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Recovering a Space-Dependent Source Term in the Fractional Diffusion Equation with the Riemann–Liouville Derivative

Songshu Liu

School of Mathematics and Statistics, Northeastern University at Qinhuangdao, Qinhuangdao 066004, China; liusongshu@neuq.edu.cn

Abstract: This research determines an unknown source term in the fractional diffusion equation with the Riemann–Liouville derivative. This problem is ill-posed. Conditional stability for the inverse source problem can be given. Further, a fractional Tikhonov regularization method was applied to regularize the inverse source problem. In the theoretical results, we propose a priori and a posteriori regularization parameter choice rules and obtain the convergence estimates.

Keywords: inverse problem; fractional diffusion equation with Riemann–Liouville derivative; ill-posed problem; fractional Tikhonov regularization method

MSC: 35R25; 35R30; 35R11; 47A52

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1. Introduction

In recent years, fractional calculus has become an important tool in mathematical modeling and has attracted the attention of researchers in various fields of science and engineering [1,2]. Research into the fractional diffusion equation has become a hot spot, attracting the interests of many researchers in fields such as elastic material mechanics, hydrology, random walking, biomedical, physics, medicine, and social sciences [3–8].

The direct problems for the time-fractional diffusion equation have been studied for many years, for example, the maximum principle, uniqueness results, existence results, numerical solutions, and analytic solutions [9–17]. In addition, various inverse problems of fractional diffusion equations have been researched extensively, such as inverse source problems [18,19], backward problems [20,21], the Cauchy problem [22,23], the inversion for parameter, or fractional order [24–29].

Let Ω be a bounded domain in $\mathbb{R}^d$, $(d = 1,2,3)$ with the sufficiently smooth boundary $\partial\Omega$. We consider an unknown source issue for the fractional diffusion equation with the Riemann–Liouville derivative

$$\begin{cases} \partial_t u(x,t) = \partial_t^{1-\alpha} Au(x,t) + F(x,t), & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & x \in \partial\Omega, \quad t \in (0,T], \\ u(x,T) = g(x), & x \in \overline{\Omega}, \end{cases} \tag{1}$$

where $T > 0$ is a given time. The symbol $\partial_t^{1-\alpha}$ is the Riemann–Liouville derivative of the order of $1-\alpha \in (0,1)$ defined in [30]

$$\partial_t^{1-\alpha} u(x,t) = \frac{1}{\Gamma(\alpha)} \frac{d}{dt} \int_0^t (t-s)^{\alpha-1} u(x,s)ds, \quad t > 0, \tag{2}$$

where $\Gamma(\cdot)$ is the Gamma function. The operator A is a symmetric uniformly elliptic operator defined on $D(A) := H^2(\Omega) \cap H_0^1(\Omega)$ in [31]

$$Au(x,t) = \sum_{i=1}^{d} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{d} a_{ij}(x)\frac{\partial}{\partial x_j}u(x,t)\right) + b(x)u(x,t), \quad x \in \Omega. \tag{3}$$

Moreover, the coefficients in (3) satisfy

$$a_{ij} = a_{ji}, \quad 1 \le i,j \le d, \tag{4}$$

$$\mu \sum_{i=1}^{d} \xi_i^2 \le \sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j, \quad x \in \overline{\Omega}, \quad \xi \in \mathbb{R}^d, \quad \mu > 0, \tag{5}$$

$$a_{ij} \in C^1(\overline{\Omega}), \quad b \in C(\overline{\Omega}), \quad b(x) \le 0, \quad x \in \overline{\Omega}. \tag{6}$$

The purpose of our article is to determine the source term $F(x,t)$ from the measured data $u(x,t) = g(x)$. The measurement is always noise-contaminated; thus, we have the measurement data $g^\delta \in L^2(\Omega)$ satisfying

$$\|g^\delta - g\| \le \delta, \tag{7}$$

where the constant δ denotes the noise level, and $\|\cdot\|$ denotes the L^2-norm.

For $\alpha = 1$, the problem (1) is an inverse source problem of the classical diffusion equation

$$\begin{cases} \partial_t u(x,t) = Au(x,t) + F(x,t), & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & x \in \partial\Omega, \quad t \in (0,T], \\ u(x,T) = g(x), & x \in \overline{\Omega}. \end{cases} \tag{8}$$

Obviously, Problem (8) has been researched extensively, see [32–37] for details. Recently, the source term identifications of fractional diffusion equations have been studied in different ways. If $F(x,t) = f(x)$, Zhang and Xu [18] proved the unique result of the source term identification problem using Laplace transform and analytic continuation. In [38–40], the authors studied inverse source problems of time-fractional diffusion equation using different methods, such as the Tikhonov regularization method, the simplified Tikhonov regularization method, the modified quasi-boundary value method, and the quasi-reversibility method. In [41], the authors recovered a space-dependent source term of a time-fractional diffusion equation using an iterative regularization method. In [42], the authors considered an inverse space-dependent source problem for a time-fractional diffusion equation via a new fractional Tikhonov regularization method. If $F(x,t) = f(t)$, Zhang and Wei [43] studied an inverse time-dependent source problem for the time-fractional diffusion equation by a truncation method. Yang and his group [44–46] considered the inverse time-dependent source problem for a fractional diffusion equation via several methods, such as the mollification regularization method, the quasi-reversibility regularization method, and the Fourier regularization method. If $F(x,t) = f(x)h(t)$, Nguyen et al. [47] applied the Tikhonov method to solve the inverse source problem of a time-fractional diffusion equation. In [48], the authors used the integral equation method and the standard Tikhonov regularization method to identify a time-dependent source term for a time-fractional diffusion equation. In [49,50], the authors investigated a source term identification problem in a time-fractional diffusion equation by using the Landweber iterative regularization method. In [51], the authors solved the inverse space-dependent source term in a time-fractional diffusion equation by using generalized and revised generalized Tikhonov regularization methods. In [52], the authors identified the source function in the time-fractional diffusion equation with non-local in-time conditions by using the modified fractional Landweber method.

