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An Application to Fixed-Point Results in Tricomplex-Valued Metric Spaces Using Control Functions

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Abstract: In the present work, we establish fixed-point results for a pair of mappings satisfying some contractive conditions on rational expressions with coefficients as point-dependent control functions in the setting of tricomplex-valued metric spaces. The proven results are extension and generalisation of some of the literature's well-known results. We also explore some of the applications to our key results.

Keywords: common fixed point; TVMS; complete TVMS; Cauchy sequence; fixed point

MSC: 47H9; 47H10; 30G35; 46N99; 54H25

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1. Introduction

The works of [1–4] paved the way for the study of bicomplex numbers. These numbers offer a commutative alternative to the skew field of quaternions. As they generalise complex numbers more closely and accurately than quaternions, they are a topic of interest. The book [5] of Price will give the reader a deep insight to the concept of bicomplex algebra and function theory. Extensions in this area have resulted in some significant applications in several fields of mathematical sciences as well as some components of science and technology, as reported in [6]. In this area, various researchers have reported interesting results. Amongst all of them, Luna-Elizaarrarás et al. [7] developed an important work on the elementary functions of bicomplex numerals. They introduced the algebra of bicomplex numbers as a generalisation of the field of complex numbers, describing how to define elementary functions in such algebra as polynomials, exponential functions, and trigonometric functions, as well as their inverse functions such as roots, logarithms, and inverse trigonometric functions.

The Banach contraction principle [8] paved the way for development of metric fixed-point theory. Subsequently, fixed point results were reported by various researchers for conformal mappings in the setting of various topological spaces (see [9–23]). The introduction of complex valued metric spaces and the fixed-point results proved there on by Azam et al. [24] gave the scope for further research in these spaces, and various fixed-point results were reported by many researchers in the recent past. Most recently, Gunaseelan et al. [25] established CFPT on CPMS, and Rajagopalan et al. [26] reported the application of fixed-point results on CPMS to find analytical solutions to integral equations.

In the sequel work of [27], Choi et al. defined bicomplex-valued metric space and proved some fixed-point results connected with two weakly compatible maps. Later, Jebril et al. [28] proved some common fixed-point theorems under rational contractions for a pair of mappings in bicomplex-valued metric spaces. In 2021, Datta et al. [29,30] and Beg et al. [31] established fixed-point results for bicomplex-valued metric spaces. Recently, Gunaseelan et al. [32] introduced and established fixed-point results in the setting of tricomplex-valued metric spaces. In the present work, some new fixed-point results with their applications are established using control functions in the setting of tricomplex-valued metric spaces.

This paper is organised as follows. Section 2 presents some basic concepts and definitions with suitable examples which are vital for establishing the main results. In Section 3, we present our main results and some corollaries that are consequences of our main results. Our results are supported by suitable examples. An application based on the derived fixed-point results is given in Section 4.

2. Preliminaries

Throughout this paper, $\mathbb{C}_0$, $\mathbb{C}_1$, $\mathbb{C}_2$, and $\mathbb{C}_3$ denote the families of real, complex, bicomplex, and tricomplex numbers, respectively. Price [5] defined the bicomplex numbers as

$$\mu = \mathfrak{q}_1 + \mathfrak{q}_2 i_1 + \mathfrak{q}_3 i_2 + \mathfrak{q}_4 i_1 i_2,$$

where $\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_4 \in \mathbb{C}_0$ and i_1 and i_2 are independent units such that $i_1^2 = i_2^2 = -1$ and $i_1 i_2 = i_2 i_1$.

We define the set of bicomplex numbers as

$$\mathbb{C}_2 = \{\mu : \mu = \mathfrak{q}_1 + \mathfrak{q}_2 i_1 + \mathfrak{q}_3 i_2 + \mathfrak{q}_4 i_1 i_2, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_4 \in \mathbb{C}_0\},$$

