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Article

On Several Bounds for Types of Angular Distances

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Abstract: In this study, we introduce the expression $d_\lambda(x,y) := \lambda\|x\| + (1-\lambda)\|y\| - \|\lambda x + (1-\lambda)y\|$ on the real normed space $\mathscr{X}\,(\mathscr{X}, \|\cdot\|)$, where $x, y \in \mathscr{X}$ and $\lambda \in \mathbb{R}$. We characterize this expression and find various estimates of it. We also obtain a generalization and some refinements of Maligranda's inequality. Finally, we give some relations between $d_\lambda(x,y)$ and several types of angular distances between two nonzero vectors in a real normed space.

Keywords: normed space; triangle inequality; Maligranda inequality

MSC: 46C05; 26D15; 26D10

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1. Introduction

In the literature related to the theory of inequalities, many recently published papers contain studies of certain inequalities in different normed spaces.

An important inequality in a real or complex inner product space is the inequality of Cauchy–Schwarz (see [1,2]), namely:

$$|\langle x, y\rangle| \leq \|x\|\|y\| \tag{1}$$

for all $x, y \in \mathscr{X}$, where $\mathscr{X}$ is a real or complex inner product space. Moreover, equality (1) holds if and only if x and y are linearly dependent. Among those who have studied this inequality, we can mention Aldaz [3] and Dragomir [4].

Let $\mathscr{X}$ be a complex normed space. A classical inequality that characterizes a normed space is the triangle inequality, which is given by

$$\|x+y\| \leq \|x\| + \|y\| \tag{2}$$

for all $x, y \in \mathscr{X}$. Other results about this inequality were given by Pečarić and Rajić in [5]. In [6], a characterization for a generalized triangle inequality in normed spaces was proven, and in [7], we observed some estimates of the triangle inequality using integrals. A recent study of the variational inequalities was given in [8,9].

In [10], Maligranda proved the following interesting inequality:

$$A \cdot \min\{\|x\|, \|y\|\} \leq \|x\| + \|y\| - \|x+y\| \leq A \cdot \max\{\|x\|, \|y\|\}, \tag{3}$$

where $A = 2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\| \geq 0$, and x and y are nonzero vectors in the normed space $\mathscr{X} = (\mathscr{X}, \|\cdot\|)$. Applying this inequality deduced by Maligranda [11], a lower bound and an upper bound for the *norm-angular distance* or *Clarkson distance* can be found (see e.g., [12]) between nonzero vectors x and y, $\alpha[x,y] = \left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|$; thus,

$$\frac{\|x-y\| - |\|x\| - \|y\||}{\min\{\|x\|, \|y\|\}} \leq \alpha[x,y] \leq \frac{\|x-y\| + |\|x\| - \|y\||}{\max\{\|x\|, \|y\|\}}. \tag{4}$$

The right side of inequality (4) represents an improvement of the Massera–Schäffer inequality proved in [13]:

$$\alpha[x,y] \leq \frac{2\|x-y\|}{\max\{\|x\|, \|y\|\}} \tag{5}$$

for every nonzero vector x and y in $\mathscr{X}$, which is stronger than the Dunkl–Williams inequality, which was given in [14]. Other results for bounds to the angular distance, named Dunkl–Williams-type theorems (see [14]), were given by Moslehian et al. [15].

In [16], Dehghan presented a refinement of the triangle inequality, and a simple characterization of inner product spaces was obtained by using the skew angular distance between nonzero vectors x and y; thus: $\beta[x,y] = \left\| \frac{x}{\|y\|} - \frac{y}{\|x\|} \right\|$. In [16], we found the following double inequality:

$$\frac{\|x-y\|}{\min\{\|x\|, \|y\|\}} - \frac{|\|x\| - \|y\||}{\max\{\|x\|, \|y\|\}} \leq \beta[x,y] \leq \frac{\|x-y\|}{\max\{\|x\|, \|y\|\}} + \frac{|\|x\| - \|y\||}{\min\{\|x\|, \|y\|\}} \tag{6}$$

for any nonzero vectors x and y in a real normed linear space $\mathscr{X} = (\mathscr{X}, \|\cdot\|)$.

