mathematics

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Some New Mathematical Integral Inequalities Pertaining to Generalized Harmonic Convexity with Applications

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Abstract: The subject of convex analysis and integral inequalities represents a comprehensive and absorbing field of research within the field of mathematical interpretation. In recent times, the strategies of convex theory and integral inequalities have become the subject of intensive research at historical and contemporary times because of their applications in various branches of sciences. In this work, we reveal the idea of a new version of generalized harmonic convexity i.e., an m–polynomial p–harmonic s–type convex function. We discuss this new idea by employing some examples and demonstrating some interesting algebraic properties. Furthermore, this work leads us to establish some new generalized Hermite–Hadamard- and generalized Ostrowski-type integral identities. Additionally, employing Hölder's inequality and the power-mean inequality, we present some refinements of the H–H (Hermite–Hadamard) inequality and Ostrowski inequalities. Finally, we investigate some applications to special means involving the established results. These new results yield us some generalizations of the prior results in the literature. We believe that the methodology and concept examined in this paper will further inspire interested researchers.

Keywords: Hermite–Hadamard inequality; Hölder's inequality; convex function; harmonic convex function; m–polynomial harmonic convex function; s–type convex function

MSC: 26A51; 26A33; 26D10

Citation: Tariq, M.; Sahoo, S.K.; Ntouyas, S.K.; Alsalami, O.M.; Shaikh, A.A.; Nonlaopon, K. Some New Mathematical Integral Inequalities Pertaining to Generalized Harmonic Convexity with Applications. *Mathematics* **2022**, *10*, 3286. https://doi.org/10.3390/math10183286

Academic Editor: Marius Radulescu

Received: 4 August 2022
Accepted: 6 September 2022
Published: 10 September 2022

1. Introduction

During the last century, the notion of convexity and its generalizations have emerged as an interesting field of pure and applied mathematics. This theory plays a crucial and consequential role in applied mathematics, especially in control theory, optimization theory, nonlinear programming and functional analysis. In economics, this theory plays a fundamental role in equilibrium and duality theory. The concept of a convex function is expressed as follows:

A real valued function $\mathbf{Q} : \Im \subseteq \Re \to \Re$ (set of real numbers) is said to be convex iff the following inequality is satisfied (see [1])

$$\mathbf{Q}(\omega\varkappa_1 + (1-\omega)\varkappa_2) \leq \omega\mathbf{Q}(\varkappa_1) + (1-\omega)\mathbf{Q}(\varkappa_2), \tag{1}$$

for all $\varkappa_1, \varkappa_2 \in \Im, \omega \in [0,1]$.

Let $\mathbf{Q} : \Im \subseteq \Re \to \Re$ be a convex function with $\varkappa_1 < \varkappa_2$ and $\varkappa_1, \varkappa_2 \in \Im$. Then the H–H inequality is expressed as follows: (see [2]):

$$\mathbf{Q}\left(\frac{\varkappa_1 + \varkappa_2}{2}\right) \leq \frac{1}{\varkappa_2 - \varkappa_1} \int_{\varkappa_1}^{\varkappa_2} \mathbf{Q}(x)dx \leq \frac{\mathbf{Q}(\varkappa_1) + \mathbf{Q}(\varkappa_2)}{2}. \tag{2}$$

The study of inequality theory gives a huge system for managing symmetrical aspects in real-life circumstances. The well-known features of integral inequalities have a strong chance to manage consistent issues with high capability. This manuscript contributes to a captivating association of integral calculus, special functions and convex functions. The authors foster a novel methodology for examining another class of convex function which is known as an m–polynomial p–harmonic s–type convex function.

Several mathematicians have put their insight into this field, presenting new versions of different types of inequalities with convex sets and convex functions. It is seen that the modern and amazing view point on convexity always provides ideas and fruitful applications in every field and branch of pure and applied mathematics. Among all the inequalities, most extensively used are H–H (Hermite–Hadamard)-type and Ostrowski-type inequalities. These inequalities involving convex functions play a consequential and fundamental role in mathematical analysis as well as in other areas of pure and applied mathematics. Thus, convex analysis and inequalities have been referred to as an absorbing field for the mathematicians due their wide applications in different branches of sciences. The reader can refer to [3–8]. Recently, Toplu et al. [9], investigated a generalized form of convexity called n–polynomial convex function and obtained a corresponding H–H inequality.

Harmonic mean is used to define the harmonic convex set. In 2003, the concept of harmonic set was introduced by Shi [10] and consequently, Anderson et al. [11] and Noor et al. [12] introduced harmonic and p–harmonic convex functions, respectively. Noor [13] generalized the class of n–polynomial convex function, called an n–polynomial harmonic convex function. Recently, İşcan et al. [14] introduced s–type and n–polynomial s–type convex functions.

The focal length f is one-half of the harmonic mean of the distances of the image v and object u from the lens. The thin lens equation is presented as

$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v} = \frac{u+v}{uv} \implies f = \frac{1}{2}\mathcal{H}(u,v)$$

As far as the importance of harmonic mean is concerned, in [15], the authors have examined its significant role in Asian investment opportunities. Curiously, harmonic means have been applied in electric circuits to determine the overall resistance of electrical resistors connected in parallel. That, the absolute obstruction/resistance of several resistors is only half of the harmonic mean of all of the resistors. For instance, in the event that $\mathcal{X}$ and $\mathcal{Y}$ are the resistances offered by two resistors, then the final resistance is given by the equation:

$$\frac{1}{\mathcal{R}_{Total}} = \frac{1}{\mathcal{X}} + \frac{1}{\mathcal{Y}} = \frac{\mathcal{X}+\mathcal{Y}}{\mathcal{X}\mathcal{Y}} \implies \mathcal{R} = \frac{1}{2}\mathcal{H}(\mathcal{X},\mathcal{Y})$$

In 1938, A. Ostrowski presented an inequality, the Ostrowski inequality, to the world of mathematics. This inequality has an extraordinary range of applications in likelihood, mathematical coordination, and numerical investigation. In the present scenario, nobody can disregard and reject its significance and meaning.

As of late, various extensions and generalizations of Ostrowski's inequality utilizing various strategies are composed by many scientists. For instance, Alomari et al. [16] employed s–convex function to get Ostrowski-type disparities. Consequently, Ardic et al. [17] used GA–convex and GG–convex functions. Budak and Sarikaya [18] likewise obtained some weighted Ostrowski-type inequalities for differentiable convex functions. Iscan [19] acquired some Ostrowski type inequalities utilizing the class of harmonically s–convex functions. Mohsin et al. [20] got new variants of Ostrowski-type inequalities utilizing harmonically h–convex functions.

The motivation of this work reverberates in all aspects of this article. This paper has numerous reasons. Our first aim is to present the idea of the m-polynomial p-harmonic s–type convex functions. Considering two identities, we determined a few H–H type inequalities involving notable integral inequalities such as Hölder's inequality and power-mean inequality. The second fundamental goal is to gather the results from our discoveries for special means such as arithmetic mean, geometric mean and harmonic mean.

The paper is coordinated as follows. In Section 2, we review some essential and principal definitions related to the new convex function. In Section 3, we give the definition of m-polynomial p-harmonic s–type convex functions and explain their algebraic properties and formulate some examples. In Section 4, a H–H-type inequality employing the newly introduced harmonic convexity is presented. In Section 5, we build up the H–H type inequalities for differentiable functions as refinements. In Section 6, we present one new identity and, employing this, some Ostrowski-type inequalities for the aforementioned strategy are established. In Section 7, we present the applications of our outcomes to special means.

Before we start, we need the following necessary known definitions and literature references. Throughout the paper, for brevity we have used "poly" for polynomial and H–H for Hermite–Hadamard.

2. Preliminaries

For the sake of completeness, it will be better to explore and investigate the preliminary section due to the number of definitions. In this section, we will discuss some known concepts and definitions which we need in our investigation in further sections. We begin by introducing harmonic convex functions, p-harmonic convex functions and s-type convex functions. We conclude this section with recalling the m-polynomial convex function, which will be required in our studies.

Definition 1 ([21])**.** *A function $\boldsymbol{Q} : \mathfrak{I} = [\varkappa_1, \varkappa_2] \subseteq \mathfrak{R} \setminus \{0\} \to \mathfrak{R}$ is said to be harmonic convex, if*

$$\boldsymbol{Q}\left(\frac{\varkappa_1 \varkappa_2}{\omega \varkappa_2 + (1-\omega)\varkappa_1}\right) \leq \omega \boldsymbol{Q}(\varkappa_1) + (1-\omega)\boldsymbol{Q}(\varkappa_2), \tag{3}$$

holds for all $\varkappa_1, \varkappa_2 \in \mathfrak{I}$ and $\omega \in [0,1]$.

Definition 2 ([22])**.** *A function $\boldsymbol{Q} : \mathfrak{I} = [\varkappa_1, \varkappa_2] \subseteq \mathfrak{R} \setminus \{0\} \to \mathfrak{R}$ is said to be p–harmonic convex, if*

$$\boldsymbol{Q}\left[\frac{\varkappa_1^p \varkappa_2^p}{\omega \varkappa_2^p + (1-\omega)\varkappa_1^p}\right]^{\frac{1}{p}} \leq \omega \boldsymbol{Q}(\varkappa_1) + (1-\omega)\boldsymbol{Q}(\varkappa_2), \tag{4}$$

holds for all $\varkappa_1, \varkappa_2 \in \mathfrak{I}$ and $\omega \in [0,1]$.

Note that if we choose $\omega = \frac{1}{2}$ in (4), we get the following Jensen p–harmonic convex function.

$$\mathbf{Q}\left[\frac{2\varkappa_1^p \varkappa_2^p}{\varkappa_1^p + \varkappa_2^p}\right]^{\frac{1}{p}} \leq \frac{\mathbf{Q}(\varkappa_1) + \mathbf{Q}(\varkappa_2)}{2},$$

holds for all $\varkappa_1, \varkappa_2 \in \mathfrak{I}$.

If we choose $(p = 1)$, then p–harmonic convex functions reduce to classical harmonic convex functions.

Definition 3 ([14])**.** *A function $\boldsymbol{Q} : \mathfrak{I} \to \mathfrak{R}$, is said to be an s–type convex function if*

$$\boldsymbol{Q}(\omega \varkappa_1 + (1-\omega)\varkappa_2) \leq [1 - s(1-\omega)]\boldsymbol{Q}(\varkappa_1) + [1 - s\omega]\boldsymbol{Q}(\varkappa_2), \tag{5}$$

holds $\forall\ \varkappa_1, \varkappa_2 \in \Im$, $s \in [0,1]$ *and* $\omega \in [0,1]$.

Definition 4 ([9]). *A nonnegative function* $Q : \Im \to \Re$ *is called m–poly convex function if for every* $\varkappa_1, \varkappa_2 \in \Im$, $m \in \mathbb{N}$ *and* $\omega \in (0,1]$, *if*

$$Q(\omega\varkappa_1 + (1-\omega)\varkappa_2) \le \frac{1}{m}\sum_{j=1}^{m}\left[1-(1-\omega)^j\right]Q(\varkappa_1) + \frac{1}{m}\sum_{j=1}^{m}\left[1-\omega^j\right]Q(\varkappa_2). \tag{6}$$

Now, we recall the hypergeometric function [23], which is defined by Euler in integral form:

$${}_2F_1(a,b,c,z) = \frac{1}{\beta(b,c-b)}\int_0^1 \omega^{b-1}(1-\omega)^{c-b-1}(1-z\omega)^{-a}d\omega,\ \ c > b > 0, |z| < 1.$$

3. New *m*–Poly *p*–Harmonic *s*–Type Convex Function and Its Properties

The concept of harmonic convexity and its applications have been intensively investigated for a long time by many researchers in numerous disciplines, and attention to this subject has grown tremendously. By making use of the concept of the harmonic convexity and integral inequalities, various estimations and refinements of them have been introduced, and authors have gained numerous perspectives in many research directions such as economics, probability, statistics, engineering and physics.

