati equation.*

$$
\begin{aligned}
\langle \frac{dR(t)}{dt}\xi, x\rangle = &-\langle L(t)\xi, L(t)x\rangle - \langle R(t)T_0M_1(t)\xi, x\rangle - \langle M_1^*(t)T_0^*R(t)\xi, x\rangle \\
&-\langle M_2^*(t)T_0^*R(t)T_0M_2(t)\xi, x\rangle \\
&+\langle (M^2(t) + N_2^*(t)T_0^*R(t)T_0N_2(t))^{-1}(N_1^*(t)T_0^*R(t) + N_2^*(t)T_0^*R(t)T_0M_2(t))\xi, \\
&(N_1^*(t)T_0^*R(t) + N_2^*(t)T_0^*R(t)T_0M_2(t))x\rangle
\end{aligned} \tag{13}
$$

for all $\xi, x \in \mathbb{D}_0$

4.1. Some Lemmas

First, we define the operator P, which is given by

$$
P = \int_s^t T(t,r)N_1(r)(M^2(r) + N_2^*(r)T_0^*R(r)T_0N_2(r))^{-1} \cdot
$$
$$
(N_1^*(r)T_0^*R(r) + N_2^*(r)T_0^*R(r)T_0M_2(r))dr.
$$

Similar to the proof of [28,29], we can obtain the following lemma.

Lemma 3. *(i)* $\|Pf\|_{C([s,a],\mathbb{D})} \le c_P(a-s)\|f\|_{C([s,a],\mathbb{D})}$, *where c_P is a constant.*

(ii) The operator $I + P$ is invertible on $C([s,a],\mathbb{D})$, and the inverse satisfies

$$
\|(I+P)^{-1}f\|_{C([s,a],\mathbb{D})} \le c_1(a-s)\|f\|_{C([s,a],\mathbb{D})},
$$

where I denotes the identical operator, and c_1 is a constant.

(iii) The GE-evolution operator $U(t,s)$ satisfies

$$
U(\cdot,s)E\xi = (I+P)^{-1}T(\cdot,s)E\xi, \forall \xi \in \mathbb{D}.
$$

According to Lemma 3, we can obtain the following lemma.

Lemma 4. *(i)* $U(t,s)N_1(s)\eta \in C([s,a],\mathbb{D})$ *for* $\forall \eta \in \mathbb{U}$.

(ii) The derivative of $U(t,s)E\xi$ with respect to s in the weak sense is

$$
\frac{\partial U(\cdot,s)}{\partial s}E\xi = -U(\cdot,s)[M_1(s) - N_1(s)(M^2(s) + N_2^*(s)T_0^*R(s)T_0N_2(s))^{-1} \cdot
$$
$$
(N_1^*(s)T_0^*R(s) + N_2^*(s)T_0^*R(s)T_0M_2(s))]\xi \in C([s,a],\mathbb{D}), \forall \xi \in \mathbb{D}_0
$$

and satisfies

$$
\|\frac{\partial U(\cdot,s)}{\partial s}E\xi\| \le c_2\|\xi\|_{\mathbb{D}_0},
$$

where c_2 is a constant.

4.2. Proof of Theorem 4

Proof. Assume $\xi, x \in \mathbb{D}_0$ and consider the generalized integral Riccati equation satisfied by $R(t)$ in (5). Taking the derivative with respect to t and using Lemma 4 (ii), we can obtain

$$\langle \frac{dR(t)}{dt}\xi, x\rangle = -\langle L^*(t)L(t)\xi, x\rangle - \langle M_2^*(t)T_0^*R(t)T_0M_2(t)\xi, x\rangle$$

$$+\langle M_2^*(t)T_0^*R(t)T_0N_2(t)(M^2(t) + N_2^*(t)T_0^*R(t)T_0N_2(t))^{-1}\cdot$$
$$(N_1^*(t)T_0^*R(t) + N_2^*(t)T_0^*R(t)T_0M_2(t))\xi, x\rangle$$
$$-\langle M_1^*(t)T_0R(t)\xi, x\rangle + \langle \int_t^a E^*T^*(r,t)L^*(r)L(r)\frac{\partial U(r,t)}{\partial t}E\xi dr, x\rangle$$
$$+\langle \int_t^a E^*T^*(r,t)M_2^*(r)T_0^*R(r)T_0M_2(r)\frac{\partial U(r,t)}{\partial t}E\xi dr, x\rangle$$
$$-\langle \int_t^a E^*T^*(r,t)M_2^*(r)T_0^*R(r)T_0N_2(r)(M^2(r) + N_2^*(r)T_0^*R(r)T_0N_2(r))^{-1}\cdot$$
$$(N_1^*(r)T_0^*R(r) + N_2^*(r)T_0^*R(r)T_0M_2(r))\frac{\partial U(r,t)}{\partial t}E\xi dr, x\rangle$$
$$= -\langle L^*(t)L(t)\xi, x\rangle - \langle M_2^*(t)T_0^*R(t)T_0M_2(t)\xi, x\rangle$$
$$+\langle M_2^*(t)T_0^*R(t)T_0N_2(t)(M^2(t) + N_2^*(t)T_0^*R(t)T_0N_2(t))^{-1}\cdot$$
$$(N_1^*(t)T_0^*R(t) + N_2^*(t)T_0^*R(t)T_0M_2(t))\xi, x\rangle - \langle M_1^*(t)T_0R(t)\xi, x\rangle$$
$$-\langle R(t)(T_0M_1(t) - T_0N_1(t)(M^2(t) + N_2^*(t)T_0^*R(t)T_0N_2(t))^{-1})\cdot$$
$$(N_1^*(t)T_0^*R(t) + N_2^*(t)T_0^*R(t)T_0M_2(t))\xi, x\rangle$$
$$= -\langle L^*(t)L(t)\xi, x\rangle - \langle R(t)T_0M_1(t)\xi, x\rangle - \langle M_1^*(t)T_0^*R(t)\xi, x\rangle$$
$$-\langle M_2^*(t)T_0^*R(t)T_0M_2(t)\xi, x\rangle$$
$$+\langle (M^2(t) + N_2^*(t)T_0^*R(t)T_0N_2(t))^{-1}(N_1^*(t)T_0^*R(t) + N_2^*(t)T_0^*R(t)T_0M_2(t))\xi,$$
$$(N_1^*(t)T_0^*R(t) + N_2^*(t)T_0^*R(t)T_0M_2(t))x\rangle.$$