In other words, they are defined as

$$\mathbb{C}_2 = \{\mu : \mu = \theta_1 + i_2\theta_2, \theta_1, \theta_2 \in \mathbb{C}_1\},$$

where $\theta_1 = \mathfrak{q}_1 + \mathfrak{q}_2 i_1 \in \mathbb{C}_1$ and $\theta_2 = \mathfrak{q}_3 + \mathfrak{q}_4 i_1 \in \mathbb{C}_1$. Price [5] defined the tricomplex numbers as

$$\omega = \mathfrak{q}_1 + \mathfrak{q}_2 i_1 + \mathfrak{q}_3 i_2 + \mathfrak{q}_4 j_1 + \mathfrak{q}_5 i_3 + \mathfrak{q}_6 j_2 + \mathfrak{q}_7 j_3 + \mathfrak{q}_8 i_4,$$

where $\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_4, \mathfrak{q}_5, \mathfrak{q}_6, \mathfrak{q}_7, \mathfrak{q}_8 \in \mathbb{C}_0$ and independent units $i_1, i_2, i_3, i_4, j_1, j_2$ and j_3 are such that $i_1^2 = i_4^2 = -1, i_4 = i_1 j_3 = i_1 i_2 i_3, j_2 = i_1 i_3 = i_3 i_1, j_2^2 = 1, j_1 = i_1 i_2 = i_2 i_1$, and $j_1^2 = 1$.

We define the set of tricomplex numbers as

$$\mathbb{C}_3 = \{\omega : \omega = \mathfrak{q}_1 + \mathfrak{q}_2 i_1 + \mathfrak{q}_3 i_2 + \mathfrak{q}_4 j_1 + \mathfrak{q}_5 i_3 + \mathfrak{q}_6 j_2 + \mathfrak{q}_7 j_3 + \mathfrak{q}_8 i_4, \\ \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_4, \mathfrak{q}_5, \mathfrak{q}_6, \mathfrak{q}_7, \mathfrak{q}_8 \in \mathbb{C}_0\},$$

In other words, they are defined as

$$\mathbb{C}_3 = \{\omega : \omega = \mu_1 + i_3\mu_2, \mu_1, \mu_2 \in \mathbb{C}_2\},$$

where $\mu_1 = \theta_1 + \theta_2 i_2 \in \mathbb{C}_2$ and $\mu_2 = \theta_3 + \theta_4 i_2 \in \mathbb{C}_2$. If $\omega = \mu_1 + i_3\mu_2$ and $\varrho = \mathfrak{L}_1 + i_3\mathfrak{L}_2$ are any two tricomplex numbers, then their sum is

$$\omega \pm \varrho = (\mu_1 + i_3\mu_2) \pm (\mathfrak{L}_1 + i_3\mathfrak{L}_2) = \mu_1 \pm \mathfrak{L}_1 + i_3(\mu_2 \pm \mathfrak{L}_2)$$

and their product is

$$\omega.\varrho = (\mu_1 + i_3\mu_2)(\mathfrak{L}_1 + i_3\mathfrak{L}_2) = (\mu_1\mathfrak{L}_1 - \mu_2\mathfrak{L}_2) + i_3(\mu_1\mathfrak{L}_2 + \mu_2\mathfrak{L}_1).$$

There are four idempotent elements in $\mathbb{C}_3$, which are $0, 1, \mathfrak{f}_1 = \frac{1+j_3}{2}$, and $\mathfrak{f}_2 = \frac{1-j_3}{2}$. Hence, $\mathfrak{f}_1$ and $\mathfrak{f}_2$ are nontrivial such that $\mathfrak{f}_1 + \mathfrak{f}_2 = 1$ and $\mathfrak{f}_1\mathfrak{f}_2 = 0$. Every tricomplex numeral $\mu_1 + i_3\mu_2$ can be expressed as an union of $\mathfrak{f}_1$ and $\mathfrak{f}_2$, where

$$\omega = \mu_1 + i_3\mu_2 = (\mu_1 - i_2\mu_2)\mathfrak{f}_1 + (\mu_1 + i_2\mu_2)\mathfrak{f}_2.$$

This notation of ω represents the idempotent of the tricomplex numeral, and the coefficients of the complex numerals $\omega_1 = (\mu_1 - i_2\mu_2)$ and $\omega_2 = (\mu_1 + i_2\mu_2)$ are the idempotent components of the bicomplex numeral ω.

An element $\omega = \mu_1 + i_3\mu_2 \in \mathbb{C}_3$ is invertible if there exists ϱ in $\mathbb{C}_3$ such that $\omega\varrho = 1$, and ϱ is called the inverse (multiplicative) of ω.

Therefore, ω is the inverse (multiplicative) of ϱ. An element having an inverse in $\mathbb{C}_3$ is called non-singular, and the element not having an inverse in $\mathbb{C}_3$ is the singular element of $\mathbb{C}_3$.

An element $\omega = \mu_1 + i_3\mu_2 \in \mathbb{C}_3$ is nonsingular if $|\mu_1^2 + \mu_2^2| \neq 0$ and singular if $|\mu_1^2 + \mu_2^2| = 0$.