The objective goal of this section is to introduce a new family of harmonic convexity, namely m–poly p–harmonic s–type convex functions, and to discuss some of its algebraic properties. Further to enhance the quality and utility of this paper, we add lemmas and some interesting and amazing propositions. Some examples pertaining to this newly introduced idea are investigated.

Definition 5. *A nonnegative real-valued function* $Q : \Im \subseteq (0,+\infty) \to [0,+\infty)$ *is called m–poly p–harmonic s–type convex, if*

$$Q\left[\frac{\varkappa_1^p\varkappa_2^p}{\omega\varkappa_2^p + (1-\omega)\varkappa_1^p}\right]^{\frac{1}{p}} \le \frac{1}{m}\sum_{j=1}^{m}[1-(s(1-\omega))^j]Q(\varkappa_1) + \frac{1}{m}\sum_{j=1}^{m}[1-(s\omega)^j]Q(\varkappa_2), \tag{7}$$

holds for every $\varkappa_1, \varkappa_2 \in \Im$, $m \in \mathbb{N}$, $s \in [0,1]$ *and* $\omega \in [0,1]$.

Remark 1. (i) *Choosing* $m = 1$ *in Definition 5, we obtain the following new definition for a p–harmonically s–type convex function:*

$$Q\left[\frac{\varkappa_1^p\varkappa_2^p}{\omega\varkappa_2^p + (1-\omega)\varkappa_1^p}\right]^{\frac{1}{p}} \le [1-(s(1-\omega))]Q(\varkappa_1) + [1-s\omega]Q(\varkappa_2). \tag{8}$$

(ii) *Choosing* $p = 1$ *in Definition 5, we obtain new definition about m–poly harmonically s–type convex function:*

$$Q\left[\frac{\varkappa_1\varkappa_2}{\omega\varkappa_2 + (1-\omega)\varkappa_1}\right] \le \frac{1}{m}\sum_{j=1}^{m}[1-(s(1-\omega))^j]Q(\varkappa_1) + \frac{1}{m}\sum_{j=1}^{m}[1-(s\omega)^j]Q(\varkappa_2). \tag{9}$$

(iii) *When we put* $p = -1$, *Definition 5, yields the definition of an m–poly s–type convex function, which is defined by İşcan (see [14]) .*
(iv) *When we put* $m = 1$, $p = 1$ *and* $s = 0$, *Definition 5, yields the definition of a harmonically P–function (see [24]).*
(v) *When we put* $m = 1$, $p = -1$ *and* $s = 0$, *Definition 5, yields the definition of a P–function [25].*
(vi) *When we put* $m = 1$, $p = 1$ *and* $s = 1$, *Definition 5, yields the definition of a harmonically convex function, which is defined by İşcan (see [21]).*

(vii) *When we put $m=1$, $p=-1$ and $s=1$, Definition 5, yields the Definition (1).*
$(viii)$ *When we put $p=-1$ and $s=1$, Definition 5, yields the definition of an m–poly convex function, which is defined by Kadakal (see [9]).*
(ix) *When we put $m=1$ and $p=1$, Definition 5 yields the following new definition of a harmonically s–type convex function:*

$$\mathbf{Q}\left(\frac{\varkappa_1\varkappa_2}{\omega\varkappa_2+(1-\omega)\varkappa_1}\right)\le[1-(s(1-\omega))]\mathbf{Q}(\varkappa_1)+[1-s\omega]\mathbf{Q}(\varkappa_2). \tag{10}$$

(x) *When we put $m=1$ and $p=-1$, Definition 5, yields the definition namely s–type convex function, which is defined by İşcan (see [14]).*

The best part of this newly introduced definition is that, if we choose different values for m, s and p, it yields new inequalities and explains its relation with some classical established results.

Lemma 1. *Let $m\in\mathbb{N}$ and $s\in[0,1]$, then the following inequalities $\frac{1}{m}\sum_{j=1}^{m}[1-(s(1-\omega))^j]\ge\omega$ and $\frac{1}{m}\sum_{j=1}^{m}[1-(s\omega)^j]\ge(1-\omega)$ hold true for all $\omega\in[0,1]$.*

Proof. The proof is evident. □

Proposition 1. *Let $\mathfrak{I}\subset(0,+\infty)$ be a p–harmonic convex set. Every p–harmonic convex function on a p–harmonic convex set is an m–poly p–harmonic s–type convex function.*

Proof. Using the definition of p–harmonic convex function and from the Lemma 1, since $\omega\le\frac{1}{m}\sum_{j=1}^{m}[1-(s(1-\omega))^j]$ and $1-\omega\le\frac{1}{m}\sum_{j=1}^{m}[1-(s\omega)^j]$ for all $\omega\in[0,1]$, we have

$$\begin{aligned}\mathbf{Q}\left[\frac{\varkappa_1^p\varkappa_2^p}{\omega\varkappa_2^p+(1-\omega)\varkappa_1^p}\right]^{\frac{1}{p}}&\le\omega\mathbf{Q}(\varkappa_1)+(1-\omega)\mathbf{Q}(\varkappa_2)\\&\le\frac{1}{m}\sum_{j=1}^{m}[1-(s(1-\omega)^j]\mathbf{Q}(\varkappa_1)+\frac{1}{m}\sum_{j=1}^{m}[1-(s\omega)^j]\mathbf{Q}(\varkappa_2).\end{aligned}$$

□

Proposition 2. *Every m–poly p–harmonic convex function is an m–poly p–harmonic s–type convex function.*

Proof. Using the definition of m–poly p–harmonic convex function and from (Remark 3, see [14]), we have $\frac{1}{m}\sum_{j=1}^{m}[1-(1-\omega)^j]\le\frac{1}{m}\sum_{j=1}^{m}[1-(s(1-\omega))^j]$ and $\frac{1}{m}\sum_{j=1}^{m}[1-\omega^j]\le\frac{1}{m}\sum_{j=1}^{m}[1-(s\omega)^j]$ for all $\omega\in[0,1]$, $m\in\mathbb{N}$ and $s\in[0,1]$.

$$\begin{aligned}\mathbf{Q}\left[\frac{\varkappa_1^p\varkappa_2^p}{\omega\varkappa_2^p+(1-\omega)\varkappa_1^p}\right]^{\frac{1}{p}}&\le\frac{1}{m}\sum_{j=1}^{m}[1-(1-\omega)^j]\mathbf{Q}(\varkappa_1)+\frac{1}{m}\sum_{j=1}^{m}[1-\omega^j]\mathbf{Q}(\varkappa_2)\\&\le\frac{1}{m}\sum_{j=1}^{m}[1-(s(1-\omega))^j]\mathbf{Q}(\varkappa_1)+\frac{1}{m}\sum_{j=1}^{m}[1-(s\omega)^j]\mathbf{Q}(\varkappa_2).\end{aligned}$$

□

Proposition 3. *Every m–poly p–harmonic s–type convex function is a p–harmonic h–convex function with* $h(\omega) = \frac{1}{m}\sum_{j=1}^{m}[1-(s(1-\omega))^j]$.

Proof.

$$\mathbf{Q}\left[\frac{\varkappa_1^p \varkappa_2^p}{\omega\varkappa_2^p + (1-\omega)\varkappa_1^p}\right]^{\frac{1}{p}} \leq \frac{1}{m}\sum_{j=1}^{m}[1-(s(1-\omega))^j]\mathbf{Q}(\varkappa_1) + \frac{1}{m}\sum_{j=1}^{m}[1-(s\omega)^j]\mathbf{Q}(\varkappa_2)$$
$$\leq h(\omega)\mathbf{Q}(\varkappa_1) + h(1-\omega)\mathbf{Q}(\varkappa_2).$$

□

Remark 2. (i) *If we put* $p = 1$ *in Proposition 2, then it yields the harmonically h–convex function, which was introduced by Noor et al. [26].*
(ii) *If we put* $p = -1$ *in Proposition 2, then it yields the h–convex function, which was defined by Varošanec et al. [27].*

Now, to enhance the stability of the newly introduced definition of m–poly p–harmonic s–type convex functions, we give some examples.

Example 1. *If* $\mathbf{Q}(x) = x^p$ *is a non-decreasing harmonic convex function and* $p \geq 1$, *then it is a p–harmonic convex function (see [28]). So, taking Proposition 1 into consideration, it is an m–poly p–harmonic s–type convex function.*

Example 2. *Using the literature of a published paper (see [28]), we say that if* $\mathbf{Q}(x) = x^2 e^{x^2}$ *is a non-decreasing harmonic convex function on* $(0,1)$ *and* $p \geq 1$, *then it is p–harmonic convex function. So, taking Proposition 1, into consideration it is an m–poly p–harmonic s–type convex function.*

Example 3. *İşcan proved that in (see [28]),* $\mathbf{Q}(x) = e^x$ *is p–harmonic convex function for* $p \geq 1$ *. So, according to Proposition 1, it is an m–poly p–harmonic s–type convex function.*

Example 4. *Every non-decreasing harmonic convex function and* $p \geq 1$ *is p–harmonic convex function (see [28]). Therefore* $\mathbf{Q}(x) = 1/x^2$ *is p–harmonic for nonnegative values of x. So, taking Proposition 1, into consideration it is an m–poly p–harmonic s–type convex function.*

Now, before presenting our main results, we study some algebraic properties of the newly introduced function.

Theorem 1. *Let* $\mathbf{Q}, \mathbf{O} : \mathfrak{I} = [\varkappa_1, \varkappa_2] \to \mathfrak{R}$. *If* $\mathbf{Q}$ *and* $\mathbf{O}$ *are two m–poly p–harmonic s–type convex functions, then*

1. $\mathbf{Q} + \mathbf{O}$ *is m–poly p–harmonic s–type convex function.*
2. *For nonnegative real numbers c,* $c\mathbf{Q}$ *is an m–poly p–harmonic s–type convex function.*

Proof. (1) Let $\mathbf{Q}$ and $\mathbf{O}$ be m–poly p–harmonic s–type convex, then

$$\begin{aligned}
&(\mathbf{Q}+\mathbf{O})\left[\frac{\varkappa_1^p\varkappa_2^p}{\omega\varkappa_2^p+(1-\omega)\varkappa_1^p}\right]^{\frac{1}{p}} \\
&= \mathbf{Q}\left[\frac{\varkappa_1^p\varkappa_2^p}{\omega\varkappa_2^p+(1-\omega)\varkappa_1^p}\right]^{\frac{1}{p}} + \mathbf{O}\left[\frac{\varkappa_1^p\varkappa_2^p}{\omega\varkappa_2^p+(1-\omega)\varkappa_1^p}\right]^{\frac{1}{p}} \\
&\leq \frac{1}{m}\sum_{j=1}^{m}[1-(s(1-\omega))^j]\mathbf{Q}(\varkappa_1) + \frac{1}{m}\sum_{j=1}^{m}[1-(s\omega)^j]\mathbf{Q}(\varkappa_2) \\
&+ \frac{1}{m}\sum_{j=1}^{m}[1-(s(1-\omega))^j]\mathbf{O}(\varkappa_1) + \frac{1}{m}\sum_{j=1}^{m}[1-(s\omega)^j]\mathbf{O}(\varkappa_2) \\
&= \frac{1}{m}\sum_{j=1}^{m}[1-(s(1-\omega))^j][\mathbf{Q}(\varkappa_1)+\mathbf{O}(\varkappa_1)] + \frac{1}{m}\sum_{j=1}^{m}[1-(s\omega)^j][\mathbf{Q}(\varkappa_2)+\mathbf{O}(\varkappa_2)] \\
&= \frac{1}{m}\sum_{j=1}^{m}[1-(s(1-\omega))^j](\mathbf{Q}+\mathbf{O})(\varkappa_1) + \frac{1}{m}\sum_{j=1}^{m}[1-(s\omega)^j](\mathbf{Q}+\mathbf{O})(\varkappa_2).
\end{aligned}$$