This is the generalized Riccati differential equation. □

5. The Generalized Riccati Equation and the Optimal Control

In this section, we consider the relation between the optimization problem and the solution of the generalized differential equation by using Ito's formula.

Theorem 5. *The quadratic cost function (2) has the following form:*

$$F(t, \xi_0, \eta(\cdot)) = \langle R(t)\xi_0, \xi_0\rangle + \mathbb{E}(\int_t^a \|(M^2(r) + N_2^*(r)T_0^*R(r)T_0N_2(r))^{1/2}\eta(r)$$
$$+(M^2(r) + N_2^*(r)T_0^*R(r)T_0N_2(r))^{-1/2}\cdot$$
$$(N_1^*(r)T_0^*R(r) + N_2^*(r)T_0^*R(r)T_0M_2(r))\xi(r)\|_{\mathbb{U}}^2 dr) \tag{14}$$

for $s \leq t \leq a$ and $s = a_{max}$. Here, $R(t)$ is a solution to the generalized differential Riccati Equation (13), and $\xi(\cdot)$ is the mild solution of the linear stochastic generalized System (1) corresponding to $\eta(\cdot) \in L^2_{sm}([s,a], \Omega, \mathbb{F}_t), \mathbb{U})$.

Proof. Suppose that $\xi(t)$ is the mild solution of System (1) corresponding to $\eta(\cdot) \in L^2_{sm}([s,a],\Omega,\mathbb{F}_t),\mathbb{U})$. First of all, we assume that $\xi(t)$ is a strong solution of System (1). If $R(t)$ satisfies (13), then, by Ito's formula, we have

$$\begin{aligned} d\langle R(t)\xi(t),\xi(t)\rangle &= \langle \frac{dR(t)}{dt}\xi(t),\xi(t)\rangle dt + \langle R(t)\xi(t), T_0M_1(t)\xi(t)+T_0N_1(t)\eta(t)\rangle dt \\ &+\langle R(t)\xi(t), T_0M_2(t)\xi(t)+T_0N_2(t)\eta(t)\rangle dw(t) \\ &+\langle R(t)(T_0M_1(t)\xi(t)+T_0N_1(t)\eta(t)),\xi(t)\rangle dt \\ &+\langle R(t)(T_0M_2(t)\xi(t)+T_0N_2(t)\eta(t)),\xi(t)\rangle dw(t) \\ &+\langle R(t)(T_0M_2(t)\xi(t)+T_0N_2(t)\eta(t)), T_0M_2(t)\xi(t)+T_0N_2(t)\eta(t)\rangle dt \\ &= -\langle L^*(t)L(t)\xi(t),\xi(t)\rangle dt + \langle (N_1^*(t)T_0^*R(t)+N_2^*(t)T_0^*R(t)T_0M_2(t))^* \cdot \\ &(M^2(t)+N_2^*(t)T_0^*R(t)T_0N_2(t))^{-1} \cdot \\ &(N_1^*(t)T_0^*R(t)+N_2^*(t)T_0^*R(t)T_0M_2(t))\xi(t),\xi(t)\rangle dt \\ &+2\langle R(t)T_0N_1(t)\eta(t),\xi(t)\rangle dt + \langle (M^2(t)+N_2^*(t)T_0^*R(t)T_0N_2(t))\eta(t),\eta(t)\rangle dt \\ &+2\langle R(t)\xi(t), T_0M_2(t)\xi(t)+T_0N_2(t)\eta(t)\rangle dw(t) \\ &+2\langle N_2^*(t)T_0^*R(t)T_0M_2(t))\xi(t),\eta(t)\rangle dt - \langle M^2(t)\eta(t),\eta(t)\rangle dt \\ &= -\|L(t)\xi(t)\|^2 dt + \|(M^2(t)+N_2^*(t)T_0^*R(t)T_0N_2(t))^{-1/2} \cdot \\ &(N_1^*(t)T_0^*R(t)+N_2^*(t)T_0^*R(t)T_0M_2(t))\xi(t) \\ &+(M^2(t)+N_2^*(t)T_0^*R(t)T_0N_2(t))^{1/2}\eta(t)\|^2 dt - \langle M^2(t)\eta(t),\eta(t)\rangle dt \\ &+2\langle R(t)\xi(t), T_0M_2(t)\xi(t)+T_0N_2(t)\eta(t)\rangle dw(t). \end{aligned}$$