Many improvements of bounds for the angular distance and skew angular distance were established in [17,18]. The notion of the norm-angular distance was generalized to the p-angular distance in normed space in [10]; thus,

$$\alpha_p[x,y] = \left\| \|x\|^{p-1}x - \|y\|^{p-1}y \right\|,$$

where the nonzero vectors x and y are in a normed space $\mathscr{X} = (\mathscr{X}, \|\cdot\|)$ and $p \geq 0$. In fact, we can take $p \in \mathbb{R}$. It is easy to see that $\alpha_0[x,y] = \alpha[x,y]$.

The notion of skew p–angular distance between nonzero vectors x and y in a normed space $\mathscr{X} = (\mathscr{X}, \|\cdot\|)$ and $p \in \mathbb{R}$ was given by Rooin et al. in [19]; thus,

$$\beta_p[x,y] = \left\| \|y\|^{p-1}x - \|x\|^{p-1}y \right\|.$$

This notion generalizes the concept of skew angular distance between nonzero vectors x and y, $\beta[x,y] = \left\| \frac{x}{\|y\|} - \frac{y}{\|x\|} \right\|$, which was given by Dehghan in [16], because $\beta_0[x,y] = \beta[x,y]$.

On the real normed space $\mathscr{X} = (\mathscr{X}, \|\cdot\|)$, we introduce the following expression:

$$d_\lambda(x,y) := \lambda\|x\| + (1-\lambda)\|y\| - \|\lambda x + (1-\lambda)y\|,$$

where $x, y \in \mathscr{X}$ and $\lambda \in \mathbb{R}$. When $\lambda \in [0,1]$, from the triangle inequality, we find that $d_\lambda(x,y) \geq 0$, for all $x, y \in \mathscr{X}$, and it easy to see that $2d_{1/2}(x,y) = \|x\| + \|y\| - \|x+y\| \geq 0$. We find the following properties:

$$d_\lambda(x,0) = d_\lambda(0,y) = d_\lambda(x,x) = d_0(x,y) = d_1(x,y) = 0, \;\; d_\lambda(x,y) = d_{1-\lambda}(y,x)$$

and

$$d_\lambda\left(\frac{x}{\|x\|}, \frac{y}{\|y\|}\right) = 1 - \left\| \lambda\frac{x}{\|x\|} + (1-\lambda)\frac{y}{\|y\|} \right\|$$

for every $x, y \in \mathscr{X} - \{0\}$.

The purpose of this article is to characterize the expression $d_\lambda(x,y)$ by finding various estimates of it. We will obtain some refinements of some known inequalities. We also give some relations between $d_\lambda(x,y)$ and several types of angular distances between two nonzero vectors in a real normed space $\mathscr{X}$.

2. Main Results

Next, we give several relations that are necessary for proving some inequalities of the Maligranda type related to $d_\cdot(\cdot,\cdot)$.

Lemma 1. *Let x, y be two vectors in the real normed space $\mathscr{X} = (\mathscr{X}, \|\cdot\|)$. If $\lambda \in \mathbb{R}$, then the following equalities hold:*

$$d_\lambda(x,y) = d_{2\lambda}\left(\frac{1}{2}(x+y), y\right) + 2\lambda d_{1/2}(x,y) \tag{7}$$

and

$$d_\lambda(x,y) = d_{2\lambda-1}\left(x, \frac{1}{2}(x+y)\right) + 2(1-\lambda)d_{1/2}(x,y). \tag{8}$$