(2) Let $\mathbf{Q}$ be m–poly p–harmonic s–type convex, then

$$\begin{aligned}
&(c\mathbf{Q})\left[\frac{\varkappa_1^p\varkappa_2^p}{\omega\varkappa_2^p+(1-\omega)\varkappa_1^p}\right]^{\frac{1}{p}} \\
&\leq c\left[\frac{1}{m}\sum_{j=1}^{m}[1-(s(1-\omega))^j]\mathbf{Q}(\varkappa_1) + \frac{1}{m}\sum_{j=1}^{m}[1-(s\omega)^j]\mathbf{Q}(\varkappa_2)\right] \\
&= \frac{1}{m}\sum_{j=1}^{m}[1-(s(1-\omega))^j]c\mathbf{Q}(\varkappa_1) + \frac{1}{m}\sum_{j=1}^{m}[1-(s\omega)^j]c\mathbf{Q}(\varkappa_2) \\
&= \frac{1}{m}\sum_{j=1}^{m}[1-(s(1-\omega))^j](c\mathbf{Q})(\varkappa_1) + \frac{1}{m}\sum_{j=1}^{m}[1-(s\omega)^j](c\mathbf{Q})(\varkappa_2),
\end{aligned}$$

which completes the proof. □

Remark 3. *(i) Choosing $m = 1$ in Theorem 1, we get that $\mathbf{Q}+\mathbf{O}$ and $c\mathbf{Q}$ are p–harmonic s–type convex functions.*
(ii) Choosing $p = 1$ in Theorem 1, we get that $\mathbf{Q}+\mathbf{O}$ and $c\mathbf{Q}$ are m–poly harmonic s–type convex functions.
(iii) Choosing $m = 1$ and $p = 1$ in Theorem 1, we get that $\mathbf{Q}+\mathbf{O}$ and $c\mathbf{Q}$ are harmonic s–type convex functions.
(iv) Choosing $p = -1$ in Theorem 1, we get that $\mathbf{Q}+\mathbf{O}$ and $c\mathbf{Q}$ are m–poly s–type convex functions.
(v) Choosing $m = 1$ and $p = -1$ in Theorem 1, we get that $\mathbf{Q}+\mathbf{O}$ and $c\mathbf{Q}$ are s–type convex functions.

Theorem 2. *Let $\mathbf{Q} : \mathfrak{I} \to J$ be a p–harmonic convex function and $\mathbf{O} : J \to \mathfrak{R}$ be a non-decreasing and m–poly s–type convex function. Then the function $\mathbf{O} \circ \mathbf{Q} : \mathfrak{I} \to \mathfrak{R}$ is m–poly p–harmonic s–type convex.*

Proof. For all $\varkappa_1, \varkappa_2 \in \Im$, and $\omega \in [0,1]$, we have

$$\begin{aligned}
&(\mathbf{O} \circ \mathbf{Q})\left[\frac{\varkappa_1^p \varkappa_2^p}{\omega \varkappa_2^p + (1-\omega)\varkappa_1^p}\right]^{\frac{1}{p}} \\
&= \mathbf{O}\left(\mathbf{Q}\left[\frac{\varkappa_1^p \varkappa_2^p}{\omega \varkappa_2^p + (1-\omega)\varkappa_1^p}\right]^{\frac{1}{p}}\right) \\
&\leq \mathbf{O}(\omega \mathbf{Q}(\varkappa_1) + (1-\omega)\mathbf{Q}(\varkappa_2)) \\
&\leq \frac{1}{m}\sum_{j=1}^{m}[1-(s(1-\omega))^j]\mathbf{O}(\mathbf{Q}(\varkappa_1)) + \frac{1}{m}\sum_{j=1}^{m}[1-(s\omega)^j]\mathbf{O}(\mathbf{Q}(\varkappa_2)) \\
&= \frac{1}{m}\sum_{j=1}^{m}[1-(s(1-\omega))^j](\mathbf{O} \circ \mathbf{Q})(\varkappa_1) + \frac{1}{m}\sum_{j=1}^{m}[1-(s\omega)^j](\mathbf{O} \circ \mathbf{Q})(\varkappa_2),
\end{aligned}$$

which completes the proof. □

Remark 4. *(i) If we put $m = 1$ in Theorem 2, then we obtain the following inequality*

$$(\boldsymbol{O} \circ \boldsymbol{Q})\left[\frac{\varkappa_1^p \varkappa_2^p}{t\varkappa_2^p + (1-t)\varkappa_1^p}\right]^{\frac{1}{p}} \leq [1-(s(1-\omega))](\boldsymbol{O} \circ \boldsymbol{Q})(\varkappa_1) + [1-(s\omega)](\boldsymbol{O} \circ \boldsymbol{Q})(\varkappa_2).$$

(ii) If we put $p = 1$ in Theorem 2, then we obtain the following inequality

$$(\boldsymbol{O} \circ \boldsymbol{Q})\left[\frac{\varkappa_1 \varkappa_2}{\omega\varkappa_2 + (1-\omega)\varkappa_1}\right] \leq \frac{1}{m}\sum_{j=1}^{m}[1-(s(1-\omega))^j](\boldsymbol{O} \circ \boldsymbol{Q})(\varkappa_1) + \frac{1}{m}\sum_{j=1}^{m}[1-(s\omega)^j](\boldsymbol{O} \circ \boldsymbol{Q})(\varkappa_2).$$

(iii) If we put $m = 1$ and $p = 1$ in Theorem 2, then we obtain the following inequality

$$(\boldsymbol{O} \circ \boldsymbol{Q})\left[\frac{\varkappa_1 \varkappa_2}{\omega\varkappa_2 + (1-\omega)\varkappa_1}\right] \leq [1-(s(1-\omega))](\boldsymbol{O} \circ \boldsymbol{Q})(\varkappa_1) + [1-(s\omega)](\boldsymbol{O} \circ \boldsymbol{Q})(\varkappa_2).$$

(iv) If we put $p = -1$ in Theorem 2, then we obtain the following inequality

$$(\boldsymbol{O} \circ \boldsymbol{Q})(\omega\varkappa_1 + (1-\omega)\varkappa_2) \leq \frac{1}{m}\sum_{j=1}^{m}[1-(s(1-\omega))^j](\boldsymbol{O} \circ \boldsymbol{Q})(\varkappa_1) + \frac{1}{m}\sum_{j=1}^{m}[1-(s\omega)^j](\boldsymbol{O} \circ \boldsymbol{Q})(\varkappa_2).$$

(v) If we put $m = 1$ and $p = -1$ in Theorem 2, then we obtain the following inequality:

$$(\boldsymbol{O} \circ \boldsymbol{Q})(\omega\varkappa_1 + (1-\omega)\varkappa_2) \leq [1-(s(1-\omega))](\boldsymbol{O} \circ \boldsymbol{Q})(\varkappa_1) + [1-(s\omega)](\boldsymbol{O} \circ \boldsymbol{Q})(\varkappa_2).$$

Theorem 3. *Let $0 < \varkappa_1 < \varkappa_2$, $\boldsymbol{Q}_j : [\varkappa_1, \varkappa_2] \to [0, +\infty)$ be a class of m–poly p–harmonic s–type convex functions and $\boldsymbol{Q}(u) = \sup_j \boldsymbol{Q}_j(u)$. Then $\boldsymbol{Q}$ is an m–poly p–harmonic s–type convex function and $U = \{u \in [\varkappa_1, \varkappa_2] : \boldsymbol{Q}(u) < +\infty\}$ is an interval.*

Proof. Let $\varkappa_1, \varkappa_2 \in U$ and $\omega \in [0,1]$, then

$$\begin{aligned}
&\mathbf{Q}\left[\frac{\varkappa_1^p \varkappa_2^p}{\omega \varkappa_2^p + (1-\omega)\varkappa_1^p}\right]^{\frac{1}{p}} \\
&= \sup_j \mathbf{Q}_j\left[\frac{\varkappa_1^p \varkappa_2^p}{\omega \varkappa_2^p + (1-\omega)\varkappa_1^p}\right]^{\frac{1}{p}} \\
&\leq \frac{1}{m}\sum_{j=1}^{m}[1-(s(1-\omega))^j] \sup_j \mathbf{Q}_j(\varkappa_1) + \frac{1}{m}\sum_{j=1}^{m}[1-(s\omega)^j] \sup_j \mathbf{Q}_j(\varkappa_2) \\
&= \frac{1}{m}\sum_{j=1}^{m}[1-(s(1-\omega))^j]\mathbf{Q}(\varkappa_1) + \frac{1}{m}\sum_{j=1}^{m}[1-(s\omega)^j]\mathbf{Q}(\varkappa_2) < +\infty,
\end{aligned}$$

which completes the proof. □

Remark 5. *If we put $p = 1$ and $s = 1$ in Theorem 3, then we get Theorem (2.2) in [13].*

4. (H–H) Type Inequality for *m*–Poly *p*–Harmonic *s*–Type Convex Functions

The Hermite–Hadamard-type inequality was first studied for convex functions and has been examined and investigated extensively in different directions. This inequality plays an amazing role in the literature; no one can deny its applications and fruitful uses. Numerous extensions, generalizations and improvements have appeared in the literature of this inequality. Many researchers have collaborated on numerous concepts in the field of inequalities. This type of inequality has remained an area of great interest due to its widespread perspective and importance in the area of pure and applied sciences.

The purpose of this section is to derive a new inequality of the (H–H) type using m–poly p–harmonic s–type convexity. Further, some corollaries and remarks are presented.

Theorem 4. *Let $\mathbf{Q} : [\varkappa_1, \varkappa_2] \to [0, +\infty)$ be an m–poly p–harmonic s–type convex function. If $\mathbf{Q} \in L[\varkappa_1, \varkappa_2]$, then*

$$\frac{m}{2\sum_{j=1}^{m}[1-(\frac{s}{2})^j]}\mathbf{Q}\left[\frac{2\varkappa_1^p \varkappa_2^p}{\varkappa_1^p + \varkappa_2^p}\right]^{\frac{1}{p}} \leq \frac{p\varkappa_1^p \varkappa_2^p}{\varkappa_2^p - \varkappa_1^p}\int_{\varkappa_1}^{\varkappa_2}\frac{\mathbf{Q}(x)}{x^{p+1}}dx \leq \left[\frac{\mathbf{Q}(\varkappa_1) + \mathbf{Q}(\varkappa_2)}{m}\right]\sum_{j=1}^{m}\left[\frac{j+1-s^j}{j+1}\right]. \quad (11)$$

Proof. Since $\mathbf{Q}$ is m–poly p–harmonic s–type convex function, we have

$$\mathbf{Q}\left[\frac{x^p y^p}{\omega y^p + (1-\omega)x^p}\right]^{\frac{1}{p}} \leq \frac{1}{m}\sum_{j=1}^{m}[1-(s(1-\omega))^j]\mathbf{Q}(x) + \frac{1}{m}\sum_{j=1}^{m}[1-(s\omega)^j]\mathbf{Q}(y),$$

which leads to

$$\mathbf{Q}\left[\frac{2x^p y^p}{x^p + y^p}\right]^{\frac{1}{p}} \leq \frac{1}{m}\sum_{j=1}^{m}[1-(\frac{s}{2})^j]\mathbf{Q}(x) + \frac{1}{m}\sum_{j=1}^{m}[1-(\frac{s}{2})^j]\mathbf{Q}(y).$$

Using a change of variables, we get

$$\mathbf{Q}\left[\frac{2\varkappa_1^p \varkappa_2^p}{\varkappa_1^p + \varkappa_2^p}\right]^{\frac{1}{p}} \leq \frac{1}{m}\sum_{j=1}^{m}[1-(\frac{s}{2})^j]\left\{\mathbf{Q}\left[\frac{\varkappa_1^p \varkappa_2^p}{\left(\omega\varkappa_2^p + (1-\omega)\varkappa_1^p\right)}\right]^{\frac{1}{p}} + \mathbf{Q}\left[\frac{\varkappa_1^p \varkappa_2^p}{\left(\omega\varkappa_1^p + (1-\omega)\varkappa_2^p\right)}\right]^{\frac{1}{p}}\right\}.$$

