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Almost Sure Exponential Stability of Numerical Solutions for Stochastic Pantograph Differential Equations with Poisson Jumps

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Abstract: The stability analysis of the numerical solutions of stochastic models has gained great interest, but there is not much research about the stability of stochastic pantograph differential equations. This paper deals with the almost sure exponential stability of numerical solutions for stochastic pantograph differential equations interspersed with the Poisson jumps by using the discrete semimartingale convergence theorem. It is shown that the Euler–Maruyama method can reproduce the almost sure exponential stability under the linear growth condition. It is also shown that the backward Euler method can reproduce the almost sure exponential stability of the exact solution under the polynomial growth condition and the one-sided Lipschitz condition. Additionally, numerical examples are performed to validate our theoretical result.

Keywords: stochastic pantograph differential equation with jumps; Poisson process; Euler–Maruyama method; backward Euler–Maruyama method almost sure exponential stability; Lipschitz condition; polynomial growth condition

MSC: 60H35; 60H10; 65C30

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1. Introduction

Stochastic differential equations (SDEs) have been widely used in a variety of fields, such as physics, chemistry, engineering, biology and mathematical finance, to describe models of dynamical systems affected by uncertain factors. In order to have more realistic simulations for random systems, it is more desirable and efficient to study SDEs with delay. SDEs with delay are named stochastic functional differential equations [1] and they act better than SDEs. Hobson and Rogers [2] gave a new non-constant volatility model with past dependency in finance. Arriojas et al. [3] assumed that the stock price follows a stochastic model with delay. Recently, stochastic models with variable delay have received intensive attention [4–8] and have been used in many applications in finance, biology, control and stochastic neural networks [9–12]. These types of models are called stochastic pantograph differential equations (SPDEs), and they have received great concern and have been used in different fields of science. The pantograph model was used by Ockendon and Tayler [13] to know how the electric current is gathered by the pantograph of an electric locomotive, from where it gets the name.

On the other hand, it is desirable to incorporate jumps into stochastic models for more realistic simulations and data fitting. Thus, jump models are important and play a vital role in describing a sudden change in the system [14,15]. It is often better to use jump–diffusion models when the stochastic systems are interspersed with some randomly occurring impulses to describe them [16–18]. It is also preferable to study SDEs with delay and jump [19,20]. In this paper, we deal with the stochastic pantograph differential model interspersed with Poisson jumps.

Most of the stochastic pantograph differential equations with jumps have difficulties in their analytical solutions; therefore, numerical schemes have to be used to solve them. There is much research that focuses on the convergence of these numerical methods. For example, Fan et al. [21] presented numerical algorithms for solving SPDEs via the Razumikhin technique. Fan et al. [8] applied Euler methods on SPDEs and proved the existence, uniqueness and convergence of these numerical schemes. Moreover, Li et al. [22] applied the Euler technique on SPDEs and proved the convergence of that scheme.

The stability analysis is another important factor in the numerical analysis. There are two common concepts, namely the mean square stability and asymptotic stability. Guo and Li [23] formulated the global mean square stability of the Euler–Maruyama method. Higham et al. [24–27] studied the stability of the numerical techniques for SDEs. Mao [28–30] studied the almost sure asymptotic stability of stochastic differential equations with and without delay based on the continuous semimartingale convergence theorem. Rodkina and Schurz [31] studied the almost sure asymptotic stability of numerical solutions for linear SDEs based on the discrete semimartingale convergence theorem. Recently, Wu et al. [32] studied the almost sure exponential stability of Euler-type techniques for the nonlinear stochastic delay differential equations based on using the semimartingale convergence theorem. Zhou et al. [33] investigated the exponential stability for stochastic functional differential equations using the polynomial growth condition. Zhou [33] studied the almost sure exponential stability of numerical solutions for SPDEs. This paper extends the previous work which was concerned with the almost sure exponential stability of SPDEs and discusses the almost sure exponential stability of numerical solutions for SPDEs interspersed with Poisson jumps with the help of the discrete semimartingale convergence theorem.

The structure of this paper is arranged as follows. Section 2 gives some important notations and discusses the global and almost sure exponential stability of the analytical solution. The almost sure exponential stability of the Euler–Maruyama method is presented in Section 3. Then, Section 4 discusses the almost sure exponential stability of the backward Euler method when imposing the one-sided Lipschitz condition. Numerical examples are given in Section 5 to validate our theoretical results. Finally, the conclusions are given in Section 6.

2. Almost Sure Exponential Stability of the Analytical Solution

Throughout this paper, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with filtration $\{\mathcal{F}_t\}_{t\geq 0}$, satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all P-null sets). Let $W(t)$ be a $d-$dimension Brownian motion defined on the probability space and $N(t)$ be a scalar Poisson process independent of $W(t)$ with parameter $\lambda > 0$ defined on the same probability space. Let $|\cdot|$ denote the Euclidean vector norm or Frobenius matrix norm and let $\langle x, y\rangle$ be the inner product of x, y in $\mathbb{R}^m$ and for $a \in \mathbb{R}$, $[a]$ denotes the integer part of a. $a \bigvee b$ represents $\max(a, b)$ and $a \bigwedge b$ represents $\min(a, b)$.

Consider the following $m-$dimensional stochastic pantograph differential equation interspersed with Poisson jumps of the form

$$\begin{aligned} dx(t) = f(x(t^-), x(qt^-))dt + g(x(t^-), x(qt^-))dW(t) \\ + h(x(t^-), x(qt^-))dN(t), \qquad t > 0, \end{aligned} \tag{1}$$

with initial data $x(0^-) = x_0$, where $0 < q < 1$, $x(t)$ is $m-$dimensional state process, $x(t^-) := lim_{s\to t^-} x(s)$, $x(qt^-) := lim_{s\to qt^-} x(s)$, $f : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$, $g : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^{m\times d}$ and $h : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ are Borel-measurable functions. Let the initial data x_0 be a bounded $\mathcal{F}_0-$measurable random variable and $E|x_0|^2 < \infty$ and $f(0, 0) = g(0, 0) = h(0, 0) = 0$ which indicate that Equation (1) has a trivial solution. An important thing in our analysis is the compensated Poisson process

$$\widetilde{N}(t) := N(t) - \lambda t, \tag{2}$$

which is considered as a martingale.

Proof. Let $s < t$, then we have

$$\begin{aligned}\mathbb{E}\left[\widetilde{N}_t|\mathcal{F}_s\right] &= \mathbb{E}[N_t - \lambda t|\mathcal{F}_s] \\ &= \mathbb{E}\left[\widetilde{N}_s + (N_t - N_s) - \lambda(t-s)|\mathcal{F}_s\right] \\ &= \widetilde{N}_s + \mathbb{E}[(N_t - N_s)] - \lambda(t-s) \\ &= \widetilde{N}_s + \lambda(t-s) - \lambda(t-s) = \widetilde{N}_s.\end{aligned}$$

It is also clear that $\widetilde{N}_t$ is a function of N_t and it is integrable as N_t has a Poisson distribution. Therefore, we conclude that (2) is a martingale. By defining

$$f_\lambda(x,y) = f(x,y) + \lambda h(x,y) \tag{3}$$

it can be easily seen that Equation (1) may be written in the form

$$\begin{aligned} dx(t) = f_\lambda(x(t^-), x(qt^-))dt + g(x(t^-), x(qt^-))dW(t) \\ + h(x(t^-), x(qt^-))d\widetilde{N}(t). \end{aligned} \tag{4}$$

□

Assumption 1. *The functions f, g and h satisfy the local Lipschitz condition, that is, for each integer $j \geq 1$, there exists a positive constant L_j such that*

$$|v(x_1,y_1) - v(x_2,y_2)|^2 \leq L_j(|x_1 - x_2|^2 + |y_1 - y_2|^2), \tag{5}$$

for all $t \geq 0$, $v = f, g$, or h, and $x_k, y_k \in \mathbb{R}^m$ with $|x_k| \vee |y_k| \leq j (k = 1, 2)$

Assumption 2. *The polynomial growth conditions. For all $x \in \mathbb{R}^m$, there exist positive constants $\alpha, \beta, \gamma, \delta, a, \bar{a}, \widetilde{a}, b, \bar{b}, \widetilde{b}, c, \bar{c}, \widetilde{c}$ such that*

$$\begin{aligned} &\langle x(s), f(x(s), x(qs))\rangle \\ &\quad \leq -a|x(s)|^{\alpha+2} + \bar{a}(|x(qs)|^{\beta+2} + |x(s)|^{\beta+2}) - \widetilde{a}|x(s)|^2, \end{aligned} \tag{6}$$

$$|g(x(s), x(qs))|^2 \leq b|x(s)|^{\gamma+2} + \bar{b}(|x(qs)|^{\gamma+2} + \widetilde{b}|x(s)|^2, \tag{7}$$

$$|h(x(s), x(qs))|^2 \leq c|x(s)|^{\delta+2} + \bar{c}(|x(qs)|^{\delta+2} + \widetilde{c}|x(s)|^2. \tag{8}$$

Theorem 1. *Let Assumptions 1 and 2 hold with $2\widetilde{a} > \widetilde{b} + \lambda(\widetilde{c} + 1)$, $2a > 2\bar{a}(1 + \frac{1}{q}) + b + \frac{\bar{b}}{q} + \lambda(c + \frac{\bar{c}}{q})$ and $\alpha \geq \beta \vee \gamma \vee \delta$. Then, for any initial data x_0, there almost surely exists unique global solution $x(t)$ to Equation (1) on $t \geq 0$.*

Proof. Under Assumption 1, applying the standing truncation technique to Equation (1) for any initial data x_0, there exists a unique maximal local strong solution $0 < t < \tau_e$, where τ_e is the explosion time. In order to show that the solution is global, it is only needed to show that $\tau_e = \infty$ a.s. Let n_0 be sufficiently large such that $n_0 > |x_0|$. For each integer $n \geq n_0$, define the stopping time

$$\tau_n = \inf\{t \in [0, \tau_e) : |x(t)| \geq n\}, \quad n \in N \tag{9}$$

where, throughout this paper, we set $\inf \phi = \infty$ (ϕ is the empty set). It is clear that $\tau_n, n \geq n_0$ is an increasing sequence; therefore, $\tau_n \to \tau_\infty \leq \tau_e(n \to \infty)a.s.$ If it can be shown that

$\tau_\infty = \infty$ *a.s.*, then $\tau_e = \infty$ *a.s.* which indicates that $x(t)$ is global. In other words, our target is to prove that $P(\tau_n \le t) \to 0 (n \to \infty, t > 0)$.