Proof. Using the definition of $d_\lambda(x,y)$, by regrouping the terms, we obtain

$$\begin{aligned} &d_{2\lambda}\left(\frac{1}{2}(x+y), y\right) = 2\lambda\frac{\|x+y\|}{2} + (1-2\lambda)\|y\| - \|2\lambda\frac{x+y}{2} + (1-2\lambda)y\| \\ &= \lambda\|x+y\| + (1-2\lambda)\|y\| - \|\lambda x + (1-\lambda)y\| \\ &= \lambda\|x\| + (1-\lambda)\|y\| - \|\lambda x + (1-\lambda)y\| - \lambda(\|x\| + \|y\| - \|x+y\|) \\ &= d_\lambda(x,y) - \lambda(\|x\| + \|y\| - \|x+y\|), \end{aligned}$$

which implies the first relation of the statement. In the same way, we have

$$\begin{aligned} &d_{2\lambda-1}\left(x, \frac{1}{2}(x+y)\right) = (2\lambda-1)\|x\| + (2-2\lambda)\frac{\|x+y\|}{2} - \left\|(2\lambda-1)x + (2-2\lambda)\frac{x+y}{2}\right\| \\ &= (2\lambda-1)\|x\| + (1-\lambda)\|x+y\| - \|\lambda x + (1-\lambda)y\| \\ &= \lambda\|x\| + (1-\lambda)\|y\| - \|\lambda x + (1-\lambda)y\| - (1-\lambda)(\|x\| + \|y\| - \|x+y\|) \\ &= d_\lambda(x,y) - (1-\lambda)(\|x\| + \|y\| - \|x+y\|), \end{aligned}$$

which implies the second relation of the statement. □

For $\lambda \in [0,1]$, it is easy to see that $d_\lambda(x,y) \geq 0$ for every $x, y \in \mathscr{X}$. Next, we study the case when $\lambda \notin (0,1)$.

Lemma 2. *If $\lambda \in \mathbb{R} - (0,1)$, then the following inequality holds:*

$$d_\lambda(x,y) \leq 0 \tag{9}$$

for all $x, y \in \mathscr{X}$.

Proof. We study two cases:

(I) If $\lambda \leq 0$, then, by applying the triangle inequality, we obtain

$$d_\lambda(x,y) = \lambda\|x\| + (1-\lambda)\|y\| - \|\lambda x + (1-\lambda)y\|$$

$$= -(\|-\lambda x\| + \|\lambda x + (1-\lambda)y\| - \|(1-\lambda)y\|) \leq 0;$$

(II) If $\lambda \geq 1$, then, by using the triangle inequality, we deduce that

$$
\begin{aligned}
d_\lambda(x,y) &= \lambda\|x\| + (1-\lambda)\|y\| - \|\lambda x + (1-\lambda)y\| \\
&= -(\| - (1-\lambda)y\| + \|\lambda x + (1-\lambda)y\| - \|\lambda x\|) \leq 0.
\end{aligned}
$$

Therefore, the inequality of the statement is true. □

Proposition 1. *If $\lambda \in [0,1]$, then the following inequality holds:*

$$2\min\{\lambda, 1-\lambda\}d_{1/2}(x,y) \leq d_\lambda(x,y) \leq 2\max\{\lambda, 1-\lambda\}d_{1/2}(x,y) \tag{10}$$

for all $x, y \in \mathscr{X}$.

Proof. For $\lambda \in [0, \frac{1}{2}]$, we have $2\lambda \in [0,1]$ and $2\lambda - 1 \in [-1,0]$, so we show that $d_{2\lambda}\left(\frac{1}{2}(x+y), y\right) \geq 0$, and using Lemma 2, we have the inequality $d_{2\lambda-1}\left(x, \frac{1}{2}(x+y)\right) \leq 0$. From equalities (7) and (8), we obtain

$$2\lambda d_{1/2}(x,y) \leq d_\lambda(x,y) \leq 2(1-\lambda)d_{1/2}(x,y). \tag{11}$$

For $\lambda \in [\frac{1}{2}, 1]$, in the same way, we prove that

$$2(1-\lambda)d_{1/2}(x,y) \leq d_\lambda(x,y) \leq 2\lambda d_{1/2}(x,y). \tag{12}$$

Consequently, combining inequalities (11) and (12), the inequality of the statement is true. □

Remark 1. *For $\lambda \in (0,1)$, inequality (10) can be written as*

$$0 \leq \frac{d_\lambda(x,y)}{\max\{\lambda, 1-\lambda\}} \leq \|x\| + \|y\| - \|x+y\| \leq \frac{d_\lambda(x,y)}{\min\{\lambda, 1-\lambda\}} \tag{13}$$

for all $x, y \in \mathscr{X}$. For $\lambda = \frac{\|y\|}{\|x\|+\|y\|}$, we obtain Maligranda's inequality; thus, inequality (10) represents a generalization of Maligranda's inequality.