Integrating the above inequality with respect to ω on $[0,1]$, we obtain

$$\frac{m}{2\sum_{j=1}^{m}[1-(\frac{s}{2})^j]}\mathbf{Q}\left[\frac{2\varkappa_1^p\varkappa_2^p}{\varkappa_1^p+\varkappa_2^p}\right]^{\frac{1}{p}} \leq \frac{p\varkappa_1^p\varkappa_2^p}{\varkappa_2^p-\varkappa_1^p}\int_{\varkappa_1}^{\varkappa_2}\frac{\mathbf{Q}(x)}{x^{p+1}}dx,$$

which completes the left side inequality. For the right side inequality, changing the variable of integration as $x = \left[\frac{\varkappa_1^p\varkappa_2^p}{\omega\varkappa_2^p+(1-\omega)\varkappa_1^p}\right]^{\frac{1}{p}}$ and using Definition 5 for the function $\mathbf{Q}$, we have

$$\begin{aligned}
&\frac{p\varkappa_1^p\varkappa_2^p}{\varkappa_2^p-\varkappa_1^p}\int_{\varkappa_1}^{\varkappa_2}\frac{\mathbf{Q}(x)}{x^{p+1}}dx \\
&= \int_0^1 \mathbf{Q}\left[\frac{\varkappa_1^p\varkappa_2^p}{\omega\varkappa_2^p+(1-\omega)\varkappa_1^p}\right]^{\frac{1}{p}} d\omega \\
&\leq \int_0^1 \left[\frac{1}{m}\sum_{j=1}^{m}[1-(s(1-\omega))^j]\mathbf{Q}(\varkappa_1) + \frac{1}{m}\sum_{j=1}^{m}[1-(s\omega)^j]\mathbf{Q}(\varkappa_2)\right] d\omega \\
&= \frac{\mathbf{Q}(\varkappa_1)}{m}\sum_{j=1}^{m}\int_0^1[1-(s(1-\omega))^j]d\omega + \frac{\mathbf{Q}(\varkappa_2)}{m}\sum_{j=1}^{m}\int_0^1[1-(s\omega)^j]d\omega \\
&= \left[\frac{\mathbf{Q}(\varkappa_1)+\mathbf{Q}(\varkappa_2)}{m}\right]\sum_{j=1}^{m}\left[\frac{j+1-s^j}{j+1}\right],
\end{aligned}$$

which completes the proof. □

Corollary 1. *Choosing $m = 1$ in Theorem 4, then we get the following new H–H type inequality for p–harmonic s–type convex functions:*

$$\frac{1}{2-s}\mathbf{Q}\left[\frac{2\varkappa_1^p\varkappa_2^p}{\varkappa_1^p+\varkappa_2^p}\right]^{\frac{1}{p}} \leq \frac{p\varkappa_1^p\varkappa_2^p}{\varkappa_2^p-\varkappa_1^p}\int_{\varkappa_1}^{\varkappa_2}\frac{\mathbf{Q}(x)}{x^{p+1}}dx \leq \frac{2-s}{2}[\mathbf{Q}(\varkappa_1)+\mathbf{Q}(\varkappa_2)].$$

Corollary 2. *Choosing $p = 1$ in Theorem 4, then we get the following new H–H type inequality for m–poly harmonic s–type convex functions:*

$$\frac{m}{2\sum_{j=1}^{m}[1-(\frac{s}{2})^j]}\mathbf{Q}\left[\frac{2\varkappa_1\varkappa_2}{\varkappa_1+\varkappa_2}\right] \leq \frac{\varkappa_1\varkappa_2}{\varkappa_2-\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\frac{\mathbf{Q}(x)}{x^2}dx \leq \left[\frac{\mathbf{Q}(\varkappa_1)+\mathbf{Q}(\varkappa_2)}{\boldsymbol{n}}\right]\sum_{j=1}^{m}\left[\frac{j+1-s^j}{j+1}\right]. \tag{12}$$

Corollary 3. *Choosing $p = -1$ in Theorem 4, then we get the following new H–H type inequality for m–poly s–type convex functions:*

$$\frac{m}{2\sum_{j=1}^{m}[1-(\frac{s}{2})^j}\mathbf{Q}\left(\frac{\varkappa_1+\varkappa_2}{2}\right) \leq \frac{1}{\varkappa_2-\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\mathbf{Q}(x)dx \leq \left(\frac{\mathbf{Q}(\varkappa_1)+\mathbf{Q}(\varkappa_2)}{m}\right)\sum_{j=1}^{m}\left[\frac{j+1-s^j}{j+1}\right].$$

Remark 6. *Choosing $m = 1$ and $s = 1$ in Theorem 4, we get Theorem (2.5) in [29].*

Remark 7. *Choosing $p = 1$ and $s = 1$ in Theorem 4, we get Theorem (2.3) in [13].*

Remark 8. *Choosing $p = -1$ and $s = 1$ in Theorem 4, we get Theorem (4) in [9].*

Remark 9. *Choosing $m = 1$, $p = 1$ and $s = 1$ in Theorem 4, we get Theorem (3) in [30].*

Remark 10. *Choosing $m = 1$, $p = -1$ and $s = 1$ in Theorem 4, we get the simple H–H inequality in [2].*

5. Refinements of (H–H) Type Inequality

The main objective and goal of this section is to investigate and examine the refinements of H–H type inequality via the newly introduced idea, namely the m–poly p–harmonic s–convex function. Further, here a lemma, power mean and Hölder-type inequality will be required in our studies. We conclude this section by adding some corollaries.

Lemma 2 ([31])**.** *Let $\boldsymbol{Q} : \mathfrak{I} = [\varkappa_1, \varkappa_2] \subseteq \mathfrak{R} \setminus \{0\} \to \mathfrak{R}$ be differentiable function on the I° of $\mathfrak{I}$. If $\boldsymbol{Q}' \in L[\varkappa_1, \varkappa_2]$, then*

$$\frac{\boldsymbol{Q}(\varkappa_1) + \boldsymbol{Q}(\varkappa_2)}{2} - \frac{p\varkappa_1^p \varkappa_2^p}{\varkappa_2^p - \varkappa_1^p} \int_{\varkappa_1}^{\varkappa_2} \frac{\boldsymbol{Q}(x)}{x^{1+p}} dx = \frac{\varkappa_1 \varkappa_2 (\varkappa_2^p - \varkappa_1^p)}{2p} \int_0^1 \frac{\mu(\omega)}{A_\omega^{p+1}} \boldsymbol{Q}'\left(\frac{\varkappa_1 \varkappa_2}{A_\omega}\right) d\omega,$$

where $A_\omega = \left[\omega \varkappa_2^p + (1-\omega)\varkappa_1^p\right]^{\frac{1}{p}}$ *and* $\mu(\omega) = (1 - 2\omega)$.

Theorem 5. *Let $\boldsymbol{Q} : \mathfrak{I} = [\varkappa_1, \varkappa_2] \subseteq \mathfrak{R} \setminus \{0\} \to \mathfrak{R}$ be a differentiable function on the $\mathfrak{I}^\circ$ of $\mathfrak{I}$. If $\boldsymbol{Q}' \in L[\varkappa_1, \varkappa_2]$ and $|\boldsymbol{Q}'|^q$ is m–poly p–harmonic s–type convex function on $\mathfrak{I}$, $q \geq 1$, then*

$$\begin{aligned} &\left|\frac{\boldsymbol{Q}(\varkappa_1) + \boldsymbol{Q}(\varkappa_2)}{2} - \frac{p\varkappa_1^p \varkappa_2^p}{\varkappa_2^p - \varkappa_1^p} \int_{\varkappa_1}^{\varkappa_2} \frac{\boldsymbol{Q}(x)}{x^{1+p}} dx\right| \\ &\leq \frac{\varkappa_1 \varkappa_2 (\varkappa_2^p - \varkappa_1^p)}{2p} \left\{ G_1^{1-\frac{1}{q}} \left[G_2 |\boldsymbol{Q}'(\varkappa_1)|^q + G_3 |\boldsymbol{Q}'(\varkappa_2)|^q \right]^{\frac{1}{q}} \right\}, \end{aligned}$$

where,

$$G_1 = \int_0^1 \frac{|1-2\omega|}{A_\omega^{p+1}} d\omega,\ G_2 = \frac{1}{m} \int_0^1 \frac{|1-2\omega| \sum_{j=1}^m [1-(s(1-\omega))^j]}{A_\omega^{1+p}} d\omega,\ G_3 = \frac{1}{m} \int_0^1 \frac{|1-2\omega| \sum_{j=1}^m [1-(s\omega)^j]}{A_\omega^{1+p}} d\omega.$$

Proof. Using Lemma 2, the power mean inequality and m–poly p–harmonic s–type convexity of the $|\mathbf{Q}'|^q$, we have

$$\begin{aligned}
&\left|\frac{\mathbf{Q}(\varkappa_1)+\mathbf{Q}(\varkappa_2)}{2}-\frac{p\varkappa_1^p\varkappa_2^p}{\varkappa_2^p-\varkappa_1^p}\int_{\varkappa_1}^{\varkappa_2}\frac{\mathbf{Q}(x)}{x^{1+p}}dx\right| \\
&\leq \frac{\varkappa_1\varkappa_2(\varkappa_2^p-\varkappa_1^p)}{2p}\int_0^1\frac{|1-2\omega|}{A_\omega^{p+1}}\left|\mathbf{Q}'\left(\frac{\varkappa_1\varkappa_2}{A_\omega}\right)\right|d\omega \\
&\leq \frac{\varkappa_1\varkappa_2(\varkappa_2^p-\varkappa_1^p)}{2p}\left(\int_0^1\frac{|1-2\omega|}{A_\omega^{p+1}}d\omega\right)^{1-\frac{1}{q}}\left(\int_0^1\frac{|1-2\omega|}{A_\omega^{p+1}}\left|\mathbf{Q}'\left(\frac{\varkappa_1\varkappa_2}{A_\omega}\right)\right|^q d\omega\right)^{\frac{1}{q}} \\
&\leq \frac{\varkappa_1\varkappa_2(\varkappa_2^p-\varkappa_1^p)}{2p}\left(\int_0^1\frac{|1-2\omega|}{A_\omega^{p+1}}d\omega\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1\frac{|1-2\omega|\left[\frac{1}{m}\sum_{j=1}^m[1-(s(1-\omega))^j]|\mathbf{Q}'(\varkappa_1)|^q+\frac{1}{m}\sum_{j=1}^m[1-(s\omega)^j]|\mathbf{Q}'(\varkappa_2)|^q\right]}{A_\omega^{1+p}}d\omega\right)^{\frac{1}{q}} \\
&\leq \frac{\varkappa_1\varkappa_2(\varkappa_2^p-\varkappa_1^p)}{2p}\left(\int_0^1\frac{|1-2\omega|}{A_\omega^{p+1}}d\omega\right)^{1-\frac{1}{q}} \\
&\times\left(\frac{1}{m}\int_0^1\frac{|1-2\omega|\sum_{j=1}^m[1-(s(1-\omega))^j]}{A_\omega^{1+p}}|\mathbf{Q}'(\varkappa_1)|^q d\omega+\frac{1}{m}\int_0^1\frac{|1-2\omega|\sum_{j=1}^m[1-(s\omega)^j]}{A_\omega^{1+p}}|\mathbf{Q}'(\varkappa_2)|^q d\omega\right)^{\frac{1}{q}} \\
&\leq \frac{\varkappa_1\varkappa_2(\varkappa_2^p-\varkappa_1^p)}{2p}\left\{G_1^{1-\frac{1}{q}}\left[G_2|\mathbf{Q}'(\varkappa_1)|^q+G_3|\mathbf{Q}'(\varkappa_2)|^q\right]^{\frac{1}{q}}\right\},
\end{aligned}$$

which completes the proof. □

Corollary 4. *Under the assumptions of Theorem 5 with $m=1$ and $p=-1$, we have the following new result:*

$$\begin{aligned}
&\left|\frac{\mathbf{Q}(\varkappa_1)+\mathbf{Q}(\varkappa_2)}{2}-\frac{1}{\varkappa_2-\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\mathbf{Q}(x)dx\right| \\
&\leq\frac{(\varkappa_2-\varkappa_1)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{s-6}{24}\right)\left\{\left[|\mathbf{Q}'(\varkappa_1)|^q+|\mathbf{Q}'(\varkappa_2)|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

