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Review

Markov Moment Problem and Sandwich Conditions on Bounded Linear Operators in Terms of Quadratic Forms

Octav Olteanu

Department of Mathematics and Informatics, University Politehnica of Bucharest, 060042 Bucharest, Romania; octav.olteanu50@gmail.com

Abstract: As is well-known, unlike the one-dimensional case, there exist nonnegative polynomials in several real variables that are not sums of squares. First, we briefly review a method of approximating any real-valued nonnegative continuous compactly supported function defined on a closed unbounded subset by dominating special polynomials that are sums of squares. This also works in several-dimensional cases. To perform this, a Hahn–Banach-type theorem (Kantorovich theorem on an extension of positive linear operators), a Haviland theorem, and the notion of a moment-determinate measure are applied. Second, completions and other results on solving full Markov moment problems in terms of quadratic forms are proposed based on polynomial approximation. The existence and uniqueness of the solution are discussed. Third, the characterization of the constraints $T_1 \leq T \leq T_2$ for the linear operator T, only in terms of quadratic forms, is deduced. Here, $T_1,\ T,$ and T_2 are bounded linear operators. Concrete spaces, operators, and functionals are involved in our corollaries or examples.

Keywords: polynomial approximation; unbounded subsets; Markov moment problem; positive operators; solution; existence; uniqueness; sums of squares; Banach lattices

MSC: 41A10; 46A22; 46B42; 46B70; 47B65

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1. Introduction

We begin by recalling a few general remarks on approximation theory and its applications. A first fact is that the results of the present review paper focus on the existence and uniqueness of the solution of the solution for a large class of Markov moment problems. The involved solutions are bounded linear operators T mapping $L^1_\nu(F)$ into an order-complete Banach lattice Y, where ν is a moment-determinate positive regular Borel measure on the closed unbounded subset $F \subseteq \mathbb{R}^n,\ n \in \{1, 2, \ldots\}$. The uniqueness follows from the density of polynomials in $L^1_\nu(F)$ (Lemma 1) via the continuity of the operator T. Of note, our first result (Lemma 1) also works for $n \geq 2$, when, unlike the case $n = 1$, there exist moment-determinate measures ν on $\mathbb{R}^n$ for which the polynomials are not dense in $L^2_\nu(F)$ (according to [1]). Thus, for $n \geq 2$, Lemmas 1, 2, and 3 are no longer valid if we turn $L^1_\nu(F)$ into $L^2_\nu(F)$. Moreover, Lemma 1 holds true for any closed (unbounded) subset of $F \subseteq \mathbb{R}^n$. Hence, the nonnegative polynomials on F are dense in the positive cone of $L^1_\nu(F)$. If $F = \mathbb{R}^n$ or $F = \mathbb{R}^n_+$, special convex cones of nonnegative polynomials (which are sums of squares) are dense in the positive cone of $L^1_\nu(F)$ (Lemmas 2 and 3). These remarks lead to the characterizations in terms of quadratic forms in the case $n \geq 2$, which is the main contribution of this review paper. Going back to our aim on the applications of approximation theory, in [2] an interesting connection of a moment problem on $[0, 1]$ (the Hausdorff moment problem) with fixed point theory was pointed out. As a rule, fixed point theorems use an iteration process. In [2], this iteration involved a rational function. The solution of the Hausdorff moment problem under attention is regarded as the fixed point of a transformation appearing naturally from the context. In [3], deep results on the uniqueness of the solutions for moment problems were carefully discussed. The article [4] provided approximation results on various locally compact spaces not necessarily related to the moment problem. In references [5] and [6], the geometric and iterative aspects of optimization theory were emphasized. The article [7] provided several interesting functional equations and new simple proofs of related inequalities involving logarithmic convexity and proposed new conjectures on the subject. In the article [8], an iterative method and its related algorithm, accompanied by a convergence analysis, for solving an optimization problem were discussed. As a general remark, recall that determining the element of minimum norm of a closed convex subset in a Hilbert space, not containing the origin, is also a passing to the limit process associated with an iteration geometrical method. This method can be adapted for a more general setting. The article [9] provides an iterative method for solving and approximating the solution of an operator equation, starting from Newton's global method for convex monotone increasing (or decreasing) operators. Sometimes, the usual iteration defining Newton's method leads to an iteration $A_{k+1} = \varphi(A_k)$, where A_k are self-adjoint operators acting on a Hilbert space and φ is a contractive convex mapping. As is well-known, the convergence of the sequence generated by Newton's method generally only works locally. For convex monotone operators of the C^1 class, it works globally, with the control of the norm of the error (providing the velocity of the convergence). The key point of the article [9] is that the convergence of the sequence of the successive approximations associated with the contraction mapping φ can be handled more easily than that provided by Newton's method. The contraction constant of φ can be determined quite easily. In particular, if the matrices have real entries, the result holds for functions of symmetric matrices. In the end, recall the connection between optimization (such as the best approximation by the elements of a closed subspace of a Hilbert space) and Fourier approximation. This is a useful remark that can be used in controlling the mean square error $g - h_2^2$ between the solutions g, h of the reduced moment problems $\langle g, \psi_j \rangle = y_j,\ \langle h, \psi_j \rangle = m_j,\ j = 0, 1, \ldots, m$ in terms of the squares of the errors $\left(m_j - y_j\right)^2,\ j = 0, 1, \ldots, m$. Here, all the involved functions $g, h,\ \psi_j$ are elements of the Hilbert space $L^2_\mu(F)$, and $F \subseteq \mathbb{R}^n$ is a closed subset:

$$\psi_j(t) = t^j := t_1^{j_1} \cdots t_n^{j_n},\ \ t = (t_1, \ldots, t_n) \in F,\ \ j = (j_1, \ldots, j_n) \in \mathbb{N}^n,$$

where y_j are the exact values of the moments, determined in the experimental stage, while m_j are the modified values for y_j, perturbed by external influences in the real-life measuring stage. Another important field in approximation theory is provided by Korovkin-type theorems and their applications. The article [10] presents such an application in approximating a Kantorovich-type rational operator by means of Korovkin's classical approximating result and completing technique. Associated inequalities are established as well. The papers [11,12] refer to the aspects related to or like those of the moment problem, being inverse problems, as the moment problem is as well. The references [13,14] contain a polynomial approximation on the unbounded subsets discussed in the beginning of this introduction. Another direction of applying these approximation results is that of characterizing sandwich conditions on bounded linear operators defined on $L^1_\nu(F)$ (where ν is moment-determinate) only in terms of quadratic forms (see below). Another well-known application of approximation theory arises from Krein-Milman theorem, which leads to approximation by convex combinations of the extreme points of a compact convex subset in a locally convex space. Such results lead to representation theorems and possible applications for optimization (see the references [14–17]).

Before stating our work on the multidimensional Markov moment problem and the related results studied in Section 3, we recall some basic notions and related terminology on compatible structures on usual spaces, which are used in the sequel. The motivation for this is that all concrete spaces of functions and self-adjoint operators have such natural structures. For complete and related information, see the monographs and books [18–27].

