mathematics

MDPI

Article

On Energy Release Rate for Propagation of a Straight Crack in a Cosserat Elastic Body

Marin Marin [1,*], Sorin Vlase [2,3] and Ioan Tuns [4]

[1] Department of Mathematics and Computer Science, Transilvania University of Brasov, 500036 Brasov, Romania
[2] Department of Mechanical Engineering, Transilvania University of Brasov, 500036 Brasov, Romania
[3] Romanian Academy of Technical Sciences, B-dul Dacia 26, 030167 Bucharest, Romania
[4] Department of Civil Engineering, Transilvania University of Brasov, 500036 Brasov, Romania
* Correspondence: m.marin@unitbv.ro

Abstract: In this paper, we extend some results involving the energy release rate in the case of the propagation of a straight crack in an elastic solid. These results, approached by Gurtin and Yatomi in classical elasticity, are generalized in order to cover a Cosserat-type elastic body. We also investigate the effects of the microinertia and the couple stresses on the energy release rate.

Keywords: Cosserat bodies; elasticity; propagation; straight crack; energy release rate

MSC: 74Q15; 74A35; 74A45

check for updates

Citation: Marin, M.; Vlase, S.; Tuns, I. On Energy Release Rate for Propagation of a Straight Crack in a Cosserat Elastic Body. *Mathematics* **2022**, *10*, 3226. https://doi.org/10.3390/math10183226

Academic Editor: Andrey Jivkov

Received: 15 August 2022
Accepted: 4 September 2022
Published: 6 September 2022

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Copyright: © 2022 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

1. Introduction

It is unanimously accepted that the non-classical theories of elasticity have as their main goal to eliminate experimental aspects that cannot be described by elasticity based on the classical theory. If a body's intimate structure is considered for behavior, classical elasticity theory does not offer a good description, for example, in polymers, human and animal bones, and graphite. This is even more pronounced in materials with specific porous or granular internal structures. For more information, we recommend [1,2].

The first researcher who approached microstructure theories in a large number of studies was Eringen, who is considered a true pioneer in this field. We suggest [3,4]. After the way was opened by Eringen, many other researchers tackled a large number of aspects of microstructure, such as [5–21]. In [7], Ciarletta investigated the bending of microstretch elastic material plates with no thermal effects using the main ideas of Eringen. In [5], Iesan and Pompei find a exact solution of the Boussinesq–Somigliana–Galerkin type for a boundary-value problem in the same context of an elastic material with microstretch structure. In [12,14], the authors present some aspects regarding the propagation of waves through bodies with different microstructures. One of the first and most important microstructures is the Cosserat type. The Cosserat continuum, proposed by the French brothers Cosserat, introduces the rotational kinematic degree of freedom in addition to the translation components of the Cauchy model. A large number of influential scientists have granted great importance to this type of structure. A long time ago, the study of crack propagation was raised, at first, obviously, in the case of elastic or thermoelastic media. The first studies dedicated to this matter include Eringen [22], Freund [23], Gurtin [24], Gurtin and Yatomi [25], and so on. For instance, Freund [18] proved that energy flux depends on the elastic field near the tip. Further, Gurtin [24] discussed some thermodynamically cohesive models of mechanical fractures.

Gurtin and Yatomi [25] provided some expressions for the energy release rates in the presence of a propagating crack for a body that is hyperelastic and is subject to finite deformation.

The authors of [26] address the physical meaning of path-independent integrals for elastodynamically propagating cracks, the relation of these integrals to the energy release rates for propagating cracks, and the relation between these integrals and time-dependent stress-intensity factors in general mixed-mode dynamic crack propagation. Recently, some studies have presented new results regarding the energy release rate for crack propagation in bodies with microstructures. For examples, Zhang and Wang [27] theoretically investigated magneto–thermo–electro–elastic crack branching in these materials. A mathematical model was presented by Tian and Rajapakse [28] to find the parameters of fracture in the case of a finite impermeable crack in a magneto–electro–elastic plane actuated by magneto–electro–mechanical loading.

A study of a semi-infinite crack that is constantly propagated in an elastic material was conducted by Gourgiotis and Piccolroaz [29]. The body is subjected to some plane-strain shear loadings.

Our present paper tackles the problem of the propagation of a straight crack in a Cosserat elastic material.

2. Preliminaries

An open region D of the Euclidean three-dimensional space R^3 is considered, and an elastic Cosserat material fills this domain. As usual, the closure of D is denoted by $\bar{D}$, and its boundary ∂D is assumed to be a regular surface so as to allow application of the divergence theorem. We adopt the Cartesian vector and tensor notation and use two types of variables: the points of D have the spatial variables x_m, and we use the temporal variable t, with $t \in [0, t_0]$. When the meaning is unambiguous, we do not specify the dependence of a function on its spatial and/or temporal variables. We use the known rule of summation in the case of repeated indices.