Therefore,

$$\begin{aligned} &\int_s^a [\|M(t)\eta(t)\|^2 + \|L(t)\xi(t)\|^2]dt + \|N\xi(a)\|^2 = \langle R(s)\xi_0,\xi_0\rangle \\ &+\int_s^a \|(M^2(t)+N_2^*(t)T_0^*R(t)T_0N_2(t))^{-1/2}(N_1^*(t)T_0^*R(t) \\ &+N_2^*(t)T_0^*R(t)T_0M_2(t))\xi(t) + (M^2(t)+N_2^*(t)T_0^*R(t)T_0N_2(t))^{1/2}\eta(t)\|^2 dt \\ &+2\int_s^a \langle R(t)\xi(t), T_0M_2(t)\xi(t)+T_0N_2(t)\eta(t)\rangle dw(t), \\ &F(s,\xi_0,\eta(\cdot)) = \langle R(s)\xi_0,\xi_0\rangle \\ &+\mathbb{E}[\int_s^a \|(M^2(t)+N_2^*(t)T_0^*R(t)T_0N_2(t))^{-1/2}(N_1^*(t)T_0^*R(t) \\ &+N_2^*(t)T_0^*R(t)T_0M_2(t))\xi(t) + (M^2(t)+N_2^*(t)T_0^*R(t)T_0N_2(t))^{1/2}\eta(t)\|^2 dt] \end{aligned}$$

since $\xi(t) \in \mathbb{D}_0$, and $\overline{\mathbb{D}_0} = \mathbb{D}$. Therefore, (14) holds for $\forall \eta(\cdot) \in L^2_{sm}([s,a],\Omega,\mathbb{F}_t),\mathbb{U})$ □

Corollary 1. *The solution of the generalized Riccati equation in Theorem 4 can be extended to a global solution on any time interval* $[s,a]$.

Proof. According to Hypotheses 3 and 4, we have

$$\langle R(s)\xi_0,\xi_0\rangle \le F(s,\xi_0,\eta=0) = \mathbb{E}(\int_s^a \|L(t)\xi(t)\|^2]dt + \|N\xi(a)\|^2) \le c_R\|\xi_0\|^2$$

for all $t \in [a_{max}, a]$, i.e., $\|R(t)\| \le c_R$, where c_R is a constant. This implies that the proofs of Lemma 1 and Theorem 2 hold on a new interval $[a_1, a_{max}]$ with $N = R^{1/2}(a_{max})$. The bound ensures that all the estimates are uniform and that c_r and the step $a_{max} - a_1$ are the same. Hence, the results can be extend to any time interval $[s, a]$ by repeating the above proof processes on equal time steps. □

Corollary 2. *The solution to the generalized differential Riccati equation is unique in the class of self-adjoint operators in* $P([s, a], L(\mathbb{D}, \mathbb{D}))$.

Proof. If there exists another solution $R_1(t)$ to the generalized differential Riccati equation in this class, then, by the same method as that of Theorem 5, we can obtain that

$$\min F(t, \xi_0, \eta(\cdot)) = \langle R(t)\xi_0, \xi_0\rangle = \langle R_1(t)\xi_0, \xi_0\rangle$$

for $\forall\xi_0 \in \mathbb{D}$. Therefore, for any $\xi_0, x \in \mathbb{D}$, we can obtain that

$$0 = \langle (R(t) - R_1(t))(\xi_0 + x), \xi_0 + x\rangle = 2\langle (R(t) - R_1(t))\xi_0, x\rangle$$

by the self-adjoints of R and R_1. This implies that $R(t) = R_1(t)$. □

6. Main Results and Proofs

Theorem 6. *Under the Hypotheses 1–4, there is a positive self-adjoint operator*

$$R(t) \in P([s, a], L(\mathbb{D}, \mathbb{D}))$$

satisfying the generalized Riccati differential equation

$$\begin{aligned}\langle \frac{dR(t)}{dt}\xi_0, x\rangle = &-\langle L^*(t)L(t)\xi_0, x\rangle - \langle R(t)T_0M_1(t)\xi_0, x\rangle - \langle M_1^*(t)T_0^*R(t)\xi_0, x\rangle \\ &-\langle M_2^*(t)T_0^*R(t)T_0M_2(t)\xi_0, x\rangle \\ &+\langle (N_1^*(t)T_0^*R(t) + N_2^*(t)T_0^*R(t)T_0M_2(t))^*(M^2(t) + N_2^*(t)T_0^*R(t)T_0N_2(t))^{-1}\cdot \\ &(N_1^*(t)T_0^*R(t) + N_2^*(t)T_0^*R(t)T_0M_2(t))\xi_0, x\rangle \end{aligned} \tag{15}$$

$$M^2(t) + N_2^*(t)T_0^*R(t)T_0N_2(t) > 0 \tag{16}$$

$$R(a)\xi_0 = N^*N\xi_0, \tag{17}$$

for $\forall\xi_0, x \in \mathbb{D}$. *Furthermore, the following statements hold:*

(i) $\min_{\eta(\cdot)\in L^2_{sm}([s,a],\Omega,\mathbb{F}_t,\mathbb{D})} F(t, \xi_0, \eta(\cdot)) = \langle R(t)\xi_0, \xi_0\rangle$.