In general, for $a, b > 0$ and using relation (10), for $\lambda = \dfrac{a}{a+b}$, we find the following inequality:

$$\frac{2\min\{a,b\}}{a+b}d_{1/2}(x,y) \leq d_{\frac{a}{a+b}}(x,y) \leq \frac{2\max\{a,b\}}{a+b}d_{1/2}(x,y) \tag{14}$$

for all $x, y \in \mathscr{X}$, which can be rewritten as

$$
\begin{aligned}
\min\{a,b\}(\|x\| + \|y\| - \|x+y\|) &\leq a\|x\| + b\|y\| - \|ax+by\| \\
&\leq \max\{a,b\}(\|x\| + \|y\| - \|x+y\|)
\end{aligned} \tag{15}
$$

for all vectors x and y in $\mathscr{X}$ and $a, b \in \mathbb{R}_+$. This inequality is given in [20]. For the nonzero vectors $x, y \in \mathscr{X}$, we take $a = \dfrac{1}{\|x\|}$ and $b = \dfrac{1}{\|y\|}$ in inequality (15), which prove inequality (3) given by Maligranda [10]. If $x = 0$ or $y = 0$, then, in relation (15), we obtain equality.

Inequality (15) has several results related to the p-angular distance in normed space in [20] and the skew p-angular distance between nonzero vectors x and y in a normed space $\mathscr{X}$. Therefore, our interest is in refining inequality (15), which can be obtained from inequality (10).

Theorem 1. *Let x, y be two vectors in a real normed space $\mathscr{X}$. If $\lambda \in [0, \frac{1}{2}]$; then, the following inequality holds:*

$$2\lambda d_{1/2}(x,y) + 2\min\{2\lambda, 1-2\lambda\} d_{1/2}\left(\frac{1}{2}(x+y), y\right) \leq d_\lambda(x,y)$$

$$\leq 2\lambda d_{1/2}(x,y) + 2\max\{2\lambda, 1-2\lambda\} d_{1/2}\left(\frac{1}{2}(x+y), y\right) \tag{16}$$

and if $\lambda \in [\frac{1}{2}, 1]$, then the inequality

$$2(1-\lambda) d_{1/2}(x,y) + 2\min\{2\lambda-1, 2-2\lambda\} d_{1/2}\left(x, \frac{1}{2}(x+y)\right) \leq d_\lambda(x,y)$$

$$\leq 2(1-\lambda) d_{1/2}(x,y) + 2\max\{2\lambda-1, 2-2\lambda\} d_{1/2}\left(x, \frac{1}{2}(x+y)\right) \tag{17}$$

holds.

Proof. For $\lambda \in [0, \frac{1}{2}]$, we have $2\lambda \in [0,1]$, and by replacing x with $\frac{1}{2}(x+y)$ in inequality (10), we deduce that

$$2\min\{2\lambda, 1-2\lambda\} d_{1/2}\left(\frac{1}{2}(x+y), y\right) \leq d_{2\lambda}\left(\frac{1}{2}(x+y), y\right)$$

$$\leq 2\max\{2\lambda, 1-2\lambda\} d_{1/2}\left(\frac{1}{2}(x+y), y\right). \tag{18}$$

Consequently, by combining equality (7) with inequality (18), we show the first inequality of the statement.