Regarding the differentiation of functions, we use the following two rules: (1) a superposed dot on a function denotes the partial differentiation of the function with respect to t; and (2) the partial differentiation with respect to spatial variables x_m will be designated with the respective subscript preceded by a comma: f, m.

Notations for the mechanical quantities:

- v_m the components of the displacement vector field;
- ϕ_m the components of the microrotation vector;
- e_{mn} the components of the strain tensor;
- ϵ_{mn} the components of the couple strain tensor;
- ε_{mnk} the permutation symbol (the Ricci's tensor);
- τ_{mn} the components of the stress tensor;
- σ_{mn} the components of the couple stress tensor;
- ϱ the reference mass density;
- J_{mn} the components of the microinertia;
- f_m the components of the body force;
- g_m the components of the couple body force;
- n_k are the director cosines of the unit normal, outward to the boundary ∂D;
- τ_m the components of the surface traction vector;
- σ_m the components of the couple surface traction vector;
- $\mathcal{E}$ the dynamic energy release rate;

The deformation of a Cosserat elastic body will be described using the following internal constitutive variables, as functions depending on the time $t \in [0, t_0]$ and on point position $x \in D$:

- $v_m(x, t)$ the component elements of the vector of displacement;
- $\phi_m(x, t)$ the component elements of the tensor of Cosserat displacement.

The strain tensors have the component elements ϵ_{mn} and e_{mn}, and these are introduced with help of the components of displacement by means of the next kinematic relations:

$$\epsilon_{mn} = v_{n,m} + \varepsilon_{nmk}\phi_k,$$
$$e_{mn} = \phi_{n,m}. \tag{1}$$

Because our subsequent considerations are made only in the context of a linear theory, it is normal to consider that the density of the internal energy E is a quadratic form regarding all its constitutive variables. Thus, by assuming that our body in its reference state is free from stress and has zero intrinsic body couples, E has the following expression:

$$E = \frac{1}{2}\, C_{klmn}\epsilon_{kl}\epsilon_{mn} + B_{klmn}\epsilon_{kl}e_{mn} + \frac{1}{2}\, A_{klmn}e_{kl}e_{mn}. \tag{2}$$

Taking into account the expression of the density of internal energy using Clausius–Duhem inequality, we can obtain the constitutive relations that give the expressions for the stress tensors τ_{mn} and σ_{mn} in terms of the strain tensors:

$$\tau_{mn} = \frac{\partial E}{\partial \epsilon_{mn}} = C_{klmn}\epsilon_{kl} + B_{klmn}e_{kl},$$
$$\sigma_{mn} = \frac{\partial E}{\partial e_{mn}} = B_{klmn}\epsilon_{kl} + A_{klmn}e_{kl}, \tag{3}$$

obtained in the case of a homogenous Cosserat elastic body.

Above tensors C_{klmn}, B_{klmn}, and A_{klmn} are elastic constants that characterize the material properties and obey these relations of symmetry:

$$C_{mnkl} = C_{klmn},\ \ A_{klmn} = A_{mnkl}. \tag{4}$$

By using the procedure of Eringen [4], we can deduce the equations of motion. If it is assumed that mass forces and the couples of forces are missing, then the motion equations have the following form:

$$\varrho\ddot{v}_m = \tau_{nm,n},$$
$$J_{mn}\ddot{\phi}_n = \sigma_{nm,n} + \varepsilon_{mnk}\tau_{nk}, \tag{5}$$

in which ϱ is the density of the body in its reference state, and J_{mn} are the components of the tensor of inertia, with $J_{mn} = J_{nm}$.

Further, ε_{mnk} is the permutation symbol (the Ricci's tensor).

We assume that Equations (1), (3), and (5) take place for all $(x,t) \in B \times [0,t_0]$.

Further, we suppose that our body D in moment t has a straight crack, denoted by C_t, and the tip of this crack at moment t is denoted by $\mathbf{y}_t$. Assume that $\mathbf{y}_t$ is a function of class C^2 regarding the time variable t, and its velocity is $\mathbf{w}_t = d\mathbf{y}_t/dt$.

Let us denote by D_0 the domain covered by $\mathbf{y}_t$ for all $t \in [0,t_0]$.

In what follows, we consider certain applications of the form $\varphi(x,t)$, depending on $x \in D \setminus C_t$ and on $t \in [0,t_0]$. According to a suggestion given by Gurtin and Yatomi in [20], we introduce the next definition.

Definition 1. *We say that a function f is called a C^m fracture field for a positive integer m if the following two conditions are met:*

(1) f admits derivatives of any order i, $i \le m$, except for the points in the crack;

(2) all derivatives $f^{(i)}$, $i \le n$, are continuous applications except for the points in the crack, and functions $f^{(i)}$ are continuous at points approaching the crack from both sides except in points at the tip of the crack.