Define $V(x) = |x|^2$. Because of $P(\tau_n \le t)V(x(\tau_n)) \le \mathbb{E}V(x(t \wedge \tau_n))$, our objective will be to prove that $\mathbb{E}V(x(t \wedge \tau_n)) < +\infty$ because $V(x(\tau_n)) = |x(\tau_n)|^2 = n^2 \to \infty$. Applying the Itô formula [34], we obtain

$$\begin{aligned} V(x(t \wedge \tau_n)) &= V(x(0)) + 2\int_0^{t\wedge\tau_n} \langle x(s), f_\lambda(x(s), x(qs))\rangle ds \\ &\quad + \int_0^{t\wedge\tau_n} |g(x(s), x(qs))|^2 ds + M(t), \end{aligned} \tag{10}$$

where

$$\begin{aligned} M(t) &= \int_0^{t\wedge\tau_n} 2\langle x(s), g(x(s), x(qs))\rangle dW(s) \\ &\quad + \int_0^{t\wedge\tau_n} 2\langle x(s), h(x(s), x(qs))\rangle d\widetilde{N}(s) \\ &\quad + \int_0^{t\wedge\tau_n} |h(x(s), x(qs))|^2 d\widetilde{N}(s) \end{aligned} \tag{11}$$

is a local martingale with $M(0) = 0$. Using Assumption 2, we may compute

$$\begin{aligned} &2\langle x(s), f_\lambda(x(s), x(qs))\rangle + |g(x(s), x(qs))|^2 \\ &\quad = 2\langle x(s), f(x(s), x(qs))\rangle \\ &\qquad + 2\lambda\langle x(s), h(x(s), x(qs))\rangle \\ &\qquad + |g(x(s), x(qs))|^2, \end{aligned} \tag{12}$$

therefore,

$$\begin{aligned} &2\langle x(s), f_\lambda(x(s), x(qs))\rangle + |g(x(s), x(qs))|^2 \\ &\quad \le -2a|x(s)|^{\alpha+2} - 2\widetilde{a}|x(s)|^2 \\ &\qquad + 2\bar{a}(|x(qs)|^{\beta+2} + |x(s)|^{\beta+2}) \\ &\qquad + \lambda|x(s)|^2 + \lambda c|x(s)|^{\delta+2} \\ &\qquad + \lambda\bar{c}|x(s)|^{\delta+2} + \lambda\widetilde{c}|x(s)|^2 \\ &\qquad + b|x(s)|^{\gamma+2} + \lambda\bar{b}|x(s)|^{\delta+2} \\ &\qquad + \lambda\widetilde{b}|x(s)|^2, \end{aligned} \tag{13}$$

which equals to

$$\begin{aligned} &2\langle x(s), f_\lambda(x(s), x(qs))\rangle + |g(x(s), x(qs))|^2 \\ &\quad \le -2a|x(s)|^{\alpha+2} \\ &\qquad + \frac{2\bar{a}}{q}(q|x(qs)|^{\beta+2} - |x(s)|^{\beta+2}) \\ &\qquad + \frac{\lambda\bar{c}}{q}(q|x(qs)|^{\delta+2} - |x(s)|^{\delta+2}) \\ &\qquad + \frac{\bar{b}}{q}(q|x(qs)|^{\gamma+2} - |x(s)|^{\gamma+2}) \\ &\qquad - (2\widetilde{a} - \widetilde{b} - \lambda(\widetilde{c} + 1))|x(s)|^2 \\ &\qquad + 2\bar{a}(1 + \frac{1}{q})|x(s)|^{\beta+2} \\ &\qquad + (b + \frac{\bar{b}}{q})|x(s)|^{\gamma+2} \\ &\qquad + \lambda(c + \frac{\bar{c}}{q})|x(s)|^{\delta+2}. \end{aligned} \tag{14}$$

Let

$$I(x(s)) = (2\widetilde{a} - \widetilde{b} - \lambda(\widetilde{c}+1))|x(s)|^2 - 2\bar{a}(1+\frac{1}{q})|x(s)|^{\beta+2} + 2a|x(s)|^{\alpha+2} - (b+\frac{\bar{b}}{q})|x(s)|^{\gamma+2} - \lambda(c+\frac{\bar{c}}{q})|x(s)|^{\delta+2} \tag{15}$$

Recall that $2\widetilde{a} > \widetilde{b} + \lambda(\widetilde{c}+1)$, $2a > 2\bar{a}(1+\frac{1}{q}) + b + \frac{\bar{b}}{q} + \lambda(c+\frac{\bar{c}}{q})$, $\alpha \geq \beta \vee \gamma \vee \delta$. By Lemma 1, in [33], there exists a positive constant ξ_0 such that $I(x(s)) \geq \xi_0|x(s)|^2$. Substituting (14) and (15) into (10) yields

$$\begin{aligned} V(x(t\wedge\tau_n)) \leq V(x(0)) &+ \frac{2\bar{a}}{q}\int_0^{t\wedge\tau_n}(q|x(qs)|^{\beta+2} - |x(s)|^{\beta+2})ds \\ &+ \frac{\bar{b}}{q}\int_0^{t\wedge\tau_n}(q|X(qs)|^{\gamma+2} - |x(s)|^{\gamma+2})ds \\ &+ \frac{\lambda\bar{c}}{q}\int_0^{t\wedge\tau_n}(q|x(qs)|^{\delta+2} - |x(s)|^{\delta+2})ds \\ &- \xi_0\int_0^{t\wedge\tau_n}|x(s)|^2 ds + M(t). \end{aligned} \tag{16}$$

Using the property of integral, we may estimate

$$\int_0^t (q|x(qs)|^{\beta+2} - |x(s)|^{\beta+2})ds = \int_0^t q|x(qs)|^{\beta+2}ds - \int_0^t |x(s)|^{\beta+2}ds. \tag{17}$$

Letting $z = qs$ and $ds = (1/q)dz$ in the first integral of the right hand side of (17) lead to

$$\int_0^t (q|x(qs)|^{\beta+2} - |x(s)|^{\beta+2})ds = \int_0^{qt} |x(z)|^{\beta+2}dz - \int_0^t |x(s)|^{\beta+2}ds. \tag{18}$$

By making change of variable for the dummy variable z in the first integral of right hand side of (18) and making it equal to s, we obtain the following

$$\begin{aligned} \int_0^t (q|x(qs)|^{\beta+2} - |x(s)|^{\beta+2})ds &= \int_0^{qt} |x(s)|^{\beta+2}ds - \int_0^t |x(s)|^{\beta+2}ds \\ &= -\int_{qt}^t |x(s)|^{\beta+2}ds \end{aligned} \tag{19}$$

By the same analogy, we obtain

$$\int_0^t (q|x(qs)|^{\gamma+2} - |x(s)|^{\gamma+2})ds \leq -\int_{qt}^t |x(s)|^{\gamma+2}ds, \tag{20}$$

and

$$\int_0^t (q|x(qs)|^{\delta+2} - |x(s)|^{\delta+2})ds \leq -\int_{qt}^t |x(s)|^{\delta+2}ds. \tag{21}$$

By plugging (19)–(21) into (16) and taking expectation, we obtain

$$\begin{aligned} \mathbb{E}V(x(t\wedge\tau_n)) \leq V(x(0)) &- \frac{2\bar{a}}{q}\mathbb{E}\int_{q(t\wedge\tau_n)}^{t\wedge\tau_n}|x(s)|^{\beta+2}ds \\ &- \frac{\bar{b}}{q}\mathbb{E}\int_{q(t\wedge\tau_n)}^{t\wedge\tau_n}|x(s)|^{\gamma+2}ds \\ &- \frac{\lambda\bar{c}}{q}\mathbb{E}\int_{q(t\wedge\nu_k)}^{t\wedge\tau_n}|x(s)|^{\delta+2}ds \\ &- \xi_0\mathbb{E}\int_0^{t\wedge\tau_n}|x(s)|^2 ds, \end{aligned} \tag{22}$$

which indicates that there exists a positive constant D such that $\mathbb{E}V(x(t \wedge \tau_n)) \leq D$. As mentioned before, $P(\tau_n \leq t)V(x(\tau_n)) \leq \mathbb{E}V(x(t \wedge \tau_n))$ and $V(x(\tau_n)) = |x(\tau_n)|^2 = n^2$, allowing $n \to \infty$ leads to

$$\limsup_{n\to\infty} P(\tau_n \leq t) = 0. \tag{23}$$

This indicates that Equation (1) has a unique global solution. □

Assumption 3. *The polynomial growth conditions. For all $x \in \mathbb{R}^m$, there exist positive constants $\alpha, \beta, \gamma, \delta, a, \bar{a}, \widetilde{a}, b, \bar{b}, \widetilde{b}, c, \bar{c}, \widetilde{c}$ such that*

$$\begin{aligned}&\langle x(s), f(x(s), x(qs))\rangle \\ &\quad \leq -a|x(s)|^{\alpha+2} + \bar{a}(e^{-(1-q)\varepsilon s}|x(qs)|^{\beta+2} + |x(s)|^{\beta+2}) - \widetilde{a}|x(s)|^2,\end{aligned} \tag{24}$$

$$|g(x(s), x(qs))|^2 \leq b|x(s)|^{\gamma+2} + \bar{b}e^{-(1-q)\varepsilon s}|x(qs)|^{\gamma+2} + \widetilde{b}|x(s)|^2, \tag{25}$$

$$|h(x(s), x(qs))|^2 \leq c|x(s)|^{\delta+2} + \bar{c}e^{-(1-q)\varepsilon s}|x(qs)|^{\delta+2} + \widetilde{c}|x(s)|^2. \tag{26}$$