Corollary 5. *Under the assumptions of Theorem 5 with $m=p=1$, we have the following new result:*

$$\begin{aligned}
&\left|\frac{\mathbf{Q}(\varkappa_1)+\mathbf{Q}(\varkappa_2)}{2}-\frac{\varkappa_1\varkappa_2}{\varkappa_2-\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\frac{\mathbf{Q}(x)}{x^2}dx\right| \\
&\leq\frac{\varkappa_1\varkappa_2(\varkappa_2-\varkappa_1)}{2}\left\{G_1'^{1-\frac{1}{q}}\left[G_2'|\mathbf{Q}'(\varkappa_1)|^q+G_3'|\mathbf{Q}'(\varkappa_2)|^q\right]^{\frac{1}{q}}\right\},
\end{aligned}$$

where

$$G_1'=\int_0^1\frac{|1-2\omega|}{A_\omega^2}d\omega,\quad G_2'=\int_0^1\frac{|1-2\omega|[1-s(1-\omega)]}{A_\omega^2}d\omega,$$
$$G_3'=\int_0^1\frac{|1-2\omega|[1-s\omega]}{A_\omega^2}d\omega.$$

Theorem 6. *Let $\mathbf{Q} : \Im = [\varkappa_1, \varkappa_2] \subseteq \Re \setminus \{0\} \to \Re$ be a differentiable function on the $\Im^\circ$ of $\Im$. If $\mathbf{Q}' \in L[\varkappa_1, \varkappa_2]$ and $|\mathbf{Q}'|^q$ is m–poly p–harmonic s–type convex function on $\Im$, $r, q \geq 1$, $\frac{1}{r} + \frac{1}{q} \geq 1$ then*

$$\left|\frac{\mathbf{Q}(\varkappa_1) + \mathbf{Q}(\varkappa_2)}{2} - \frac{p\varkappa_1^p \varkappa_2^p}{\varkappa_2^p - \varkappa_1^p} \int_{\varkappa_1}^{\varkappa_2} \frac{\mathbf{Q}(x)}{x^{1+p}} dx\right|$$
$$\leq \frac{\varkappa_1\varkappa_2(\varkappa_2^p - \varkappa_1^p)}{2p} \left\{ G_4^{\frac{1}{r}} \left[G_5 |\mathbf{Q}'(\varkappa_1)|^q + G_6 |\mathbf{Q}'(\varkappa_2)|^q \right]^{\frac{1}{q}} \right\},$$

where

$$G_4 = \int_0^1 |1 - 2\omega|^r d\omega, \ \ G_5 = \frac{1}{m} \int_0^1 \frac{\sum_{j=1}^m [1 - (s(1-\omega))^j]}{A_\omega^{(1+p)q}} d\omega, \ G_6 = \frac{1}{m} \int_0^1 \frac{\sum_{j=1}^m [1 - (s\omega)^j]}{A_\omega^{(1+p)q}} d\omega.$$

Proof. Considering the equality presented in Lemma 2, Hölder's inequality and p–harmonic s–type convexity of the $|\mathbf{Q}'|^q$, we have

$$\begin{aligned}
&\left|\frac{\mathbf{Q}(\varkappa_1) + \mathbf{Q}(\varkappa_2)}{2} - \frac{p\varkappa_1^p \varkappa_2^p}{\varkappa_2^p - \varkappa_1^p} \int_{\varkappa_1}^{\varkappa_2} \frac{\mathbf{Q}(x)}{x^{1+p}} dx\right| \\
&\leq \frac{\varkappa_1\varkappa_2(\varkappa_2^p - \varkappa_1^p)}{2p} \int_0^1 \frac{|1-2\omega|}{A_\omega^{p+1}} \left|\mathbf{Q}'\left(\frac{\varkappa_1\varkappa_2}{A_\omega}\right)\right| d\omega \\
&\leq \frac{\varkappa_1\varkappa_2(\varkappa_2^p - \varkappa_1^p)}{2p} \left\{ \left(\int_0^1 |1-2\omega|^r d\omega \right)^{\frac{1}{r}} \left(\int_0^1 \frac{1}{A_\omega^{(1+p)q}} \left|\mathbf{Q}'\left(\frac{\varkappa_1\varkappa_2}{A_\omega}\right)\right|^q d\omega \right)^{\frac{1}{q}} \right. \\
&\leq \frac{\varkappa_1\varkappa_2(\varkappa_2^p - \varkappa_1^p)}{2p} \left\{ \left(\int_0^1 |1-2\omega|^r d\omega \right)^{\frac{1}{r}} \right. \\
&\times \left. \left(\int_0^1 \frac{1}{A_\omega^{(1+p)q}} \left[\frac{1}{m} \sum_{j=1}^m [1 - (s(1-\omega))^j] |\mathbf{Q}'(\varkappa_1)|^q + \frac{1}{m} \sum_{j=1}^m [1 - (s\omega)^j] |\mathbf{Q}'(\varkappa_2)|^q \right] d\omega \right)^{\frac{1}{q}} \right\} \\
&= \frac{\varkappa_1\varkappa_2(\varkappa_2^p - \varkappa_1^p)}{2p} \left\{ G_4^{\frac{1}{r}} \left[G_5 |\mathbf{Q}'(\varkappa_1)|^q + G_6 |\mathbf{Q}'(\varkappa_2)|^q \right]^{\frac{1}{q}} \right\},
\end{aligned}$$

which completes the proof. □

Corollary 6. *Under the assumptions of Theorem 6 with $m = 1$ and $p = -1$, we have the following new result:*

$$\left|\frac{\mathbf{Q}(\varkappa_1) + \mathbf{Q}(\varkappa_2)}{2} - \frac{1}{\varkappa_2 - \varkappa_1} \int_{\varkappa_1}^{\varkappa_2} \mathbf{Q}(x) dx\right|$$
$$\leq \frac{(\varkappa_2 - \varkappa_1)}{2} \left(\int_0^1 |1-2\omega|^r d\omega \right)^{\frac{1}{r}} \left[\frac{2-s}{2}\right] \left\{ \left[|\mathbf{Q}'(\varkappa_1)|^q + |\mathbf{Q}'(\varkappa_2)|^q \right]^{\frac{1}{q}} \right\}.$$

Corollary 7. *Under the assumptions of Theorem 6 with $m = p = 1$, we have the following new result:*

$$\left|\frac{\mathbf{Q}(\varkappa_1) + \mathbf{Q}(\varkappa_2)}{2} - \frac{\varkappa_1\varkappa_2}{\varkappa_2 - \varkappa_1} \int_{\varkappa_1}^{\varkappa_2} \frac{\mathbf{Q}(x)}{x^2} dx\right|$$
$$\leq \frac{\varkappa_1\varkappa_2(\varkappa_2 - \varkappa_1)}{2} \left\{ G_4'^{\frac{1}{r}} \left[G_5' |\mathbf{Q}'(\varkappa_1)|^q + G_6' |\mathbf{Q}'(\varkappa_2)|^q \right]^{\frac{1}{q}} \right\},$$

where

$$G_4' = \int_0^1 |1-2\omega|^r d\omega, \;\; G_5' = \int_0^1 \frac{[1-s(1-\omega)]}{A_\omega^{2q}} d\omega, \;\; G_6' = \int_0^1 \frac{[1-s\omega]}{A_\omega^{2q}} d\omega.$$

6. Ostrowski-Type Inequalities

The Ostrowski inequality expresses bounds on the deviation of a function from its integral mean. The great mathematician Ostrowski investigated his famous and popular inequality in 1938. This inequality has lot of applications in the field of numerical analysis, cumulative distribution functions, probability theory and approximation theory.

The main objective and goal of this section is to introduce a new lemma. On the basis of this newly introduced lemma, we make some refinements of the Ostrowski-type inequality with the help of the power mean and Hölder type inequality.

Lemma 3. *Let $\mathbf{Q} : \mathfrak{I} \to \mathfrak{R}$ be a differential mapping on $\mathfrak{I}^\circ$ and $\varkappa_1, \varkappa_2 \in \mathfrak{I}$ with $\varkappa_1 < \varkappa_2$ and $p \in \mathbb{R} \setminus \{0\}$. If $\mathbf{Q} \in \mathcal{L}[\varkappa_1, \varkappa_2]$, then the following equality holds true.*

$$\begin{aligned}
&\mathbf{Q}(x) - p\frac{\varkappa_1^p \varkappa_2^p}{\varkappa_2^p - \varkappa_1^p} \int_{\varkappa_1}^{\varkappa_2} \frac{\mathbf{Q}(u)}{u^{1+p}} du \\
&= \frac{\varkappa_1^p \varkappa_2^p}{p(\varkappa_2^p - \varkappa_1^p)} \Bigg\{ \left(\frac{x^p - \varkappa_1^p}{\varkappa_1^p x^p}\right)^2 \int_0^1 \omega \left[\frac{\varkappa_1^p x^p}{\omega \varkappa_1^p + (1-\omega)x^p}\right]^{1+\frac{1}{p}} \mathbf{Q}'\left(\left[\frac{\varkappa_1^p x^p}{\omega \varkappa_1^p + (1-\omega)x^p}\right]^{\frac{1}{p}}\right) d\omega \\
&- \left(\frac{\varkappa_2^p - x^p}{\varkappa_2^p x^p}\right)^2 \int_0^1 \omega \left[\frac{\varkappa_2^p x^p}{\omega \varkappa_2^p + (1-\omega)x^p}\right]^{1+\frac{1}{p}} \mathbf{Q}'\left(\left[\frac{\varkappa_2^p x^p}{\omega \varkappa_2^p + (1-\omega)x^p}\right]^{\frac{1}{p}}\right) d\omega \Bigg\}.
\end{aligned}$$

Proof. Integration by parts and changing variables of integration yields

$$\begin{aligned}
&\frac{\varkappa_1^p \varkappa_2^p}{p(\varkappa_2^p - \varkappa_1^p)} \Bigg\{ \left(\frac{x^p - \varkappa_1^p}{\varkappa_1^p x^p}\right)^2 \int_0^1 \omega \left[\frac{\varkappa_1^p x^p}{\omega \varkappa_1^p + (1-\omega)x^p}\right]^{1+\frac{1}{p}} \mathbf{Q}'\left(\left[\frac{\varkappa_1^p x^p}{\omega \varkappa_1^p + (1-\omega)x^p}\right]^{\frac{1}{p}}\right) d\omega \\
&- \left(\frac{\varkappa_2^p - x^p}{\varkappa_2^p x^p}\right)^2 \int_0^1 \omega \left[\frac{\varkappa_2^p x^p}{\omega \varkappa_2^p + (1-\omega)x^p}\right]^{1+\frac{1}{p}} \mathbf{Q}'\left(\left[\frac{\varkappa_2^p x^p}{\omega \varkappa_2^p + (1-\omega)x^p}\right]^{\frac{1}{p}}\right) d\omega \Bigg\} \\
&= \frac{\varkappa_1^p \varkappa_2^p}{p(\varkappa_2^p - \varkappa_1^p)} \Bigg\{ p\left(\frac{x^p - \varkappa_1^p}{\varkappa_1^p x^p}\right)^2 \Bigg[p\frac{\varkappa_1^p x^p}{x^p - \varkappa_1^p} \omega \mathbf{Q}\left(\left[\frac{\varkappa_1^p x^p}{\omega \varkappa_1^p + (1-\omega)x^p}\right]^{\frac{1}{p}}\right)\Bigg|_0^1 \\
&- p\frac{\varkappa_1^p x^p}{x^p - \varkappa_1^p} \int_0^1 \mathbf{Q}\left(\left[\frac{\varkappa_1^p x^p}{\omega \varkappa_1^p + (1-\omega)x^p}\right]^{\frac{1}{p}}\right) d\omega \\
&- \left(\frac{\varkappa_2^p - x^p}{\varkappa_2^p x^p}\right)^2 \Bigg[p\frac{\varkappa_2^p x^p}{\varkappa_2^p - x^p} \omega \mathbf{Q}\left(\left[\frac{\varkappa_2^p x^p}{\omega \varkappa_2^p + (1-\omega)x^p}\right]^{\frac{1}{p}}\right)\Bigg|_0^1 + p\frac{\varkappa_2^p x^p}{\varkappa_2^p - x^p} \int_0^1 \mathbf{Q}\left(\left[\frac{\varkappa_2^p x^p}{\omega \varkappa_2^p + (1-\omega)x^p}\right]^{\frac{1}{p}}\right) d\omega \\
&= \frac{\varkappa_1^p \varkappa_2^p}{p(\varkappa_2^p - \varkappa_1^p)} \Bigg\{ p\left(\frac{x^p - \varkappa_1^p}{\varkappa_1^p x^p}\right) \mathbf{Q}(x) - p\frac{x^p - \varkappa_1^p}{x^p \varkappa_1^p} \int_0^1 \mathbf{Q}\left(\left[\frac{\varkappa_1^p x^p}{\omega \varkappa_1^p + (1-\omega)x^p}\right]^{\frac{1}{p}}\right) d\omega \\
&- p\frac{\varkappa_2^p - x^p}{\varkappa_2^p x^p} \mathbf{Q}(x) + p\frac{\varkappa_2^p - x^p}{\varkappa_2^p x^p} \int_0^1 \mathbf{Q}\left(\left[\frac{\varkappa_2^p x^p}{\omega \varkappa_2^p + (1-\omega)x^p}\right]^{\frac{1}{p}}\right) d\omega \\
&= \mathbf{Q}(x) - p\frac{\varkappa_1^p \varkappa_2^p}{\varkappa_2^p - \varkappa_1^p} \int_{\varkappa_1}^{\varkappa_2} \frac{\mathbf{Q}(u)}{u^{1+p}} du.
\end{aligned}$$