Let us specify some theoretical issues that we use below.

(i) The notation $f \in L^p(D)$ means that we are in the case $f(x,t) \in L^p(D)$ for any $t \in [0,t_0]$;

(ii) Let us introduce the family D_β of smooth surfaces D_β of D, having the property: *area* $\left(D \setminus D_\beta\right) \to 0$ as $\beta \to 0$. If $h \in L^p(D)$, we have

$$\lim_{\beta \to 0} \int_{D_\beta} |h(x,t)|^p dV = \int_D |h(x,t)|^p dV. \tag{6}$$

(iii) If the limit from (6) takes place uniformly with regards to $t \in [0, t_0]$, we consider that $f \in L^p(D)$ belongs uniformly in time.

(iv) Consider that $\mathbf{r}$ is the vector of the position from the tip. If f is a function that depends on t in the form $f(\mathbf{y}_t + \mathbf{r}, t)$, then the derivative of f with respect to t, denoted by f', is computed as follows:

$$f' = \frac{\partial f}{\partial t} + c_i \frac{\partial f}{\partial x_i} = \dot{f} + d_c f, \tag{7}$$

holding $\mathbf{r}$ fixed. Here, we use the notation

$$d_c f = c_i \frac{\partial f}{\partial x_i}.$$

(v) In the case of three scalar functions u, v, and w with properties $u, w \in C^1$, and $\dot{u} = v + d_c w$, we use the rule:

$$\int_D u dV = \int_D v\, dV + \int_{\partial D} w \mathbf{c}.\mathbf{n}\, dA,$$

in which $\mathbf{n}$ is a normal versor oriented outward from ∂D.

(vi) If $\mathbf{u}$ is a C^1 fracture field so that *div* $\mathbf{u} \in L^1(D)$, then the divergence theorem gives:

$$\int_{\partial D} \mathbf{u}.\mathbf{n}\, dA = \int_D div\, \mathbf{u}\, dV,$$

called the "flow-divergence formula".

Statements (v) and (vi) above are easy to prove. However, for more details, we recommend [20].

3. Basic Results

We assume that the displacement vector of components v_m and the couple displacement vectors of components ϕ_m are C^3 fracture fields in the sense of the above definition. Further, we suppose that the tensors of stress τ_{mn} and σ_{mn} are C^1 fracture fields.

We can now define the surface traction vector of components τ_l and the couple surface traction vector of components σ_l with the help of the next equations:

$$\tau_l = \tau_{kl} n_k, \ \sigma_l = \sigma_{kl} n_k, \tag{8}$$

where n_k are the director cosines of the unit normal outward to the border ∂D.

By definition, the application $\mathcal{E}$ defined in any t of the interval $[0, t_0]$ is called *a rate of the dynamic energy release* if $\mathcal{E}$ has the following expression:

$$\begin{aligned}\mathcal{E} = \int_{\partial D} (\tau_m \dot{v}_m + \sigma_m \dot{\phi}_m) dA - \frac{d}{dt} \int_D \bigg[\frac{1}{2} (\varrho \dot{v}_m \dot{v}_m + J_{mn} \dot{\phi}_m \dot{\phi}_n) \\ + \frac{1}{2} C_{klmn} \epsilon_{kl} \epsilon_{mn} + B_{klmn} \epsilon_{kl} e_{mn} + \frac{1}{2} A_{klmn} e_{kl} e_{mn} \bigg] dV. \end{aligned} \tag{9}$$

We will consider S a surface that includes its inside at time t_1 at the tip of the crack, where $t_1 \in [0, t_0]$ and so ∂S and $\mathbf{y}_t \in S$ intersects the crack C_t only once.

It can be shown that the rate of dynamic energy release does not depend on the choice of surface S.

Now, we prove a property of the crack tip.

Proposition 1. *The rate of dynamic energy release $\mathcal{E}$ is intrinsic to the tip of the crack.*

Proof. First of all, observing the expression of the internal density, we can write:

$$\dot{E} = \tau_{mn}\dot{\epsilon}_{mn} + \sigma_{mn}\dot{e}_{mn}. \tag{10}$$

Then, considering the theorem of divergence and Equation (8), we can write:

$$\int_{\partial(D\setminus S)} (\tau_m \dot{v}_m + \sigma_m \dot{\phi}_m) dA = \int_{D\setminus S} (\tau_{mn}\dot{v}_n + \sigma_{mn}\dot{\phi}_n)_{,m} dV. \tag{11}$$