An ordered vector space is a real vector space X endowed with an order relation compatible with the algebraic structure expressed by the following two properties:

$$x, y \in X, \qquad x \le y := x + z \le y + z \text{ for all } z \in X,$$
$$x \le y := \alpha x \le \alpha y \text{ for all real } \alpha \in [0, \infty).$$

An order relation with the above two compatibility properties is called a linear order relation on X. An ordered vector space X with the property that for any $x_1, x_2 \in X$ there exists the least upper bound $sup\{x_1, x_2\} = x_1 \vee x_2$ for the set $\{x_1, x_2\}$ is called a vector lattice. In a vector lattice X, the following basic notations are used:

$$x^+ := x \vee \mathbf{0}, \quad x^- := (-x) \vee \mathbf{0}, \quad |x| := x \vee (-x), \; x \in X.$$

All the usual vector spaces have such a natural order relation. If X is an order vector space, one denotes by X_+ the convex cone with a vertex at $\mathbf{0}$, defined by $X_+ := \{x \in X; x \ge \mathbf{0}\}$. This cone is called the positive cone of X. In the function spaces and in the spaces of symmetric matrices with real entries, as well as in the space of self-adjoint operators acting on an infinite-dimensional Hilbert space, there exist natural norms, which make them Banach spaces. Generally, the structures given by the norms are compatible with the algebraic and order structures on the Banach spaces appearing in applications. An ordered Banach space is a Banach space X endowed with a linear order relation such that the positive cone X_+ is topologically closed and the norm is monotone increasing (isotone) on X_+ :

$$x_1, x_2 \in X, \;\; \mathbf{0} \le x_1 \le x_2 := \|x_1\| \le \|x_2\|.$$

A Banach lattice is a Banach space X, which is also a vector lattice, such that the norm is solid on X :

$$x_1, x_2 \in X, \;\; |x_1| \le |x_2| := \|x_1\| \le \|x_2\|.$$

Almost all Banach function spaces have a natural structure of a Banach lattice. From the above definitions, clearly, any Banach lattice is an ordered Banach space. The converse is false. A first example of an ordered Banach spaces that is not a lattice is the space $\mathcal{SM}(n \times n)$ of all symmetric $n \times n$ matrices with real entries. The order relation on this space is given by:

$$A, B \in \mathcal{SM}(n \times n), \; A \le B \text{ if and only if } \langle Ah, h\rangle \;\le\; \langle Bh, h\rangle \text{ for all } h \in \mathbb{R}^n.$$

From this definition, we infer that $A \le B$ if and only if $B - A$ is positive semidefinite. The norm of the symmetric matrix A is: $A = \sup_{h \le 1} |\langle Ah, h\rangle|$. Here, by $\|h\|$ we denote the Euclidean norm of the vector h. These definitions and notations make sense and have motivations in the infinite-dimensional case. Namely, if H is an arbitrary infinite-dimensional real or complex Hilbert space, a linear operator $A : H \to H$ is called a symmetric operator if $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all $x, y \in H$. A linear symmetric (continuous) operator is called a self-adjoint operator. Of note, any symmetric linear operator acting on H is continuous and therefore self-adjoint thanks to the closed graph theorem. The last definition makes sense for linear operators $A : D(A) \to H$, where $D(A) \subseteq H$ is a vector subspace of H, called the domain of definition of A. In this case, $\langle Ax, y\rangle = \langle x, Ay\rangle$ holds for all $x, y \in D(A)$. To avoid the inconvenience arising from the fact that the real vector space of self-adjoint operators is not a lattice as well as the noncommutativity of the multiplication (composition) of self-adjoint operators (and of symmetric square matrices), the following subspace has been studied. Let $A \in \mathcal{A}(H)$, where $\mathcal{A}(H)$ is the real vector space of all self-adjoint operators acting on H. We define:

$$Y_1(A) := \{V \in \mathcal{A}(H); AV = VA\}, \;\; Y(A) := \{W \in Y_1(A); UW = WU \;\; \forall U \in Y_1(A)\}.$$

Then, $Y(A)$ is an order complete Banach lattice and a commutative real algebra of self-adjoint operators (according to [22]). $\mathcal{P} = \mathbb{R}[t_1, \ldots, t_n]$ is the real vector space of all polynomial functions with real coefficients of n real variables $t_1, \ldots, t_n$.In what follows, F is a closed, unbounded subset of $\mathbb{R}^n$, and $\mathcal{P}_+(F)$ is the convex cone of polynomials $p : F \to \mathbb{R}$, with $p(t) \geq 0$ for all $t \in F$. We denote by $\mathcal{P}_{++}(F)$ a convex subcone of $\mathcal{P}_+(F)$ whose elements are special nonnegative polynomials. For example, $\mathcal{P}_{++}(\mathbb{R}^n)$ can be the convex cone of all sums of polynomials of the form $p_1 \otimes \cdots \otimes p_n$, where:

$$(p_1 \otimes \cdots \otimes p_n)(t_1, \ldots, t_n) := p_1(t_1) \cdots p_n(t_n),\ t = (t_1, \ldots, t_n) \in \mathbb{R}^n, \\ p_i \in \mathcal{P}_+(\mathbb{R}),\ \ i = 1, \ldots, n. \tag{1}$$

We recall that:

$$p \in \mathcal{P}_+(\mathbb{R}) \Leftrightarrow p = q^2 + r^2 \tag{2}$$

for some polynomials q, r and

$$p \in \mathcal{P}_+(\mathbb{R}_+) \Leftrightarrow p(t) = q(t)^2 + tr(t)^2 \text{ for all } t \in \mathbb{R}_+ := [0, \infty). \tag{3}$$

for some $q, r \in \mathbb{R}[t]$. We denote by $\mathbb{N} := \{0, 1, \ldots\}$ the set of all nonnegative integers. If F is a closed unbounded subset of $\mathbb{R}^n$, then $C_c(F)$ is the vector space of all real-valued continuous compactly supported functions defined on F. In the sequel, all the involved vector space and linear operators (or functionals) are considered over the real field.

The classical moment problem can be written as follows: being given a sequence $(y_j)_{j \in \mathbb{N}^n}$ of real numbers and a closed subset $F \subseteq \mathbb{R}^n$, $n \in \{1, 2, \ldots\}$, find a positive regular Borel measure μ on F such that $\int_F t^j d\mu = y_j,\ j \in \mathbb{N}^n$. This is the full moment problem. The existence, uniqueness, and construction of the unknown solution μ are the focus of attention. The truncated (or reduced) moment problem requires the interpolation moment conditions only for $j_k \leq d,\ k = 1, \ldots, n,,\ j = (j_1, \ldots, j_n)$, where d is a given positive integer. The numbers $y_j,\ j \in \mathbb{N}^n$ are called the moments of the measure μ. When a sandwich condition on the solution is required, we have a Markov moment problem. The moment problem is an inverse problem since the measure μ is not known. It must be found, starting from its moments. Instead of real number moments, one can work with elements $y_j \in Y,\ j \in \mathbb{N}^n$, where Y is an order complete Banach lattice of functions or self-adjoint operators. If the y_j are operators, we have an operator-valued moment problem. When Y is a Banach lattice of functions, we have a vector-valued moment problem. The requirement for Y to be order-complete is motivated by the necessity of applying Hahn–Banach-type theorems in order to obtain a linear positive extension $T : X \to Y$ of the linear operator $T_0 : \mathcal{P} \to Y$, satisfying the moment conditions $T_0(\varphi_j) := y_j,\ j \in \mathbb{N}^n,\ \varphi_j(t) = t^j = t_1^{j_1} \cdots t_n^{j_n}$ from $\mathcal{P}$ to an ordered Banach space X containing both spaces $\mathcal{P}$ and $C_c(F)$. When a sandwich condition $T_1 \leq T \leq T_2$ is required on the extension T, where $T_i,\ i = 1, 2$ are given bounded linear operators mapping X into Y, we have a Markov moment problem. In this case the positivity of T on X_+ is replaced by the condition $T_1 \leq T$, while the requirement $T \leq T_2$ controls the norm of the solution T. As in the case of a scalar-valued linear solution, we now study the existence, the uniqueness, and eventually the construction of a/the linear solution T satisfying the interpolation moment conditions and the sandwich condition. A basic result in solving the classical moment on unbounded closed subsets is the Haviland theorem [28]. In [29], the result of Kantorovich on the extension of positive linear operators preserving linearity and positivity was reviewed and proven. This a Hahn–Banach-type result. The references [30–43] point out various aspects of the moment and related problems. Unlike other unbounded subsets of $\mathbb{R}^n$, $n \geq 2$, the expression of nonnegative polynomials on a strip in terms of sums of squares is known due to M. Marshall's theorem [39]. Using the polynomial approximation ensured by Lemma 1 and Theorem 1, proven below, the Markov moment problem in terms of quadratic forms is solved (see Theorem 3 below). Applications of Hahn–Banach-type extension theorems to the study of the isotonicity (increasing monotonicity) of continuous convex operators on the positive cone X_+ were published in the article [44]. References [45–48] focus mainly on several aspects of the truncated or full Markov moment problem. The rest of this paper is organized as follows. Section 2 summarizes the basic methods and results used along the proofs of the theorems in the present paper. Section 3 is devoted to the results: polynomial approximation on unbounded subsets in some L_ν^1 spaces, applications of such results accompanied by other theorems to the existence and uniqueness of the solution of the Markov moment problem on an unbounded closed subset, and characterizations of the sandwich condition for bounded linear operators. All these applications of approximation-type results are partially or completely formulated in terms of quadratic forms. Section 4 concludes the paper.