Inverse source problems have applications in geophysical prospecting and pollutant detection [53,54]. As far as we know, there are few articles on inverse source problems of the fractional diffusion equation with the Riemann–Liouville derivative; see [55,56]. In this article, we use a new fractional Tikhonov regularization method to solve the inverse source problem (1).

The fractional Tikhonov regularization method was firstly proposed in [57]. Compared with the Tikhonov regularization method, the fractional Tikhonov regularization method has a better numerical effect. It was also used to solve some ill-posed problems, such as the inverse source problem of the time-fractional diffusion equation [42], the inverse time-fractional diffusion problem in a two-dimensional space[58], the initial value problem for a time-fractional diffusion equation [59], the backward problem for the space fractional diffusion equation [60], and the Cauchy problem of the Helmholtz equation [61].

The outline of this manuscript is as follows. In Section 2, we provide some preliminaries. The ill-posedness and conditional stability results are given in Section 3. In Section 4, we propose a fractional Tikhonov regularization method and obtain the convergence results based on a priori and a posteriori choice rules. The conclusion is given in Section 5.

2. Preliminaries

Throughout this article, we use the following definitions and lemmas.

Definition 1. *Let λ_p, e_p be the eigenvalues and corresponding eigenvectors of the operator A in Ω. The family of eigenvalues $\{\lambda_p\}_{p=1}^{\infty}$ satisfy $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_p \le \cdots$, where $\lambda_p \to \infty$ as $p \to \infty$:*

$$\begin{cases} Ae_p(x) = -\lambda_p e_p(x), & x \in \Omega, \\ e_p(x) = 0, & x \in \partial\Omega. \end{cases} \tag{9}$$

Definition 2. *Let $(\cdot,\cdot)$ be an inner product in $L^2(\Omega)$. The notation $\|\cdot\|_X$ stands for the norm in the Banach space. For any $k \ge 0$, we define the space*

$$H^k(\Omega) := \left\{ u \in L^2(\Omega) \,\middle|\, \sum_{p=1}^{\infty} \lambda_p^{2k} |(u, e_p)|^2 < +\infty \right\} \tag{10}$$

equipped with the norm

$$\|u\|_{H^k(\Omega)} = \left(\sum_{p=1}^{\infty} \lambda_p^{2k} |(u, e_p)|^2 \right)^{\frac{1}{2}}. \tag{11}$$

Definition 3 ([30])**.** *The Mittag–Leffler function $E_{\alpha,\beta}(\cdot)$ is*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C}, \tag{12}$$

where $\alpha > 0$ and $\beta \in \mathbb{R}$ are arbitrary constants.

Lemma 1 ([30])**.** *For $\alpha > 0$ and $\beta \in \mathbb{R}$, one has*

$$E_{\alpha,\beta}(z) = zE_{\alpha,\alpha+\beta}(z) + \frac{1}{\Gamma(\beta)}, \quad z \in \mathbb{C}. \tag{13}$$

Lemma 2 ([31])**.** *Let $\lambda > 0$, $\alpha > 0$, then we have*

$$\frac{d^m}{dt^m} E_{\alpha,1}(-\lambda t^\alpha) = -\lambda t^{\alpha-m} E_{\alpha,\alpha-m+1}(-\lambda t^\alpha), \quad t > 0, \tag{14}$$

$$\frac{d}{dt}(tE_{\alpha,2}(-\lambda t^\alpha)) = E_{\alpha,1}(-\lambda t^\alpha), \quad t > 0. \tag{15}$$

Lemma 3 ([30]). *Let $0 < \alpha < 1$ and $\lambda, a > 0$, then we have*

$$\frac{d}{dt}(E_{\alpha,1}(-\lambda t^{\alpha})) = -\lambda t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^{\alpha}), \quad for\ t > 0, \tag{16}$$

$$\frac{d}{dt}(t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^{\alpha})) = t^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda t^{\alpha}), \quad for\ t > 0, \tag{17}$$

$$\int_0^{\infty} e^{-st}E_{\alpha,1}(-at^{\alpha})dt = \frac{s^{\alpha-1}}{s^{\alpha}+a}, \quad for\ \Re(s) > a^{\frac{1}{\alpha}}. \tag{18}$$

Lemma 4 ([30]). *Let $0 < \alpha_0 < \alpha_1 < 1$. Then there exists positive constants M_1, M_2, M_3, depending only on α_0, α_1, such that for all $\alpha \in [\alpha_0, \alpha_1]$,*

$$\frac{M_1}{1-z} \leq E_{\alpha,1}(z) \leq \frac{M_2}{1-z}, \quad E_{\alpha,\beta}(x) \leq \frac{M_3}{1-z}, \quad for\ all\ z \leq 0, \alpha \in \mathbb{R}. \tag{19}$$