Further, if we consider the motion Equations (5) and (10), we deduce:

$$\begin{aligned}
&\int_{\partial(D\setminus S)} (\tau_m \dot{v}_m + \sigma_m \dot{\phi}_m) dA \\
&= \int_{D\setminus S} (\varrho \ddot{v}_m \dot{v}_m + J_{mn}\ddot{\phi}_m \dot{\phi}_n + \tau_{mn}\dot{\epsilon}_{mn} + \sigma_{mn}\dot{e}_{mn}) dV \\
&= \frac{d}{dt}\int_{D\setminus S} \frac{1}{2}(\varrho \dot{v}_m \dot{v}_m + J_{mn}\dot{\phi}_m \dot{\phi}_n + C_{klmn}\epsilon_{kl}\epsilon_{mn} + 2B_{klmn}\epsilon_{kl}e_{mn} + A_{klmn}e_{kl}e_{mn}) dV.
\end{aligned} \tag{12}$$

After these calculations, dynamic energy release rate receives the form:

$$\begin{aligned}
\mathcal{E} = \int_{\partial D} (\tau_m \dot{v}_m + \sigma_m \dot{\phi}_m) dA - \frac{d}{dt}\int_D \Big[\frac{1}{2}(\varrho \dot{v}_m \dot{v}_m + J_{mn}\dot{\phi}_m \dot{\phi}_n) \\
+ \frac{1}{2}C_{klmn}\epsilon_{kl}\epsilon_{mn} + B_{klmn}\epsilon_{kl}e_{mn} + \frac{1}{2}A_{klmn}e_{kl}e_{mn}\Big] dV,
\end{aligned}$$

which is even (9), and thus we conclude the proof of this proposition. □

In our next result, we obtain another expression for the rate of dynamic energy release.

Theorem 1. *For any $t \in [0, t_0]$, the rate of dynamic energy release can be written in the following form:*

$$\begin{aligned}
\mathcal{E}(t) = \lim_{S\to \mathbf{y}_t} \int_S \Big\{ \Big[\frac{1}{2}(\varrho d_c v_m d_c v_m + J_{mn} d_c \phi_m d_c \phi_n) \\
+ \frac{1}{2}C_{klmn}\epsilon_{kl}\epsilon_{mn} + B_{klmn}\epsilon_{kl}e_{mn} + \frac{1}{2}A_{klmn}e_{kl}e_{mn}\Big] c_s \\
- \tau_{sm} d_c v_m - \sigma_{sm} d_c \phi_m \Big\} n_s dA,
\end{aligned} \tag{13}$$

where S is a surface that includes the inside of the tip of the crack. Further, c_s represent the elements of the speed of the tip $\mathbf{y}_t$, and d_c appears in (7).

Proof. If we consider Rule (7), we obtain

$$\int_{\partial D} (\tau_m \dot{v}_m + \sigma_m \dot{\phi}_m) dA = \int_{\partial D} (\tau_m v'_m + \sigma_m \phi'_m) dA - \int_{\partial D} (\tau_m d_c v_m + \sigma_m d_c \phi_m) dA. \tag{14}$$

If we take into account the kinematic Equation (1), we are led to

$$\left(\tau_{mn} u'_m + \sigma_{mn}\phi'_m\right)_{,n} = \varrho \ddot{v}_m v'_m + J_{mn}\ddot{\phi}_m \phi'_n + \tau_{mn}\epsilon'_{mn} + \sigma_{mn}e'_{mn}, \tag{15}$$

where the following notations are used:

$$\epsilon'_{mn} = v'_{n,m} + \varepsilon_{nmk}\phi'_k, \; e'_{mn} = \phi'_{n,m}.$$

Applying the second derivative, according to the rule in (7), we deduce:

$$\ddot{v}_m = v''_m - 2d_c v'_m + d_c^2 v_m - d_c v_m,\ d_c^2 v_m = v_{m,kl} c_l c_k,$$
$$\ddot{\phi}_m = \phi''_m - 2d_c \phi'_m + d_c^2 \phi_m - d_c \phi_m,\ d_c^2 \phi_{mm} = \phi_{m,kl} c_l c_k. \tag{16}$$

We obtain a relation similar to the one in (11) using (7) again:

$$\int_{\partial D} \left(\tau_m v'_m + \sigma_m \phi'_m\right) dA$$
$$= \int_D \left(\varrho \ddot{v}_m v'_m + J_{mn} \ddot{\phi}_m \phi'_n + \tau_{mn} \epsilon'_{mn} + \sigma_{mn} e'_{mn}\right) dV, \tag{17}$$

and by using (16), we also deduce that

$$\frac{1}{2}\frac{d}{dt}(\varrho \dot{v}_m \dot{v}_m + J_{mn} \dot{\phi}_m \dot{\phi}_n) = \varrho \ddot{v}_m v'_m + J_{mn} \ddot{\phi}_m \phi'_n$$
$$-\varrho\left(v''_m - 2d_c v'_m - d_c v_m\right) d_c v_m - J_{mn}\left(\phi''_m - 2d_c \phi'_m - d_c \phi_m\right) d_c \phi_n \tag{18}$$
$$-\frac{1}{2}\, d_c(\varrho\, d_c v_m d_c v_m + J_{mn} d_c \phi_m d_c \phi_n).$$