Theorem 2. *Let Assumptions 1–3 hold with $2\widetilde{a} > \widetilde{b} + \lambda(\widetilde{c}+1)$, $2a > 2\bar{a}(1+\frac{1}{q}) + b + \frac{\bar{b}}{q} + \lambda(c + \frac{\bar{c}}{q})$ and $\alpha \geq \beta \vee \gamma \vee \delta$. Then, for any initial data x_0, the solution $x(t)$ to Equation (1) is almost sure exponentially stable, that is,*

$$\limsup_{t\to\infty} \frac{1}{t}\log|x(t)| \leq -\frac{\varepsilon}{2}, \tag{27}$$

where $\varepsilon \leq 2\widetilde{a} - \widetilde{b} - \lambda(\widetilde{c}+1)$.

Proof. Let $V(x) = |x|^2$ and for any $\varepsilon > 0$, we obtain the following by applying the Itô formula

$$\begin{aligned}e^{\varepsilon t}V(x(t)) = V(x(0)) + \int_0^t e^{\varepsilon s}(&\varepsilon V(x(s)) + 2\langle x(s), f_\lambda(x(s), x(qs))\rangle \\ &+ |g(x(s), x(qs))|^2)ds + \bar{M}(t),\end{aligned} \tag{28}$$

where

$$\begin{aligned}\bar{M}(t) = &\int_0^t 2e^{\varepsilon s}\langle x(s), g(x(s), x(qs))\rangle dW(s) \\ &+ \int_0^t 2e^{\varepsilon s}\langle x(s), h(x(s), x(qs))\rangle d\widetilde{N}(s) \\ &+ \int_0^t e^{\varepsilon s}|h(x(s), x(qs))|^2 d\widetilde{N}(s)\end{aligned} \tag{29}$$

is a local martingale with $\bar{M}(0) = 0$. Using Assumption 3, we obtain

$$\begin{aligned}\varepsilon V(x(s)) + &2\langle x(s), f_\lambda(x(s), x(qs))\rangle + |g(x(s), x(qs))|^2 \\ &\leq \varepsilon|x(s)|^2 - 2a|x(s)|^{\alpha+2} - 2\widetilde{a}|x(s)|^2 \\ &\quad + 2\bar{a}(e^{-(1-q)\varepsilon s}|x(qs)|^{\beta+2} + |x(s)|^{\beta+2}) \\ &\quad + \lambda|x(s)|^2 + \lambda c|x(s)|^{\delta+2} + \lambda\bar{c}e^{-(1-q)\varepsilon s}|x(qs)|^{\delta+2} \\ &\quad + \lambda\widetilde{c}|x(s)|^2 + b|x(s)|^{\gamma+2} + \bar{b}e^{-(1-q)\varepsilon s}|x(qs)|^{\delta+2} \\ &\quad + \widetilde{b}|x(s)|^2,\end{aligned} \tag{30}$$

which equals to

$$
\begin{aligned}
\varepsilon V(x(s)) &+ 2\langle x(s), f_\lambda(x(s), x(qs))\rangle + |g(x(s), x(qs))|^2 \\
&\leq \frac{2\bar{a}}{q}(qe^{-(1-q)\varepsilon s}|x(qs)|^{\beta+2} - |x(s)|^{\beta+2}) \\
&\quad + \frac{2\bar{a}}{q}(qe^{-(1-q)\varepsilon s}|x(qs)|^{\beta+2} - |x(s)|^{\beta+2}) \\
&\quad + \frac{\bar{b}}{q}(qe^{-(1-q)\varepsilon s}|x(qs)|^{\gamma+2} - |x(s)|^{\gamma+2}) \\
&\quad + \frac{\lambda\bar{c}}{q}(qe^{-(1-q)\varepsilon s}|x(qs)|^{\delta+2} - |x(s)|^{\delta+2}) - 2a|x(s)|^{\alpha+2} \\
&\quad - (2\widetilde{a} - \widetilde{b} - \lambda(\widetilde{c}+1) - \varepsilon)|x(s)|^2 + 2\bar{a}(1+\frac{1}{q})|x(s)|^{\beta+2} \\
&\quad + (b+\frac{\bar{b}}{q})|x(s)|^{\gamma+2} + \lambda(c+\frac{\bar{c}}{q})|x(s)|^{\delta+2}.
\end{aligned}
\tag{31}
$$

Let

$$
\begin{aligned}
I(x(s)) = (2\widetilde{a} - \widetilde{b} - \lambda(\widetilde{c}+1) - \varepsilon)|x(s)|^2 - 2\bar{a}(1+\frac{1}{q})|x(s)|^{\beta+2} \\
+ 2a|x(s)|^{\alpha+2} - (b+\frac{\bar{b}}{q})|x(s)|^{\gamma+2} - \lambda(c+\frac{\bar{c}}{q})|x(s)|^{\delta+2}
\end{aligned}.
\tag{32}
$$

Recall that $2\widetilde{a} > \widetilde{b} + \lambda(\widetilde{c}+1)$, $2a > 2\bar{a}(1+\frac{1}{q}) + b + \frac{\bar{b}}{q} + \lambda(c+\frac{\bar{c}}{q})$, $\varepsilon < 2\widetilde{a} - \widetilde{b} - \lambda(\widetilde{c}+1)$, $\alpha \geq \beta \vee \gamma \vee \delta$. By Lemma 1, in [33], there exists a positive constant $\bar{\xi}_0$ such that $I(x(s)) \geq \bar{\xi}_0|x(s)|^2$. Substituting (31) and (32) into (28) yields

$$
\begin{aligned}
e^{\varepsilon t}V(x(t)) \leq V(x(0)) \\
&+ \frac{2\bar{a}}{q}\int_0^t e^{\varepsilon s}(qe^{-(1-q)\varepsilon s}|x(qs)|^{\beta+2} - |x(s)|^{\beta+2})ds \\
&+ \frac{\bar{b}}{q}\int_0^t e^{\varepsilon s}(qe^{-(1-q)\varepsilon s}|x(qs)|^{\gamma+2} - |x(s)|^{\gamma+2})ds \\
&+ \frac{\lambda\bar{c}}{q}\int_0^t e^{\varepsilon s}(qe^{-(1-q)\varepsilon s}|x(qs)|^{\delta+2} - |x(s)|^{\delta+2})ds \\
&- \bar{\xi}_0\int_0^t e^{\varepsilon s}|X(s)|^2 ds + \bar{M}(t).
\end{aligned}
\tag{33}
$$

Using the property of the integral, the following is obtained

$$
\begin{aligned}
\int_0^t e^{\varepsilon s}(qe^{-(1-q)\varepsilon s}|x(qs)|^{\beta+2} - |x(s)|^{\beta+2})ds \\
&\leq \int_0^{qt} e^{\varepsilon s}|x(s)|^{\beta+2}ds - \int_0^t |x(s)|^{\beta+2}ds \\
&= -\int_{qt}^t e^{\varepsilon s}|x(s)|^{\beta+2}ds,
\end{aligned}
\tag{34}
$$

by the same analogy, the following are obtained

$$
\int_0^t e^{\varepsilon s}(qe^{-(1-q)\varepsilon s}|x(qs)|^{\gamma+2} - |x(s)|^{\gamma+2})ds \leq -\int_{qt}^t e^{\varepsilon s}|x(s)|^{\gamma+2}ds,
\tag{35}
$$

and

$$
\int_0^t e^{\varepsilon s}(qe^{-(1-q)\varepsilon s}|x(qs)|^{\delta+2} - |x(s)|^{\delta+2})ds \leq -\int_{qt}^t e^{\varepsilon s}|x(s)|^{\delta+2}ds.
\tag{36}
$$

By plugging (34)–(36) into (33), we obtain

$$\begin{aligned} e^{\varepsilon t}V(x(t)) \leq V(x(0)) &- \frac{2\bar{a}}{q}\int_{qt}^{t} e^{\varepsilon s}|x(s)|^{\beta+2}ds \\ &- \frac{\bar{b}}{q}\int_{qt}^{t} e^{\varepsilon s}|x(s)|^{\gamma+2}ds - \frac{\lambda\bar{c}}{q}\int_{qt}^{t} e^{\varepsilon s}|x(s)|^{\delta+2}ds \\ &- \bar{\xi}_0\int_{0}^{t} e^{\varepsilon s}|x(s)|^{2}ds + \bar{M}(t) \end{aligned} \tag{37}$$

Applying the nonnegative semimartingale convergence theorem [33], we obtain

$$\limsup_{t\to\infty} e^{\varepsilon t}V(X(t)) < \infty \quad a.s. \tag{38}$$

That is, there exists a finite positive random variable C_0 such that

$$\sup_{0\leq t<\infty} e^{\varepsilon t}V(xt) \leq C_0 \quad a.s. \tag{39}$$

Which implies

$$\sup_{0\leq t<\infty} \frac{1}{t}\log|x(t)| \leq -\frac{\varepsilon}{2} \quad a.s. \tag{40}$$

The proof is completed. □

Remark 1.