(ii) $R(t)$ *is unique in the class of self-adjoint operators in* $P([s, a], L(\mathbb{D}, \mathbb{D}))$

(iii)
$$\|R(t)\xi_0\| \le c_R\|\xi_0\|, \forall t \in [0, a], \xi_0 \in \mathbb{D}. \tag{18}$$

Proof. (i) According to (14) in Theorem 5, F satisfies

$$\min_{\eta(\cdot)\in L^2_{sm}([s,a],\Omega,\mathbb{F}_t,\mathbb{D})} F(t, \xi_0, \eta(\cdot)) = \langle R(t)\xi_0, \xi_0\rangle$$

Here, $R(t)$ is the solution to the generalized differential Riccati equation.

(ii) The existence of the solution to the generalized differential Riccati equation in $P([s, a], L(\mathbb{D}, \mathbb{D}))$ can be obtained by Theorem 4, and the uniqueness has been proved in Corollary 2.

(iii) According to Corollary 1, we can obtain (18).

□

Theorem 7. *Under Hypotheses 1–4, the optimal control Problem 1 with the linear stochastic generalized System (1) and initial condition $\xi_0 \in \mathbb{D}$ has a unique optimal input solution $\eta_o(s,\cdot,\xi_0) \in L^2_{sm}([s,a],\Omega,\mathbb{F}_t,\mathbb{U})$ and a corresponding optimal state*

$$\xi_o(s,\cdot,\xi_0) \in C^2([s,a],\Omega,\mathbb{F}_t,\mathbb{D}).$$

Furthermore, η_o has feedback characterization in terms of the ξ_o

$$\eta_o(s,t,\xi_0) = -(M^2(t) + N_2^*(t)T_0^*R(t)T_0N_2(t))^{-1}\cdot$$
$$(N_1^*(t)T_0^*R(t) + N_2^*(t)T_0^*R(t)T_0M_2(t))\xi_o(s,t,\xi_0).$$

Here, $R(t)$ is the unique solution to (15)–(17).

Proof. In order to prove that the minimum of F is realized in (14), we can construct the existence of a unique solution

$$\eta_o(s,\cdot,\xi_0) \in L^2_{sm}([s,a],\Omega,\mathbb{F}_t,\mathbb{U})$$

to the equation

$$\eta_o(s,t,\xi_0) = -(M^2(t) + N_2^*(t)T_0^*R(t)T_0N_2(t))^{-1}\cdot$$
$$(N_1^*(t)T_0^*R(t) + N_2^*(t)T_0^*R(t)T_0M_2(t))\xi(s,t,\eta_o,\xi_0),$$

via a fixed point argument on $L^2_{sm}([s,a],\Omega,\mathbb{F}_t,\mathbb{U})$. Therefore,

$$\eta_o(s,t,\xi_0) = -(M^2(t) + N_2^*(t)T_0^*R(t)T_0N_2(t))^{-1}\cdot$$
$$(N_1^*(t)T_0^*R(t) + N_2^*(t)T_0^*R(t)T_0M_2(t))\xi_o(s,t,\xi_0)$$

such that $F(s,\xi_0,\eta_o) = \langle R(s)\xi_0,\xi_0\rangle$. □

7. Linear Quadratic Optimal Control Problem for a Class of Linear Stochastic Generalized Systems

In this section, we investigate the following linear quadratic optimal control problem. We consider the following linear stochastic generalized system

$$E_1 d\xi(t) = (M_{11}\xi(t) + N_{11}\eta(t))dt + (M_{12}\xi(t) + N_{12}\eta(t))dw(t)$$
$$t \in [0,a], \xi(0) = \xi_0. \tag{19}$$
$$0 = \sigma(t)dt + N_{21}\eta(t)dt, t \in [0,a], \sigma(0) = \sigma_0. \tag{20}$$

The following quadratic cost functional is introduced:

$$F(0,\begin{bmatrix} \xi_0 \\ \sigma_0 \end{bmatrix},\eta(\cdot)) = \mathbb{E}(\int_0^a (\|L\begin{bmatrix} \xi(t) \\ \sigma(t) \end{bmatrix}\|^2 + \|M\eta(t)\|^2)dt + \|N\xi(a)\|^2). \tag{21}$$

Here, M is a strongly positive operator, and L and N are bounded linear operators. Let $\mathbb{H} = \mathbb{H}_1 \oplus \mathbb{H}_2$,

$$E = \begin{bmatrix} E_1 & 0 \\ 0 & 0 \end{bmatrix}, M_1 = \begin{bmatrix} M_{11} & 0 \\ 0 & I \end{bmatrix}, N_1 = \begin{bmatrix} N_{11} \\ N_{21} \end{bmatrix}$$

$$M_2 = \begin{bmatrix} M_{12} & 0 \\ 0 & 0 \end{bmatrix}, N_2 = \begin{bmatrix} N_{12} \\ 0 \end{bmatrix},$$

$$\mathbb{D} = \mathbb{H}_1, \xi(t) \in \mathbb{H}_1, \sigma(t) \in \mathbb{H}_2.$$

The optimal control problem considered in this section is as follows:

Problem 2. *For any given initial value* $\begin{bmatrix} \xi_0 \\ \sigma_0 \end{bmatrix} \in \mathbb{H}_1 \times \mathbb{H}_2$*, find a*

$$\eta_o(\cdot) \in L^2_{sm}([0,a],\Omega,\mathbb{F}_t,\mathbb{U}),$$

such that

$$F_o(0,\begin{bmatrix} \xi_0 \\ \sigma_0 \end{bmatrix}) = F(0,\xi_0,\eta_o(\cdot)) = \min_{\eta(\cdot)\in L^2_{sm}([0,a],\Omega,\mathbb{F}_t,\mathbb{U})} F(0,\xi_0,\eta(\cdot)).$$