Lemma 5 ([55]). *Let $0 < \alpha < 1$ and $\lambda_p > 0$. Then*

$$\frac{Q_{\alpha}^{-}(\lambda_1, M_1)}{\lambda_p} \leq \int_0^T E_{\alpha,1}(-\lambda_p(T-\tau)^{\alpha})d\tau \leq \frac{Q_{\alpha}^{+}(M_2)}{\lambda_p}, \tag{20}$$

where

$$Q_{\alpha}^{-}(\lambda_1, M_1) = \frac{M_1 T\lambda_1}{1+\lambda_1 T^{\alpha}}, \quad Q_{\alpha}^{+}(M_2) = \frac{M_2 T^{1-\alpha}}{1-\alpha}, \tag{21}$$

$M_1 and M_2$ are positive constants.

3. Ill-Posedness and Conditional Stability

First, we introduce the mild solution of the following problem:

$$\begin{cases} \partial_t u(x,t) = \partial_t^{1-\alpha} Au(x,t) + F(x,t), & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & x \in \partial\Omega, \quad t \in (0,T], \\ u(x,0) = 0, & x \in \Omega, \\ u(x,T) = g(x), & x \in \overline{\Omega}. \end{cases} \tag{22}$$

Here, $F(x,t) = \sigma(t)f(x)$.

Throughout this work, we assume that there exists a positive constant E, such that

$$\|f\|_{H^k} \leq E, \tag{23}$$

for a positive real number, k. Here,

$$0 < \sigma_0 \leq \sigma(t) \leq \sigma_1, \quad \forall t \in [0,T]. \tag{24}$$

Now, according to the reference [56], we know

$$f(x) = \sum_{p=1}^{\infty} \frac{(g(x), e_p(x))e_p(x)}{\int_0^T E_{\alpha,1}(-\lambda_p(T-\tau)^{\alpha})\sigma(\tau)d\tau}. \tag{25}$$

From [62], we know that $E_{\alpha,1}(-x)$ is a completely monotonic decreasing function for $x \geq 0$. Furthermore, $\frac{1}{\int_0^T E_{\alpha,1}(-\lambda_p(T-\tau)^{\alpha})\sigma(\tau)d\tau} \to \infty$ with respect to λ_p. So the small error in the measured data $g^{\delta}(x)$ will be amplified by the factor $\frac{1}{\int_0^T E_{\alpha,1}(-\lambda_p(T-\tau)^{\alpha})\sigma(\tau)d\tau}$. Thus, we call $\frac{1}{\int_0^T E_{\alpha,1}(-\lambda_p(T-\tau)^{\alpha})\sigma(\tau)d\tau}$ the magnifying factor of the problem.

We define

$$Kf = \sum_{p=1}^{\infty} \left[\int_0^T E_{\alpha,1}(-\lambda_p (T-\tau)^\alpha)\sigma(\tau)d\tau \right] (f(x), e_p(x))e_p(x) = \int_\Omega k(x,\xi)f(\xi)d\xi = g(x), \quad x \in \Omega. \tag{26}$$

Here,

$$k(x,\xi) = \sum_{p=1}^{\infty} \left(\int_0^T E_{\alpha,1}(-\lambda_p (T-\tau)^\alpha)\sigma(\tau)d\tau \right) e_p(x)e_p(\xi).$$

Obviously, according to reference [55], we know that the operator K is a self-adjoint compact operator and an injective. By Kirsch [63], we conclude that the problem (22) is ill-posed.

The following theorem gives conditional stability for the inverse source problem.

Theorem 1. ([55]) *Let $E > 0$ and $k > 0$, suppose $\|f\|_{H^k(\Omega)} \le E$ holds, then*

$$\|f\| \le D(k)E^{\frac{1}{k+1}} \|Kf\|^{\frac{k}{k+1}}, \tag{27}$$

where

$$D(k) = \left(\frac{1}{\sigma_0 Q_\alpha^-(\lambda_1, M_1)} \right)^{\frac{k}{k+1}}. \tag{28}$$

Remark 1. *Essentially, Theorem 1 provides the following condition stability estimate*

$$\|f_1 - f_2\| \le D(k) \|f_1 - f_2\|_{H^k(\Omega)}^{\frac{1}{k+1}} \|Kf_1 - Kf_2\|^{\frac{k}{k+1}}.$$

4. Fractional Tikhonov Regularization Method and Convergence Estimates

In this section, we prove the convergence estimates for the fractional Tikhonov regularization method under a priori and a posteriori choice rules, respectively.