1. *The solution $x(t)$ to Equation (1) is said to be almost sure exponential stable if there exists a constant $c > 0$ such that*

$$\limsup_{t\to\infty} \frac{1}{t}\log|x(t)| \leq -c \quad a.s.$$

for any initial data x_0.

2. *The solution $x(t)$ to Equation (1) is said to be mean square stable if for every $\epsilon > 0$ there exists a constant $c > 0$ such that*

$$\sup_{t_0\leq t<\infty} \mathbb{E}\|x(t)\|^2 \leq \epsilon$$

for any initial data x_0 such that $\| x_0 \|\leq c$.

3. Almost Sure Stability of Euler–Maruyama Method

For a given step-size $\Delta t \in (0,1)$, the Euler–Maruyama method for (4) is defined as follows

$$\begin{aligned} X_{n+1} = X_n + f_\lambda(X_n, X_{[qn]})\Delta t + g(X_n, X_{[qn]})\Delta W_n \\ + h(X_n, X_{[qn]})\Delta\widetilde{N}_n, \quad n = 0,1,2,... \end{aligned} \tag{41}$$

where X_n is an approximation value of $x(t_n)$, $t_n = n\Delta t$, $0 < q < 1$ and $X_0 = x_0$. $\Delta W_n = W(t_{n+1}) - W(t_n)$ represents the Brownian motion increments and $\Delta\widetilde{N}_n = \widetilde{N}_{n+1} - \widetilde{N}_n$ represents the increments of the compensated Poisson process. The delay argument may not hit the previous time step which appears in the numerical method while dealing with the pantograph delay. This problem is tackled by interpolating the unknown approximate values of the solution to the closet grid point on the left endpoint of the interval containing the delay argument using piecewise constant polynomials.

Assumption 4. *The Linear Growth Conditions. For any $x \in \mathbb{R}^m$, there exist positive constants $a, \bar{a}, b, \bar{b}, c, \bar{c}, d, \bar{d}$ such that*

$$\langle x(s), f(x(s), x(qs))\rangle \leq -a|x(s)|^2 + \bar{a}e^{-(1-q)\varepsilon s}|x(qs)|^2, \tag{42}$$

$$|f(x(s), x(qs))|^2 \leq c|x(s)|^2 + \bar{c}e^{-(1-q)\varepsilon s}|x(qs)|^2 \tag{43}$$

$$|g(x(s), X(qs))|^2 \leq b|x(s)|^2 + \bar{b}e^{-(1-q)\varepsilon s}|x(qs)|^2 \tag{44}$$

$$|h(x(s), x(qs))|^2 \leq d|x(s)|^2 + \bar{d}e^{-(1-q)\varepsilon s}|x(qs)|^2 \tag{45}$$

Theorem 3. *Let Assumption 4 hold. Then, for any given $\varepsilon > 0$, there exists a small $\Delta t^* \in (0,1)$ such that if $\Delta t < \Delta t^*$, then the approximate solution $\{X_n\}$ defined by (41) has the property*

$$\limsup_{n\to\infty} \frac{1}{n\Delta t} \log|X_n| \leq -\frac{\varepsilon}{2} \quad a.s. \tag{46}$$

Proof. Using Assumption 4 and Euler–Maruyama technique (41), we may calculate

$$\begin{aligned}|X_{n+1}|^2 = {} & |X_n|^2 + 2\langle X_n, f_\lambda(X_n, X_{[qn]})\rangle\Delta t + |f_\lambda(X_n, X_{[qn])}|^2\Delta t^2 \\ & + |g(X_n, X_{[qn]})\Delta W_n|^2 + |h(X_n, X_{[qn]})\Delta\widetilde{N}_n|^2 \\ & + 2\langle X_n + f_\lambda(X_n, X_{[qn]}, g(X_n, X_{[qn]})\Delta W_n)\rangle \\ & + 2\langle X_n + f_\lambda(X_n, X_{[qn]}, h(X_n, X_{[qn]})\Delta\widetilde{N}_n)\rangle \\ & + 2\langle g(X_n, X_{[qn]})\Delta W_n, h(X_n, X_{[qn]})\Delta\widetilde{N}_n\rangle,\end{aligned} \tag{47}$$

$$\begin{aligned}|X_{n+1}|^2 \leq {} & |X_n|^2 + 2\langle X_n, f(X_n, X_{[qn]})\rangle\Delta t + \lambda\Delta t|h(X_n, X_{[qn]})|^2 \\ & + \lambda\Delta t|X_n|^2 + 2\Delta t^2|f(X_n, X_{[qn]})|^2 + 2(\lambda\Delta t)^2|h(X_n, X_{[qn]})|^2 \\ & + |g(X_n, X_{[qn]})|^2\Delta t + |g(X_n, X_{[qn]})|^2(\Delta W_n^2 - \Delta t) \\ & + |h(X_n, X_{[qn]})|^2\lambda\Delta t + |h(X_n, X_{[qn]})|^2(\widetilde{N}_n^2 - \lambda\Delta t) \\ & + 2\langle X_n + f_\lambda(X_n, X_{[qn]}, g(X_n, X_{[qn]})\Delta W_n)\rangle \\ & + 2\langle X_n + f_\lambda(X_n, X_{[qn]}, h(X_n, X_{[qn]})\Delta\widetilde{N}_n)\rangle \\ & + 2\langle g(X_n, X_{[qn]})\Delta W_n, h(X_n, X_{[qn]})\Delta\widetilde{N}_n\rangle,\end{aligned} \tag{48}$$

$$\begin{aligned}|X_{n+1}|^2 \leq {} & (1 - 2a\Delta t + b\Delta t + 2c\Delta t^2 + \lambda\Delta t(1 + 2d(1 + \lambda\Delta t))|X_n|^2 \\ & + (2\bar{a} + 2\bar{c}\Delta t + \bar{b} + 2\lambda\bar{d}(1 + \lambda\Delta t))e^{-(1-q)\varepsilon n\Delta t}|X_{[qn]}|^2\Delta t \\ & + S_n,\end{aligned} \tag{49}$$

where

$$\begin{aligned}S_n = {} & |g(X_n, X_{[qn]})|^2(\Delta W_n^2 - \Delta t) + |h(X_n, X_{[qn]})|^2(\widetilde{N}_n^2 - \lambda\Delta t) \\ & + 2\langle X_n + f_\lambda(X_n, X_{[qn]}, g(X_n, X_{[qn]})\Delta W_n)\rangle \\ & + 2\langle X_n + f_\lambda(X_n, X_{[qn]}, h(X_n, X_{[qn]})\Delta\widetilde{N}_n)\rangle \\ & + 2\langle g(X_n, X_{[qn]})\Delta W_n, h(X_n, X_{[qn]})\Delta\widetilde{N}_n\rangle,\end{aligned} \tag{50}$$

after obtaining (50), it is easy to have

$$\begin{aligned}e^{\varepsilon(n+1)\Delta t}|X_{n+1}|^2 - e^{\varepsilon n\Delta t}|X_n|^2 \leq {} & (A - e^{-\varepsilon\Delta t})e^{\varepsilon(n+1)\Delta t}|X_n|^2 \\ & + Be^{\varepsilon(qn+1)\Delta t}|X_{[qn]}|^2\Delta t \\ & + e^{\varepsilon(n+1)\Delta t}S_n,\end{aligned} \tag{51}$$

where

$$A = 1 - 2a\Delta t + b\Delta t + 2c\Delta t^2 + \lambda\Delta t(1 + 2d(\lambda\Delta t)), \tag{52}$$

and

$$B = 2\bar{a} + 2\bar{c}\Delta t + \bar{b} + 2\lambda\bar{d}(1 + \lambda\Delta t). \quad (53)$$

By applying the recursive method, it is easy to obtain

$$\begin{aligned} e^{\varepsilon n\Delta t}|X_n|^2 \le {} & |X_0|^2 + (A - e^{-\varepsilon\Delta t})\sum_{i=0}^{n-1} e^{\varepsilon(i+1)\Delta t}|X_i|^2 \\ & + B\sum_{i=0}^{n-1} e^{\varepsilon(i+1)\Delta t}(e^{-(1-q)\varepsilon i\Delta t}|X_{[qi]}|^2 - e^{\varepsilon\Delta t}|X_i|^2)\Delta t \\ & + Be^{\varepsilon\Delta t}\sum_{i=0}^{n-1} e^{\varepsilon(i+1)\Delta t}|X_i|^2\Delta t + \sum_{i=0}^{n-1} e^{\varepsilon(i+1)\Delta t}S_i \end{aligned} \quad (54)$$

$$\begin{aligned} e^{\varepsilon n\Delta t}|X_n|^2 \le {} & |X_0|^2 - (\frac{e^{-\varepsilon\Delta t} - A}{\Delta t} - Be^{\varepsilon\Delta t})\sum_{i=0}^{n-1} e^{\varepsilon(i+1)\Delta t}|X_i|^2\Delta t \\ & + B\sum_{i=0}^{n-1} e^{\varepsilon(i+1)\Delta t}(e^{-(1-q)\varepsilon i\Delta t}|X_{[qi]}|^2 - e^{\varepsilon\Delta t}|X_i|^2)\Delta t \\ & + \sum_{i=0}^{n-1} e^{\varepsilon(i+1)\Delta t}S_i, \end{aligned} \quad (55)$$

where $\sum_{i=0}^{n-1} e^{\varepsilon(i+1)\Delta t}S_i$ is a martingale. Assume that $[qi] = j$; then $j \le qi < j+1$; therefore, $qi - 1 < j \le qi$. If $0 \le i \le n-1$, then $-1 < j \le q(n-1) \le [qn] + 1 - q \le [qn] + 1$. This leads to

$$\begin{aligned} & \sum_{i=0}^{n-1} e^{\varepsilon(i+1)\Delta t}(e^{-\varepsilon(1-q)i\Delta t}|X_{[qi]}|^2 - e^{\varepsilon\Delta t}|X_i|^2) \\ & = \sum_{i=0}^{n-1} e^{\varepsilon(1+qi)\Delta t}|X_{[qi]}|^2 - \sum_{i=0}^{n-1} e^{\varepsilon(i+2)\Delta t}|X_i|^2 \\ & = \sum_{i=0}^{[qn]+1} e^{\varepsilon(i+2)\Delta t}|X_i|^2 - \sum_{i=0}^{n-1} e^{\varepsilon(i+2)\Delta t}|X_i|^2 \\ & \le -\sum_{i=[qn]+2}^{n-1} e^{\varepsilon(i+2)\Delta t}|X_i|^2. \end{aligned} \quad (56)$$