Any $\eta_o(\cdot) \in L^2_{sm}([0,a],\Omega,\mathbb{F}_t,\mathbb{U})$ satisfying Problem 2 is called an optimal control of Problem 2 for the initial value $\begin{bmatrix} \xi_0 \\ \sigma_0 \end{bmatrix}$, and the corresponding state $\begin{bmatrix} \xi_o(\cdot) \\ \sigma_o(\cdot) \end{bmatrix} = \begin{bmatrix} \xi(0,\cdot,\xi_0,\eta_o(\cdot)) \\ \sigma(0,\cdot,\xi_0,\eta_o(\cdot)) \end{bmatrix}$ is called an optimal state process; the pair $(\begin{bmatrix} \xi_o(\cdot) \\ \sigma_o(\cdot) \end{bmatrix},\eta_o(\cdot))$ is called an optimal pair. The function $F_o(0,\cdot)$ is called the value function of Problem 2.

Definition 2. *If* $\xi(t)$ *is the mild solution of system (19),* $\sigma(t) = -N_{21}\eta(t)$*, then* $\begin{bmatrix} \xi(t) \\ \sigma(t) \end{bmatrix}$ *is called the solution of linear stochastic generalized systems (19)–(20).*

Theorem 8. *Let* E *and* M_1 *satisfy Hypothesis 1 and define*

$$L^*L = \begin{bmatrix} L_{11} & L_{12} \\ L_{12}^* & L_{22} \end{bmatrix}, \overline{L} = L_{11} - L_{12}N_{21}\overline{M}^{-1}N_{21}^*L_{12}^*, \overline{M} = M^2 + N_{21}^*L_{22}N_{21} \tag{22}$$

$$\overline{M}_{11} = M_{11} + N_{11}\overline{M}^{-1}N_{21}^*L_{12}^* \tag{23}$$

$$\overline{M}_{12} = M_{12} + N_{12}\overline{M}^{-1}N_{21}^*L_{12}^*. \tag{24}$$

Then, by the controller substitution

$$\eta(t) = u(t) + \overline{M}^{-1}N_{21}^*L_{12}^*\xi(t), \tag{25}$$

the optimal control for the linear stochastic generalized system (19)–(20) with the cost functional (21) can be converted into the following linear quadratic optimal control problem:

$$E_1 d\xi(t) = (\overline{M}_{11}\xi(t) + N_{11}u(t))dt + (\overline{M}_{12}\xi(t) + N_{12}u(t))dw(t),$$
$$t \in [0,a], \xi(0) = \xi_0. \tag{26}$$

with the cost functional

$$F(0,\begin{bmatrix} \xi_0 \\ \sigma_0 \end{bmatrix},\eta(\cdot)) = \mathbb{E}(\int_0^a (\|\overline{L}^{1/2}\xi(t)\|^2 + \|\overline{M}^{1/2}u(t)\|^2)dt + \|N\xi(a)\|^2). \tag{27}$$

Proof. According to (20), we obtain $\sigma(t) = -N_{21}\eta(t)$. From the operators $L_{11}, L_{12}, \overline{L}$ defined by (22) and $\sigma(t) = -N_{21}\eta(t)$, the cost functional (21) can be converted as

$$F(0,\begin{bmatrix} \xi_0 \\ \sigma_0 \end{bmatrix},\eta(\cdot)) = \mathbb{E}(\int_0^a \langle \begin{bmatrix} L^*L & 0 \\ 0 & M^2 \end{bmatrix} \begin{bmatrix} \xi(t) \\ \sigma(t) \\ \eta(t) \end{bmatrix}, \begin{bmatrix} \xi(t) \\ \sigma(t) \\ \eta(t) \end{bmatrix} \rangle dt + \|N\xi(a)\|^2)$$

$$= \mathbb{E}(\int_0^a \langle \begin{bmatrix} L_{11} & L_{12} & 0 \\ L_{12}^* & L_{22} & 0 \\ 0 & 0 & M^2 \end{bmatrix} \begin{bmatrix} \xi(t) \\ \sigma(t) \\ \eta(t) \end{bmatrix}, \begin{bmatrix} \xi(t) \\ \sigma(t) \\ \eta(t) \end{bmatrix} \rangle dt + \|N\xi(a)\|^2)$$

$$= \mathbb{E}(\int_0^a \langle \begin{bmatrix} L_{11} & -L_{12}N_{21} \\ -N_{21}^* L_{12}^* & \overline{M} \end{bmatrix} \begin{bmatrix} \xi(t) \\ \eta(t) \end{bmatrix}, \begin{bmatrix} \xi(t) \\ \eta(t) \end{bmatrix} \rangle dt + \|N\xi(a)\|^2)$$