For $\lambda \in [\frac{1}{2}, 1]$, we have $2\lambda - 1 \in [0,1]$, and by replacing y with $\frac{1}{2}(x+y)$ in inequality (10), we deduce that

$$2\min\{2\lambda-1, 2-2\lambda\} d_{1/2}\left(x, \frac{1}{2}(x+y)\right) \leq d_{2\lambda-1}\left(x, \frac{1}{2}(x+y)\right)$$

$$\leq 2\max\{2\lambda-1, 2-2\lambda\} d_{1/2}\left(x, \frac{1}{2}(x+y)\right). \tag{19}$$

Consequently, by combining equality (8) with inequality (19), we prove the second inequality of the statement. □

Remark 2. *We are studying the problem of comparing the upper bound from inequality (10) with the upper bounds from inequalities (16) and (17) to see which is better. For $\lambda \in [0, \frac{1}{4}]$, through simple calculations, we prove the inequality*

$$2\lambda d_{1/2}(x,y) + 2\max\{2\lambda, 1-2\lambda\} d_{1/2}\left(\frac{1}{2}(x+y), y\right) \leq 2(1-\lambda) d_{1/2}(x,y). \tag{20}$$

Therefore, for $\lambda \in [0, \frac{1}{4}]$, the upper bound from inequality (16) is better. However, for $\lambda \in [\frac{1}{4}, \frac{1}{2}]$, inequality (20) becomes

$$(1-\lambda)\|x+y\| + (4\lambda-1)\|y\| \leq (1-2\lambda)\|x\| + \lambda\|x+3y\|,$$

which is true for $y = -x$ and false for $y = -\frac{1}{3}x$. For $\lambda \in [\frac{3}{4}, 1]$, through simple calculations, we prove the inequality

$$2(1-\lambda) d_{1/2}(x,y) + 2\max\{2\lambda-1, 2-2\lambda\} d_{1/2}\left(x, \frac{1}{2}(x+y)\right) \leq 2\lambda d_{1/2}(x,y). \tag{21}$$

Consequently, for $\lambda \in [\frac{3}{4}, 1]$*, the upper bound from inequality (17) is better. However, for* $\lambda \in [\frac{1}{2}, \frac{3}{4}]$*, inequality (21) becomes*

$$\lambda\|x+y\| + (3-4\lambda)\|x\| \le (2\lambda-1)\|y\| + (1-\lambda)\|3x+y\|,$$

which is true for $y = -x$ *and false for* $y = -3x$*.*

We choose two real numbers a *and* b *such that* $0 < a \le b$*; if we use relation (16) for* $\lambda = \dfrac{a}{a+b} \le \frac{1}{2}$*, then we obtain the following inequality:*

$$2\frac{a}{a+b}d_{1/2}(x,y) + \frac{2\min\{2a, b-a\}}{a+b}d_{1/2}\left(\frac{1}{2}(x+y), y\right) \le d_{\frac{a}{a+b}}(x,y)$$
$$\le 2\frac{a}{a+b}d_{1/2}(x,y) + \frac{2\max\{2a, b-a\}}{a+b}d_{1/2}\left(\frac{1}{2}(x+y), y\right) \tag{22}$$

for all $x, y \in \mathscr{X}$*, which can be rewritten as*

$$a(\|x\| + \|y\| - \|x+y\|) + \frac{1}{2}\min\{2a, b-a\}(\|x+y\| + 2\|y\| - \|x+3y\|)$$
$$\le a\|x\| + b\|y\| - \|ax+by\|$$
$$\le a(\|x\| + \|y\| - \|x+y\|) + \frac{1}{2}\max\{2a, b-a\}(\|x+y\| + 2\|y\| - \|x+3y\|) \tag{23}$$

for all vectors x *and* y *in* $\mathscr{X}$ *and* $a, b \in \mathbb{R}_+$*,* $a \le b$*. This inequality refined the first part of inequality (15). Next, if we have* $\|y\| \le \|x\|$*, then we take* $a = \|y\|$ *and* $b = \|x\|$ *in inequality (23), and we find the inequality*