2. Methods

Here are the basic methods used directly or as background of this paper:

(1) Polynomial approximation on closed unbounded subsets $F \subseteq \mathbb{R}^n$. in spaces $L_\nu^1(F)$, where ν is a moment-determinate positive regular Borel measure on F. Here, we use notions on the determinacy of measures, Kantorovich theorem on the extension of positive linear operators, Haviland theorem, and measure theory standard results. However, the key point is the notion of a moment-determinate measure and its use in the proof of Lemma 1. Bernstein-approximating polynomials are applied in the proofs of Lemmas 2 and 3.

(2) The characterization of the existence and uniqueness of the solution for full vector-valued Markov moment problems on unbounded subsets and their consequences for scalar Markov moment problems.

(3) The characterization of the sandwich-type conditions for a large class of bounded linear operators on $L_\nu^1(\mathbb{R}^n)$, only in terms of quadratic forms.

3. Results

3.1. On Polynomial Approximation on Unbounded Closed Subsets $F \subseteq \mathbb{R}^n$ in Spaces $L_\nu^1(F)$, Where ν Is a Moment-Determinate Positive Regular Borel Measure on F

In the sequel, the following approximation lemmas are applied

Lemma 1. *Let $F \subseteq \mathbb{R}^n$ be an unbounded closed subset and ν be a moment-determinate positive regular Borel measure on F, with finite moments of all natural orders. Then, for any $x \in C_c(F)$, $x(t) \geq 0$, $\forall t \in F$, there exists a sequence $(p_m)_m, p_m \geq x$, $m \in \mathbb{N}$, $p_m \to x$ in $L_\nu^1(F)$. Consequently, we have:*

$$\lim_m \int_F p_m(t)d\nu = \int_F x(t)d\nu,$$

where $\mathcal{P}_+ = \mathcal{P}_+(F)$ is dense in $\left(L_\nu^1(F)\right)_+$, and $\mathcal{P}$ is dense in $L_\nu^1(F)$.

Proof To prove the assertions of the statement, it is sufficient to show that for any $x \in (C_c(F))_+$ we have

$$Q_1(x) := inf\left\{\int_F p(t)d\nu; p \geq x,\ p \in \mathcal{P}\right\} = \int_F x(t)d\nu.$$

Obviously, one has

$$Q_1(x) \geq \int_F x(t)d\nu.$$

To prove the converse, we define the linear form

$$T_0 : X_0 := \mathcal{P} \oplus Sp\{x\} \to \mathbb{R},\ F_0(p + \alpha x) := \int_F p(t)d\nu + \alpha Q_1(x),\ p \in \mathcal{P},\ \alpha \in \mathbb{R}.$$

We denote by X_1 the vector subspace of $X = L^1_\nu(F)$ of all functions from f from X whose absolute value $|f|$ is dominated by a polynomial p_f on F. Next, we show that F_0 is positive on X_0. In fact, for $\alpha < 0$, one has (from the definition of Q_1, which is a sublinear functional on X_1):

$$p + \alpha x \geq 0 := p \geq -\alpha x := (-\alpha)Q_1(x) = Q_1(-\alpha x) \leq \int_F p(t)d\nu := T_0(p + \alpha x) \geq 0.$$

If $a \geq 0$, we infer that:

$$0 = Q_1(0) = Q_1(\alpha x - \alpha x) \leq \alpha Q_1(x) + Q_1(-\alpha x) \Longrightarrow$$
$$\int_F p(t)d\nu \geq Q_1(-\alpha x) \geq -\alpha Q_1(x) := T_0(p + \alpha x) \geq 0,$$

where, in both possible cases, we have $x_0 \in (X_0)_+ := T_0(x_0) \geq 0$. Since X_0 contains the space of the polynomials' functions, which is a majorizing subspace of X_1, there exists a linear positive extension $T : X \to \mathbb{R}$ of T_0 (cf. [29]), which is continuous on $C_c(F)$ with respect to the sup-norm. Therefore, T has a representation by means of a positive Borel regular measure μ on F such that

$$T(x) = \int_F x(t)d\mu,\ x \in C_c(F).$$

Let $p \in \mathcal{P}_+$ be a nonnegative polynomial function. There is a nondecreasing sequence $(x_m)_m$ of continuous nonnegative function with compact support such that $x_m \nearrow p$ pointwise on F. The positivity of T and Lebesgue's dominated convergence theorem for μ yield

$$\int_F p(t)d\nu = T(p) \geq sup T(x_m) = sup \int_F x_m(t)d\mu = \int_F p(t)d\mu,\ p \in \mathcal{P}_+.$$

Thanks to Haviland's theorem [28], there exists a positive Borel regular measure λ on F such that

$$\lambda(p) = \nu(p) - \mu(p) \Leftrightarrow \nu(p) = \lambda(p) + \mu(p),\ p \in \mathcal{P}.$$

Since ν is assumed to be M-determinate, it follows that:

$$\nu(B) = \lambda(B) + \mu(B),$$

for any Borel subset B of F. From this last assertion, approximating each $x \in \left(L^1_\nu(F)\right)_+$ by a nondecreasing sequence of nonnegative simple functions and using Lebesgue's convergence theorem, one obtains, first for positive functions, then for arbitrary ν-integrable functions, φ :

$$\int_F \varphi d\nu = \int_F \varphi d\lambda + \int_F \varphi d\mu,\ \ \varphi \in L^1_\nu(F).$$

In particular, we must have

$$\int_F x d\nu \geq \int_F x d\mu = T(x) = T_0(x) = Q_1(x).$$

The conclusion is: $Q_1(x) = \int_F x(t)d\nu$. This ends the proof. □

Using Bernstein polynomial of n real variables when Lemma 1 is applied to $n = 1$, for $F = \mathbb{R}$ and Fubini's theorem we derive the following multidimensional polynomial approximation result.