Let

$$f(\Delta t) = 2a - b\Delta t - 2c\Delta t - \lambda(1 + 2d(\lambda\Delta t)) + \frac{e^{-\varepsilon\Delta t} - 1}{\Delta t} - Be^{\varepsilon\Delta t}. \quad (57)$$

Using the Taylor series, we obtain

$$e^{-\varepsilon\Delta t} = 1 - \varepsilon\Delta + \frac{(\varepsilon\Delta t)^2}{2} - \frac{(\varepsilon\Delta t)^3}{3!} + \cdots > 1 - \varepsilon\Delta t,$$

which leads to

$$\frac{e^{-\varepsilon\Delta t} - 1}{\Delta t} > -\varepsilon. \quad (58)$$

Thus,

$$f(\Delta t) > 2a - b\Delta t - 2c\Delta t - \lambda(1 + 2d(\lambda\Delta t)) - \varepsilon - Be^{\varepsilon\Delta t}. \quad (59)$$

For a given ε, pick up a very small Δt^* such that for all $\Delta t < \Delta t^*$,

$$2a - b\Delta t - 2c\Delta t - \lambda(1 + 2d(\lambda\Delta t)) - \varepsilon - Be^{\varepsilon\Delta t} > 0. \quad (60)$$

After plugging (56) and (60) into (55), the discrete semimartingale theorem which was stated in [33] implies that there exists a positive constant C_0 such that

$$e^{\varepsilon n\Delta t}|X_n|^2 \le C_0. \quad (61)$$

Which implies

$$\limsup_{n\to\infty} \frac{1}{n\Delta t} \log |X_n| \leq -\frac{\varepsilon}{2} \quad a.s. \tag{62}$$

The proof is completed. □

4. Almost Sure Stability of Backward Euler–Maruyama Method

In this section, it will be shown that the backward Euler–Maruyama technique can reproduce the almost sure exponential stability of the exact solution of SPDE interspersed with Poisson jumps.

Assumption 5. *The Polynomial Growth Conditions. For any $x \in \mathbb{R}^m$, there exist positive constants α, $a, \bar{a}, \widetilde{a}, b, \bar{b}, \widetilde{b}, c, \bar{c}, \widetilde{c}$ such that*

$$\begin{aligned}&\langle x(s), f(x(s), x(qs))\rangle \\ &\qquad \leq -a|x(s)|^{\alpha+2} + \bar{a}e^{-(1-q)\varepsilon s}|x(qs)|^{\alpha+2} - \widetilde{a}|x(s)|^2,\end{aligned} \tag{63}$$

$$|g(x(s), x(qs))|^2 \leq b|x(s)|^{\alpha+2} + \bar{b}e^{-(1-q)\varepsilon s}|x(qs)|^{\alpha+2} + \widetilde{b}|x(s)|^2, \tag{64}$$

$$|h(x(s), x(qs))|^2 \leq c|x(s)|^{\alpha+2} + \bar{c}e^{-(1-q)\varepsilon s}|x(qs)|^{\alpha+2} + \widetilde{c}|x(s)|^2. \tag{65}$$

Given a step-size $\Delta t \in (0,1)$ and for $t \in [0,T]$, let $N\Delta t = T$ for some positive integer N and $t_n = n\Delta t$ $(n \geq 0)$. Then, the backward Euler–Maruyama technique is defined as follows

$$\begin{aligned}X_{n+1} = X_n &+ f(X_{n+1}, X_{[q(n+1)]})\Delta t + g(X_n, X_{[qn]})\Delta W_n \\ &+ h(X_n, X_{[qn]})\Delta N_n.\end{aligned} \tag{66}$$

To ensure that this scheme is well-defined, the following one-sided Lipschitz condition is imposed on the drift coefficient $f(x,y)$ in x.

Assumption 6. *One-sided Lipschitz condition. There exists a constant ζ such that for any $x_1, x_2, y \in \mathbb{R}^m$ and $t \geq 0$*

$$\langle x_1 - x_2, f(x_1, y) - f(x_2, y)\rangle \leq \zeta|x_1 - x_2|^2. \tag{67}$$

Under this condition, if $\zeta\Delta t < 1$, then the backward Euler scheme (66) is well-defined (see, e.g., [35]). The following theorem shows the almost sure exponential stability of the backward Euler scheme.

Theorem 4. *Let Assumptions 5 and 6 hold. Then, there exists a small $\Delta t^* \in (0,1)$ such that if $\Delta t < \Delta t^*$, then the approximate solution $\{X_n\}$ defined by (66) has the property*

$$\limsup_{n\to\infty} \frac{1}{n\Delta t} \log |X_n| \leq -\frac{\varepsilon}{2} \quad a.s. \tag{68}$$

where $\varepsilon < 2\widetilde{a} - m\widetilde{b} - \lambda m\widetilde{c}$.

Proof. Using Assumption 5 and Equation (66), we may calculate

$$\begin{aligned}|X_{n+1}|^2 = \langle X_{n+1}, X_n &+ f(X_{n+1}, X_{[q(n+1)]})\Delta t + g(X_n, X_{[qn]})\Delta W_n \\ &+ h(X_n, X_{[qn]})\Delta\widetilde{N}_n + \lambda h(X_n, X_{[qn]})\Delta t\rangle,\end{aligned} \tag{69}$$

which equals to

$$\begin{aligned}|X_{n+1}|^2 = &\langle X_{n+1}, f(X_{n+1}, X_{[q(n+1)]})\rangle\Delta t \\ &+ \langle X_{n+1}, X_n + g(X_n, X_{[qn]})\Delta W_n \\ &+ h(X_n, X_{[qn]})\Delta\widetilde{N}_n + \lambda h(X_n, X_{[qn]})\Delta t\rangle,\end{aligned} \tag{70}$$

then

$$
\begin{aligned}
|X_{n+1}|^2 \leq\ & (-a|X_{n+1}|^{\alpha+2} + \bar{a}e^{-\varepsilon(1-q)(n+1)}|X_{[q(n+1)]}|^{\alpha+2} - \widetilde{a}|X_{n+1}|^2)\Delta t \\
& + \frac{1}{2}|X_{n+1}|^2 + |X_n|^2 + \frac{1}{2}(b|X_n|^{\alpha+2} + \bar{b}e^{-\varepsilon(1-q)n\Delta t}|X_{[qn]}|^{\alpha+2} \\
& + \widetilde{b}|X_n|^2)|\Delta W_n|^2 + \frac{1}{2}(c|X_n|^{\alpha+2} + \bar{c}e^{-\varepsilon(1-q)n\Delta t}|X_{[qn]}|^{\alpha+2} \\
& + \widetilde{c}|X_n|^2)|\Delta \widetilde{N}_n|^2 + (c|X_n|^{\alpha+2} + \bar{c}e^{-\varepsilon(1-q)n\Delta t}|X_{[qn]}|^{\alpha+2} \\
& + \widetilde{c}|X_n|^2)|\lambda\Delta t|^2 + \langle X_n, g(X_n, X_{[qn]})\rangle\Delta W_n \\
& + \langle g(X_n, X_{[qn]})\Delta W_n, h(X_n, X_{[qn]})\Delta\widetilde{N}_n\rangle \\
& + \langle g(X_n, X_{[qn]})\Delta W_n, \lambda h(X_n, X_{[qn]})\rangle\Delta t \\
& + \langle h(X_n, X_{[qn]})\Delta\widetilde{N}, \lambda h(X_n, X_{[qn]})\rangle\Delta t \\
& + \langle X_n, h(X_n, X_{[qn]})\rangle\Delta\widetilde{N}_n.
\end{aligned} \tag{71}
$$