The inverse of ω is defined as

$$\omega^{-1} = \varrho = \frac{\mu_1 - i_3\mu_2}{\mu_1^2 + \mu_2^2}.$$

The norm $||.||$ of $\mathbb{C}_3$ is a positive real-valued function, and $||.|| : \mathbb{C}_3 \to \mathbb{C}_0^+$ is defined by

$$\begin{aligned} ||\omega|| &= ||\mu_1 + i_3\mu_2|| = \{|\mu_1|^2 + |\mu_2|^2\}^{\frac{1}{2}} \\ &= \left[\frac{|(\mu_1 - i_2\mu_2)|^2 + |(\mu_1 + i_2\mu_2)|^2}{2}\right]^{\frac{1}{2}} \\ &= (\mathfrak{q}_1^2 + \mathfrak{q}_2^2 + \mathfrak{q}_1^2 + \mathfrak{q}_3^2 + \mathfrak{q}_4^2 + \mathfrak{q}_5^2 + \mathfrak{q}_6^2 + \mathfrak{q}_7^2 + \mathfrak{q}_8^2)^{\frac{1}{2}}, \end{aligned}$$

where $\omega = \mathfrak{q}_1 + \mathfrak{q}_2 i_1 + \mathfrak{q}_3 i_2 + \mathfrak{q}_4 j_1 + \mathfrak{q}_5 i_3 + \mathfrak{q}_6 j_2 + \mathfrak{q}_7 j_3 + \mathfrak{q}_8 i_4 = \mu_1 + i_3\mu_2 \in \mathbb{C}_3$. Clearly, $\mathbb{C}_3$ is a Banach space, as the linear space $\mathbb{C}_3$ is complete.

If $\omega, \varrho \in \mathbb{C}_3$, then $||\omega\varrho|| \leq 2||\omega||||\varrho||$ holds instead of $||\omega\varrho|| \leq ||\omega||||\varrho||$, and then $\mathbb{C}_3$ is not Banach algebra.

The partial order relation $\preceq_{i_3}$ on $\mathbb{C}_3$ is defined as follows. Let $\mathbb{C}_3$ be the set of tricomplex numerals and $\omega = \mu_1 + i_3\mu_2$ and $\varrho = \mathfrak{L}_1 + i_3\mathfrak{L}_2 \in \mathbb{C}_3$. Then, $\omega \preceq_{i_3} \varrho$ if $\mu_1 \preceq_{i_2} \mathfrak{L}_1$ and $\mu_2 \preceq_{i_2} \mathfrak{L}_2$; in other words, $\omega \preceq_{i_3} \varrho$ if one of the bellow axioms is fullfilled:

(a) $\mu_1 = \mathfrak{L}_1, \mu_2 = \mathfrak{L}_2$;
(b) $\mu_1 \prec_{i_2} \mathfrak{L}_1, \mu_2 = \mathfrak{L}_2$;
(c) $\mu_1 = \mathfrak{L}_1, \mu_2 \prec_{i_2} \mathfrak{L}_2$;
(d) $\mu_1 \prec_{i_2} \mathfrak{L}_1, \mu_2 \prec_{i_2} \mathfrak{L}_2$.

In particular, $\omega \lnsim_{i_3} \varrho$ if $\omega \preceq_{i_3} \varrho$, where $\omega \neq \varrho$ when one of (b), (c), or (d) holds and $\omega \prec_{i_3} \varrho$ only if (d) holds.

Given any two tricomplex numerals $\omega, \varrho \in \mathbb{C}_3$, the following holds:

(1) $\omega \preceq_{i_3} \varrho$ if $||\omega|| \leq ||\varrho||$;
(2) $||\omega + \varrho|| \leq ||\omega|| + ||\varrho||$;
(3) $||\mathfrak{q}\omega|| = |\mathfrak{q}|||\omega||$, where $\mathfrak{q} \in \mathbb{C}_0^+$;
(4) $||\omega\varrho|| \leq 2||\omega||||\varrho||$, and the equality holds only when at least one of ω and ϱ is non-singular;
(5) $||\omega^{-1}|| = ||\omega||^{-1}$ if ω is a non-singular;
(6) $||\frac{\omega}{\varrho}|| = \frac{||\omega||}{||\varrho||}$, if ϱ is non-singular.