Lemma 2. *Let $\nu = \nu_1 \times \cdots \times \nu_n$ be a product of n positive regular Borel-moment-determinate measures on $\mathbb{R}$, with finite moments of all orders. Then, we can approximate any nonnegative continuous compactly supported function $\psi \in X = (C_c(\mathbb{R}^n))_+$ with the sums of products:*

$$p_1 \otimes \cdots \otimes \ p_n,$$
$$(p_1 \otimes \cdots \otimes \ p_n)(t_1, \ldots, t_n) := p_1(t_1) \cdots p_n(t_n).t = (t_1, \ldots, t_n) \in \mathbb{R}^n,$$

where p_j is a nonnegative polynomial on the entire real line, $j = 1, \ldots, n$, and any such sum of special polynomials dominates ψon $\mathbb{R}^n$.

Lemma 3. *Let $\nu = \nu_1 \times \cdots \times \nu_n$ be a product of n positive regular Borel-moment-determinate measures on $\mathbb{R}_+$, with finite moments of all orders. Then, we can approximate any nonnegative continuous compactly supported function $\psi \in \left(C_c\left(\mathbb{R}^n_+\right)\right)_+$ with the sums of products:*

$$p_1 \otimes \cdots \otimes \ p_n,$$
$$(p_1 \otimes \cdots \otimes p_n)(t_1, \ldots, t_n) := p_1(t_1) \cdots p_n(t_n).t = (t_1, \ldots, t_n) \in \mathbb{R}^n_+,$$

where p_j is a nonnegative polynomial on the entire nonnegative semi axes, $j = 1, \ldots, n$, and any such sum of special polynomials dominates ψ on $\mathbb{R}^n_+$.

Proof. Let $f \in \left(C_c\left(\mathbb{R}^n_+\right)\right)_+, K_i = pr_i(supp(f)),\ a_i = infK_i, b_i = supK_i,\ i = 1, \ldots, n,$ $K = [a_1, b_1] \times \cdots \times [a_n, b_n]$.

The restriction of f to the parallelepiped K can be approximated uniformly on K by Bernstein polynomials B_m in n variables. Any such polynomial B_m is a sum of the products of the form $q_{m,1} \otimes \cdots \otimes \ q_{m,n}$, where each $q_{m,i}$ is a polynomial nonnegative on $[a_i, b_i],\ i = 1, \ldots, n, m \in \mathbb{N}$. B_m can be written as:

$$B_m = \sum_{\substack{k_i = 0, \ldots, m, \\ i = 1, \ldots, n}} q_{m,k_1} \otimes \cdots \otimes q_{m,k_n},$$

where q_{m,k_i} is a nonnegative polynomial on $[a_i, b_i],\ i = 1, \ldots, n, m \in \mathbb{N}$. By the Weierstrass–Bernstein uniform approximation theorem, we have:

$$\|f - B_m\|_\infty := \sup_{t \in K} |f(t) - B_m(t)| \to 0,\ m \to \infty.$$

By an abuse of notation, we write $q_{m,i} = q_{m,k_i}$. We need a similar approximation, with sums of tensor products of nonnegative polynomials $p_i,\ p_i(t_i) \geq 0$, for all $t_i \in \mathbb{R}_+,\ i = 1, \ldots, n$ in the space $L^1_\nu\left(\mathbb{R}^n_+\right)$. To this aim, the idea is to use Lemma 18 for $n = 1,\ F = \mathbb{R}_+$, followed by Fubini's theorem. We define $q_{0,m,i} = q_{m,i} \cdot \chi_{[a_i, b_i]}$, $i = 1, \ldots, n$ and $f_i(t) = q_{m,i}(t), t \in [a_i, b_i],\ f_i(t) = 0$ for t outside an interval $[a_i - \varepsilon, b_i + \varepsilon]$ with small $\varepsilon > 0$, the graph of f_i on $[b_i, b_i + \varepsilon]$ being the line segment of the ends of the points $(b_i, q_i(b_i))$ and $(b_i + \varepsilon, 0)$. We proceed similarly on an interval $[a_i - \varepsilon, a_i]$. Clearly, for $\varepsilon > 0$ small enough, f_i approximates $q_{0,\, m,i}$ in $L^1_{\nu_i}(\mathbb{R}_+)$ as accurate as we wish. On the other hand, f_i is nonnegative, compactly supported, and continuous on $\mathbb{R}_+$, so that Lemma 1 ensures the existence of an approximating polynomial p_i with respect to the norm of $L^1_{d\nu_i}(\ \mathbb{R}_+),\ p_i(t) \geq 0$ for all $t \in\ \mathbb{R}_+,\ i = 1, \ldots, n$. According to Fubini's theorem, the preceding reasoning yields $p_1 \otimes \cdots \otimes p_n$, which approximates $f_1 \otimes \cdots \otimes f_n$, and $f_1 \otimes \cdots \otimes f_n$, which approximates $q_{0,m,1} \otimes \cdots \otimes q_{0,m,n} = q_{0,m,k_1} \otimes \cdots \otimes q_{0,m,k_n}$. The approximations hold for finite sums of these products in $L^1_\nu\left(\mathbb{R}^n_+\right)$. Moreover, finite sums of functions $q_{0,m,1,} \otimes \cdots \otimes q_{0,m,n}$ approximate f uniformly on K because their restrictions to K define the restriction to K of approximating Bernstein polynomials $(B_m)_{m \in \mathbb{N}}$ associated to

f. Since f and $q_{0,m,1,} \otimes \cdots \otimes q_{0,m,n}$ vanish outside K, we infer that the following norm $\| \|_1$ in $L^1_\nu(\mathbb{R}^n_+)$ is evaluated as:

$$\|f - \sum_{\substack{k_i=0,\ldots,m,\\ i=1,\ldots,n}} q_{0,m,k_1,} \otimes \cdots \otimes q_{0,m,k_n}\|_1 = \int_K \left| f - \sum_{\substack{k_i=0,\ldots,m,\\ i=1,\ldots,n}} q_{m,,k_1} \otimes \cdots \otimes q_{m,k_n} \right| d\nu \le$$

$$\sup_{t\in K} |f(t) - B_m(t)| \cdot \nu(K) \to 0,\ m \to \infty.$$

The conclusion is that f can be approximated in $L^1_\nu(\mathbb{R}^n_+)$ by the sums of products $p_1 \otimes \cdots \otimes p_n$, where p_i is nonnegative on $\mathbb{R}_+$ for all $i = 1, \ldots, n$. This ends the proof. □

Example 1. *For any $\alpha \in (0, \infty)$, $d\nu = e^{-\alpha t}dt$ is a moment-determinate positive Borel measure on $\mathbb{R}_+$, according to* [14]. *The application of Lemma 3 shows that for the product measure:*

$$d\nu = exp\left(-\sum_{j=1}^{n} \alpha_j t_j\right) dt_1 \cdots dt_n =$$

$$exp(-\alpha_1 t_1)dt_1 \times \cdots \times exp(-\alpha_n t_n)dt_n,\ \alpha_j > 0,\ j = 1, \ldots, n,$$

the polynomials are dense in $L^1_\nu(\mathbb{R}^n_+)$. In particular, the measureν is moment-determinate on $\mathbb{R}^n_+$. A similar consequence follows from Lemma 2, for the measure

$$d\mu = exp\left(-\sum_{j=1}^{n} \alpha_j t_j^2\right) dt_1 \cdots dt_n,\ \alpha_j > 0,\ j = 1, \ldots, n.$$

In this case, the polynomials are dense in $L^1_\mu(\mathbb{R}^n)$; in particular, μ is a moment-determinate measure on $\mathbb{R}^n$.

3.2. Solving Markov Moment Problems in Terms of Signatures of Quadratic Forms

The approximation results reviewed in Section 3.1 allow the extension of sandwich conditions on the solution T, preserving the interpolation moment conditions, from the subspace of polynomials to the entire space $L^1_\nu(F)$ for moment-determinate measures ν. The results stated in the sequel complete theorems previously published in [13,14,16].