The fractional Tikhonov regularized solution (with exact data) is given by

$$f_\omega(x) = \sum_{p=1}^{\infty} \frac{(\int_0^T E_{\alpha,1}(-\lambda_p (T-\tau)^\alpha)\sigma(\tau)d\tau)^{2\gamma-1}}{\omega + (\int_0^T E_{\alpha,1}(-\lambda_p (T-\tau)^\alpha)\sigma(\tau)d\tau)^{2\gamma}} (g(x), e_p(x))e_p(x), \quad \frac{1}{2} \le \gamma \le 1, \tag{29}$$

and the fractional Tikhonov regularized solution (with noisy data) is given by

$$f_\omega^\delta(x) = \sum_{p=1}^{\infty} \frac{(\int_0^T E_{\alpha,1}(-\lambda_p (T-\tau)^\alpha)\sigma(\tau)d\tau)^{2\gamma-1}}{\omega + (\int_0^T E_{\alpha,1}(-\lambda_p (T-\tau)^\alpha)\sigma(\tau)d\tau)^{2\gamma}} (g^\delta(x), e_p(x))e_p(x), \quad \frac{1}{2} \le \gamma \le 1, \tag{30}$$

where $\omega > 0$ is a regularization parameter, γ is called the fractional parameter.

Case 1. When $\gamma = \frac{1}{2}$, it is the quasi-boundary value method [55].

Case 2. When $\gamma = 1$, it is the classical Tikhonov regularization method [56].

Before proving the main results, several useful and important lemmas are given.

Lemma 6. *For $\frac{1}{2} \le \gamma \le 1$, we can obtain*

$$T(b) = \frac{b^{2\gamma-1}}{\omega + b^{2\gamma}} \le c_1(\gamma)\omega^{-\frac{1}{2\gamma}}, \tag{31}$$

where $c_1(\gamma) = \frac{(2\gamma-1)^{\frac{2\gamma-1}{2\gamma}}}{2\gamma} > 0$.

Proof. For $\frac{1}{2} \leq \gamma \leq 1$, then $\lim_{b\to 0} T(b) = \lim_{b\to\infty} T(b) = 0$. Thus, there exists a $b^* = [(2\gamma-1)\omega]^{\frac{1}{2\gamma}} \geq 0$, which is a global maximizer, such that $T'(b^*) = 0$. So we have

$$T(b) \leq T(b^*) = c_1(\gamma)\omega^{-\frac{1}{2\gamma}}, \tag{32}$$

where $c_1(\gamma) = \frac{(2\gamma-1)^{\frac{2\gamma-1}{2\gamma}}}{2\gamma} > 0$. □

Lemma 7. *For constants $\omega > 0, a > 0, b \geq \lambda_1 > 0$, we have*

$$G(b) = \frac{\omega b^{2\gamma-k}}{\omega b^{2\gamma} + a^{2\gamma}} \leq \begin{cases} c_2(a,k,\gamma)\omega^{\frac{k}{2\gamma}}, & 0 < k < 2\gamma, \\ c_3(a,k,\gamma,\lambda_1)\omega, & k \geq 2\gamma, \end{cases}$$

where $c_2(a,k,\gamma) = \frac{1}{2\gamma a^k}(2\gamma-k)^{1-\frac{k}{2\gamma}} k^{\frac{k}{2\gamma}}$ *and* $c_3(a,k,\gamma,\lambda_1) = \frac{1}{a^{2\gamma}\lambda_1^{k-2\gamma}}$.

Proof. If $0 < k < 2\gamma$, we know that $\lim_{b\to 0} G(b) = \lim_{b\to\infty} G(b) = 0$. We have

$$0 \leq \sup_{b\in(0,+\infty)} G(b) \leq G(b_0). \tag{33}$$

Let $G'(b_0) = 0$, we have $b_0 = a(\frac{2\gamma-k}{\omega k})^{\frac{1}{2\gamma}} > 0$, then we obtain

$$G(b) \leq G(b_0) = \frac{\omega a^{-k}(\frac{2\gamma-k}{\omega k})^{1-\frac{k}{2\gamma}}}{\frac{2\gamma}{k}} := c_2(a,k,\gamma)\omega^{\frac{k}{2\gamma}}. \tag{34}$$

If $k > 2\gamma$, then we have

$$G(b) \leq \frac{\omega b^{2\gamma-k}}{a^{2\gamma}} = \frac{1}{a^{2\gamma} b^{k-2\gamma}}\omega \leq \frac{1}{a^{2\gamma}\lambda_1^{k-2\gamma}}\omega := c_3(a,k,\gamma,\lambda_1)\omega. \tag{35}$$

□

Lemma 8. *For constants $\omega > 0, a > 0, b \geq \lambda_1 > 0$, then we have*

$$L(b) = \frac{\omega b^{2\gamma-k-1}}{\omega b^{2\gamma} + a^{2\gamma}} \leq \begin{cases} c_4(a,k,\gamma)\omega^{\frac{k+1}{2\gamma}}, & 0 < k < 2\gamma-1, \\ c_5(a,k,\gamma,\lambda_1)\omega, & k \geq 2\gamma-1, \end{cases}$$

where $c_4(a,k,\gamma) = \frac{1}{2\gamma a^{k+1}}(2\gamma-k-1)^{1-\frac{k+1}{2\gamma}}(k+1)^{\frac{k+1}{2\gamma}}$ *and* $c_5(a,k,\gamma,\lambda_1) = \frac{1}{a^{2\gamma}\lambda_1^{k+1-2\gamma}}$.