We now recall some basic concepts and notations which will be used in the sequel:

Definition 1 ([32])**.** *Let $\Re$ be non-empty set. A map $\varsigma\colon \Re \times \Re \to C_3$ is a tricomplex-valued metric if the following holds:*

(CM1) $0 \preceq_{i_3} \varsigma(\tau,\eta)$ *for all* $\tau,\eta \in \Re$*, and* $\varsigma(\tau,\eta) = 0 \Leftrightarrow \tau = \eta$*;*
(CM2) $\varsigma(\tau,\eta) = \varsigma(\eta,\tau)$ *for all* $\tau,\eta \in \Re$*;*
(CM3) $\varsigma(\tau,\eta) \preceq_{i_3} \varsigma(\tau,\varrho) + \varsigma(\varrho,\eta)$ *for all* $\tau,\eta,\varrho \in \Re$*.*

In this case, we say $(\Re,\varsigma)$ *is a TVMS.*

Example 1. *Let* $\Re = \mathcal{C}_3$ *be a set of tricomplex numbers. Define* $\varsigma\colon \mathcal{C}_3 \times \mathcal{C}_3 \to \mathcal{C}_3$*. Under*

$$\varsigma(\varrho_1,\varrho_2) = |\tau_1 - \tau_2| + i_3|\eta_1 - \eta_2|,$$

where $\varrho_1 = \tau_1 + i_3\eta_1$ *and* $\varrho_2 = \tau_2 + i_3\eta_2$*, then* $(\mathcal{C}_3,\varsigma)$ *is a TVMS.*

Example 2. *Let* $\Re = \mathcal{C}_3$*. Define a mapping* $\varsigma\colon \Re \times \Re \to \mathcal{C}_3$ *by* $\varsigma(\varrho_1,\varrho_2) = \mathfrak{e}^{i_3\iota}|\varrho_1 - \varrho_2|$*, where* $\iota \in [0,\frac{\pi}{2}]$*. Then,* $(\Re,\varsigma)$ *is a complete valued metric space.*

Definition 2 ([32]). *Suppose that* $(\Re,\varsigma)$ *is a TVMS:*

1. *A sequence* $\{\tau_\alpha\}$ *is a Cauchy if for every* $0 \prec_{i_3} \mathfrak{z} \in \mathcal{C}_3$*, there exists an integer* $\mathcal{N}$ *such that* $\varsigma(\tau_\alpha,\tau_\mathfrak{m}) \prec_{i_3} \mathfrak{z}$ *for all* $\alpha,\mathfrak{m} \geq \mathcal{N}$*;*
2. $\{\tau_\alpha\}$ *converges to* $\tau \in \Re$ *if for every* $0 \prec_{i_3} \mathfrak{z} \in \mathcal{C}_3$*, and there exists an integer* $\mathcal{N}$ *such that* $\varsigma(\tau_\alpha,\tau) \prec_{i_3} \mathfrak{z}$ *for all* $\alpha \geq \mathcal{N}$*. We denote this as* $\tau_\alpha \xrightarrow{\varsigma} \tau$*;*
3. $(\Re,\varsigma)$ *is complete if every Cauchy sequence in* $\Re$ *converges in* $\Re$*.*

Lemma 1 ([32]). *A sequence* $\{\tau_\alpha\}$ *in a TVMS* $(\Re,\varsigma)$ *converges to* τ *if and only if* $||\varsigma(\tau_\alpha,\tau)|| \to 0$ *as* $\alpha \to +\infty$*.*

Lemma 2 ([32]). *Let* $(\Re,\varsigma)$ *be a TVMS, and let* $\{\tau_\alpha\}$ *be a sequence in* $\Re$*. Then,* $\{\tau_\alpha\}$ *is Cauchy if and only if* $||\varsigma(\tau_\alpha,\tau_{\alpha+\mathfrak{m}})|| \to 0$ *as* $\alpha \to +\infty$*.*

In the next section, we present our main results, where we establish fixed-point results in the setting of a TVMS using control functions.