□

Theorem 7. *Let $\mathbf{Q} : \mathfrak{I} = [\varkappa_1, \varkappa_2] \subseteq \mathfrak{R} \setminus \{0\} \to \mathfrak{R}$ be a differentiable mapping on $\mathfrak{I}^\circ$, where $\varkappa_1, \varkappa_2 \in \mathfrak{I}$ with $\varkappa_1 < \varkappa_2, \mathbf{Q}' \in \mathcal{L}[\varkappa_1, \varkappa_2]$ and $p \in \mathfrak{R} \setminus \{0\}$. If $|\mathbf{Q}|'$ is an m–poly p–harmonic s–type convex function on $\mathfrak{I}$ for $q \geq 1$ and $s \in [0,1]$, then*

$$\begin{aligned}&\left|\boldsymbol{Q}(x)-p\frac{\varkappa_1^p\varkappa_2^p}{\varkappa_2^p-\varkappa_1^p}\int_{\varkappa_1}^{\varkappa_2}\frac{\boldsymbol{Q}(u)}{u^{1+p}}du\right|\\&\le\frac{\varkappa_1^p\varkappa_2^p}{p(\varkappa_2^p-\varkappa_1^p)}\left\{\left(\frac{x^p-\varkappa_1^p}{\varkappa_1^px^p}\right)^2\left[\left(k_1(p,\varkappa_1,x)\right)^{1-\frac{1}{q}}\left\{|\boldsymbol{Q}^{'}(x)|^qk_3(p,\varkappa_1,x)+|\boldsymbol{Q}^{'}(\varkappa_1)|^qk_4(p,\varkappa_1,x)\right\}^{\frac{1}{q}}\right]\right.\\&\left.-\left(\frac{\varkappa_2^p-x^p}{\varkappa_2^px^p}\right)^2\left[\left(k_2(p,\varkappa_2,x)\right)^{1-\frac{1}{q}}\left\{|\boldsymbol{Q}^{'}(x)|^qk_5(p,\varkappa_2,x)+|\boldsymbol{Q}^{'}(\varkappa_2)|^qk_6(p,\varkappa_2,x)\right\}^{\frac{1}{q}}\right]\right\},\end{aligned}$$

where

$$\begin{aligned}&K_1(p,\varkappa_1,x)=\int_0^1\omega\left[\frac{\varkappa_1^px^p}{\omega\varkappa_1^p+(1-\omega)x^p}\right]^{1+\frac{1}{p}}\\&=\frac{\varkappa_1^{1+p}}{2}\,{}_2F_1\left(1+\frac{1}{p},2,3,1-(\frac{\varkappa_1}{x})^p\right),\\&K_2(p,\varkappa_2,x)=\int_0^1\omega\left[\frac{\varkappa_2^px^p}{\omega\varkappa_2^p+(1-\omega)x^p}\right]^{1+\frac{1}{p}}\\&=\frac{\varkappa_2^{1+p}}{2}\,{}_2F_1\left(1+\frac{1}{p},2,3,1-(\frac{\varkappa_2}{x})^p\right),\\&K_3(p,\varkappa_1,x)=\frac{1}{m}\sum_{j=1}^m\int_0^1\omega(1-(s(1-\omega))^j)\left[\frac{\varkappa_1^px^p}{\omega\varkappa_1^p+(1-\omega)x^p}\right]^{1+\frac{1}{p}}d\omega\\&=\frac{1}{m}\sum_{j=1}^m\frac{\varkappa_1^{1+p}}{2}\,{}_2F_1\left(1+\frac{1}{p},2,3,1-(\frac{a}{x})^p\right)-\frac{s^j\varkappa_1^{1+p}}{(j+1)(j+2)}\,{}_2F_1\left(1+\frac{1}{p},2,j+3,1-(\frac{\varkappa_1}{x})^p\right),\\&K_4(p,\varkappa_1,x)=\frac{1}{m}\sum_{j=1}^m\int_0^1\omega(1-(s\omega)^j)\left[\frac{\varkappa_1^px^p}{\omega\varkappa_1^p+(1-\omega)x^p}\right]^{1+\frac{1}{p}}d\omega\\&=\frac{1}{m}\sum_{j=1}^m\frac{\varkappa_1^{1+p}}{2}\,{}_2F_1\left(1+\frac{1}{p},2,3,1-(\frac{\varkappa_1}{x})^p\right)-\frac{s^j{\varkappa_1}^{1+p}}{(j+2)}\,{}_2F_1\left(1+\frac{1}{p},j+2,j+3,1-(\frac{a}{x})^p\right),\\&K_5(p,\varkappa_2,x)=\frac{1}{m}\sum_{j=1}^m\int_0^1\omega(1-(s(1-\omega))^j)\left[\frac{\varkappa_2^px^p}{\omega\varkappa_2^p+(1-\omega)x^p}\right]^{1+\frac{1}{p}}d\omega\\&=\frac{1}{m}\sum_{j=1}^m\frac{\varkappa_2^{1+p}}{2}\,{}_2F_1\left(1+\frac{1}{p},2,3,1-(\frac{\varkappa_2}{x})^p\right)-\frac{s^j\varkappa_2^{1+p}}{(j+1)(j+2)}\,{}_2F_1\left(1+\frac{1}{p},2,j+3,1-(\frac{\varkappa_2}{x})^p\right),\\&K_6(p,\varkappa_2,x)=\frac{1}{m}\sum_{j=1}^m\int_0^1\omega(1-(s\omega)^j)\left[\frac{\varkappa_2^px^p}{\omega\varkappa_2^p+(1-\omega)x^p}\right]^{1+\frac{1}{p}}d\omega\\&=\frac{1}{m}\sum_{j=1}^m\frac{\varkappa_2^{1+p}}{2}\,{}_2F_1\left(1+\frac{1}{p},2,3,1-(\frac{\varkappa_2}{x})^p\right)-\frac{s^j\varkappa_2^{1+p}}{(j+2)}\,{}_2F_1\left(1+\frac{1}{p},2,j+3,1-(\frac{\varkappa_2}{x})^p\right).\end{aligned}$$

Proof. Considering the equality presented in Lemma 3, applying power-mean inequality and m–poly p–harmonic s–type convexity of the $|\mathbf{Q}'|^q$, we have

$$
\begin{aligned}
&\left|\mathbf{Q}(x)-p\frac{\varkappa_1^p\varkappa_2^p}{\varkappa_2^p-\varkappa_1^p}\int_{\varkappa_1}^{\varkappa_2}\frac{\mathbf{Q}(u)}{u^{1+p}}du\right|\\
&\le \frac{\varkappa_1^p\varkappa_2^p}{p(\varkappa_2^p-\varkappa_1^p)}\Bigg\{\left(\frac{x^p-\varkappa_1^p}{\varkappa_1^p x^p}\right)^2\int_0^1\left|\omega\left[\frac{\varkappa_1^p x^p}{\omega\varkappa_1^p+(1-\omega)x^p}\right]^{1+\frac{1}{p}}\right|\left|\mathbf{Q}'\left(\left[\frac{\varkappa_1^p x^p}{\omega\varkappa_1^p+(1-\omega)x^p}\right]^{\frac{1}{p}}\right)\right|d\omega\\
&-\left(\frac{\varkappa_2^p-x^p}{\varkappa_2^p x^p}\right)^2\int_0^1\left|\omega\left[\frac{\varkappa_2^p x^p}{\omega\varkappa_2^p+(1-\omega)x^p}\right]^{1+\frac{1}{p}}\right|\left|\mathbf{Q}'\left(\left[\frac{\varkappa_2^p x^p}{\omega\varkappa_2^p+(1-\omega)x^p}\right]^{\frac{1}{p}}\right)\right|d\omega\Bigg\}\\
&\le \frac{\varkappa_1^p\varkappa_2^p}{p(\varkappa_2^p-\varkappa_1^p)}\Bigg\{\left(\frac{x^p-\varkappa_1^p}{\varkappa_1^p x^p}\right)^2\Bigg[\left(\int_0^1\omega\left[\frac{\varkappa_1^p x^p}{\omega\varkappa_1^p+(1-\omega)x^p}\right]^{1+\frac{1}{p}}d\omega\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1\omega\left[\frac{\varkappa_1^p x^p}{\omega\varkappa_1^p+(1-\omega)x^p}\right]^{1+\frac{1}{p}}\left|\mathbf{Q}'\left(\left[\frac{\varkappa_1^p x^p}{\omega\varkappa_1^p+(1-\omega)x^p}\right]^{\frac{1}{p}}\right)\right|^q d\omega\right)\Bigg]^{\frac{1}{q}}\\
&-\left(\frac{\varkappa_2^p-x^p}{\varkappa_2^p x^p}\right)^2\Bigg[\left(\int_0^1\omega\left[\frac{\varkappa_2^p x^p}{\omega\varkappa_2^p+(1-\omega)x^p}\right]^{1+\frac{1}{p}}d\omega\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1\omega\left[\frac{\varkappa_2^p x^p}{\omega\varkappa_2^p+(1-\omega)x^p}\right]^{1+\frac{1}{p}}\left|\mathbf{Q}'\left(\left[\frac{\varkappa_2^p x^p}{\omega\varkappa_2^p+(1-\omega)x^p}\right]^{\frac{1}{p}}\right)\right|^q d\omega\right)\Bigg]^{\frac{1}{q}}\Bigg\}\\
&\le \frac{\varkappa_1^p\varkappa_2^p}{p(\varkappa_2^p-\varkappa_1^p)}\times\Bigg\{\left(\frac{x^p-\varkappa_1^p}{\varkappa_1^p x^p}\right)^2\Bigg[\left(\int_0^1\omega\left[\frac{\varkappa_1^p x^p}{\omega\varkappa_1^p+(1-\omega)x^p}\right]^{1+\frac{1}{p}}d\omega\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1\omega\left[\frac{\varkappa_1^p x^p}{\omega\varkappa_1^p+(1-\omega)x^p}\right]^{1+\frac{1}{p}}\left(\frac{1}{m}\sum_{j=1}^m\Big(1-(s(1-\omega))^j\Big)|\mathbf{Q}'(x)|^q+\frac{1}{m}\sum_{j=1}^m\Big(1-(s\omega)^j\Big)|\mathbf{Q}'(\varkappa_1)|^q\right)d\omega\right)\Bigg]^{\frac{1}{q}}\\
&-\left(\frac{\varkappa_2^p-x^p}{\varkappa_2^p x^p}\right)^2\Bigg[\left(\int_0^1\omega\left[\frac{\varkappa_2^p x^p}{\omega\varkappa_2^p+(1-\omega)x^p}\right]^{1+\frac{1}{p}}d\omega\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1\omega\left[\frac{\varkappa_2^p x^p}{\omega\varkappa_2^p+(1-\omega)x^p}\right]^{1+\frac{1}{p}}\left(\frac{1}{m}\sum_{j=1}^m\Big(1-(s(1-\omega))^j\Big)|\mathbf{Q}'(x)|^q+\frac{1}{m}\sum_{j=1}^m\Big(1-(s\omega)^j\Big)|\mathbf{Q}'(\varkappa_2)|^q\right)d\omega\right)\Bigg]^{\frac{1}{q}}\Bigg\}\\
&\le \frac{\varkappa_1^p\varkappa_2^p}{p(\varkappa_2^p-\varkappa_1^p)}\times\Bigg\{\left(\frac{x^p-\varkappa_1^p}{\varkappa_1^p x^p}\right)^2\Bigg[\left(\int_0^1\omega\left[\frac{\varkappa_1^p x^p}{\omega\varkappa_1^p+(1-\omega)x^p}\right]^{1+\frac{1}{p}}d\omega\right)^{1-\frac{1}{q}}\\
&\times\Bigg(\frac{|\mathbf{Q}'(x)|^q}{m}\sum_{j=1}^m\int_0^1\omega(1-(s(1-\omega))^j)\left[\frac{\varkappa_1^p x^p}{\omega\varkappa_1^p+(1-\omega)x^p}\right]^{1+\frac{1}{p}}d\omega\\
&+\frac{|\mathbf{Q}'(\varkappa_1)|^q}{m}\sum_{j=1}^m\int_0^1\omega(1-(s\omega)^j)\left[\frac{\varkappa_1^p x^p}{\omega\varkappa_1^p+(1-\omega)x^p}\right]^{1+\frac{1}{p}}d\omega\Bigg)\Bigg]^{\frac{1}{q}}\\
&-\left(\frac{\varkappa_2^p-x^p}{\varkappa_2^p x^p}\right)^2\Bigg[\left(\int_0^1\omega\left[\frac{\varkappa_2^p x^p}{\omega\varkappa_2^p+(1-\omega)x^p}\right]^{1+\frac{1}{p}}d\omega\right)^{1-\frac{1}{q}}\\
&\times\Bigg(\frac{|\mathbf{Q}'(x)|^q}{m}\sum_{j=1}^m\int_0^1\omega(1-(s(1-\omega))^j)\left[\frac{\varkappa_2^p x^p}{\omega\varkappa_2^p+(1-\omega)x^p}\right]^{1+\frac{1}{p}}d\omega\\
&+\frac{|\mathbf{Q}'(\varkappa_2)|^q}{m}\sum_{j=1}^m\int_0^1\omega(1-(s\omega)^j)\left[\frac{\varkappa_2^p x^p}{\omega\varkappa_2^p+(1-\omega)x^p}\right]^{1+\frac{1}{p}}d\omega\Bigg)\Bigg]^{\frac{1}{q}}\Bigg\}\\
&\le \frac{\varkappa_1^p\varkappa_2^p}{p(\varkappa_2^p-\varkappa_1^p)}\Bigg\{\left(\frac{x^p-\varkappa_1^p}{\varkappa_1^p x^p}\right)^2\Big[\big(k_1(p,\varkappa_1,x)\big)^{1-\frac{1}{q}}\big\{|\mathbf{Q}'(x)|^q k_3(p,\varkappa_1,x)+|\mathbf{Q}'(\varkappa_1)|^q k_4(p,\varkappa_1,x)\big\}^{\frac{1}{q}}\Big]\\
&-\left(\frac{\varkappa_2^p-x^p}{\varkappa_2^p x^p}\right)^2\Big[\big(k_2(p,\varkappa_2,x)\big)^{1-\frac{1}{q}}\big\{|\mathbf{Q}'(x)|^q k_5(p,\varkappa_2,x)+|\mathbf{Q}'(\varkappa_2)|^q k_6(p,\varkappa_2,x)\big\}^{\frac{1}{q}}\Big]\Bigg\}.
\end{aligned}
$$