Proof. The proof is similar to Lemma 7, so we omit it. □

4.1. A Priori Convergence Estimate

Theorem 2. *Assume that conditions (7) and (23) hold. Let $f(x)$ be the exact solution of problem (22), and $f_\omega^\delta(x)$ be the fractional Tikhonov regularized solution of problem (22).*

(a) If $0 < k \leq 2\gamma$, and if we choose

$$\omega = \left(\frac{\delta}{E}\right)^{\frac{2\gamma}{k+1}}, \tag{36}$$

then we can obtain the following convergence estimates

$$\|f_\omega^\delta(x) - f(x)\| \leq (c_1 + c_2)\delta^{\frac{k}{k+1}} E^{\frac{1}{k+1}}. \tag{37}$$

(b) If $k \geq 2\gamma$, and if we choose

$$\omega = \left(\frac{\delta}{E}\right)^{\frac{2\gamma}{2\gamma+1}}, \tag{38}$$

then we can obtain the following convergence estimates

$$\|f_\omega^\delta(x) - f(x)\| \leq (c_1 + c_3)\delta^{\frac{2\gamma}{2\gamma+1}} E^{\frac{1}{2\gamma+1}}, \tag{39}$$

where $c_1, c_2, andc_3$ are defined in Lemma 6 and Lemma 7.

Proof. According to the triangle inequality, we have

$$\|f_\omega^\delta(x) - f(x)\| \leq \|f_\omega^\delta(x) - f_\omega(x)\| + \|f_\omega(x) - f(x)\|. \tag{40}$$

Now, using (7) and Lemma 6, we estimate the first term

$$\begin{aligned} \|f_\omega^\delta(x) - f_\omega(x)\| &= \left\| \sum_{p=1}^{\infty} \frac{(\int_0^T E_{\alpha,1}(-\lambda_p(T-\tau)^\alpha)\sigma(\tau)d\tau)^{2\gamma-1}}{\omega + (\int_0^T E_{\alpha,1}(-\lambda_p(T-\tau)^\alpha)\sigma(\tau)d\tau)^{2\gamma}} (g^\delta(x) - g(x), e_p(x)) e_p(x) \right\| \\ &\leq \delta \sup_p \frac{(\int_0^T E_{\alpha,1}(-\lambda_p(T-\tau)^\alpha)\sigma(\tau)d\tau)^{2\gamma-1}}{\omega + (\int_0^T E_{\alpha,1}(-\lambda_p(T-\tau)^\alpha)\sigma(\tau)d\tau)^{2\gamma}} \\ &\leq c_1 \delta \omega^{-\frac{1}{2\gamma}}. \end{aligned} \tag{41}$$

For the second term, using the Parseval identity and (23), we obtain

$$\begin{aligned} &\|f_\omega(x) - f(x)\|^2 \\ = &\sum_{p=1}^{\infty} \left(\frac{(\int_0^T E_{\alpha,1}(-\lambda_p(T-\tau)^\alpha)\sigma(\tau)d\tau)^{2\gamma-1}}{\omega + (\int_0^T E_{\alpha,1}(-\lambda_p(T-\tau)^\alpha)\sigma(\tau)d\tau)^{2\gamma}} - \frac{1}{\int_0^T E_{\alpha,1}(-\lambda_p(T-\tau)^\alpha)\sigma(\tau)d\tau} \right)^2 |(g(x), e_p(x))|^2 \\ = &\sum_{p=1}^{\infty} \left(\frac{-\omega}{(\omega + (\int_0^T E_{\alpha,1}(-\lambda_p(T-\tau)^\alpha)\sigma(\tau)d\tau)^{2\gamma}) \int_0^T E_{\alpha,1}(-\lambda_p(T-\tau)^\alpha)\sigma(\tau)d\tau} \right)^2 |(g(x), e_p(x))|^2 \\ = &\sum_{p=1}^{\infty} \frac{\omega^2 |(g(x), e_p(x))|^2}{[\omega + (\int_0^T E_{\alpha,1}(-\lambda_p(T-\tau)^\alpha)\sigma(\tau)d\tau)^{2\gamma}]^2 |\int_0^T E_{\alpha,1}(-\lambda_p(T-\tau)^\alpha)\sigma(\tau)d\tau|^2} \\ = &\sum_{p=1}^{\infty} \frac{\omega^2 \lambda_p^{-2k} \lambda_p^{2k} |(g(x), e_p(x))|^2}{[\omega + (\int_0^T E_{\alpha,1}(-\lambda_p(T-\tau)^\alpha)\sigma(\tau)d\tau)^{2\gamma}]^2 |\int_0^T E_{\alpha,1}(-\lambda_p(T-\tau)^\alpha)\sigma(\tau)d\tau|^2} \\ \leq &\sup_p |M(p)|^2 \sum_{p=1}^{\infty} \frac{\lambda_p^{2k} |(g(x), e_p(x))|^2}{|\int_0^T E_{\alpha,1}(-\lambda_p(T-\tau)^\alpha)\sigma(\tau)d\tau|^2} \\ \leq &\sup_p |M(p)|^2 \|f\|_{H^k(\Omega)}^2. \end{aligned} \tag{42}$$