3. Main Results

Henceforth, let $(\Re,\varsigma)$ be a TVMS:

Proposition 1. *Let* $\mathcal{S},\mathcal{Q}\colon \Re \to \Re$*. Let* $\tau_0 \in \Re$ *be a map. Define the sequence* $\{\tau_\alpha\}$ *by*

$$\begin{aligned}\tau_{2\alpha+1} &= \mathcal{S}\tau_{2\alpha},\\ \tau_{2\alpha+2} &= \mathcal{Q}\tau_{2\alpha+1}, for all \alpha = 0,1,2,\ldots\end{aligned} \tag{1}$$

Let there exist a map $\mathfrak{f}\colon \Re \times \Re \times \Re \to [0,1)$ *such that* $\mathfrak{f}(\mathcal{QS}\tau,\eta,\mathfrak{x}) \leq \mathfrak{f}(\tau,\eta,\mathfrak{x})$ *and* $\mathfrak{f}(\tau,\mathcal{SQ}\eta,\mathfrak{x}) \leq \mathfrak{f}(\tau,\eta,\mathfrak{x}), for all \tau,\eta \in \Re$*, and for a fixed element,* $\mathfrak{x} \in \Re$ *and* $\alpha = 0,1,2,\ldots$ *Then, we have*

$$\mathfrak{f}(\tau_{2\alpha},\eta,\mathfrak{x}) \leq \mathfrak{f}(\tau_0,\eta,\mathfrak{x}) \quad and \quad \mathfrak{f}(\tau,\tau_{2\alpha+1},\mathfrak{x}) \leq \mathfrak{f}(\tau,\tau_1,\mathfrak{x}).$$

Proof. Let $\tau,\eta \in \Re$ and $\alpha = 0,1,2,\ldots$

Then, we have

$$\begin{aligned}\mathfrak{f}(\tau_{2\alpha},\eta,\mathfrak{x}) &= \mathfrak{f}(\mathcal{QS}\tau_{2\alpha-2},\eta,\mathfrak{x}) \leq \mathfrak{f}(\tau_{2\alpha-2},\eta,\mathfrak{x})\\ &= \mathfrak{f}(\mathcal{QS}\tau_{2\alpha-4},\eta,\mathfrak{x}) \leq \ldots \leq \mathfrak{f}(\tau_0,\eta,\mathfrak{x}).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\mathfrak{f}(\tau,\tau_{2\alpha+1},\mathfrak{x}) &= \mathfrak{f}(\tau,\mathcal{SQ}\tau_{2\alpha-1},\mathfrak{x}) \leq \mathfrak{f}(\tau,\tau_{2\alpha-1},\mathfrak{x})\\ &= \mathfrak{f}(\tau,\mathcal{SQ}\tau_{2\alpha-3},\mathfrak{x}) \leq \ldots \leq \mathfrak{f}(\tau,\tau_1,\mathfrak{x}).\end{aligned}$$

□

The above proposition is validated through the following example:

Example 3. *Let $\Re = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\}$. Define $\varsigma\colon \Re \times \Re \to \mathcal{C}_3$ as $\varsigma(\tau, \eta) = i_3|\tau - \eta|$. Then, clearly $(\Re, \varsigma)$ is a TVMS. Define self-mappings $\mathcal{S}$ and $\mathcal{Q}$ by*

$$\mathcal{S}\left(\frac{1}{\alpha+1}\right) = \frac{1}{\alpha+2} = \mathcal{Q}\left(\frac{1}{\alpha+1}\right), \alpha = 0, 1, 2, 3, \dots$$

Choose $\{\tau_\alpha\}$ as $\tau_\alpha = \frac{1}{\alpha+1}, \alpha = 0, 1, 2, 3, \dots$, and then $\tau_0 = 1 \in \Re$.

Clearly, $\mathcal{S}\tau_{2\alpha} = \tau_{2\alpha+1}$ and $\mathcal{Q}\tau_{2\alpha+1} = \tau_{2\alpha+2}$.

Consider a mapping $\mathfrak{f}\colon \Re \times \Re \times \Re \to [0, 1)$ by $\mathfrak{f}(\tau, \eta, \mathfrak{x}) = \frac{\tau}{7} + \frac{\eta}{9} + \mathfrak{x}$ for all $\tau, \eta \in \Re$ and for fixed $\mathfrak{x} = \frac{1}{3} \in \Re$. Then, $\mathfrak{f}(\tau, \eta, \mathfrak{x}) = \frac{\tau}{7} + \frac{\eta}{9} + \frac{1}{2}$.

Undoubtedly, we have

$$\mathfrak{f}(\mathcal{QS}\tau, \eta, \mathfrak{x}) \leq \mathfrak{f}(\tau, \eta, \mathfrak{x}) \quad \text{and} \quad \mathfrak{f}(\tau, \mathcal{SQ}\eta, \mathfrak{x}) \leq \mathfrak{f}(\tau, \eta, \mathfrak{x}).$$

for all $\tau, \eta \in \Re$ and for fixed $\mathfrak{x} \in \Re$.