Theorem 1. *Let F be a closed unbounded subset of $\mathbb{R}^n$, Y an order-complete Banach lattice, $(y_j)_{j\in\mathbb{N}^n}$ a given sequence in Y, and ν a positive regular moment-determinate Borel measure on F, with finite moments of all orders. Let $T_1, T_2 \in B(L^1_\nu(F), Y)$ be two linear bounded operators from $L^1_\nu(F)$ to Y. The following statements are equivalent:*

(a) *there exists a unique bounded linear operator $T \in B(L^1_\nu(A), Y)$ such that $T(\varphi_j) = y_j$, $j \in \mathbb{N}^n$, and T is between T_1 and T_2 on the positive cone of L^1_ν;*

(b) *for any finite subset $J_0 \subset \mathbb{N}^n$ and any $\{a_j\}_{j\in J_0} \subset \mathbb{R}$, we have*

$$\sum_{j\in J_0} a_j \varphi_j \ge 0 \, \mathrm{on} F \Rightarrow$$

$$\sum_{j\in J_0} a_j T_1(\varphi_j) \le \sum_{j\in J_0} a_j y_j \le \sum_{j\in J_0} a_j T_2(\varphi_j).$$

Proof. We define $T_0 : \mathcal{P} \to Y$ by

$$T_0\left(\sum_{j\in J_0} \lambda_j \varphi_j\right) := \sum_{j\in J_0} \lambda_j y_j. \quad (4)$$

Here, $J_0 \subset \mathbb{N}^n$ is an arbitrary finite subset, and $\lambda_j,\ j \in J_0$ are real coefficients. With this notation, point (b) says that

$$T_1(p) \le T_0(p) \le T_2(p),\ p \in \mathcal{P}_+(F). \tag{5}$$

In other words, $U_1 := T_0 - T_1,\ U_2 := T_2 - T_1,\ U_i : \mathcal{P} \to Y,\ i = 1,2$ are positive linear operators on the positive cone $\mathcal{P}_+(F)$ of the ordered vector space $\mathcal{P}$, and $U_1\Big|_{\mathcal{P}_+(F)} \le U_2\Big|_{\mathcal{P}_+(F)}$. According to the Kantorovich extension result for positive linear operators, there exists a positive linear extension V_1 of U_1 from $\mathcal{P}$ to a dense subspace X_1 of $X := L^1_\nu(F)$ since $\mathcal{P}$ is a majorizing subspace of $X_1 := \{f \in X;\ \exists p \in \mathcal{P},\ |f| \le p\}$. Clearly, the space X_1 contains both subspaces $C_c(F)$ and $\mathcal{P}$. Then, $V_1 + T_1$ extends T_0 to a linear operator:

$$W_1 : X_1 \to Y,\ W_1 := V_1 + T_1 \ge T_1 \text{ on } \mathcal{P}_+(F).$$

Using Lemma 1, the continuity of T_1, T_2, and the inequalities $\mathbf{0} \le U_1 \le U_2$ on $\mathcal{P}_+$, we infer that for any sequence of nonnegative compactly supported functions $(g_l)_l,\ g_l \to \mathbf{0}$, there exists a sequence of polynomials $(p_l)_l,\ \mathbf{0} \le g_l \le p_l$ for all $l,\ p_l - g_l \to \mathbf{0},\ l \to \infty$. These yield:

$$p_l = (p_l - g_l) + g_l \to \mathbf{0},\ l \to \infty. \tag{6}$$

On the other hand, (5) and (6) lead to:

$$\mathbf{0} \leftarrow T_1(p_l) \le T_0(p_l) \le T_2(p_l) \to 0.$$

Thus, $W_1(p_l) = T_0(p_l) \to \mathbf{0}$, which further implies

$$\mathbf{0} \le W_1(g_l) \le W_1(p_l) \to 0.$$

Thus, $W_1(g_l) \to \mathbf{0}$ for any convergent to zero sequence of elements from $(C_c(F))_+$. Now, let $(g_l)_l$ be an arbitrary sequence in $C_c(F),\ g_l \to \mathbf{0}$. Then, $g_l^+ \to 0,\ g_l^- \to 0,$ and the preceding reasons imply $W_1\left(g_l^+\right) \to \mathbf{0},\ W_1\left(g_l^-\right) \to \mathbf{0}$. Therefore, $W_1(g_l) = W_1\left(g_l^+\right) - W_1\left(g_l^-\right) \to \mathbf{0}$. The conclusion is that the linear operator W_1 is continuous on $C_c(F)$. It admits a unique linear continuous extension $T \in B(X, Y)$, since $C_c(F)$ is dense in X. Hence, T is continuous and defined on the entire space $X = L^1_\nu(F)$, verifying $T(\varphi_j) = T_0(\varphi_j) = y_j,\ j \in \mathbb{N}^n$. If $\psi \in X_+$, there exists a sequence $(g_l)_l$ of functions in $(C_c(F))_+$ such that $g_l \to \psi$ in X. If $(p_l)_l$ is a sequence of polynomial functions, $g_l \le p_l$ for all $l,\ p_l - g_l \to \mathbf{0}$, then the continuity of the operators T_1, T, T_2 on X and the inequalities (5) yield:

$$T_1(\psi) = \lim_l T_1(p_l) \le \lim_l T_0(p_l) = \lim_l T(p_l) \le \lim_l T_2(p_l) = T_2(\psi),\ \psi \in X_+.$$

This ends the proof. □

If the nonnegative polynomials on F are expressible in terms of sums of squares, theorem 1 allows the characterization of the existence and uniqueness of the solution in terms of quadratic forms. The following consequences hold. We start with the simplest case, when $F = \mathbb{R}$.

Corollary 1. *Let $X = L^1_\nu(\mathbb{R})$, where ν is a positive regular moment-determinate Borel measure on $\mathbb{R}$, with finite moments of all orders. Assume that Y is an arbitrary order complete Banach lattice and $(y_n)_{n\ge 0}$ is a given sequence with its terms in Y. Let T_1, T_2 be two linear operators from X to Y such that $\mathbf{0} \le T_1 \le T_2$ on X_+. The following statements are equivalent:*

(a) *There exists a unique bounded linear operator T from X to Y, $T_1 \le T \le T_2$ on X_+, $\|T_1\| \le \|T\| \le \|T_2\|$ such that $T(\varphi_n) = y_n$ for all $n \in \mathbb{N}$;*

(b) *If $J_0 \subset \mathbb{N}$ is a finite subset and $\{\lambda_j; j \in J_0\} \subset \mathbb{R}$, then*

$$\sum_{i,j\in J_0} \lambda_i\lambda_j T_1(\varphi_{i+j}) \le \sum_{i,j\in J_0} \lambda_i\lambda_j y_{i+j} \le \sum_{i,j\in J_0} \lambda_i\lambda_j T_2(\varphi_{i+j}).$$

Proof. We apply Theorem 1 to $F = \mathbb{R}$ as well as the explicit form of nonnegative polynomials on the real axes (2). One uses the obvious equality:

$$q = \sum_{j \in J_0} \lambda_j \varphi_j \Rightarrow q^2 = \sum_{i,j \in J_0} \lambda_i \lambda_j \varphi_i \varphi_j = \sum_{i,j \in J_0} \lambda_i \lambda_j \varphi_{i+j},$$

Here, $J_0 \subset \mathbb{N}$ is an arbitrary finite subset, $\lambda_j \in \mathbb{R},\ j \in J_0$. It remains to prove that

$$\|T_1\| \leq \|T\| \leq \|T_2\|.$$

The positivity of the linear operators $T_1, T, T_2,\ T - T_1,\ T_2 - T$ on X_+ and their continuity yields:

$$\pm T_1(x) = T_1(\pm x) \leq T_1(|x|) \leq T(|x|),$$

which implies $|T_1(x)| \leq T(|x|),\ x \in X$. Since Y is a Banach lattice, we infer that the inequalities:

$$\|T_1(x)\| \leq \|T(|x|)\| \leq \|T\|\|x\|,$$

hold for all $x \in X$. This proves that $\|T_1\| \leq \|T\|$. Similarly, we show that $\|T\| \leq \|T_2\|$. This ends the proof. □

Here is the scalar-valued version of Corollary 1.