Here,

$$\begin{aligned} M(p) &= \frac{\omega \lambda_p^{-k}}{\omega + (\int_0^T E_{\alpha,1}(-\lambda_p(T-\tau)^\alpha)\sigma(\tau)d\tau)^{2\gamma}} \\ &\leq \frac{\omega \lambda_p^{-k}}{\omega + (\sigma_0 \int_0^T E_{\alpha,1}(-\lambda_p(T-\tau)^\alpha)d\tau)^{2\gamma}} \\ &\leq \frac{\omega \lambda_p^{2\gamma-k}}{\omega \lambda_p^{2\gamma} + (\sigma_0 Q_\alpha^-(\lambda_1, M_1))^{2\gamma}}. \end{aligned} \tag{43}$$

Using Lemma 7, we have

$$M(p) \leq \begin{cases} c_2\omega^{\frac{k}{2\gamma}}, & 0 < k < 2\gamma, \\ c_3\omega, & k \geq 2\gamma. \end{cases} \tag{44}$$

Combining (42) and (44), we have

$$\|f_\omega(x) - f(x)\| \leq \begin{cases} c_2\omega^{\frac{k}{2\gamma}}E, & 0 < k < 2\gamma, \\ c_3\omega E, & k \geq 2\gamma. \end{cases} \tag{45}$$

Therefore, combining (40), (41) and (45), we have

$$\|f_\omega^\delta(x) - f(x)\| \leq c_1\delta\omega^{-\frac{1}{2\gamma}} + \begin{cases} c_2\omega^{\frac{k}{2\gamma}}E, & 0 < k < 2\gamma, \\ c_3\omega E, & k \geq 2\gamma. \end{cases} \tag{46}$$

If we choose the regularization parameter by (36) and (38), we can have the following convergence estimates

$$\|f_\omega^\delta(x) - f(x)\| \leq \begin{cases} (c_1 + c_2)\delta^{\frac{k}{k+1}}E^{\frac{1}{k+1}}, & 0 < k < 2\gamma, \\ (c_1 + c_3)\delta^{\frac{2\gamma}{2\gamma+1}}E^{\frac{1}{2\gamma+1}}, & k \geq 2\gamma. \end{cases} \tag{47}$$

□

4.2. A Posteriori Convergence Estimate

In this subsection, we give the convergence estimate based on the a posteriori choice rule. According to Morozov's discrepancy principle [63], we choose the regularization parameter ω as the solution of the following equation:

$$\|Kf_\omega^\delta(x) - g^\delta(x)\| = \left\| \sum_{p=1}^{\infty} \frac{(\int_0^T E_{\alpha,1}(-\lambda_p(T-\tau)^\alpha)\sigma(\tau)d\tau)^{2\gamma}}{\omega + (\int_0^T E_{\alpha,1}(-\lambda_p(T-\tau)^\alpha)\sigma(\tau)d\tau)^{2\gamma}}(g^\delta(x), e_p(x))e_p(x) - g^\delta(x) \right\| = \tau\delta, \tag{48}$$

where $\tau > 1$ is a constant.

Lemma 9. *Let*

$$\rho(\omega) = \left\| \sum_{p=1}^{+\infty} \frac{(\int_0^T E_{\alpha,1}(-\lambda_p(T-\tau)^\alpha)\sigma(\tau)d\tau)^{2\gamma}}{\omega + (\int_0^T E_{\alpha,1}(-\lambda_p(T-\tau)^\alpha)\sigma(\tau)d\tau)^{2\gamma}}(g^\delta(x), e_p(x))e_p(x) - g^\delta(x) \right\|,$$

then the following results hold:
(a) $\rho(\omega)$ *is a continuous function;*
(b) $\lim_{\omega\to 0}\rho(\omega) = 0$;
(c) $\lim_{\omega\to +\infty}\rho(\omega) = \|g^\delta\|$;
(d) $\rho(\omega)$ *is a strictly increasing function over* $(0, +\infty)$.

The proof is obvious, and we omit it here.

Remark 2. *According to Lemma 9, we know that there exists a unique solution for (48) if* $0 < \tau\delta < \|g^\delta\|$.

Lemma 10. *If* ω *is the solution of (48), we can obtain the following inequality*

$$\omega^{-\frac{1}{2\gamma}} \leq \begin{cases} (\frac{c_4 q}{\tau-1})^{\frac{1}{k+1}}(\frac{E}{\delta})^{\frac{1}{k+1}}, & 0 < k < 2\gamma - 1, \\ (\frac{c_5 q}{\tau-1})^{\frac{1}{2\gamma}}(\frac{E}{\delta})^{\frac{1}{2\gamma}}, & k \geq 2\gamma - 1, \end{cases} \tag{49}$$

where c_4, c_5 *are defined in Lemma 8.*

Proof. From (48), there holds

$$\begin{aligned}
\tau\delta &= \left\| \sum_{p=1}^{+\infty} \frac{\omega}{\omega + (\int_0^T E_{\alpha,1}(-\lambda_p(T-\tau)^\alpha)\sigma(\tau)d\tau)^{2\gamma}} (g^\delta(x), e_p(x))e_p(x) \right\| \\
&\le \left\| \sum_{p=1}^{+\infty} \frac{\omega}{\omega + (\int_0^T E_{\alpha,1}(-\lambda_p(T-\tau)^\alpha)\sigma(\tau)d\tau)^{2\gamma}} (g^\delta(x) - g(x), e_p(x))e_p(x) \right\| \\
&\quad + \left\| \sum_{p=1}^{+\infty} \frac{\omega}{\omega + (\int_0^T E_{\alpha,1}(-\lambda_p(T-\tau)^\alpha)\sigma(\tau)d\tau)^{2\gamma}} (g(x), e_p(x))e_p(x) \right\| \\
&\le \delta + \left\| \sum_{p=1}^{+\infty} \frac{\omega}{\omega + (\int_0^T E_{\alpha,1}(-\lambda_p(T-\tau)^\alpha)\sigma(\tau)d\tau)^{2\gamma}} (g(x), e_p(x))e_p(x) \right\|. \qquad (50)
\end{aligned}$$