Consider the following:

$$\begin{aligned} \mathfrak{f}(\tau_{2\alpha}, \eta, \mathfrak{x}) &= \frac{1}{6(2\alpha+1)} + \frac{\eta}{8} + \frac{1}{2} \leq \frac{1}{6} + \frac{\eta}{8} + \frac{1}{2} \\ &= \mathfrak{f}(\tau_0, \eta, \mathfrak{x}). \end{aligned}$$

That is, $\mathfrak{f}(\tau_{2\alpha}, \eta, \mathfrak{x}) \leq \mathfrak{f}(\tau_0, \eta, \mathfrak{x}), \alpha = 0, 1, 2, \dots\dots, \forall \eta \in \Re$, and for $\mathfrak{x} = \frac{1}{3} \in \Re$. One should also consider

$$\begin{aligned} \mathfrak{f}(\tau, \tau_{2\alpha+1}, \mathfrak{x}) &= \frac{\tau}{7} + \frac{1}{9(2\alpha+2)} + \frac{1}{3} \leq \frac{\tau}{7} + \frac{\frac{1}{2}}{9} + \frac{1}{3} \\ &= \mathfrak{f}(\tau, \tau_1, \mathfrak{x}). \end{aligned}$$

That is, $\mathfrak{f}(\tau, \tau_{2\alpha+1}, \mathfrak{x}) \leq \mathfrak{f}(\tau, \tau_1, \mathfrak{x}), \alpha = 0, 1, 2, \dots \forall \tau \in \Re$, and for fixed $\mathfrak{x} = \frac{1}{3} \in \Re$.

Thus, Proposition 1 is verified.

Lemma 3. *Let $\{\tau_\alpha\}$ be a sequence in $\Re$ and $\mathfrak{h} \in [0, 1)$. If $\mathfrak{x}_\alpha = ||\varsigma(\tau_\alpha, \tau_{\alpha+1})||$ satisfies*

$$\mathfrak{x}_\alpha \leq \mathfrak{h}\mathfrak{x}_{\alpha-1}, forall \alpha \in \mathbb{N}.$$

then $\{\tau_\alpha\}$ is a Cauchy sequence.

Proof. Let $\mathfrak{h} \in [0, 1)$. Then, we have

$$\mathfrak{x}_\alpha \leq \mathfrak{h}\mathfrak{x}_{\alpha-1} \leq \mathfrak{h}^2\mathfrak{x}_{\alpha-2} \leq \dots\dots \leq \mathfrak{h}^\alpha\mathfrak{x}_0, \forall \alpha \in \mathbb{N}.$$

For $\mathfrak{m}, \alpha \in \mathbb{N}$ such that $\mathfrak{m} > \alpha$, we have

$$\begin{aligned} ||\varsigma(\tau_\alpha, \tau_\mathfrak{m})|| &\leq \mathfrak{x}_\alpha + \mathfrak{x}_{\alpha+1} + \dots\dots + \mathfrak{x}_{\mathfrak{m}-1} \\ &\leq \mathfrak{h}^\alpha(1 + \mathfrak{h} + \mathfrak{h}^2 + \dots + \mathfrak{h}^{\mathfrak{m}-\alpha-1})\mathfrak{x}_0 \\ &\leq \frac{\mathfrak{h}^\alpha}{1-\mathfrak{h}}\mathfrak{x}_0. \end{aligned}$$

Thus, we have $||\varsigma(\tau_\alpha, \tau_\mathfrak{m})|| \to 0$ as $\alpha \to +\infty$, and so $\{\tau_\alpha\}$ is Cauchy. □

Now, we present our main theorem.

Theorem 1. *Let $(\Re, \varsigma)$ be a CTVMS (short for complete TVMS) and $\mathcal{S}, \mathcal{Q}\colon \Re \to \Re$. If there exist mappings $\mathfrak{f}, \mathfrak{z}, \beta, \gamma\colon \Re \times \Re \times \Re \to [0,1)$ such that for all $\tau, \eta \in \Re$, we have*