Corollary 2. *Let ν be a positive regular moment-determinate Borel measure on $\mathbb{R}$, with finite moments of all orders. Assume that h_1, h_2 are two functions in $L^\infty_\nu(\mathbb{R})$ such that $0 \leq h_1 \leq h_2$ almost everywhere. Let $(y_n)_{n \geq 0}$ be a given sequence of real numbers. The following statements are equivalent:*

(a) *There exists a unique $h \in L^\infty_\nu(\mathbb{R})$ such that $h_1 \leq h \leq h_2$ $\nu-$almost everywhere and $\int_{\mathbb{R}} t^j h(t) d\nu = y_j$ for all $j \in \mathbb{N}$.*

(b) *If $J_0 \subset \mathbb{N}$ is a finite subset, and $\{\lambda_j; j \in J_0\} \subset \mathbb{R}$, then:*

$$\sum_{i,j \in J_0} \lambda_i \lambda_j \int_{\mathbb{R}} t^{i+j} h_1(t) d\nu \leq \sum_{i,j \in J_0} \lambda_i \lambda_j y_{i+j} \leq \sum_{i,j \in J_0} \lambda_i \lambda_j \int_{\mathbb{R}} t^{i+j} h_2(t) d\nu.$$

Proof. The implication $(a) := (b)$ is obvious. To prove the converse, we apply Corollary 1 to the case $Y = \mathbb{R},\ T_i(f) := \int_{\mathbb{R}} h_i(t) f(t) d\nu,\ i = 1, 2$. The linear positive (hence, continuous) functional T is represented by a function $h \in L^\infty_\nu(\mathbb{R})$ according to the measure theory results from [9]. The moment interpolation conditions from Corollary 1 must be written as

$$\int_{\mathbb{R}} h(t) t^j d\nu = T(\varphi_j) = y_j,\ j \in \mathbb{N}.$$

To finish the proof, we must show that $h_1 \leq h \leq h_2$ $\nu-$almost everywhere in $\mathbb{R}$. According to Corollary 1, we already know that:

$$\int_{\mathbb{R}} h_1(t) f(t) d\nu \leq \int_{\mathbb{R}} h(t) f(t) d\nu \leq \int_{\mathbb{R}} h_2(t) f(t) d\nu,$$

for all $f \in \left(L^1_\nu(\mathbb{R})\right)_+$. Writing this for any $f = \chi_B$, where $B \subseteq \mathbb{R}$ is an arbitrary Borel subset with $\nu(B) \in (0, \infty)$, the following conclusion holds:

$$\int_B (h(t) - h_1(t)) d\nu \geq 0,\ \int_B (h_2(t) - h(t)) d\nu \geq 0,\ B \in \mathcal{B},\ \nu(B) > 0.$$

Here, $\mathcal{B}$ is the sigma algebra of all Borel subsets of $\mathbb{R}$. Now, a well-known measure theory argument [9] leads to $h_1(t) \le h(t) \le h_2(t)$ for almost all $t \in \mathbb{R}$ with respect to the measure $d\nu$. This ends the proof. □

If in Corollaries 1 and 2 we take $\mathbb{R}_+$ instead of $\mathbb{R}$, the following statements hold, via proofs like those shown above.

Corollary 3. *Let $X = L^1_\nu(\mathbb{R}_+)$, where ν is a positive regular moment-determinate Borel measure on $\mathbb{R}_+$. Assume that Y is an arbitrary order-complete Banach lattice and $(y_n)_{n\ge 0}$ is a given sequence with its terms in Y. Let T_1, T_2 be two linear operators from X to Y such that $0 \le T_1 \le T_2$ on X_+. The following statements are equivalent:*

(c) *There exists a unique bounded linear operator T from X to Y, $T_1 \le T \le T_2$ on X_+, $\|T_1\| \le \|T\| \le \|T_2\|$ such that $T(\varphi_n) = y_n$ for all $n \in \mathbb{N}$;*

(d) *If $J_0 \subset \mathbb{N}$ is a finite subset and $\{\lambda_j; j \in J_0\} \subset \mathbb{R}$, then*

$$\sum_{i,j\in J_0} \lambda_i\lambda_j T_1\left(\varphi_{i+j+k}\right) \le \sum_{i,j\in J_0} \lambda_i\lambda_j y_{i+j+k} \le \sum_{i,j\in J_0} \lambda_i\lambda_j T_2\left(\varphi_{i+j+k}\right),\ k \in \{0,1\}.$$

Corollary 4. *Let ν be a positive regular moment-determinate Borel measure on $\mathbb{R}_+$, with finite moments of all orders. Assume that h_1, h_2 are two functions in $L^\infty_\nu(\mathbb{R}_+)$ such that $\mathbf{0} \le h_1 \le h_2$ almost everywhere. Let $(y_n)_{n\ge 0}$ be a given sequence of real numbers. The following statements are equivalent:*

(c) *There exists a unique $h \in L^\infty_\nu(\mathbb{R}_+)$ such that $h_1 \le h \le h_2$ $\nu-$almost everywhere, and*

$$\int_{\mathbb{R}_+} t^j h(t) d\nu = y_j \text{for all } j \in \mathbb{N}.$$

(d) *If $J_0 \subset \mathbb{N}$ is a finite subset and $\{\lambda_j; j \in J_0\} \subset \mathbb{R}$, then:*

$$\sum_{i,j\in J_0} \lambda_i\lambda_j \int_{\mathbb{R}_+} t^{i+j+k} h_1(t) d\nu \le \sum_{i,j\in J_0} \lambda_i\lambda_j y_{i+j+k} \le \sum_{i,j\in J_0} \lambda_i\lambda_j \int_{\mathbb{R}_+} t^{i+j+k} h_2(t) d\nu,\ k \in \{0,1\}.$$

Example 2. *If, in Corollary 4, we take $d\nu = e^{-t}dt$, $h_1(t) := te^{-t}$, $h_2(t) :=$ **1/2**, then $d\nu$ is moment-determinate* [14],

$$\int_{\mathbb{R}_+} t^{i+j+k} h_1(t) d\nu = \int_0^\infty t^{i+j+k+1} e^{-2t} dt = 2^{-(i+j+k+2)} \int_0^\infty u^{i+j+k+1} e^{-u} du = 2^{-(i+j+k+2)}(i+j+k+1)!,$$

$$\int_{\mathbb{R}_+} t^{i+j+k} h_2(t) d\nu = 2^{-1}(i+j+k)!.$$

Thus, condition (b) must be written as follows:

$$\sum_{i,j\in J_0} \lambda_i\lambda_j 2^{-(i+j+k+2)}(i+j+k+1)! \le \sum_{i,j\in J_0} \lambda_i\lambda_j y_{i+j+k} \le \sum_{i,j\in J_0} \lambda_i\lambda_j 2^{-1}(i+j+k)!,\ k \in \{0,1\},$$

where $J_0 \subset \mathbb{N}$ is an arbitrary finite subset and λ_j, $j \in J_0$ are arbitrary real numbers.