$$\|x\| + \|y\| - \|x+y\| + \min\{1, \frac{\|x\| - \|y\|}{2\|y\|}\}(\|x+y\| + 2\|y\| - \|x+3y\|)$$
$$\le \|x\|\left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right)$$
$$\le \|x\| + \|y\| - \|x+y\| + \max\{1, \frac{\|x\| - \|y\|}{2\|y\|}\}(\|x+y\| + 2\|y\| - \|x+3y\|) \tag{24}$$

for all nonzero vectors x *and* y *in* $\mathscr{X}$*. Thus, we prove an improvement of the second part of the inequality of Maligranda given in relation (3). Replacing* y *with* $-y$ *and taking* $\|y\| \le \|x\|$*, through simple calculations in inequality (24), we obtain a refinement of the second part of inequality (4); thus,*

$$\alpha[x,y] \le \frac{\|x-y\| + \|x\| - \|y\|}{\|x\|} - E = \frac{\|x-y\| + |\|x\| - \|y\||}{\max\{\|x\|, \|y\|\}} - E, \tag{25}$$

where $E = \dfrac{1}{2\|x\|\|y\|}\min\{2\|y\|, \|x\| - \|y\|\}(\|x-y\| + 2\|y\| - \|x-3y\|) \ge 0.$

A generalization of the equalities from Lemma 1 is given below.

Theorem 2. *Let* x, y *be two vectors in the real normed space* $\mathscr{X}$ *and a natural number* $n \ge 1$*. If* $\lambda \in \mathbb{R}$*, then the following equalities hold:*

$$d_\lambda(x,y) = \lambda\sum_{k=1}^{n} 2^k d_{1/2}\left(\frac{1}{2^{k-1}}x + \left(1 - \frac{1}{2^{k-1}}\right)y, y\right) + d_{2^n\lambda}\left(\frac{1}{2^n}x + \left(1 - \frac{1}{2^n}\right)y, y\right) \tag{26}$$

and

$$d_\lambda(x,y) = (1-\lambda)\sum_{k=1}^{n} 2^k d_{1/2}\left(x, \left(1-\frac{1}{2^{k-1}}\right)x + \frac{1}{2^{k-1}}y\right) + d_{2^n\lambda+1-2^n}\left(x, \left(1-\frac{1}{2^n}\right)x + \frac{1}{2^n}y\right). \tag{27}$$

Proof. Using Lemma 1 for $\lambda \in \mathbb{R}$, we have

$$d_\lambda(x,y) = d_{2\lambda}\left(\frac{1}{2}(x+y), y\right) + 2\lambda d_{1/2}(x,y).$$

Replacing λ with 2λ and x with $\frac{1}{2}(x+y)$ in the above equality, we get

$$d_{2\lambda}\left(\frac{1}{2}(x+y), y\right) = d_{2^2\lambda}\left(\frac{1}{2}(\frac{1}{2}(x+y)+y), y\right) + 2^2\lambda d_{1/2}\left(\frac{1}{2}(x+y), y\right).$$

If we inductively repeat the above substitutions, for $k \geq 1$, we have

$$d_{2^{k-1}\lambda}\left(\frac{1}{2^{k-1}}x + \left(1-\frac{1}{2^{k-1}}\right)y, y\right) = d_{2^k\lambda}\left(\frac{1}{2^k}x + \left(1-\frac{1}{2^k}\right)y, y\right)$$
$$+2^k\lambda d_{1/2}\left(\frac{1}{2^{k-1}}x + \left(1-\frac{1}{2^{k-1}}\right)y, y\right).$$

Therefore, summarizing the above relations for $k \in \{1,\ldots,n\}$, we obtain the relation of the statement. Applying equality (26) and taking into account that $d_\lambda(x,y) = d_{1-\lambda}(y,x)$, we deduce equality (27). □

These equalities offer the possibility of refining inequalities (16) and (17), giving the following.