(where (22) and $\sigma(t) = -N_{21}\eta(t)$ are used)

$$= \mathbb{E}(\int_0^a (\langle L_{11}\xi(t), \xi(t)\rangle - \langle N_{21}^* L_{12}^* \xi(t), \eta(t)\rangle - \langle L_{12}N_{21}\eta(t), \xi(t)\rangle$$
$$+\langle \overline{M}\eta(t), \eta(t)\rangle)dt + \|N\xi(a)\|^2)$$
$$= \mathbb{E}(\int_0^a (\langle L_{11}\xi(t), \xi(t)\rangle - \langle L_{12}N_{21}\overline{M}^{-1}N_{21}^* L_{12}^* \xi(t), \xi(t)\rangle$$
$$+\langle L_{12}N_{21}\overline{M}^{-1}N_{21}^* L_{12}^* \xi(t), \xi(t)\rangle)dt$$
$$+ \int_0^a (-\langle N_{21}^* L_{12}^* \xi(t), \eta(t)\rangle - \langle L_{12}N_{21}\eta(t), \xi(t)\rangle$$
$$+\langle \overline{M}\eta(t), \eta(t)\rangle)dt + \|N\xi(a)\|^2)$$
$$= \mathbb{E}(\int_0^a (\langle (L_{11} - L_{12}N_{21}\overline{M}^{-1}N_{21}^* L_{12}^*)\xi(t), \xi(t)\rangle$$
$$+\langle \overline{M}(\eta(t) - M^{-1}N_{21}^* L_{12}^* \xi(t)),$$
$$\eta(t) - M^{-1}N_{21}^* L_{12}^* \xi(t)\rangle)dt + \|N\xi(a)\|^2)$$
$$= \mathbb{E}(\int_0^a (\|\overline{L}^{1/2}\xi(t)\|^2 + \|\overline{M}^{1/2}u(t)\|^2)dt + \|N\xi(a)\|^2), \tag{28}$$

where

$$\overline{L} = L_{11} - L_{12}N_{21}\overline{M}^{-1}N_{21}^* L_{12}^*$$

is used, and

$$u(t) = \eta(t) - \overline{M}^{-1}N_{21}^* L_{12}^* \xi(t),$$

i.e.,

$$\eta(t) = u(t) + \overline{M}^{-1}N_{21}^* L_{12}^* \xi(t).$$

This is the same as (27). Substituting (25) into (19), we can obtain System (26) given by (23) and (24).

In order to finish the proof, we need to show that $\overline{L}$ is a symmetric nonnegative operator. Since

$$\begin{bmatrix} \overline{L} & 0 \\ 0 & \overline{M} \end{bmatrix} = \begin{bmatrix} I & -L_{12}N_{21}\overline{M} \\ 0 & I \end{bmatrix} \begin{bmatrix} L_{11} & L_{12}N_{21} \\ N_{21}^* L_{12}^* & \overline{M} \end{bmatrix} \begin{bmatrix} I & 0 \\ -N_{21}^* L_{12}^* \overline{M} & I \end{bmatrix},$$

we obtain that operator $\overline{L}$ is symmetrically nonnegative. □

As can be seen from the above, Problem 2 is transformed into the following problem:

Problem 3. *For any given initial value $\xi_0 \in \mathbb{H}_1$, find a*

$$u_o(\cdot) \in L^2_{sm}([0,a], \Omega, \mathbb{F}_t, \mathbb{U}),$$

such that

$$F_o(0, \xi_0) = F(0, \xi_0, u_o(\cdot)) = \min_{u(\cdot)\in L^2_{sm}([0,a],\Omega,\mathbb{F}_t,\mathbb{U})} F(0, \xi_0, u(\cdot)),$$

where

$$F(0, \xi_0, u(\cdot)) = \mathbb{E}(\int_0^a (\|\overline{L}^{1/2}\xi(t)\|^2 + \|\overline{M}^{1/2}u(t)\|^2)dt + \|N\xi(a)\|^2).$$

Any $u_o(\cdot) \in L^2_{sm}([0,a], \Omega, \mathbb{F}_t, \mathbb{U})$ satisfying Problem 3 is called an optimal control of Problem 3 for the initial value ξ_0, and the corresponding state $\xi_o(\cdot) = \xi(0, \cdot, \xi_0, u_o(\cdot))$ is called an optimal state process; the pair $(\xi_o(\cdot), u_o(\cdot))$ is called an optimal pair. The function $F_o(0,\cdot)$ is called the value function of Problem 3.

Obviously, Problem 3 can be solved by the method of Problem 1. According to Theorem 7, we can obtain the following results:

Corollary 3. *Under Hypotheses 1–4, corresponding to Problem 3, the optimal control Problem 3 with the linear stochastic generalized System (26) and initial condition $\xi_0 \in \mathbb{D}$ have a unique optimal input solution $u_o(0,\cdot,\xi_0) \in L^2_{sm}([0,a],\Omega,\mathbb{F}_t,\mathbb{U})$ and a corresponding optimal state*

$$\xi_o(0,\cdot,\xi_0) \in C^2([0,a],\Omega,\mathbb{F}_t,\mathbb{D}).$$

Furthermore, u_o has feedback characterization in terms of the ξ_o

$$u_o(0,t,\xi_0) = -(\overline{M} + N_{12}^* T_0^* R(t) T_0 N_{12})^{-1}\cdot$$

$$(N_{11}^* T_0^* R(t) + N_{12}^* T_0^* R(t) T_0 \overline{M}_{12})\xi_o(0,t,\xi_0).$$

Here, $R(t)$ is the unique solution to (15)–(17) corresponding to Problem 3.

8. Application Examples

In this section, three application examples are given to illustrate the effectiveness of the theoretical results obtained in this paper.