We go on with the two-dimensional case, starting with the Markov moment problem on a strip. The motivation is that the explicit expression of nonnegative polynomials on a strip in terms of sums of squares is known due to following M. Marshall's result [39].

Theorem 2. *If* $p(t_1, t_2) \in \mathbb{R}[t_1, t_2]$ *is nonnegative on the strip* $F = [0,1] \times \mathbb{R}$, *then* $p(t_1, t_2)$ *is expressible as:*

$$p(t_1, t_2) = \sigma(t_1, t_2) + \tau(t_1, t_2)t_1(1 - t_1),$$

where $\sigma(t_1, t_2), \tau(t_1, t_2)$ *are sums of squares in* $\mathbb{R}[t_1, t_2]$.

From Theorems 1 and 2, the next result also holds. Let $F = [0,1] \times \mathbb{R}$, ν be a positive regular Borel M-determinate (moment-determinate) measure on F, and $X = L^1_\nu(F), \varphi_j(t_1, t_2) := t_1^{j_1} t_2^{j_2},\ j = (j_1, j_2) \in \mathbb{N}^2,\ (t_1, t_2) \in F$. Let Y be an order-complete Banach lattice and $(y_j)_{j \in \mathbb{N}^2}$ be a sequence of given elements in Y.

Theorem 3. *Let* $T_1, T_2 \in B_+(X, Y)$ *be two linear (bounded) positive operators mapping* X *into* Y. *The following statements are equivalent:*

(a) *There exists a unique (bounded) linear operator* $T : X \to Y$ *such that* $T(\varphi_j) = y_j,\ j \in \mathbb{N}^2$, *where* T *is between* T_1 *and* T_2 *on the positive cone of* X, $\|T_1\| \le \|T\| \le \|T_2\|$;

(b) *For any finite subset* $J_0 \subset \mathbb{N}^2$ *and any* $\{\lambda_j; j \in J_0\} \subset \mathbb{R}$, *we have:*

$$\sum_{i,j \in J_0} \lambda_i \lambda_j T_1(\varphi_{i+j}) \le \sum_{i,j \in J_0} \lambda_i \lambda_j y_{i+j} \le \sum_{i,j \in J_0} \lambda_i \lambda_j T_2(\varphi_{i+j}),$$
$$\sum_{i,j \in J_0} \lambda_i \lambda_j \left(T_1\left(\varphi_{i_1+j_1+1,\ i_2+j_2} - \varphi_{i_1+j_1+2,\ i_2+j_2}\right)\right) \le$$
$$\sum_{i,j \in J_0} \lambda_i \lambda_j \left(y_{i_1+j_1+1,\ i_2+j_2} - y_{i_1+j_1+2,\ i_2+j_2}\right) \le$$
$$\sum_{i,j \in J_0} \lambda_i \lambda_j \left(T_2\left(\varphi_{i_1+j_1+1,\ i_2+j_2} - \varphi_{i_1+j_1+2,\ i_2+j_2}\right)\right),\ i = (i_1, i_2),\ j = (j_1, j_2) \in J_0.$$

Unfortunately, similar results cannot be proven for moment problems on $\mathbb{R}^n$ and $\mathbb{R}^n_+$. This is a motivation for reviewing the following result [13].

If $F \subseteq \mathbb{R}^n$ is an arbitrary closed unbounded subset, then we denote, by $\mathcal{P}_{++}$, a subcone of $\mathcal{P}_+$ generated by special nonnegative polynomials expressible in terms of sums of squares.

Theorem 4. *Let* $F \subseteq \mathbb{R}^n$ *be a closed unbounded subset;* ν *be a positive regular Borel-moment-determinate measure on* F, *having finite moments of all orders; and* $X = L^1_\nu(F),\ \varphi_j(t) = t^j$, $t \in F,\ j \in \mathbb{N}^n$. *Let* Y *be an order-complete Banach lattice,* $(y_j)_{j \in \mathbb{N}^n}$ *be a given sequence of elements in* Y, *and* T_1 *and* T_2 *be two bounded linear operators mapping* X *into* Y. *Assume that there exists a subcone* $\mathcal{P}_{++} \subseteq \mathcal{P}_+$ *such that each* $f \in (C_c(F))_+$ *can be approximated in* X *by a sequence* $(p_l)_l,\ p_l \in \mathcal{P}_{++},\ p_l \ge f$ *for all* l. *The following statements are equivalent:*

(a) *There exists a unique (bounded) linear operator*

$$T : X \to Y,\ T(\varphi_j) = y_j,\ j \in \mathbb{N}^n, \mathbf{0} \le T_1 \le T \le T_2\ \ X_+,\ \|T_1\| \le \|T\| \le \|T_2\|;$$

(b) *For any finite subset* $J_0 \subset \mathbb{N}^n$ *and any* $\{\lambda_j; j \in J_0\} \subset \mathbb{R}$, *the following implications hold true:*

$$\sum_{j \in J_0} \lambda_j \varphi_j \in \mathcal{P}_+(F) := \sum_{j \in J_0} \lambda_j T_1(\varphi_j) \le \sum_{j \in J_0} \lambda_j y_j,$$
$$\sum_{j \in J_0} \lambda_j \varphi_j \in \mathcal{P}_{++} := \sum_{j \in J_0} \lambda_j T_1(\varphi_j) \ge \mathbf{0},\ \sum_{j \in J_0} \lambda_j y_j \ \ \le \sum_{j \in J_0} \lambda_j T_2(\varphi_j).$$

The application of Theorem 4 and Lemma 2 yields the following result.

Theorem 5. *Let* $\nu = \nu_1 \times \cdots \times \nu_n,\ n \ge 2, \nu_j$ *being a positive regular* $M-$*determinate (moment-determinate) Borel measure on* $\mathbb{R},\ j = 1, \ldots, n,\ X = L^1_\nu(\mathbb{R}^n),\ \varphi_j(t) = t^j, t \in \mathbb{R}^n,\ j \in \mathbb{N}^n$. *Additionally, assume that* ν_j *has finite moments of all orders,* $j = 1, \ldots, n$. *Let* Y *be an order-*

complete Banach lattice, $(y_j)_{j\in\mathbb{N}^n}$ *a given sequence of elements in* Y, *and* T_1 *and* T_2 *two bounded linear operators mapping* X *into* Y. *The following statements are equivalent:*

(a) *There exists a unique (bounded) linear operator* $T : X \to Y$, $T(\varphi_j) = y_j$, $j \in \mathbb{N}^n$, $\mathbf{0} \leq T_{\mathbf{1}} \leq T \leq T_2$. *on* X_+, $\|T_1\| \leq \|T\| \leq \|T_2\|$;

(b) *For any finite subset* $J_0 \subset \mathbb{N}^n$ *and any* $\{\lambda_j; j \in J_0\} \subset \mathbb{R}$, *the following implication holds true:*

$$\sum_{j\in J_0}\lambda_j\varphi_j \in \mathcal{P}_+ \Rightarrow \sum_{j\in J_0}\lambda_j T_1(\varphi_j) \leq \sum_{j\in J_0}\lambda_j y_j.$$

For any finite subsets $J_k \subset \mathbb{N}$, $k = 1,\ldots,n$ *and any* $\{\lambda_{j_k}\}_{j_k\in J_k} \subset \mathbb{R}$, *the following inequalities hold true:*

$$\mathbf{0} \leq \sum_{i_1,j_1\in J_1}\left(\cdots\left(\sum_{i_n,j_n\in J_n}\lambda_{i_1}\lambda_{j_1}\cdots\lambda_{i_n}\lambda_{j_n}T_1(\varphi_{i_1+j_1,\ldots,i_n+j_n})\right)\cdots\right),$$
$$\sum_{i_1,j_1\in J_1}\left(\cdots\left(\sum_{i_n,j_n\in J_n}\lambda_{i_1}\lambda_{j_1}\cdots\lambda_{i_n}\lambda_{j_n}y_{i_1+j_1,\ldots,i_n+j_n}\right)\cdots\right) \leq$$
$$\sum_{i_1,j_1\in J_1}\left(\cdots\left(\sum_{i_n,j_n\in J_n}\lambda_{i_1}\lambda_{j_1}\cdots\lambda_{i_n}\lambda_{j_n}T_2(\varphi_{i_1+j_1,\ldots,i_n+j_n})\right)\cdots\right).$$

A similar result holds for products of n moment-determinate measures on $\mathbb{R}_+$, $n \geq 2$ via Theorem 4 and Lemma 3, also using the explicit form of nonnegative polynomials on $\mathbb{R}_+$ written in (3).