Now, if we take into account Equation (2) and use the relation (7), the following equality is obtained

$$\dot{E} = E' - d_c E, \tag{19}$$

so that from previous equations we deduce:

$$\frac{d}{dt}\int_D \left[\frac{1}{2}(\varrho \dot{v}_m \dot{v}_m + J_{mn} \dot{\phi}_m \dot{\phi}_n)\right.$$
$$\left. + \frac{1}{2} C_{klmn} \epsilon_{mn} \epsilon_{mn} + B_{ijmn} \epsilon_{mn} e_{mn} + \frac{1}{2} A_{klmn} \epsilon_{mn} e_{mn}\right] dV$$
$$= \frac{d}{dt}\int_D (E + \varrho \dot{v}_m \dot{v}_m + J_{mn} \dot{\phi}_m \dot{\phi}_n) dV \tag{20}$$
$$= \int_D \left[E' + \frac{1}{2}\frac{d}{dt}(\varrho \dot{v}_m \dot{v}_m + J_{mn} \dot{\phi}_m \dot{\phi}_n) + \frac{1}{2}\left(\varrho\, d_c v_m d_c v_m + J_{mn} d_c \phi_m d_c \phi_n\right)\right] dV$$
$$- \int_{\partial D}\left[E + \frac{1}{2}\,\left(\varrho\, d_c v_m d_c v_m + J_{mn} d_c \phi_m d_c \phi_n\right)\right] c_s n_s dA,$$

where c_s represent the elements of the speed of the tip, and $n = (n_s)$ is the outward normal to the border ∂D.

By taking into account Equations (11), (12), (17) and (20), the following relation is obtained:

$$\mathcal{E}(t) = \int_{\partial D}\left\{\left[\frac{1}{2}(\varrho d_c v_m d_c v_m + J_{mn} d_c \phi_m d_c \phi_n)\right.\right.$$
$$\left. + \frac{1}{2} C_{klmn} \epsilon_{kl} \epsilon_{mn} + B_{klmn} \epsilon_{kl} e_{mn} + \frac{1}{2} A_{klmn} e_{kl} e_{mn}\right] c_s \tag{21}$$
$$\left. - \tau_{mk} d_c v_m - \sigma_{mk} d_c \phi_m\right\} n_k dA$$
$$- \int_D \left[E' + \frac{1}{2}(\varrho \dot{v}_m \dot{v}_m + J_{mn} \dot{\phi}_m \dot{\phi}_n) - \tau_{mn} \epsilon'_{mn} + \sigma_{mn} e'_{mn} - \varrho \ddot{v}_m v'_m - J_{mn} \ddot{\phi}_m \phi'_n\right] dV.$$

Finally, we take into account that surface S includes the tip of its crack in its inside at time t. Based on Proposition 1, we know that the rate of the dynamic energy release $\mathcal{E}$ is intrinsic relative to the tip of the crack, as such, $\mathcal{E}$ can be written in the form:

$$\mathcal{E}(t) = \int_S \left\{ \left[\frac{1}{2}(\varrho d_c v_m d_c v_m + J_{mn} d_c \phi_m d_c \phi_n) + \frac{1}{2} C_{klmn} \epsilon_{kl} \epsilon_{mn} + B_{klmn} \epsilon_{kl} e_{mn} + \frac{1}{2} A_{klmn} e_{kl} e_{mn} \right] c_s - \tau_{mk} d_c v_m - \sigma_{mk} d_c \phi_m \right\} n_k dA$$
$$- \iint_S \left(E' + \frac{1}{2}(\varrho \dot{v}_m \dot{v}_m + J_{mn} \dot{\phi}_m \dot{\phi}_n) - \tau_{mn} \epsilon'_{mn} - \sigma_{mn} e'_{mn} - \varrho \ddot{v}_m v'_m - J_{mn} \ddot{\phi}_m \phi'_n \right) dV.$$

In this last relation, we take into account that the area of S is tending to zero, and so we obtain (13), which concludes the demonstration of Theorem 1. □

In Theorem 2 that follows, we obtain another expression of the rate of dynamic energy release at an arbitrary fixed time $t \in [0, t_0]$.