Example 1. *Consider the following linear quadratic optimal control problem.*

The linear stochastic generalized system is the generalized heat equation:

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} d\xi(t,\tau) \\ d\sigma(t,\tau) \end{bmatrix} = \begin{bmatrix} M_{11} & 0 \\ 0 & (1+t^2)I \end{bmatrix}\begin{bmatrix} \xi(t,\tau) \\ \sigma(t,\tau) \end{bmatrix}dt + \begin{bmatrix} (1+t)I \\ 0 \end{bmatrix}\eta(t,x)dt$$

$$+\begin{bmatrix} M_{21}(t) & 0 \\ 0 & M_{22}(t) \end{bmatrix}\begin{bmatrix} \xi(t,\tau) \\ \sigma(t,\tau) \end{bmatrix}dt + \begin{bmatrix} N_{21}(t) \\ 0 \end{bmatrix}\eta(t,x)dw(t),$$

$$\xi(0,x) = \xi_0(x), \sigma(0,x) = \sigma_0(x), 0 \le t \le a, 0 \le x \le \pi. \tag{29}$$

Here, $M_{11}\xi = \frac{d^2\xi}{d\tau^2}$ with domain $\mathrm{dom} M_{11} = \{\xi \in H_1, \xi, \frac{d\xi}{d\tau}$ *are absolutely continuous, $\frac{d^2\xi}{d\tau^2} \in H_1, \xi(0) = \xi(\pi) = 0\}, H_1 = L^2(0,\pi)$. The cost function takes the following form:*

$$F(0, \begin{bmatrix} \xi_0 \\ \sigma_0 \end{bmatrix}, \eta(\cdot)) = \mathbb{E}(\int_0^a (\| \begin{bmatrix} \xi(t) \\ \sigma(t) \end{bmatrix} \|^2 + \|\eta(t)\|^2)dt + \|\xi(a)\|^2). \tag{30}$$

Find η to minimize (30). This kind of optimal control problem can be classified as Problem 1 and solved.

Let

$$\mathbb{H} = H_1 \oplus H_1, \mathbb{U} = H_1, E = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},$$

M_1: $\mathrm{dom} M_1 \subseteq \mathbb{H} \to \mathbb{H}$ *be an operator defined by*

$$M_1(t) = \begin{bmatrix} M_{11} & 0 \\ 0 & (1+t^2)I \end{bmatrix}, \mathbb{D} = H_1$$

$$N_1(t) = \begin{bmatrix} (1+t)I \\ 0 \end{bmatrix}, M_2(t) = \begin{bmatrix} M_{21}(t) & 0 \\ 0 & M_{22}(t) \end{bmatrix}, N_2(t) = \begin{bmatrix} N_{21}(t) \\ 0 \end{bmatrix}.$$

Then, (29) can be rewritten as

$$Ed\xi(t) = (M_1(t)\begin{bmatrix} \xi(t) \\ \sigma(t) \end{bmatrix} + N_1(t)\eta(t))dt + (M_2(t)\begin{bmatrix} \xi(t) \\ \sigma(t) \end{bmatrix} + N_2(t)\eta(t))dw(t)$$

$$t \in [0, a], \begin{bmatrix} \xi(0) \\ \sigma(0) \end{bmatrix} = \begin{bmatrix} \xi_0 \\ \sigma_0 \end{bmatrix}. \tag{31}$$

It is obvious that Hypotheses 1 and 3 are true for (30) and (31). As long as $M_2(t)$ and $N_2(t)$ satisfy Hypotheses 2 and 4, then (30) and (31) satisfy Hypotheses 1–4. Hence, Theorem 7 is true for (30) and (31).

Example 2. *In Example 1, we take the state equation and cost functional as the following, respectively:*

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} d\xi(t,\tau) \\ d\sigma(t,\tau) \end{bmatrix} = \begin{bmatrix} M_{11} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \xi(t,\tau) \\ \sigma(t,\tau) \end{bmatrix} dt + \begin{bmatrix} I \\ I \end{bmatrix} \eta(t,x)dt$$

$$+ \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \xi(t,\tau) \\ \sigma(t,\tau) \end{bmatrix} dt + \begin{bmatrix} \alpha I \\ 0 \end{bmatrix} \eta(t,x)dw(t),$$

$$\xi(0,x) = \xi_0(x), \sigma(0,x) = \sigma_0(x), 0 \le t \le a, 0 \le x \le \pi. \tag{32}$$

$$F(0, \begin{bmatrix} \xi_0 \\ \sigma_0 \end{bmatrix}, \eta(\cdot)) = \mathbb{E}(\int_0^a (\| \begin{bmatrix} \xi(t) \\ \sigma(t) \end{bmatrix} \|^2 + \|\eta(t)\|^2)dt + \|\xi(a)\|^2). \tag{33}$$

Here, α is an appropriate constant. Find η to minimize (33). This kind of optimal control problem can be classified as Problem 3 and solved.

It is obvious that E and M_1 satisfy the condition of Theorem 8. According to Theorem 8, (32) and (33) can be converted to the following state equation and cost functional, respectively:

$$d\xi(t) = (M_{11}\xi(t) + \eta(t))dt + (\xi(t) + \alpha\eta(t))dw(t) \tag{34}$$

with the cost functional

$$F(0, \xi_0, \eta(\cdot)) = \mathbb{E}(\int_0^a (\|\xi(t)\|^2 + 2\|\eta(t)\|^2)dt + \|\xi(a)\|^2). \tag{35}$$

It is obvious that Corollary 3 is applicable to this kind of optimal control problem.