3.3. Characterizing Sandwich Conditions on Bounded Linear Operators in Terms of Quadratic Forms

Lemma 2 leads to the following characterization.

Theorem 6. *Let* ν, X *be as in the statement of Theorem 5,* Y *a Banach lattice, and* $T_{\mathbf{1}}, T, T_2$ *bounded linear operators mapping* X *into* Y. *The following statements are equivalent:*

(a) $T_1 \leq T \leq T_2$ *on the positive cone* X_+;

(b) *For any finite subsets* $J_k \subset \mathbb{N}$, $k = 1,\ldots,n$ *and any* $\{\lambda_{j_k}\}_{j_k\in J_k} \subset \mathbb{R}$, $k = 1,\ldots,n$, *the following inequalities hold:*

$$\sum_{i_1,j_1\in J_1}\left(\cdots\left(\sum_{i_n,j_n\in J_n}\lambda_{i_1}\lambda_{j_1}\cdots\lambda_{i_n}\lambda_{j_n}T_1(\varphi_{i_1+j_1},\ldots,{}_{i_n+j_n})\right)\cdots\right)$$
$$\leq \sum_{i_1,j_1\in J_1}\left(\cdots\left(\sum_{i_n,j_n\in J_n}\lambda_{i_1}\lambda_{j_1}\cdots\lambda_{i_n}\lambda_{j_n}T(\varphi_{i_1+j_1},\ldots,{}_{i_n+j_n})\right)\cdots\right) \leq$$
$$\sum_{i_1,j_1\in J_1}\left(\cdots\left(\sum_{i_n,j_n\in J_n}\lambda_{i_1}\lambda_{j_1}\cdots\lambda_{i_n}\lambda_{j_n}T_2(\varphi_{i_1+j_1},\ldots,{}_{i_n+j_n})\right)\cdots\right).$$

Proof. Statement (b) says that $\mathrm{T}_1(\mathrm{p}) \leq T(p) \leq \mathrm{T}_2(p)$ for all $p \in \mathcal{P}_{++}(\mathbb{R}^n)$, where $\mathcal{P}_{++}(\mathbb{R}^n)$ is the subcone of $\mathcal{P}_+(\mathbb{R}^n)$ formed by all polynomials that can be written as finite sums of the polynomial defined by (1), with $p_i \in \mathcal{P}_+(\mathbb{R})$, $i = 1,2,\ldots,n$. Hence, the implication $(a) := (b)$ is obvious. For the converse, according to a measure-type result [9], for any $\psi \in X_+$ there exists a sequence $(g_l)_{l\in\mathbb{N}}$ of functions from $(C_c(\mathbb{R}^n))_+$, with $\psi = \lim_l g_l$. On the other hand, Lemma 2 implies that there is a sequence of polynomials $(p_l)_{l\in\mathbb{N}}$, $p_l \in \mathcal{P}_{++}(\mathbb{R}^n)$ for all l such that $p_l - g_l \to \mathbf{0}$, $l \to \infty$. Thus,

$$\psi - p_l = (\psi - g_l) + (g_l - p_l) \to 0.$$

This means that $\psi = \lim_{l\to\infty} p_l$. From (b), we know that $T_1(p_l) \le T(p_l) \le \mathrm{T}_2(p_l)$ for all $l \in \mathbb{N}$. Now, the continuity of the three involved operators T_1, T, T_2 yields

$$T_1(\psi) = \lim_l T_1(p_l) \le \lim_l T(p_l) = T(\psi) \le \lim_l T_2(p_l) = T_2(\psi),\ \psi \in X_+.$$

This ends the proof. □

Using Lemma 3 and the form of nonnegative polynomials on $\mathbb{R}_+$ (3), the next result holds too.

Theorem 7. *Let $X = L^1_\nu\left(\mathbb{R}^n_+\right)$, $\varphi_j(t) = t^j, t \in \mathbb{R}^n_+,\ j \in \mathbb{N}^n$, where ν is as in Lemma 3, Y is a Banach lattice, and T_1, T, T_2 are bounded linear operators mapping X into Y. The following statements are equivalent:*

(a) *$T_1 \le T \le T_2$ on the positive cone X_+;*

(b) *For any finite subsets $J_k \subset \mathbb{N},\ k = 1,\ldots,n$ and any $\left\{\lambda_{j_k}\right\}_{j_k\in J_k} \subset \mathbb{R},\ k = 1,\ldots,n$, the following inequalities hold:*

$$\sum_{i_1,j_1\in J_1}\left(\cdots\left(\sum_{i_n\cdot j_n\in J_n}\lambda_{i_1}\lambda_{j_1}\cdots\lambda_{i_n}\lambda_{j_n}T_1\left(\varphi_{l_1+i_1+j_1},\ldots,_{l_n+i_n+j_n}\right)\right)\cdots\right)$$

$$\le \sum_{i_1,j_1\in J_1}\left(\cdots\left(\sum_{i_n\cdot j_n\in J_n}\lambda_{i_1}\lambda_{j_1}\cdots\lambda_{i_n}\lambda_{j_n}T\left(\varphi_{l_1+i_1+j_1},\ldots,_{l_n+i_n+j_n}\right)\right)\cdots\right) \le$$

$$\sum_{i_1,j_1\in J_1}\left(\cdots\left(\sum_{i_n\cdot j_n\in J_n}\lambda_{i_1}\lambda_{j_1}\cdots\lambda_{i_n}\lambda_{j_n}T_2\left(\varphi_{l_1+i_1+j_1},\ldots,_{l_n+i_n+j_n}\right)\right)\cdots\right),$$

for all $(l_1,\ldots,l_n) \in \{0,1\}^n$.

4. Discussion

The present paper provides recently published results and a new way to present them. Such results refer to the Markov moment problem, which motivated the polynomial approximation on unbounded subsets stated in the beginning of the previous section. Instead of looking for the explicit form of nonnegative polynomials on unbounded closed subsets F of $\mathbb{R}^n,\ n \ge 2$ (which has been proven to not always be expressible in terms of sums of squares), the approximation by finite sums of special polynomials pointed out in Lemmas 2 and 3, followed by the passing to the limit process, solved partially or completely, respectively, the problems discussed in the present work. With respect to our own previous similar results, this review paper comes with generalizations and improvements in the theorems, which clearly needed to be improved. We did not see a simpler method in the literature that was able to solve polynomial approximation on unbounded subsets (which is important as a separate subject) and the applications emphasized in this paper. It is a work in the settings of analysis and functional analysis over the real field. The presentation of some statements completes or generalizes the published results on the subject. As a direction for future work, it would be interesting to study what these theorems say in the cases when the codomains Y are concrete Banach lattices.

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