So, we have

$$(\tau - 1)\delta \le \left\| \sum_{p=1}^{+\infty} \frac{\omega}{\omega + (\int_0^T E_{\alpha,1}(-\lambda_p(T-\tau)^\alpha)\sigma(\tau)d\tau)^{2\gamma}} (g(x), e_p(x))e_p(x) \right\|. \qquad (51)$$

Using condition (23), we have

$$\begin{aligned}
&\left\| \sum_{p=1}^{+\infty} \frac{\omega}{\omega + (\int_0^T E_{\alpha,1}(-\lambda_p(T-\tau)^\alpha)\sigma(\tau)d\tau)^{2\gamma}} (g(x), e_p(x))e_p(x) \right\| \\
\le\ & \left\| \sum_{p=1}^{+\infty} \frac{\omega \int_0^T E_{\alpha,1}(-\lambda_p(T-\tau)^\alpha)\sigma(\tau)d\tau \cdot \lambda_p^{-k}}{\omega + (\int_0^T E_{\alpha,1}(-\lambda_p(T-\tau)^\alpha)\sigma(\tau)d\tau)^{2\gamma}} \frac{\lambda_p^k (g(x), e_p(x))e_p(x)}{\int_0^T E_{\alpha,1}(-\lambda_p(T-\tau)^\alpha)\sigma(\tau)d\tau} \right\| \\
=\ & \left\{ \sum_{p=1}^{+\infty} \left[\frac{\omega \int_0^T E_{\alpha,1}(-\lambda_p(T-\tau)^\alpha)\sigma(\tau)d\tau \cdot \lambda_p^{-k}}{\omega + (\int_0^T E_{\alpha,1}(-\lambda_p(T-\tau)^\alpha)\sigma(\tau)d\tau)^{2\gamma}} \right]^2 \left[\frac{\lambda_p^k (g(x), e_p(x))e_p(x)}{\int_0^T E_{\alpha,1}(-\lambda_p(T-\tau)^\alpha)\sigma(\tau)d\tau} \right]^2 \right\}^{\frac{1}{2}} \\
\le\ & \sup_p \frac{\omega\sigma_1 Q_\alpha^+(M_2)\lambda_p^{-k-1}}{\omega + (\frac{\sigma_0 Q_\alpha^-(\lambda_1, M_1)}{\lambda_p})^{2\gamma}} \left(\sum_{p=1}^{+\infty} \frac{\lambda_p^{2k}(g(x), e_p(x))^2}{(\int_0^T E_{\alpha,1}(-\lambda_p(T-\tau)^\alpha)\sigma(\tau)d\tau)^2} \right)^{\frac{1}{2}} \\
\le\ & \begin{cases} c_4 q \omega^{\frac{k+1}{2\gamma}} \|f\|_{H^k(\Omega)}, & 0 < k < 2\gamma - 1, \\ c_5 q \omega \|f\|_{H^k(\Omega)}, & k \ge 2\gamma - 1, \end{cases} \\
\le\ & \begin{cases} c_4 q \omega^{\frac{k+1}{2\gamma}} E, & 0 < k < 2\gamma - 1, \\ c_5 q \omega E, & k \ge 2\gamma - 1, \end{cases} \qquad (52)
\end{aligned}$$

where $q = \sigma_1 Q_\alpha^+(M_2)$. Therefore, combining (50)–(52), we have

$$\omega^{-\frac{1}{2\gamma}} \le \begin{cases} (\frac{c_4 q}{\tau - 1})^{\frac{1}{k+1}} (\frac{E}{\delta})^{\frac{1}{k+1}}, & 0 < k < 2\gamma - 1, \\ (\frac{c_5 q}{\tau - 1})^{\frac{1}{2\gamma}} (\frac{E}{\delta})^{\frac{1}{2\gamma}}, & k \ge 2\gamma - 1. \end{cases}$$

□

Theorem 3. *Suppose the conditions (7) and (23) hold, and take the solution of (48) as the regularization parameter, then*

(a) If $0 < k < 2\gamma - 1$, the following error estimate holds

$$\|f_\omega^\delta(x) - f(x)\| \le (c_1(\frac{c_4 q}{\tau - 1})^{\frac{1}{k+1}} + c_6)\delta^{\frac{k}{k+1}} E^{\frac{1}{k+1}}. \qquad (53)$$

(b) If $k \ge 2\gamma - 1$, the following error estimate holds

$$\|f_\omega^\delta(x) - f(x)\| \le c_1(\frac{c_5 q}{\tau - 1})^{\frac{1}{2\gamma}} \delta^{\frac{2\gamma-1}{2\gamma}} E^{\frac{1}{2\gamma}} + c_6 \delta^{\frac{k}{k+1}} E^{\frac{1}{k+1}}, \qquad (54)$$

where c_1 is defined in Lemma 6, c_4, c_5 are defined in Lemma 8, $c_6 = (\frac{\tau+1}{\sigma_0 Q_\alpha^-(\lambda_1, M_1)})^{\frac{k}{k+1}}$.