Theorem 2. *For a fixed time $t_1 \in [0, t_0]$, the rate of dynamic energy release can be written in the form:*

$$\mathcal{E}(t_1) = \left(\frac{d}{dt} \right) \int_D \int_{t_1}^{t} [(\tau_{mn}(t) - \tau_{mn}(s)) \dot{\epsilon}_{mn}(s) + (\sigma_{mn}(t) - \sigma_{mn}(s)) \dot{e}_{ij}(s) + \varrho(\ddot{v}_m(t) - \ddot{v}_m(s)) \dot{v}_m(s) + J_{mn}(\ddot{\phi}_m(t) - \ddot{\phi}_m(s)) \dot{\phi}_m(s)] ds dV. \tag{22}$$

Proof. We start with the observation that:

$$\mathcal{E} = \frac{d}{dt} \left(\int_{t_1}^{t} \int_{\partial D} (\tau_m \dot{v}_m + \sigma_m \dot{\phi}_m) dA ds \right) - \frac{d}{dt} \left(\int_D \left[E + \frac{1}{2}(\varrho \dot{v}_m \dot{v}_m + J_{mn} \dot{\phi}_m \dot{\phi}_n) \right] dV \right). \tag{23}$$

Then, we have

$$\int_{t_1}^{t} \int_{\partial D} (\tau_m \dot{v}_m + \sigma_m \dot{\phi}_m) dA ds = \int_{t_1}^{t} \int_{\partial D} [\tau_m(t) \dot{v}_m(s) + \sigma_m(t) \dot{\phi}_m(s)] dA ds + \int_{t_1}^{t} \int_{\partial D} \{ [\tau_m(s) - \tau_m(t)] \dot{v}_m(s) + [\sigma_m(s) - \sigma_m(t)] \dot{\phi}_m(s) \} dA ds.$$

Now, it is easy to verify that we have:

$$\frac{d}{dt} \left(\int_{t_1}^{t} \int_{\partial D} \{ [\tau_m(s) - \tau_m(t)] \dot{v}_m(s) + [\sigma_m(s) - \sigma_m(t)] \dot{\phi}_m(s) \} dA ds \right) = - \int_{t_1}^{t} \int_{\partial D} [\tau_m(t) \dot{v}_m(s) + \sigma_m(t) \dot{\phi}_m(s)] dA ds.$$

According to these last two equations, we deduce that:

$$\left(\frac{d}{dt} \right)_{t_1} \left(\int_{t_1}^{t} \int_{\partial D} (\tau_m \dot{v}_m + \sigma_m \dot{\phi}_m) dA ds \right) = \left(\frac{d}{dt} \right)_{t_1} \left(\int_{\partial D} \{ [v_m(t) - v_m(t_1)] \tau_m(t) + [\phi_m(t) - \phi_m(t_1)] \sigma_m(t) \} dA \right).$$

Now, we consider the motion Equation (5), the kinematic relations (1) and the tractions on the surface from (8), so that we obtain the next identity:

$$\left(\frac{d}{dt}\right)_{t_1}\left(\int_{\partial D}\{[v_m(t)-v_m(t_1)]\tau_m(t)+[\phi_m(t)-\phi_m(t_1)]\sigma_m(t)\}dA\right)$$
$$=\left(\frac{d}{dt}\right)_{t_1}\left(\int_D\{\varrho[v_m(t)-v_m(t_1)]\ddot{v}_m(t)+J_{mn}[\phi_m(t)-\phi_m(t_1)]\ddot{\phi}_n(t)\}dV\right) \quad (24)$$
$$+\left(\frac{d}{dt}\right)_{t_1}\left(\int_D\{[\epsilon_{mn}(t)-\epsilon_{mn}(t_1)]\tau_{mn}(t)+[e_{mn}(t)-e_{mn}(t_1)]\sigma_{mn}(t)\}dV\right).$$

According to the Equation (24), one can write:

$$\mathcal{E}(t_1)=\left(\frac{d}{dt}\right)_{t_1}\left(\int_D\{\varrho[v_m(t)-v_m(t_1)]\ddot{v}_m(t)+J_{mn}[\phi_m(t)-\phi_m(t_1)]\ddot{\phi}_n(t)\}dV\right)$$
$$+\left(\frac{d}{dt}\right)_{t_1}\left(\int_D\{[\epsilon_{mn}(t)-\epsilon_{mn}(t_1)]\tau_{mn}(t)+[e_{mn}(t)-e_{mn}(t_1)]\sigma_{mn}(t)\}dV\right) \quad (25)$$
$$-\left(\frac{d}{dt}\right)_{t_1}\left(\int_D\left[E+\frac{1}{2}(\varrho\dot{v}_m\dot{v}_m+J_{mn}\dot{\phi}_m\dot{\phi}_n)\right]dV\right).$$

Further, it is not difficult to verify that:

$$\int_{t_1}^{t}[\tau_{mn}(t)\dot{\epsilon}_{mn}(s)+\sigma_{mn}(t)\dot{e}_{mn}(s)]ds$$
$$=[\epsilon_{mn}(t)-\epsilon_{mn}(t_1)]\tau_{mn}(t)+[e_{mn}(t)-e_{mn}(t_1)]\sigma_{mn}(t). \quad (26)$$