This completes the proof. □

Theorem 8. *Let $\mathbf{Q} : \mathfrak{I} = [\varkappa_1, \varkappa_2] \subseteq \Re \setminus \{0\} \to \Re$ be a differentiable mapping on $\mathfrak{I}^\circ$, where $\varkappa_1, \varkappa_2 \in \mathfrak{I}$ with $\varkappa_1 < \varkappa_2, \mathbf{Q}' \in \mathcal{L}[\varkappa_1, \varkappa_2]$ and $p \in \Re \setminus \{0\}$. If $|\mathbf{Q}|'$ is an m–poly p–harmonic s–type convex function on $\mathfrak{I}$ for $q > 1, \frac{1}{r} + \frac{1}{q} = 1$ and $s \in [0,1]$, then*

$$\begin{aligned}
&\left|\mathbf{Q}(x) - p\frac{\varkappa_1^p \varkappa_2^p}{\varkappa_2^p - \varkappa_1^p}\int_{\varkappa_1}^{\varkappa_2} \frac{\mathbf{Q}(u)}{u^{1+p}} du\right| \\
&\leq \frac{\varkappa_1^p \varkappa_2^p}{p(\varkappa_2^p - \varkappa_1^p)}\left\{\left(\frac{x^p - \varkappa_1^p}{\varkappa_1^p x^p}\right)^2 \left(\frac{1}{r+1}\right)^{\frac{1}{r}} \left[|\mathbf{Q}^{'}(x)|^q K_7(\varkappa_1, x, p) + |\mathbf{Q}^{'}(\varkappa_1)|^q K_8(\varkappa_1, x, p)\right]^{\frac{1}{q}}\right. \\
&\left. \quad - \left(\frac{\varkappa_2^p - x^p}{\varkappa_2^p x^p}\right)^2 \left(\frac{1}{r+1}\right)^{\frac{1}{r}} \left[|\mathbf{Q}^{'}(x)|^q K_9(\varkappa_2, x, p) + |\mathbf{Q}^{'}(\varkappa_2)|^q K_{10}(\varkappa_2, x, p)\right]^{\frac{1}{q}}\right\}, \quad (13)
\end{aligned}$$

where,

$$\begin{aligned}
K_7(\varkappa_1, x, p) &= \frac{1}{m}\sum_{j=1}^{m}\int_0^1 [1 - (s(1-\omega))^j]\left[\frac{\varkappa_1^p x^p}{\omega\varkappa_1^p + (1-\omega)x^p}\right]^{q(1+\frac{1}{p})} d\omega \\
&= \frac{\varkappa_1^{q(1+p)}}{m}\sum_{j=1}^{m}\left[{}_2F_1\left(q(1+\frac{1}{p}), 1, 2, 1 - (\frac{\varkappa_1}{x})^p\right) - \frac{s^j}{j+1}\,{}_2F_1\left(q(1+\frac{1}{p}), 1, j+2, 1 - (\frac{\varkappa_1}{x})^p\right)\right], \\
K_8(\varkappa_1, x, p) &= \frac{1}{m}\sum_{j=1}^{m}\int_0^1 [1 - (s\omega)^j]\left[\frac{\varkappa_1^p x^p}{\omega\varkappa_1^p + (1-\omega)x^p}\right]^{q(1+\frac{1}{p})} d\omega \\
&= \frac{\varkappa_1^{q(1+p)}}{m}\sum_{j=1}^{m}\left[{}_2F_1\left(q(1+\frac{1}{p}), 1, 2, 1 - (\frac{\varkappa_1}{x})^p\right) - \frac{s^j}{j+1}\,{}_2F_1\left(q(1+\frac{1}{p}), j+1, j+2, 1 - (\frac{\varkappa_1}{x})^p\right)\right], \\
K_9(\varkappa_2, x, p) &= \frac{1}{m}\sum_{j=1}^{m}\int_0^1 [(1 - (s(1-\omega))^j)]\left[\frac{\varkappa_1^p x^p}{\omega\varkappa_2^p + (1-\omega)x^p}\right]^{q(1+\frac{1}{p})} d\omega \\
&= \frac{\varkappa_2^{q(1+p)}}{m}\sum_{j=1}^{m}\left[{}_2F_1\left(q(1+\frac{1}{p}), 1, 2, 1 - (\frac{\varkappa_2}{x})^p\right) - \frac{s^j}{(j+1)}\,{}_2F_1\left(q(1+\frac{1}{p}), 1, j+2, 1 - (\frac{\varkappa_2}{x})^p\right)\right], \\
K_{10}(\varkappa_2, x, p) &= \frac{1}{m}\sum_{j=1}^{m}\int_0^1 [1 - (s\omega)^j]\left[\frac{\varkappa_1^p x^p}{\omega\varkappa_2^p + (1-\omega)x^p}\right]^{q(1+\frac{1}{p})} d\omega \\
&= \frac{\varkappa_2^{q(1+p)}}{m}\sum_{j=1}^{m}\left[{}_2F_1\left(q(1+\frac{1}{p}), 1, 2, 1 - (\frac{\varkappa_2}{x})^p\right) - \frac{s^j}{(j+1)}\,{}_2F_1\left(q(1+\frac{1}{p}), j+1, j+2, 1 - (\frac{\varkappa_2}{x})^p\right)\right].
\end{aligned}$$

Proof. Considering the equality presented in Lemma 3 and applying Hölder's inequality, we have

$$\begin{aligned}
&\left|\mathbf{Q}(x)-p\frac{\varkappa_1^p\varkappa_2^p}{\varkappa_2^p-\varkappa_1^p}\int_{\varkappa_1}^{\varkappa_2}\frac{\mathbf{Q}(u)}{u^{1+p}}du\right| \\
&\leq \frac{\varkappa_1^p\varkappa_2^p}{p(\varkappa_2^p-\varkappa_1^p)}\left\{\left(\frac{x^p-\varkappa_1^p}{\varkappa_1^p x^p}\right)^2\int_0^1\left|\omega\left[\frac{\varkappa_1^p x^p}{\omega\varkappa_1^p+(1-\omega)x^p}\right]^{1+\frac{1}{p}}\right|\left|\mathbf{Q}'\left(\left[\frac{\varkappa_1^p x^p}{\omega\varkappa_1^p+(1-\omega)x^p}\right]^{\frac{1}{p}}\right)\right|d\omega\right. \\
&\left.-\left(\frac{\varkappa_2^p-x^p}{\varkappa_2^p x^p}\right)^2\int_0^1\left|\omega\left[\frac{\varkappa_2^p x^p}{\omega\varkappa_2^p+(1-\omega)x^p}\right]^{1+\frac{1}{p}}\right|\left|\mathbf{Q}'\left(\left[\frac{\varkappa_2^p x^p}{\omega\varkappa_2^p+(1-\omega)x^p}\right]^{\frac{1}{p}}\right)\right|d\omega\right\} \\
&\leq \frac{\varkappa_1^p\varkappa_2^p}{p(\varkappa_2^p-\varkappa_1^p)} \\
&\times\left\{\left(\frac{x^p-\varkappa_1^p}{\varkappa_1^p x^p}\right)^2\left(\int_0^1\omega^r d\omega\right)^{\frac{1}{r}}\left(\int_0^1\left[\frac{\varkappa_1^p x^p}{\omega\varkappa_1^p+(1-\omega)x^p}\right]^{q(1+\frac{1}{p})}\left|\mathbf{Q}'\left(\left[\frac{\varkappa_1^p x^p}{\omega\varkappa_1^p+(1-\omega)x^p}\right]^{\frac{1}{p}}\right)\right|^q d\omega\right)^{\frac{1}{q}}\right. \\
&\left.-\left(\frac{\varkappa_2^p-x^p}{\varkappa_2^p x^p}\right)^2\left(\int_0^1\omega^r d\omega\right)^{\frac{1}{r}}\left(\int_0^1\left[\frac{\varkappa_2^p x^p}{\omega\varkappa_2^p+(1-\omega)x^p}\right]^{q(1+\frac{1}{p})}\left|\mathbf{Q}'\left(\left[\frac{\varkappa_2^p x^p}{\omega\varkappa_2^p+(1-\omega)x^p}\right]^{\frac{1}{p}}\right)\right|^q d\omega\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