Proof. Due to the triangle inequality, we have

$$\|f_\omega^\delta(x) - f(x)\| \le \|f_\omega^\delta(x) - f_\omega(x)\| + \|f_\omega(x) - f(x)\|. \tag{55}$$

Now, using (41) and Lemma 10, we estimate the first term

$$\|f_\omega^\delta(x) - f_\omega(x)\| \le \begin{cases} c_1(\frac{c_4 q}{\tau-1})^{\frac{1}{k+1}} \delta^{\frac{k}{k+1}} E^{\frac{1}{k+1}}, & 0 < k < 2\gamma - 1, \\ c_1(\frac{c_5 q}{\tau-1})^{\frac{1}{2\gamma}} \delta^{\frac{2\gamma-1}{2\gamma}} E^{\frac{1}{2\gamma}}, & k \ge 2\gamma - 1. \end{cases} \tag{56}$$

In the following, we estimate the second term from Theorem 1. We know

$$\|f_\omega(x) - f(x)\| \le D(k)\widetilde{E}^{\frac{1}{k+1}} \|K(f_\omega(x) - f(x))\|^{\frac{k}{k+1}} = D(k)\widetilde{E}^{\frac{1}{k+1}} \|Kf_\omega(x) - Kf(x)\|^{\frac{k}{k+1}}. \tag{57}$$

Here, $\widetilde{E}$ is an upper bound of $\|f_\omega - f\|_{H^k(\Omega)}$.
Now, we estimate

$$\begin{aligned} \|Kf_\omega(x) - Kf(x)\| &= \left\| \sum_{p=1}^{\infty} \frac{\omega}{\omega + (\int_0^T E_{\alpha,1}(-\lambda_p(T-\tau)^\alpha)\sigma(\tau)d\tau)^{2\gamma}} (g(x), e_p(x)) e_p(x) \right\| \\ &\le \left\| \sum_{p=1}^{\infty} \frac{\omega}{\omega + (\int_0^T E_{\alpha,1}(-\lambda_p(T-\tau)^\alpha)\sigma(\tau)d\tau)^{2\gamma}} (g(x) - g^\delta(x), e_p(x)) e_p(x) \right\| \\ &\quad + \left\| \sum_{p=1}^{\infty} \frac{\omega}{\omega + (\int_0^T E_{\alpha,1}(-\lambda_p(T-\tau)^\alpha)\sigma(\tau)d\tau)^{2\gamma}} (g^\delta(x), e_p(x)) e_p(x) \right\| \\ &\le (\tau+1)\delta. \end{aligned} \tag{58}$$

Moreover, we know

$$\begin{aligned} \|f_\omega(x) - f(x)\|_{H^k(\Omega)}^2 &= \sum_{p=1}^{\infty} \left(\frac{\omega}{\omega + (\int_0^T E_{\alpha,1}(-\lambda_p(T-\tau)^\alpha)\sigma(\tau)d\tau)^{2\gamma}} \right)^2 \frac{\lambda_p^{2k} |(g(x), e_p(x))|^2}{|\int_0^T E_{\alpha,1}(-\lambda_p(T-\tau)^\alpha)\sigma(\tau)d\tau|^2} \\ &\le \sum_{p=1}^{\infty} \frac{\lambda_p^{2k} |(g(x), e_p(x))|^2}{|\int_0^T E_{\alpha,1}(-\lambda_p(T-\tau)^\alpha)\sigma(\tau)d\tau|^2} = \|f\|_{H^k(\Omega)}^2 \le E^2. \end{aligned} \tag{59}$$

From (57), we know $\|f_\omega - f\|_{H^k(\Omega)} \le \widetilde{E}$. Here, we can use E instead of $\widetilde{E}$. Combining (55)–(59), the following convergence estimates hold

$$\|f_\omega^\delta(x) - f(x)\| \le \begin{cases} (c_1(\frac{c_4 q}{\tau-1})^{\frac{1}{k+1}} + c_6)\delta^{\frac{k}{k+1}} E^{\frac{1}{k+1}}, & 0 < k < 2\gamma - 1, \\ c_1(\frac{c_5 q}{\tau-1})^{\frac{1}{2\gamma}} \delta^{\frac{2\gamma-1}{2\gamma}} E^{\frac{1}{2\gamma}} + c_6 \delta^{\frac{k}{k+1}} E^{\frac{1}{k+1}}, & k \ge 2\gamma - 1, \end{cases}$$

where $c_6 = (\frac{\tau+1}{\sigma_0 Q_\alpha^-(\lambda_1, M_1)})^{\frac{k}{k+1}}$. □

5. Conclusions

In this article, a fractional Tikhonov regularization method for the inverse source problem of the fractional diffusion equation with the Riemann–Liouville derivative is given, and we overcome its ill-posedness and prove the conditional stability result. Furthermore, the convergence estimates were obtained under a priori and a posteriori regularization parameter choice rules. In future work, we will focus on solving such an inverse source problem by using other regularization methods.

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