Next, we will explain Problem 1 through the input–output problem in economics.

Example 3. *From [14], in input–output economics, many models were established to describe the real economics. The economics Leontief dynamic input–output model can be extended as an ordinary differential equation of the form:*

$$E\frac{d\xi(t)}{dt} = M_1(t)\xi(t) + N_1(t)\eta(t), t \in [s, a], \xi(s) = \xi_0 \tag{36}$$

in the Hilbert space $\mathbb{H}$, where $E \in L(\mathbb{H}, \mathbb{H})$ $M_1(t) : \mathrm{dom} M_1(t) \subseteq \mathbb{H} \to \mathbb{H}$ is a linear operator, and $N_1(t) \in P([s,a], L(\mathbb{H}, \mathbb{H}))$, while $\xi(t), \eta(t) \in \mathbb{H}$ for $t \ge s \ge 0$. However, in reality, there are many unpredicted parameters and different types of uncertainties that have not been implemented in the mathematical modelling process of this system. Nonetheless, we can consider a stochastic version of the generalized System (36) with the standard Wiener process $w(t)$ used to model the uncertainties of the form:

$$Ed\xi(t) = (M_1(t)\xi(t) + N_1(t)\eta(t))dt + (M_2(t)\xi(t) + N_2(t)\eta(t))dw(t),$$

$$t \in [s, a], \xi(s) = \xi_0 \tag{37}$$

This stochastic version of the input–output model is a linear stochastic generalized system in the Hilbert space $\mathbb{H}$ of the Form (1). The following quadratic cost functional is introduced:

$$F(s, \xi_0, \eta(\cdot)) = \mathbb{E}(\int_s^a (\|\xi(t)\|^2 + \|\eta(t)\|^2)dt + \|\xi(a)\|^2), \tag{38}$$

Find η to minimize (38). This kind of optimal control problem can be classified as Problem 1 and solved.

The unforced linear stochastic generalized system, i.e., $\eta(t) = 0$ in (37) is the linear homogeneous system of (37):

$$Ed\xi(t) = M_1(t)\xi(t) + M_2(t)\xi(t)dw(t),$$
$$t \in [s,a], \xi(s) = \xi_0. \tag{39}$$

The linear stochastic generalized System (39) is the form of System (3). In what follows, we will verify the effectiveness of Theorem 7.

If, for some concrete engineering practices, the following data are taken in (37):

$$E = \begin{bmatrix} I_1 & 0 \\ 0 & 0 \end{bmatrix}, M_1(t) = \begin{bmatrix} -tI_1 & 0 \\ 0 & t^2 I_2 \end{bmatrix}, N_1(t) = \begin{bmatrix} (t^2+1)I_1 \\ 0 \end{bmatrix},$$

$$M_2(t) = \begin{bmatrix} I_1 & 0 \\ 0 & (1+t^3)I_2 \end{bmatrix}, N_2(t) = \begin{bmatrix} (2t^2+1)I_1 \\ 0 \end{bmatrix},$$

where I_1, I_2 are identical operators in the Hilbert spaces H_1 and H_2, respectively. Systems (37) and (39) can be written as (40) and (41), respectively:

$$\begin{bmatrix} I_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} dx_1(t) \\ dx_2(t) \end{bmatrix} = \begin{bmatrix} -tI_1 & 0 \\ 0 & t^2 I_2 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}dt + \begin{bmatrix} (t^2+1)I_1 \\ 0 \end{bmatrix}\eta(t)dt$$
$$+\begin{bmatrix} I_1 & 0 \\ 0 & (1+t^3)I_2 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}dw(t)$$
$$+\begin{bmatrix} (2t^2+1)I_1 \\ 0 \end{bmatrix}\eta(t)dw(t), \tag{40}$$

where

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \xi(t) \in H_1 \oplus H_2 = H,$$

$$t \in [0,a], \xi(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}.$$

$$\begin{bmatrix} I_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} dx_1(t) \\ dx_2(t) \end{bmatrix} = \begin{bmatrix} -tI_1 & 0 \\ 0 & t^2 I_2 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}dt$$
$$+\begin{bmatrix} I_1 & 0 \\ 0 & (1+t^3)I_2 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}dw(t), \tag{41}$$

where $t \in [0,a]$, $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$. We can obtain that $\mathbb{D} = H_1$; the GE-evolution operator $T(t,s)$ induced by E with generator $M_1(t)$ is

$$T(t,s) = \begin{bmatrix} e^{-\frac{1}{2}(t^2-s^2)} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix};$$

and the stochastic GE-evolution operator $V(t,r)$ induced by E related to the linear homogeneous Equation (41) is

$$V(t,r) = \begin{bmatrix} e^{-\frac{1}{2}(t^2+t-r^2-r)+w(t)-w(r)} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix}.$$

It is obvious that Hypotheses 1–4 hold. Therefore, according to Theorem 7 we can obtain the optimal control, optimal state, and minimum of (38).

9. Conclusions

We have investigated the linear quadratic optimal control problem for linear stochastic generalized systems by using the GE-evolution operator in the sense of the mild solution in Hilbert spaces. Sufficient conditions have been proposed for the linear quadratic optimal control problem of the linear stochastic generalized systems. These results are very convenient and effective for judging the existence and uniqueness of the optimal control and for giving the state feedback expression of the optimal control. If System (1) is a nonlinear stochastic generalized system, the results of this paper need to be considered again. This is our next research goal.

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