By using m–poly p–harmonic s–type convexity of the $|\mathbf{Q}'|^q$, we have

$$\begin{aligned}
&\leq \frac{\varkappa_1^p\varkappa_2^p}{p(\varkappa_2^p-\varkappa_1^p)}\left\{\left(\frac{x^p-\varkappa_1^p}{\varkappa_1^p x^p}\right)^2\left(\int_0^1\omega^r d\omega\right)^{\frac{1}{r}}\left(\int_0^1\left[\frac{\varkappa_1^p x^p}{\omega\varkappa_1^p+(1-\omega)x^p}\right]^{q(1+\frac{1}{p})}\right.\right. \\
&\times\left.\left(\frac{1}{m}\sum_{j=1}^m\left(1-(s(1-\omega))^j\right)|\mathbf{Q}'(x)|^q+\frac{1}{m}\sum_{j=1}^m\left(1-(s\omega)^j\right)|\mathbf{Q}'(\varkappa_1)|^q\right)d\omega\right)^{\frac{1}{q}} \\
&-\left(\frac{\varkappa_2^p-x^p}{\varkappa_2^p x^p}\right)^2\left(\int_0^1\omega^r d\omega\right)^{\frac{1}{r}}\left(\int_0^1\left[\frac{\varkappa_2^p x^p}{\omega\varkappa_2^p+(1-\omega)x^p}\right]^{q(1+\frac{1}{p})}\right. \\
&\times\left.\left.\left(\frac{1}{m}\sum_{j=1}^m\left(1-(s(1-\omega))^j\right)|\mathbf{Q}'(x)|^q+\frac{1}{m}\sum_{j=1}^m\left(1-(s\omega)^j\right)|\mathbf{Q}'(\varkappa_2)|^q\right)d\omega\right)^{\frac{1}{q}}\right\} \\
&\leq \frac{\varkappa_1^p\varkappa_2^p}{p(\varkappa_2^p-\varkappa_1^p)}\left\{\left(\frac{x^p-\varkappa_1^p}{\varkappa_1^p x^p}\right)^2\left(\frac{1}{r+1}\right)^{\frac{1}{r}}\left[|\mathbf{Q}'(x)|^q K_7(\varkappa_1,x,p)+|\mathbf{Q}'(\varkappa_1)|^q K_8(\varkappa_1,x,p)\right]^{\frac{1}{q}}\right. \\
&\left.-\left(\frac{\varkappa_2^p-x^p}{\varkappa_2^p x^p}\right)^2\left(\frac{1}{r+1}\right)^{\frac{1}{r}}\left[|\mathbf{Q}'(x)|^q K_9(\varkappa_2,x,p)+|\mathbf{Q}'(\varkappa_2)|^q K_{10}(\varkappa_2,x,p)\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

This completes the proof. □

7. Applications

The main objective and goal of this section is to attain some new inequalities for the arithmetic, geometric and harmonic means. The following means are well-known and popular due to research background because these means have fruitful importance and magnificent applications in numerical approximation, machine learning, statistics and probability. The harmonic mean is a special case of the power mean. This mean has a lot of importance in numerous field of pure and applied sciences i.e., electric circuit theory, probability, finance, computer science, geometry, trigonometry and statistics. This mean equalizes the weights of each data point because this mean is the most appropriate measure for rates and ratios. In this section, we recall the following special means of two positive numbers $\varkappa_1, \varkappa_2$ with $\varkappa_1 < \varkappa_2$:

(1) The arithmetic mean

$$A = A(\varkappa_1, \varkappa_2) = \frac{\varkappa_1 + \varkappa_2}{2}.$$

(2) The geometric mean

$$G = G(\varkappa_1, \varkappa_2) = \sqrt{\varkappa_1 \varkappa_2}.$$

(3) The harmonic mean

$$H = H(\varkappa_1, \varkappa_2) = \frac{2\varkappa_1 \varkappa_2}{\varkappa_1 + \varkappa_2}.$$

(4) The logarithmic mean

$$L = L(\varkappa_1, \varkappa_2) = \frac{\varkappa_2 - \varkappa_1}{\ln \varkappa_2 - \ln \varkappa_1}.$$

$$H(\varkappa_1, \varkappa_2) \leq G(\varkappa_1, \varkappa_2) \leq L(\varkappa_1, \varkappa_2) \leq A(\varkappa_1, \varkappa_2).$$

Proposition 4. *Let $0 < \varkappa_1 < \varkappa_2$ and $p \geq 1$. Then we have the following inequality*

$$\frac{m}{2\sum_{j=1}^{m}[1-(\frac{s}{2})^j]} H_p(\varkappa_1^p, \varkappa_2^p) \leq \frac{p\varkappa_1^p \varkappa_2^p}{\varkappa_2^p - \varkappa_1^p}\left(\frac{\varkappa_2^{1-p} - \varkappa_1^{1-p}}{1-p}\right) \leq A(\varkappa_1, \varkappa_2)\frac{2}{m}\sum_{j=1}^{m}\left[\frac{j+1-s^j}{j+1}\right]. \tag{14}$$

Proof. When we choose $\mathbf{Q}(x) = x$ for $x > 0$, Theorem 4 yields the inequality (14). □

Proposition 5. *Let $0 < \varkappa_1 < \varkappa_2$ and $p \geq 1$. Then we have the following inequality*

$$\frac{m}{2\sum_{j=1}^{m}[1-(\frac{s}{2})^j]} H_{2p}(\varkappa_1^p, \varkappa_2^p) \leq \frac{p\varkappa_1^p \varkappa_2^p}{\varkappa_2^p - \varkappa_1^p}\left(\frac{\varkappa_2^{\frac{1}{2}-p} - \varkappa_1^{\frac{1}{2}-p}}{\frac{1}{2}-p}\right) \leq A(\sqrt{\varkappa_1}, \sqrt{\varkappa_2})\frac{2}{m}\sum_{j=1}^{m}\left[\frac{j+1-s^j}{j+1}\right]. \tag{15}$$

Proof. When we choose $\mathbf{Q}(x) = \sqrt{x}$ for $x > 0$, Theorem 4 yields inequality (15). □

Proposition 6. *Let $0 < \varkappa_1 < \varkappa_2$ and $p \geq 1$. Then we have the following inequality*

$$\frac{m}{2\sum_{j=1}^{m}[1-(\frac{s}{2})^j]} H(\varkappa_1^p, \varkappa_2^p) \leq \frac{p\varkappa_1^p \varkappa_2^p}{\varkappa_2^p - \varkappa_1^p}\left(\frac{\varkappa_2 - \varkappa_1}{L(\varkappa_1, \varkappa_2)}\right) \leq A(\varkappa_1^p, \varkappa_2^p)\frac{2}{m}\sum_{j=1}^{m}\left[\frac{j+1-s^j}{j+1}\right]. \tag{16}$$

Proof. When we choose $\mathbf{Q}(x) = x^p$ for $x > 0$, Theorem 4 yields inequality (16). □

Proposition 7. *Let $0 < \varkappa_1 < \varkappa_2$ and $p \geq 1$. Then we have the following inequality*

$$\frac{m}{2\sum_{j=1}^{m}[1-(\frac{s}{2})^j]} H_p^2(\varkappa_1^p, \varkappa_2^p) \leq \frac{p\varkappa_1^p \varkappa_2^p}{\varkappa_2^p - \varkappa_1^p}\left(\frac{\varkappa_2^{2-p} - \varkappa_1^{2-p}}{2-p}\right) \leq A(\varkappa_1^2, \varkappa_2^2)\frac{2}{m}\sum_{j=1}^{m}\left[\frac{j+1-s^j}{j+1}\right]. \tag{17}$$

Proof. When we choose $\mathbf{Q}(x) = x^2$ for $x > 0$, Theorem 4 yields inequality (17). □

Proposition 8. *Let $0 < \varkappa_1 < \varkappa_2$ and $p \geq 1$. Then we have the following inequality*

$$\frac{m}{2\sum_{j=1}^{m}[1-(\frac{s}{2})^{j}]}\ln H_p(\varkappa_1^p,\varkappa_2^p) \le \frac{p\varkappa_1^p\varkappa_2^p}{\varkappa_2^p-\varkappa_1^p}\int_{\varkappa_1}^{\varkappa_2}\frac{\ln x}{x^{p+1}}dx \le \ln G(\varkappa_1,\varkappa_2)\frac{2}{m}\sum_{j=1}^{m}\left[\frac{j+1-s^j}{j+1}\right]. \quad (18)$$

Proof. When we choose $\mathbf{Q}(x) = \ln x$ for $x > 0$, Theorem 4 yields inequality (18). □

Proposition 9. *Let $0 < \varkappa_1 < \varkappa_2$. Then we have the following inequality*

$$\frac{m}{2\sum_{j=1}^{m}[1-(\frac{s}{2})^{j}]}e^{H(\varkappa_1,\varkappa_2)} \le \frac{p\varkappa_1^p\varkappa_2^p}{\varkappa_2^p-\varkappa_1^p}\int_{\varkappa_1}^{\varkappa_2}\frac{e^x}{x^{p+1}}dx \le A(e^{\varkappa_1},e^{\varkappa_2})\frac{2}{m}\sum_{j=1}^{m}\left[\frac{j+1-s^j}{j+1}\right]. \quad (19)$$

Proof. When we choose $\mathbf{Q}(x) = e^x$ for $x > 0$, Theorem 4, yields inequality (19). □

Proposition 10. *Let $0 < \varkappa_1 < \varkappa_2$. Then we have the following inequality*

$$A(\sin\varkappa_1,\sin\varkappa_2)\frac{2}{m}\sum_{j=1}^{m}\left[\frac{j+1-s^j}{j+1}\right] \le \frac{p\varkappa_1^p\varkappa_2^p}{\varkappa_2^p-\varkappa_1^p}\int_{\varkappa_1}^{\varkappa_2}\frac{\sin x}{x^{p+1}}dx \le \frac{m}{2\sum_{j=1}^{m}[1-(\frac{s}{2})^{j}]}\sin H_p(\varkappa_1,\varkappa_2). \quad (20)$$

Proof. When we choose $\mathbf{Q}(x) = \sin(-x)$ for $x \in (0, \frac{\pi}{2})$, Theorem 4 yields inequality (20). □

8. Conclusions

The theory of convex analysis and integral inequalities are fruitful and have amazing applications in statistical problems, statistical theory, optimization theory, probability, functional analysis, physics and numerical quadrature formulas. In this article,

(1) we addressed a novel idea of generalized harmonic convex function, namely m–polynomial p–harmonic s–type convex function.
(2) Some nice algebraic properties of the proposed definition are examined.
(3) In the mode of the newly proposed definition, we investigated a new sort of H–H-type inequality.
(4) In addition, we obtained refinements of the H–H type inequality.
(5) Further, a new lemma is presented. By considering this new lemma, several refinements and remarkable extensions of the Ostrowski type inequality are established.
(6) Some applications to special means are attained as well.
In the future, we hope the results of this paper and the new idea can be extended in different directions such as fractional calculus, quantum calculus and time scale calculus.

Author Contributions: Conceptualization, M.T., S.K.S.; methodology, S.K.S., M.T., S.K.N. and O.M.A.; software, S.K.S. and M.T.; validation, S.K.S., M.T, S.K.N. and A.A.S.; formal analysis, M.T., S.K.S., O.M.A. and K.N.; investigation, M.T., S.K.S. and S.K.N.; resources, S.K.S.; data curation, M.T., O.M.A., A.A.S. and K.N.; writing—original draft preparation, S.K.S. and M.T.; writing—review and editing, S.K.S. and M.T.; supervision, S.K.N. and A.A.S. and K.N.; project administration, M.T., S.K.S., K.N.; funding acquisition, K.N. All authors have read and agreed to the published version of the manuscript.

Funding: This research was supported by the Fundamental Fund of Khon Kaen University, Thailand.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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