This leads to the following

$$
\begin{aligned}
(1 + 2\widetilde{a}\Delta t)|X_{n+1}|^2 \leq\ & (2 + m\widetilde{b}\Delta t + \lambda\widetilde{c}(m + 2\lambda\Delta t)\Delta t)|X_n|^2 \\
& - 2(a|X_{n+1}|^{\alpha+2} - \bar{a}e^{-\varepsilon(1-q)n\Delta t}|X_{[q(n+1)]}|^{\alpha+2})\Delta t \\
& + (mb|X_n|^{\alpha+2} + m\bar{b}e^{-\varepsilon(1-q)n\Delta t}|X_{[qn]}|^{\alpha+2})\Delta t \\
& + \lambda(mc|X_n|^{\alpha+2} + m\bar{c}e^{-\varepsilon(1-q)n\Delta t}|X_{[qn]}|^{\alpha+2})\Delta t \\
& + 2\lambda^2(c|X_n|^{\alpha+2} + \bar{c}e^{-\varepsilon(1-q)n\Delta t}|X_{[qn]}|^{\alpha+2})\Delta t^2 \\
& + S_n,
\end{aligned} \tag{72}
$$

where

$$
\begin{aligned}
S_n =\ & (b|X_n|^{\alpha+2} + \bar{b}e^{-\varepsilon(1-q)n\Delta t}|X_{[qn]}|^{\alpha+2} + \widetilde{b}|X_n|^2)(|\Delta W_n|^2 - m\Delta t) \\
& + (c|X_n|^{\alpha+2} + \bar{c}e^{-\varepsilon(1-q)n\Delta t}|X_{[qn]}|^{\alpha+2} + \widetilde{c}|X_n|^2)(|\Delta\widetilde{N}_n|^2 - \lambda m\Delta t) \\
& + 2\langle X_n, g(X_n, X_{[qn]})\rangle\Delta W_n \\
& + 2\langle g(X_n, X_{[qn]})\Delta W_n, h(X_n, X_{[qn]})\Delta\widetilde{N}_n\rangle \\
& + 2\langle g(X_n, X_{[qn]})\Delta W_n, \lambda h(X_n, X_{[qn]})\rangle\Delta t \\
& + 2\langle h(X_n, X_{[qn]})\Delta\widetilde{N}, \lambda h(X_n, X_{[qn]})\rangle\Delta t \\
& + 2\langle X_n, h(X_n, X_{[qn]})\rangle\Delta\widetilde{N}_n.
\end{aligned} \tag{73}
$$

Then, we may obtain

$$
\begin{aligned}
& (1 + 2\widetilde{a}\Delta t)(e^{\varepsilon(n+1)\Delta t}|X_{n+1}|^2 - e^{\varepsilon n\Delta t}|X_n|^2) \\
& \leq (2 + m\widetilde{b}\Delta t + \lambda\widetilde{c}(m + 2\lambda\Delta t)\Delta t - (1 + 2\widetilde{a}\Delta t)e^{-\varepsilon\Delta t})e^{\varepsilon(n+1)\Delta t}|X_n|^2 \\
& \quad - 2ae^{\varepsilon(n+1)\Delta t}|X_{n+1}|^{\alpha+2}\Delta t + 2\bar{a}e^{\varepsilon q(n+1)\Delta t}|X_{[q(n+1)]}|^{\alpha+2}\Delta t \\
& \quad + mbe^{\varepsilon(n+1)\Delta t}|X_n|^{\alpha+2}\Delta t + m\bar{b}e^{-\varepsilon(qn+1)\Delta t}|X_{[qn]}|^{\alpha+2}\Delta t \\
& \quad + \lambda(mce^{\varepsilon(n+1)\Delta t}|X_n|^{\alpha+2} + m\bar{c}e^{-\varepsilon(qn+1)\Delta t}|X_{[qn]}|^{\alpha+2})\Delta t \\
& \quad + 2\lambda^2(ce^{\varepsilon(n+1)\Delta t}|X_n|^{\alpha+2} + \bar{c}e^{-\varepsilon(qn+1)\Delta t}|X_{[qn]}|^{\alpha+2})\Delta t^2 \\
& \quad + e^{\varepsilon(n+1)\Delta t}S_n.
\end{aligned} \tag{74}
$$

By using the recursive method, we obtain

$$
\begin{aligned}
&(1+2\widetilde{a}\Delta t)e^{\varepsilon n\Delta t}|X_n|^2 \\
&\quad \le (1+2\widetilde{a}\Delta t)|X_0|^2 + (2+m\widetilde{b}\Delta t + \lambda\widetilde{c}(m+2\lambda\Delta t)\Delta t \\
&\qquad - (1+2\widetilde{a}\Delta t)e^{-\varepsilon\Delta t})\sum_{i=0}^{n-1} e^{\varepsilon(i+1)\Delta t}|X_i|^2 \\
&\qquad - 2a\sum_{i=0}^{n-1} e^{\varepsilon(i+1)\Delta t}|X_{i+1}|^{\alpha+2}\Delta t \\
&\qquad + 2\bar{a}\sum_{i=0}^{n-1} e^{\varepsilon q(i+1)\Delta t}|X_{[q(i+1)]}|^{\alpha+2}\Delta t \\
&\qquad + mb\sum_{i=0}^{n-1} e^{\varepsilon(i+1)\Delta t}|X_i|^{\alpha+2}\Delta t + m\bar{b}\sum_{i=0}^{n-1} e^{\varepsilon(qi+1)\Delta t}|X_{[qi]}|^{\alpha+2}\Delta t \\
&\qquad + \lambda c(m+2\lambda\Delta t)\sum_{i=0}^{n-1} e^{\varepsilon(i+1)\Delta t}|X_i|^{\alpha+2}\Delta t \\
&\qquad + \lambda\bar{c}(m+2\lambda\Delta t)\sum_{i=0}^{n-1} e^{\varepsilon(qi+1)\Delta t}|X_{[qi]}|^{\alpha+2}\Delta t \\
&\qquad + \sum_{i=0}^{n-1} e^{\varepsilon(i+1)\Delta t}S_i,
\end{aligned}
\tag{75}
$$

which equals to

$$
\begin{aligned}
&(1+2\widetilde{a}\Delta t)(e^{\varepsilon n\Delta t}|X_n|^2 \\
&\quad \le (1+2\widetilde{a}\Delta t)|X_0|^2 + ((2+m\widetilde{b}\Delta t + \lambda\widetilde{c}(m+2\lambda\Delta t)\Delta t)e^{\varepsilon\Delta t} \\
&\qquad - (1+2\widetilde{a}\Delta t))\sum_{i=0}^{n-1} e^{\varepsilon i\Delta t}|X_i|^2 - 2a\sum_{i=1}^{n} e^{\varepsilon i\Delta t}|X_i|^{\alpha+2}\Delta t \\
&\qquad + mb\sum_{i=0}^{n-1} e^{\varepsilon(i+1)\Delta t}|X_i|^{\alpha+2}\Delta t + \lambda c(m+2\lambda\Delta t)\sum_{i=0}^{n-1} e^{\varepsilon(i+1)\Delta t}|X_i|^{\alpha+2}\Delta t \\
&\qquad + 2\bar{a}\sum_{i=0}^{n-1} e^{\varepsilon(i+1)\Delta t}|X_i|^{\alpha+2}\Delta t + m\bar{b}\sum_{i=0}^{n-1} e^{\varepsilon(i+2)\Delta t}|X_i|^{\alpha+2}\Delta t \\
&\qquad + \lambda\bar{c}(m+2\lambda\Delta t)\sum_{i=0}^{n-1} e^{\varepsilon(i+2)\Delta t}|X_i|^{\alpha+2}\Delta t \\
&\qquad + 2\bar{a}\sum_{i=0}^{n-1} e^{\varepsilon(i+1)\Delta t}(e^{-\varepsilon(1-q)(i+1)\Delta t}|X_{[q(i+1)]}|^{\alpha+2} - |X_i|^{\alpha+2})\Delta t \\
&\qquad + m\bar{b}\sum_{i=0}^{n-1} e^{\varepsilon(i+1)\Delta t}(e^{-\varepsilon(1-q)i\Delta t}|X_{[qi]}|^{\alpha+2} - e^{\varepsilon\Delta t}|X_i|^{\alpha+2})\Delta t \\
&\qquad + \lambda\bar{c}(m+2\lambda\Delta t)\sum_{i=0}^{n-1} e^{\varepsilon(i+1)\Delta t}(e^{-\varepsilon(1-q)i\Delta t}|X_{[qi]}|^{\alpha+2} - e^{\varepsilon\Delta t}|X_i|^{\alpha+2})\Delta t \\
&\qquad + \sum_{i=0}^{n-1} e^{\varepsilon(i+1)\Delta t}S_i,
\end{aligned}
\tag{76}
$$

where $\sum_{i=0}^{n-1} e^{\varepsilon(i+1)\Delta t}S_i$ is a martingale. Then, we could proceed as we did before in (56) and obtain the following

$$\begin{aligned}
&\sum_{i=0}^{n-1} e^{\varepsilon(i+1)\Delta t}(e^{-\varepsilon(1-q)i\Delta t}|X_{[qi]}|^{\alpha+2} - e^{\varepsilon\Delta t}|X_i|^{\alpha+2}) \\
&\quad = \sum_{i=0}^{n-1} e^{\varepsilon(1+qi)\Delta t}|X_{[qi]}|^{\alpha+2} - \sum_{i=0}^{n-1} e^{\varepsilon(i+2)\Delta t}|X_i|^{\alpha+2} \\
&\quad = \sum_{i=0}^{[qn]+1} e^{\varepsilon(i+2)\Delta t}|X_i|^{\alpha+2} - \sum_{i=0}^{n-1} e^{\varepsilon(i+2)\Delta t}|X_i|^{\alpha+2} \\
&\quad \leq -\sum_{i=[qn]+2}^{n-1} e^{\varepsilon(i+2)\Delta t}|X_i|^{\alpha+2}.
\end{aligned} \tag{77}$$

By the same analogy, we obtain

$$\begin{aligned}
&\sum_{i=0}^{n-1} e^{\varepsilon(i+1)\Delta t}(e^{-\varepsilon(1-q)(i+1)\Delta t}|X_{[q(i+1)]}|^{\alpha+2} - |X_i|^{\alpha+2}) \\
&\quad \leq -\sum_{i=[qn]+2}^{n-1} e^{\varepsilon(i+1)\Delta t}|X_i|^{\alpha+2}.
\end{aligned} \tag{78}$$