Now, by integrating over D for both sides of Equation (26), we deduce:

$$\int_D\left\{\int_{t_1}^{t}[\tau_{mn}(t)\dot{\epsilon}_{mn}(s)+\sigma_{mn}(t)\dot{e}_{mn}(s)]ds\right\}dV$$
$$=\int_D\{[\epsilon_{mn}(t)-\epsilon_{mn}(t_1)]\tau_{mn}(t)+[e_{mn}(t)-e_{mn}(t_1)]\sigma_{mn}(t)\}dV. \quad (27)$$

By using some similar calculus, we can obtain an analogous relation:

$$\int_D\int_{t_1}^{t}[\varrho\ddot{v}_m(t)\dot{v}_m(s)+J_{mn}\ddot{\phi}_m(t)\dot{\phi}_n(s)]ds\;dV$$
$$=\int_D\{\varrho[v_m(t)-v_m(t_1)]\ddot{v}_m(t)+J_{mn}[\phi_m(t)-\phi_m(t_1)]\ddot{\phi}_n(t)\}dV. \quad (28)$$

Now, we consider Equation (10) in order to obtain:

$$\frac{d}{dt}\int_D\left[E+\frac{1}{2}(\varrho\dot{v}_m\dot{v}_m+J_{mn}\dot{\phi}_m\dot{\phi}_n)\right]dV$$
$$=\frac{d}{dt}\int_D\int_{t_1}^{t}[\tau_{mn}(t)\dot{\epsilon}_{mn}(s)+t_{mn}(t)\dot{e}_{mn}(s) \quad (29)$$
$$+\varrho\ddot{v}_m(t)\dot{v}_m(s)+J_{mn}\ddot{\phi}_m(t)\dot{\phi}_n(s)]dsdV.$$

Ultimately, we consider the relations (27)–(29) so that from (25), we arrive at identity (22), and the proof of the theorem ends. □

Our results are placed within a linear theory of Cosserat elastic media. After formulating the basic equations and conditions in this context, we approached the rate of dynamic energy release due to the propagation of a straight crack in this type of material. We insisted on highlighting the effect of microinertia and the couples stresses on this rate, these being contributions of the Cosserat structure. Now, we emphasize the fact that compared to classical elasticity:

a. There was only one balance equation, namely $(5)_1$; now there is another one, namely $(5)_2$, in which microinertia appears.

b. There was only one stress tensor; now there is another one that appears in both balance equations.

c. There was only one strain tensor; now there is another one that appears in both constitutive equations.

d. There was only one constitutive equation; now there is another one.

It is not difficult to observe that these contributions are not so obvious as to cause spectacular and meaningful changes in the propagation of a straight crack. In our basic results, concretized in estimations (13) and (22), we deduced certain alternative formulations for the rate of dynamic energy release. These refer to some boundary concrete integrals calculated on a surface that includes the crack in its inside, and this surface shrinks to the tip of the crack.

Author Contributions: Conceptualization: M.M. and S.V.; methodology: M.M., S.V. and I.T.; validation: M.M. and S.V.; formal analysis: S.V. and I.T.; investigation: M.M., S.V. and I.T.; resources: S.V.; data curation: M.M. and S.V.; writing—original draft preparation: M.M.; writing—review and editing: S.V.; visualization: S.V. and I.T.; supervision: M.M.; project administration: M.M. All authors have read and agreed to the published version of the manuscript.

Funding: No funds were allocated for the development of this project.

Institutional Review Board Statement: Not Applicable.

Informed Consent Statement: Not Applicable.

Data Availability Statement: The authors confirm that the data supporting the findings of this study are available within the article.