Theorem 3. *Let n a natural number, $n \geq 1$. If $\lambda \in \left[0, \frac{1}{2^n}\right]$, then the following inequality holds:*

$$\lambda\sum_{k=1}^{n} 2^k d_{1/2}\left(\frac{1}{2^{k-1}}x + \left(1-\frac{1}{2^{k-1}}\right)y, y\right) + 2\min\{2^n\lambda, 1-2^n\lambda\} d_{1/2}\left(\frac{1}{2^n}x + \left(1-\frac{1}{2^n}\right)y, y\right)$$
$$\leq d_\lambda(x,y)$$
$$\leq \lambda\sum_{k=1}^{n} 2^k d_{1/2}\left(\frac{1}{2^{k-1}}x + \left(1-\frac{1}{2^{k-1}}\right)y, y\right) + 2\max\{2^n\lambda, 1-2^n\lambda\} d_{1/2}\left(\frac{1}{2^n}x + \left(1-\frac{1}{2^n}\right)y, y\right) \tag{28}$$

and if $\lambda \in \left[1-\frac{1}{2^n}, 1\right]$, then the following inequality holds:

$$(1-\lambda)\sum_{k=1}^{n} 2^k d_{1/2}\left(x, \left(1-\frac{1}{2^{k-1}}\right)x + \frac{1}{2^{k-1}}y\right) + 2\min\{\lambda', 1-\lambda'\} d_{1/2}\left(x, \left(1-\frac{1}{2^n}\right)x + \frac{1}{2^n}y\right)$$
$$\leq d_\lambda(x,y)$$
$$\leq (1-\lambda)\sum_{k=1}^{n} 2^k d_{1/2}\left(x, \left(1-\frac{1}{2^{k-1}}\right)x + \frac{1}{2^{k-1}}y\right) + 2\max\{\lambda', 1-\lambda'\} d_{1/2}\left(x, \left(1-\frac{1}{2^n}\right)x + \frac{1}{2^n}y\right), \tag{29}$$

where $\lambda' = 2^n\lambda + 1 - 2^n$.

Proof. Using the inequalities from Proposition 1 and combining them with equalities (26) and (27), we deduce that the inequalities of the statement are true. □

Finally, we present some applications.

(1) In terms of expression $d_\lambda(x,y)$, we show a relation that involves the p–angular distance:

$$2d_{1/2}\left(\|x\|^{p-1}(x-y), (\|x\|^{p-1} - \|y\|^{p-1})y\right)$$

$$= \|x\|^{p-1}\|x-y\| + |\|x\|^{p-1} - \|y\|^{p-1}|\|y\| - \alpha_p[x,y]. \quad (30)$$

Given that $d_{1/2}(\|x\|^{p-1}(x-y), (\|x\|^{p-1} - \|y\|^{p-1})y) \geq 0$, we have the following inequality:

$$\alpha_p[x,y] \leq \|x\|^{p-1}\|x-y\| + |\|x\|^{p-1} - \|y\|^{p-1}|\|y\| \quad (31)$$

for all nonzero vectors $x, y \in \mathscr{X}$ and $p \in \mathbb{R}$.

(2) For $p = 0$ in relation (30), we deduce the equality:

$$2d_{1/2}\left(\frac{1}{\|x\|}(x-y), \left(\frac{1}{\|x\|} - \frac{1}{\|y\|}\right)y\right) = \frac{\|x-y\| + |\|x\| - \|y\||}{\|x\|} - \alpha[x,y]. \quad (32)$$

Because $d_{1/2}\left(\frac{1}{\|x\|}(x-y), \left(\frac{1}{\|x\|} - \frac{1}{\|y\|}\right)y\right) \geq 0$, we prove the inequality

$$\alpha[x,y] \leq \frac{\|x-y\| + |\|x\| - \|y\||}{\|x\|}. \quad (33)$$

Interchanging the roles of x and y, in (33), we get the second inequality from relation (4).