(a)

$$\begin{aligned} \mathfrak{f}(\mathcal{QS}\tau, \eta, \mathfrak{x}) &\le \mathfrak{f}(\tau, \eta, \mathfrak{x}) \text{ and } \mathfrak{f}(\tau, \mathcal{SQ}\eta, \mathfrak{x}) \le \mathfrak{f}(\tau, \eta, \mathfrak{x}), \\ \mathfrak{z}(\mathcal{QS}\tau, \eta, \mathfrak{x}) &\le \mathfrak{z}(\tau, \eta, \mathfrak{x}) \text{ and } \mathfrak{z}(\tau, \mathcal{SQ}\eta, \mathfrak{x}) \le \mathfrak{z}(\tau, \eta, \mathfrak{x}), \\ \beta(\mathcal{QS}\tau, \eta, \mathfrak{x}) &\le \beta(\tau, \eta, \mathfrak{x}) \text{ and } \beta(\tau, \mathcal{SQ}\eta, \mathfrak{x}) \le \beta(\tau, \eta, \mathfrak{x}), \\ \gamma(\mathcal{QS}\tau, \eta, \mathfrak{x}) &\le \gamma(\tau, \eta, \mathfrak{x}) \text{ and } \gamma(\tau, \mathcal{SQ}\eta, \mathfrak{x}) \le \gamma(\tau, \eta, \mathfrak{x}), \end{aligned}$$

(b)

$$\begin{aligned} \varsigma(\mathcal{S}\tau, \mathcal{Q}\eta) \preceq_{i_3} & \mathfrak{f}(\tau, \eta, \mathfrak{x})\varsigma(\tau, \eta) + \mathfrak{z}(\tau, \eta, \mathfrak{x}) \\ & \times \frac{\varsigma(\tau, \mathcal{S}\tau)\varsigma(\eta, \mathcal{Q}\eta)}{1 + \varsigma(\tau, \eta)} + \beta(\tau, \eta, \mathfrak{x}) \frac{\varsigma(\eta, \mathcal{S}\tau)\varsigma(\tau, \mathcal{Q}\eta)}{1 + \varsigma(\tau, \eta)} \\ & + \gamma(\tau, \eta, \mathfrak{x}) \left\{ \frac{\varsigma(\tau, \mathcal{S}\tau)\varsigma(\tau, \mathcal{Q}\eta) + \varsigma(\eta, \mathcal{Q}\eta)\varsigma(\eta, \mathcal{S}\tau)}{1 + \varsigma(\tau, \mathcal{Q}\eta) + \varsigma(\eta, \mathcal{S}\tau)} \right\}; \end{aligned} \tag{2}$$

(c)

$$\mathfrak{f}(\tau, \eta, \mathfrak{x}) + \mathfrak{z}(\tau, \eta, \mathfrak{x}) + \mathfrak{f}(\tau, \eta, \mathfrak{x}) + \gamma(\tau, \eta, \mathfrak{x}) < 1, \tag{3}$$

then $\mathcal{S}$ and $\mathcal{Q}$ have a unique common fixed point.

Proof. Let $\tau, \eta \in \Re$. From Equation (2), we have

$$\begin{aligned} \varsigma(\mathcal{S}\tau, \mathcal{QS}\tau) \preceq_{i_3} & \mathfrak{f}(\tau, \mathcal{S}\tau, \mathfrak{x})\varsigma(\tau, \mathcal{S}\tau) + \mathfrak{z}(\tau, \mathcal{S}\tau, \mathfrak{x}) \frac{\varsigma(\tau, \mathcal{S}\tau)\varsigma(\mathcal{S}\tau, \mathcal{QS}\tau)}{1 + \varsigma(\tau, \mathcal{S}\tau)} \\ & + \beta(\tau, \mathcal{S}\tau, \mathfrak{x}) \frac{\varsigma(\mathcal{S}\tau, \mathcal{S}\tau)\varsigma(\tau, \mathcal{QS}\tau)}{1 + \varsigma(\tau, \mathcal{S}\tau)} + \gamma(\tau, \mathcal{S}\tau, \mathfrak{x}) \\ & \times \left\{ \frac{\varsigma(\tau, \mathcal{S}\tau)\varsigma(\tau, \mathcal{QS}\tau) + \varsigma(\mathcal{S}\tau, \mathcal{QS}\tau)\varsigma(\mathcal{S}\tau, \mathcal{S}\tau)}{1 + \varsigma(\tau, \mathcal{QS}\tau) + \varsigma(\mathcal{S}\tau, \mathcal{S}\tau)} \right\} \\ & = \mathfrak{f}(\tau, \mathcal{S}\tau, \mathfrak{x})\varsigma(\tau, \mathcal{S}\tau) + \mathfrak{z}(\tau, \mathcal{S}\tau, \mathfrak{x}) + \frac{\varsigma(\tau, \mathcal{S}\tau)\varsigma(\mathcal{S}\tau, \mathcal{QS}\tau)}{1 + \varsigma(\tau, \mathcal{S}\tau)} \\ & + \gamma(\tau, \mathcal{S}\tau, \mathfrak{x}) \frac{\varsigma(\tau, \mathcal{S}\tau)\varsigma(\tau, \mathcal{QS}\tau)}{1 + \varsigma(\tau, \mathcal{QS}\tau)}. \end{aligned}$$