After plugging (77) and (78) into (76), we obtain the following

$$\begin{aligned}
&(1+2\widetilde{a}\Delta t)e^{\varepsilon n\Delta t}|X_n|^2 \\
&\leq (1+2\widetilde{a}\Delta t)|X_0|^2 + \sum_{i=1}^{n-1} e^{\varepsilon(i+1)\Delta t}|X_i|^2 - 2ae^{\varepsilon n\Delta t}|X_n|^{\alpha+2}\Delta t \\
&\quad - (1+2\widetilde{a}\Delta t - (1+m\widetilde{b}\Delta t + \lambda\widetilde{c}(m+2\lambda\Delta t)\Delta t)e^{\varepsilon\Delta t})\sum_{i=0}^{n-1} e^{\varepsilon i\Delta t}|X_i|^2 \\
&\quad + (-2a + mbe^{\varepsilon\Delta t} + 2\bar{a}e^{\varepsilon\Delta t} + m\bar{b}e^{2\varepsilon\Delta t})\sum_{i=1}^{n-1} e^{\varepsilon i\Delta t}|X_i|^{\alpha+2}\Delta t \\
&\quad + \lambda e^{\varepsilon\Delta t}(m+2\lambda\Delta t)(c+\bar{c}e^{\varepsilon\Delta t})\sum_{i=1}^{n-1} e^{\varepsilon i\Delta t}|X_i|^{\alpha+2}\Delta t \\
&\quad + (2\bar{a} + mb + m\bar{b}e^{\varepsilon\Delta t} + \lambda(m+2\lambda\Delta t)(c+\bar{c}e^{\varepsilon\Delta t}))e^{\varepsilon\Delta t}|X_0|^{\alpha+2}\Delta t \\
&\quad - 2\bar{a}\sum_{[qi]+2}^{n-1} e^{\varepsilon(i+1)\Delta t}|X_i|^{\alpha+2}\Delta t - m\bar{b}\sum_{[qi]+2}^{n-1} e^{\varepsilon(i+2)\Delta t}|X_i|^{\alpha+2}\Delta t \\
&\quad - \lambda\bar{c}(m+2\lambda\Delta t)\sum_{[qi]+2}^{n-1} e^{\varepsilon(i+2)\Delta t}|X_i|^{\alpha+2}\Delta t + \sum_{i=0}^{n-1} e^{\varepsilon(i+1)\Delta t}S_i.
\end{aligned} \tag{79}$$

We follow the same procedures as in [33] and denote

$$f(\Delta t) = 1 + 2\widetilde{a}\Delta t - (1+m\widetilde{b}\Delta t + \lambda\widetilde{c}(m+2\lambda\Delta t)\Delta t)e^{\varepsilon\Delta t}. \tag{80}$$

Upon differentiating with respect to Δt yields

$$\begin{aligned}
f'(\Delta t) = {} & 2\widetilde{a} - (m\widetilde{b} + \lambda\widetilde{c}(m+4\lambda\Delta t))e^{\varepsilon\Delta t} \\
& - (1+m\widetilde{b}\Delta t + \lambda\widetilde{c}(m+2\lambda\Delta t)\Delta t)\varepsilon e^{\varepsilon\Delta t},
\end{aligned} \tag{81}$$

and

$$\begin{aligned}
f''(\Delta t) = {} & -(1+m\widetilde{b}\Delta t + \lambda\widetilde{c}(m+2\lambda\Delta t)\Delta t)\varepsilon^2 e^{\varepsilon\Delta t} - (4\lambda^2\widetilde{c})e^{\varepsilon\Delta t} \\
& - 2(m\widetilde{b} + \lambda\widetilde{c}(m+4\lambda\Delta t))\varepsilon e^{\varepsilon\Delta t}.
\end{aligned} \tag{82}$$

Clearly, $f^{'}(0) = 2\widetilde{a} - m\widetilde{b} - \lambda m\widetilde{c} - \varepsilon > 0$, $f^{''}(0) < 0$, then there exists a $\overline{\Delta t}$ such that $f^{'}(\overline{\Delta t}) = 0$. $f(\Delta t)$ is non-decreasing function for values of Δt less than $\overline{\Delta t}$ and noting that $f(0) = 0$; therefore, there exists a small Δt^* less than $\overline{\Delta t}$ such that for all $\Delta t < \Delta t^*$

$$1 + 2\widetilde{a}\Delta t - (1 + m\widetilde{b}\Delta t + \lambda\widetilde{c}(m + 2\lambda\Delta t)\Delta t)e^{\varepsilon\Delta t} > 0. \tag{83}$$

On the other hand, because $2a > \frac{(2\bar{a}+m\bar{b}+\lambda m\bar{c})}{q} + b + \lambda c$, then there exists a small $\Delta t < \Delta t^*$ such that

$$2a - 2\bar{a}e^{\varepsilon\Delta t} - m(b + \bar{b}e^{\varepsilon\Delta t})e^{\varepsilon\Delta t} - \lambda(m + 2\lambda\Delta t)(c + \bar{c}e^{\varepsilon\Delta t})e^{\varepsilon\Delta t} > 0. \tag{84}$$

Then, after plugging (83) and (84) into (79), the discrete semimartingale theorem which was stated in [33] implies that there exists a positive constant C_0 such that

$$(1 + 2\widetilde{a}\Delta t)e^{\varepsilon n\Delta t}|X_n|^2 \leq C_0. \tag{85}$$

Which implies

$$\limsup_{n\to\infty} \frac{1}{n\Delta t} \log|X_n| \leq -\frac{\varepsilon}{2} \quad a.s. \tag{86}$$

The proof is completed. □

5. Numerical Examples

In this section, we will present examples to illustrate our theory.

Example 1. *Consider the following nonlinear SPDE with Poisson jumps*

$$dx(t) = [-0.5x(t) - 4x^5(t) + 2x^5(qt)]dt + x^3(t)dW(t) + x^3(t)dN(t), \tag{87}$$

where $W(t)$ is Brownian motion and $N(t)$ is Poisson process. Define $f(x,y) = -0.5x - 4x^5 + 2y^5$ and $g(x,y) = h(x,y) = x^3$. Then, we compute the following

$$\begin{aligned} f(x_1,y) - f(x_2,y) &= -0.5(x_1 - x_2) - 4(x_1^5 - x_2^5) \\ &\leq -0.5(x_1 - x_2)[1 + 8(x_1^4 + x_1^3x_2 + x_1^2x_2^2 + x_1x_2^3 + x_2^4)]. \end{aligned}$$

Noting that $A^2 + B^2 \geq \frac{(A+B)^2}{2}$, calculate

$$\begin{aligned} x_1^4 + x_1^3x_2 + x_1^2x_2^2 + x_1x_2^3 + x_2^4 &\geq \frac{(x_1^2 + x_2^2)^2}{2} + x_1x_2(x_1^2 + x_2^2) + (x_1x_2)^2 \\ &\geq \frac{(x_1^2 + x_2^2)^2}{4} + x_1x_2(x_1^2 + x_2^2) + (x_1x_2)^2 \\ &= [\frac{x_1^2 + x_2^2}{2} + x_1x_2]^2, \end{aligned}$$

which implies

$$\langle x_1 - x_2, f(x_1,y) - f(x_2,y)\rangle \leq -0.5(x_1 - x_2)^2.$$

This indicates that $f(x,y)$ satisfies the one-sided Lipschitz condition, and upon using the inequality $A^pB^q \leq \frac{p}{p+q}A^{p+q} + \frac{q}{p+q}B^{p+q}$, it is easy to calculate

$$\begin{aligned} \langle x, f(x,y)\rangle &\leq -0.5x^2 - 4x^6 + 2y^5x \\ &\leq -0.5x^2 - \frac{11}{3}x^6 + \frac{5}{3}y^6, \end{aligned}$$

and $|g(x(t),y(t))|^2 = |h(x(t),y(t))|^2 \leq x^6$. By Theorems 1, 2 and 4, Equation (87) has a unique global solution and the solution is almost surely exponentially stable.

Example 2. *Consider the following nonlinear SPDE with Poisson jumps*

$$dx(t) = [-0.4x(t) - 5x^3(t) + x^3(qt)]dt + x^2(t)dW(t) + x^2(t)dN(t). \quad (88)$$

Define $f(x,y) = -0.4x - 5x^3 + y^3$, $g(x,y) = x^2$ *and* $h(x,y) = x^2$. *Then, we compute the following*

$$\begin{aligned} f(x_1,y) - f(x_2,y) &= -0.4(x_1 - x_2) - 5(x_1^3 - x_2^3) \\ &= -0.4(x_1 - x_2)[1 + 12.5(x_1^2 + x_1x_2 + x_2^2)]. \end{aligned}$$

Now, we test the one-sided Lipschitz condition

$$\begin{aligned} \langle x_1 - x_2, f(x_1,y) - f(x_2,y)\rangle &= -0.4(x_1 - x_2)^2[1 + 12.5(x_1^2 + x_1x_2 + x_2^2)] \\ &\le -0.4(x_1 - x_2)^2. \end{aligned}$$

This indicates that $f(x,y)$ *satisfies the one-sided Lipschitz condition and it is easy to calculate*

$$\begin{aligned} \langle x, f(x,y)\rangle &\le -0.4x^2 - 5x^4 + xy^3 \\ &\le -0.4x^2 - 4.75x^4 + 0.75y^4, \end{aligned}$$

and $|g(x(t),y(t))|^2 \le x^4$ *and* $|h(x(t),y(t))|^2 \le x^4$. *By Theorems 1, 2 and 4, Equation (88) has a unique global solution and the solution is almost surely exponentially stable, and the backward Euler technique can reproduce the almost sure exponential stability.*