(3) Next, we show a relation that involves the skew p–angular distance in terms of expression $d_\lambda(x,y)$, namely:

$$2d_{1/2}\left(\|y\|^{p-1}(x-y), (\|y\|^{p-1} - \|x\|^{p-1})y\right)$$

$$= \|y\|^{p-1}\|x-y\| + |\|x\|^{p-1} - \|y\|^{p-1}|\|y\| - \beta_p[x,y]; \quad (34)$$

(4) For $p = 0$ in relation (34), we deduce the equality:

$$2d_{1/2}\left(\frac{1}{\|y\|}(x-y), \left(\frac{1}{\|y\|} - \frac{1}{\|x\|}\right)y\right) = \frac{\|x-y\|}{\|y\|} + \frac{|\|x\| - \|y\||}{\|x\|} - \beta[x,y]. \quad (35)$$

Given that $d_{1/2}\left(\frac{1}{\|y\|}(x-y), \left(\frac{1}{\|y\|} - \frac{1}{\|x\|}\right)y\right) \geq 0$, we find the inequality

$$\beta[x,y] \leq \frac{\|x-y\|}{\|y\|} + \frac{|\|x\| - \|y\||}{\|x\|}. \quad (36)$$

Interchanging the roles of x and y, in (36), we get the second inequality from relation (6).

(5) From inequality (10), we deduce that

$$\frac{d_\lambda(x,y)}{\max\{\lambda, 1-\lambda\}} \leq 2d_{1/2}(x,y) \quad (37)$$

for all $x, y \in \mathscr{X}$ and $\lambda \in [0,1]$, and applying equality (30), we find that

$$\frac{\lambda\|x\|^{p-1}\|x-y\| + (1-\lambda)|\|x\|^{p-1} - \|y\|^{p-1}|\|y\| - F}{\max\{\lambda, 1-\lambda\}}$$

$$= \frac{d_\lambda\left(\|x\|^{p-1}(x-y), (\|x\|^{p-1} - \|y\|^{p-1})y\right)}{\max\{\lambda, 1-\lambda\}}$$

$$\leq \|x\|^{p-1}\|x-y\| + |\|x\|^{p-1} - \|y\|^{p-1}|\|y\| - \alpha_p[x,y], \quad (38)$$

where $F = \|\lambda\|x\|^{p-1}(x-y) + (1-\lambda)(\|x\|^{p-1} - \|y\|^{p-1})y\|$.

(6) From inequality (37) and by applying equality (34), we prove that

$$\frac{\lambda\|y\|^{p-1}\|x-y\| + (1-\lambda)|\|x\|^{p-1} - \|y\|^{p-1}|\|y\| - G}{\max\{\lambda, 1-\lambda\}}$$

$$= \frac{d_\lambda\left(\|y\|^{p-1}(x-y), (\|y\|^{p-1} - \|x\|^{p-1})y\right)}{\max\{\lambda, 1-\lambda\}}$$
$$\leq \|y\|^{p-1}\|x-y\| + |\|x\|^{p-1} - \|y\|^{p-1}|\|y\| - \beta_p[x,y], \quad (39)$$

where $G = \|\lambda\|y\|^{p-1}(x-y) + (1-\lambda)(\|y\|^{p-1} - \|x\|^{p-1})y\|$.

3. Conclusions

Maligranda's inequality is an important inequality in real normed spaces. This is used to estimate the angular distance, the p–angular distance, the skew angular distance, and the skew p–angular distance between two nonzero vectors in a real normed space. Therefore, the study of some refinements of this inequality helps with a better estimation of the mentioned previously angular distances. In addition, in this paper, we obtained other inequalities of the Maligranda type. Using equality (7), we proved two inequalities in Theorem 1, which improved the second part of Maligranda's inequality. These results can be used to obtain better estimates of angular distances. In Theorem 3, we also presented a result that refined the inequalities obtained in Theorem 1. By replacing the parameter λ with various values or by replacing the vectors x, y with linear combinations of these vectors, we obtained other applications for $d_\lambda(x,y)$, where $x, y \in \mathscr{X}$. It remains for the reader to find other estimates of the expression $d_\lambda(x,y)$, where $x, y \in \mathscr{X}$ and $\lambda \in \mathbb{R}$.

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