Acknowledgments: The authors thank the anonymous reviewers who have read the manuscript carefully and have proposed pertinent corrections that have improved our manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Altenbach, H.; Öchsner, A. *Cellular and Porous Materials in Structures and Processes*; Springer: Berlin/Heidelberg, Germany, 2010.
2. Öchsner, A.; Augustin, C. *Multifunctional Metallic Hollow Sphere Structures*; Springer: Berlin/Heidelberg, Germany, 2009.
3. Eringen, A.C. Theory of thermo-microstretch elastic solids. *Int. J. Eng. Sci.* **1990**, *28*, 1291–1301. [CrossRef]
4. Eringen, A.C. *Microcontinuum Field Theories*; Springer: Berlin/Heidelberg, Germany, 1999.
5. Iesan, D.; Pompei, A. Equilibrium theory of microstretch elastic solids. *Int. J. Eng. Sci.* **1995**, *33*, 399–410. [CrossRef]
6. Iesan, D.; Quintanilla, R. Thermal stresses in microstretch bodies. *Int. J. Eng. Sci.* **2005**, *43*, 885–907. [CrossRef]
7. Ciarletta, M. On the bending of microstretch elastic plates. *Int. J. Eng. Sci.* **1995**, *37*, 1309–1318. [CrossRef]
8. Abouelregal, A.E.; Marin, M. The size-dependent thermoelastic vibrations of nanobeams subjected to harmonic excitation and rectified sine wave heating. *Mathematics* **2020**, *8*, 1128. [CrossRef]
9. Abouelregal, A.E.; Marin, M. The response of nanobeams with temperature-dependent properties using state-space method via modified couple stress theory. *Symmetry* **2020**, *12*, 1276 [CrossRef]
10. Zhang, L.; Bhatti, M.M.; Michaelides, E.E.; Marin, M.; Ellahi, R. Hybrid nanofluid flow towards an elastic surface with tantalum and nickel nanoparticles, under the influence of an induced magnetic field. *Eur. Phys. J. Spec. Top.* **2022**, *231*, 521–533. [CrossRef]
11. Ghita, C.; PopbIleana, N.; Popescu, N. Existence result of an effective stress for an isotropic visco-plastic composite. *Comput. Mater. Sci.* **2012**, *64*, 52–56. [CrossRef]
12. Marin, M.; Florea, O. On temporal behaviour of solutions in thermoelasticity of porous micropolar bodies. *An. St. Univ. Ovidius Constanta* **2014**, *22*, 169–188 [CrossRef]
13. Pop, N. An algorithm for solving nonsmooth variational inequalities arising in fricti-onal quasistatic contact problems. *Carpathian J. Math.* **2008**, *24*, 110–119
14. Marin, M.; Ellahi, R.; Vlase, S.; Bhatti, M.M. On the decay of exponential type for the solutions in a dipolar elastic body. *J. Taibah Univ. Sci.* **2020**, *14*, 534–540. [CrossRef]
15. Vlase, S.; Năstac, C.; Marin, M.; Mihălcică, M. A Method for the Study of teh Vibration of Mechanical Bars Systems with Symmetries. *Acta Tech. Napoc. Ser. Appl. Math. Mech. Eng.* **2017**, *60*, 539–544.
16. Marin, M.; Agarwal, R.P.; Mahmoud, S.R. Nonsimple material problems addressed by the Lagrange's identity. *Bound. Value Probl.* **2013**, *2013*, 135. [CrossRef]
17. Vlase, S.; Marin, M.; Scutaru, M.L.; Munteanu, R. Coupled transverse and torsional vibrations in a mechanical system with two identical beams. *Aip Adv.* **2017**, *7*, 065301. [CrossRef]

18. Craciun, E.-M.; Barbu, L. Compact closed form solution of the incremental plane states in a prestressed elastic composite with an elliptical hole. *Z. Angew. Math. Mech. ZAMM* **2015**, *95*, 193–199. [CrossRef]
19. Craciun, E.-M.; Sadowski, T.; Rabaea, A. Stress concentration in an anisotropic body with three equal collinear cracks in Mode II of fracture. I. Analytical study. *Z. Angew. Math. Mech. ZAMM* **2014**, *94*, 721–729. [CrossRef]
20. Straughan, B. Heat Waves. In *Applied Mathematical Sciences*; Springer: New York, NY, USA, 2011; Volume 177.
21. Marin, M.; Chirilă, A.; Othman, M.I.A. An extension of Dafermos's results for bodies with a dipolar structure. *Appl. Math. Comput.* **2019**, *361*, 680–688. [CrossRef]
22. Eringen, A.C. *Fracture*; Liebowitz, H., Ed.; Academic Press: New York, NY, USA, 1968; Volume 2.
23. Freund, L.B. Energy flux into the tip of an extending crack in an elastic solid. *J. Elast.* **1972**, *2*, 341–349. [CrossRef]
24. Gurtin, M.E. Thermodynamics and choesive zone in fracture. *ZAMP* **1979**, *30*, 991–1003.
25. Gurtin, M.E.; Yatomi, C. On the energy release rate in elastodynamic crack propagation. *Arch. Rat. Mech. Anal.* **1980**, *74*, 231–247. [CrossRef]
26. Nishioka, T.; Alturi, S.N. Path-independent integrals, energy release rates, and general solutions of near-tip fields in mixed-mode dynamic fracture mechanics. *Eng. Fract. Mech.* **1983**, *18*, 1–22. [CrossRef]
27. Zhang, A.B.; Wang, L. Theoretical model of crack branching in magnetoelectric thermoelastic materials. *Int. J. Solids Struct.* **2014**, *51*, 1340–1349. [CrossRef]
28. Tian, W.; Rajapakse, R.K. Theoretical model for crack branching in magnetoelectroelastic solids. *Int. J. Appl. Electromagn. Mech.* **2008**, *27*, 53–74. [CrossRef]
29. Gourgiotis, P.A.; Piccolroaz, A. Steady-state propagation of a mode II crack in couple stress elasticity. *Int. J. Fract.* **2014**, 188, 119–145. [CrossRef]