Example 3. *Consider the following nonlinear SPDE with Poisson jumps*

$$dx(t) = [-x(t) - 2x^5(t) + 10x^5(qt)]dt + x^3(t)sin(x(qt))dW(t) + x^3(t)cos(x(qt))dN(t). \quad (89)$$

Define $f(x,y) = -x - 2x^5 + 10y^5$, $g(x,y) = x^3 sin(y)$ *and* $h(x,y) = x^3 cos(y)$. *Then, we compute the following*

$$\begin{aligned} f(x_1,y) - f(x_2,y) &= -(x_1 - x_2) - 2(x_1^5 - x_2^5) \\ &= -(x_1 - x_2)[1 + 2(x_1^4 + x_1^3x_2 + x_1^2x_2^2 + x_1x_2^3 + x_2^4)]. \end{aligned}$$

Now, we test the one-sided Lipschitz condition

$$\begin{aligned} \langle x_1 - x_2, f(x_1,y) - f(x_2,y)\rangle &= -(x_1 - x_2)^2[1 + 2(x_1^4 + x_1^3x_2 + x_1^2x_2^2 + x_1x_2^3 + x_2^4)] \\ &\le -(x_1 - x_2)^2. \end{aligned}$$

This indicates that $f(x,y)$ *satisfies the one-sided Lipschitz condition and it is easy to calculate*

$$\begin{aligned} \langle x, f(x,y)\rangle &\le -x^2 - 2x^6 + 10xy^5 \\ &\le -x^2 - \frac{1}{3}x^6 + \frac{25}{3}y^6, \end{aligned}$$

and $|g(x,y)|^2 = |x^3 sin(y)|^2 \le x^6$ *and* $|h(x,y)|^2 = |x^3 cos(y)|^2 \le x^6$. *By Theorems 1, 2 and 4, Equation (89) has a unique global solution and the solution is almost surely exponentially stable, and the backward Euler technique can reproduce the almost sure exponential stability.*

6. Conclusions

The conclusions of this paper can be summarized as follows:

- The almost sure exponential stability of the analytical solution of SPDEs interspersed with the Poisson jumps has been proved with the help of the continuous semimartingale convergence theorem.

- The existence and the uniqueness of the global solution of the exact solution have also been proven.
- In using the discrete semimartingale convergence theorem, it has been shown that the explicit Euler–Maruyama technique reproduces the almost sure exponential stability of the exact solution under the assumption of the linear growth condition.
- By replacing the linear growth condition with the polynomial growth condition, imposing the one-sided Lipschitz condition on the drift coefficient and using the discrete semimartingale convergence theorem, it has been demonstrated that the backward Euler technique is capable of reproducing the almost sure exponential stability.

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References

1. Li, G.; Yang, Q. Stability analysis of the θ-method for hybrid neutral stochastic functional differential equations with jumps. *Chaos Solitons Fractals* **2021**, *150*, 111062. [CrossRef]
2. Hobson, D.G.; Rogers, L.C. Complete models with stochastic volatility. *Math. Financ.* **1998**, *8*, 27–48. [CrossRef]
3. Arriojas, M.; Hu, Y.; Mohammed, S.E.; Pap, G. A delayed Black and Scholes formula. *Stoch. Anal. Appl.* **2007**, *25*, 471–492. [CrossRef]
4. Zhou, S.; Wang, Z.; Feng, D. Stochastic functional differential equations with infinite delay. *J. Math. Anal. Appl.* **2009**, *357*, 416–426. [CrossRef]
5. Appleby, J.A. Decay and growth rates of solutions of scalar stochastic delay differential equations with unbounded delay and state dependent noise. *Stochastics Dyn.* **2005**, *5*, 133–147. [CrossRef]
6. Baker, C.T.; Buckwar, E. Continuous θ-methods for the stochastic pantograph equation. *Electron. Trans. Numer. Anal.* **2000**, *11*, 131–151.
7. Balasubramaniam, P.; Ntouyas, S. Controllability for neutral stochastic functional differential inclusions with infinite delay in abstract space. *J. Math. Anal. Appl.* **2006**, *324*, 161–176. [CrossRef]
8. Fan, Z.; Liu, M.; Cao, W. Existence and uniqueness of the solutions and convergence of semi-implicit Euler methods for stochastic pantograph equations. *J. Math. Anal. Appl.* **2007**, *325*, 1142–1159. [CrossRef]
9. Huang, C.; Cao, J. Almost sure exponential stability of stochastic cellular neural networks with unbounded distributed delays. *Neurocomputing* **2009**, *72*, 3352–3356. [CrossRef]
10. Li, X.; Fu, X. Stability analysis of stochastic functional differential equations with infinite delay and its application to recurrent neural networks. *J. Comput. Appl. Math.* **2010**, *234*, 407–417. [CrossRef]
11. Meng, X.; Hu, S.; Wu, P. Pathwise estimation of stochastic differential equations with unbounded delay and its application to stochastic pantograph equations. *Acta Appl. Math.* **2011**, *113*, 231–246. [CrossRef]
12. Meng, X.; Tian, M.; Hu, S. Stability analysis of stochastic recurrent neural networks with unbounded time-varying delays. *Neurocomputing* **2011**, *74*, 949–953. [CrossRef]
13. Ockendon, J.R.; Tayler, A.B. The dynamics of a current collection system for an electric locomotive. *Proc. R. Soc. Lond. Math. Phys. Sci.* **1971**, *322*, 447–468.
14. Kou, S.G. A jump-diffusion model for option pricing. *Manag. Sci.* **2002**, *48*, 1086–1101. [CrossRef]
15. Svishchuk, A.; Kalemanova, A. The stochastic stability of interest rates with jump changes. *Theory Probab. Math. Stat.* **2000**, *61*, 1–12.
16. Tankov, P. *Financial Modelling with Jump Processes*; Chapman and Hall/CRC: Boca Raton, FL, USA, 2003.

17. Bruti-Liberati, N.; Platen, E. Strong approximations of stochastic differential equations with jumps. *J. Comput. Appl. Math.* **2007**, *205*, 982–1001. [CrossRef]
18. Kim, N.; Lee, Y. Estimation and prediction under local volatility jump–diffusion model. *Phys. A Stat. Mech. Its Appl.* **2018**, *491*, 729–740. [CrossRef]
19. Agrawal, N.; Hu, Y. Jump Models with Delay—Option Pricing and Logarithmic Euler–Maruyama Scheme. *Mathematics* **2020**, *8*, 1932. [CrossRef]
20. Ahmadian, D.; Farkhondeh Rouz, O. Exponential mean-square stability of numerical solutions for stochastic delay integro-differential equations with Poisson jump. *J. Inequalities Appl.* **2020**, *2020*, 1–33. [CrossRef]
21. Fan, Z.; Song, M.; Liu, M. The αth moment stability for the stochastic pantograph equation. *J. Comput. Appl. Math.* **2009**, *233*, 109–120. [CrossRef]
22. Ronghua, L.; Min, L.; Wan-Kai, P. Convergence of numerical solutions to stochastic pantograph equations with Markovian switching. *Appl. Math. Comput.* **2009**, *215*, 414–422. [CrossRef]
23. Guo, P.; Li, C.J. Razumikhin-type technique on stability of exact and numerical solutions for the nonlinear stochastic pantograph differential equations. *BIT Numer. Math.* **2019**, *59*, 77–96. [CrossRef]
24. Higham, D.J.; Mao, X.; Yuan, C. Almost sure and moment exponential stability in the numerical simulation of stochastic differential equations. *SIAM J. Numer. Anal.* **2007**, *45*, 592–609. [CrossRef]
25. Higham, D.J.; Mao, X.; Yuan, C. Preserving exponential mean-square stability in the simulation of hybrid stochastic differential equations. *Numer. Math.* **2007**, *108*, 295–325. [CrossRef]
26. Higham, D.J.; Mao, X.; Stuart, A.M. Exponential mean-square stability of numerical solutions to stochastic differential equations. *LMS J. Comput. Math.* **2003**, *6*, 297–313. [CrossRef]
27. Higham, D.J. Mean-square and asymptotic stability of the stochastic theta method. *SIAM J. Numer. Anal.* **2000**, *38*, 753–769. [CrossRef]
28. Mao, X. *Stochastic Differential Equations and Applications*; Elsevier: Amsterdam, The Netherlands, 2007.
29. Mao, X. LaSalle-type theorems for stochastic differential delay equations. *J. Math. Anal. Appl.* **1999**, *236*, 350–369. [CrossRef]
30. Mao, X. A note on the LaSalle-type theorems for stochastic differential delay equations. *J. Math. Anal. Appl.* **2002**, *268*, 125–142. [CrossRef]
31. Rodkina, A.; Schurz, H. Almost sure asymptotic stability of drift-implicit θ-methods for bilinear ordinary stochastic differential equations in R1. *J. Comput. Appl. Math.* **2005**, *180*, 13–31. [CrossRef]
32. Wu, F.; Mao, X.; Szpruch, L. Almost sure exponential stability of numerical solutions for stochastic delay differential equations. *Numer. Math.* **2010**, *115*, 681–697. [CrossRef]
33. Zhou, S. Almost surely exponential stability of numerical solutions for stochastic pantograph equations. *Abstr. Appl. Anal.* **2014**, *2014*, 751209. [CrossRef]
34. Gardoń, A. The order of approximations for solutions of Itô-type stochastic differential equations with jumps. *Stoch. Anal. Appl.* **2004**, *22*, 679–699. [CrossRef]
35. Mao, X.; Shen, Y.; Gray, A. Almost sure exponential stability of backward Euler–Maruyama discretizations for hybrid stochastic differential equations. *J. Comput. Appl. Math.* **2011**, *235*, 1213–1226. [CrossRef]