so that

$$\begin{aligned} ||\varsigma(\mathcal{S}\tau, \mathcal{QS}\tau)|| \le & \mathfrak{f}(\tau, \mathcal{S}\tau, \mathfrak{x})||\varsigma(\tau, \mathcal{S}\tau)|| + \mathfrak{z}(\tau, \mathcal{S}\tau, \mathfrak{x}) \left|\left| \frac{\varsigma(\tau, \mathcal{S}\tau)\varsigma(\mathcal{S}\tau, \mathcal{QS}\tau)}{1 + \varsigma(\tau, \mathcal{S}\tau)} \right|\right| \\ & + \gamma(\tau, \mathcal{S}\tau, \mathfrak{x}) \left|\left| \frac{\varsigma(\tau, \mathcal{S}\tau)\varsigma(\tau, \mathcal{QS}\tau)}{1 + \varsigma(\tau, \mathcal{QS}\tau)} \right|\right| \\ & = \mathfrak{f}(\tau, \eta, \mathfrak{x})||\varsigma(\tau, \mathcal{S}\tau)|| + \mathfrak{z}(\tau, \mathcal{S}\tau, \mathfrak{x}) \left|\left| \frac{\varsigma(\tau, \mathcal{S}\tau)}{1 + \varsigma(\tau, \mathcal{S}\tau)} \right|\right| ||\varsigma(\mathcal{S}\tau, \mathcal{QS}\tau)|| \\ & + \gamma(\tau, \mathcal{S}\tau, \mathfrak{x}) \left|\left| \frac{\varsigma(\tau, \mathcal{QS}\tau)}{1 + \varsigma(\tau, \mathcal{QS}\tau)} \right|\right| ||\varsigma(\tau, \mathcal{S}\tau)||, \end{aligned}$$

which implies that

$$\begin{aligned} ||\varsigma(\mathcal{S}\tau, \mathcal{QS}\tau)|| \le & \mathfrak{f}(\tau, \mathcal{S}\tau, \mathfrak{x})||\varsigma(\tau, \mathcal{S}\tau)|| \\ & + \mathfrak{z}(\tau, \mathcal{S}\tau, \mathfrak{x})||\varsigma(\mathcal{S}\tau, \mathcal{QS}\tau)|| \\ & + \gamma(\tau, \mathcal{S}\tau, \mathfrak{x})||\varsigma(\tau, \mathcal{S}\tau)||. \end{aligned} \tag{4}$$

Similarly, from Equation (2) we have

$$\begin{aligned}\varsigma(\mathcal{S}\mathcal{Q}\eta,\mathcal{Q}\eta) \preceq_{i_3} & \mathfrak{f}(\mathcal{Q}\eta,\eta,\mathfrak{x})\varsigma(\mathcal{Q}\eta,\eta)+\mathfrak{z}(\mathcal{Q}\eta,\eta,\mathfrak{x}) \\ & \times \frac{\varsigma(\mathcal{Q}\eta,\mathcal{S}\mathcal{Q}\eta)}{1+\varsigma(\eta,\mathcal{Q}\eta)}+\gamma(\mathcal{Q}\eta,\eta,\mathfrak{x})\frac{\varsigma(\eta,\mathcal{S}\mathcal{Q}\eta)\varsigma(\mathcal{Q}\eta,\mathcal{Q}\eta)}{1+\varsigma(\mathcal{Q}\eta,\eta)} \\ & +\gamma(\mathcal{Q}\eta,\eta,\mathfrak{x})\left\{\frac{\varsigma(\mathcal{Q}\eta,\mathcal{S}\mathcal{Q}\eta)\varsigma(\mathcal{Q}\eta,\mathcal{Q}\eta)+\varsigma(\eta,\mathcal{Q}\eta)\varsigma(\eta,\mathcal{S}\mathcal{Q}\eta)}{1+\varsigma(\mathcal{Q}\eta,\mathcal{Q}\eta)+\varsigma(\eta,\mathcal{S}\mathcal{Q}\eta)}\right\}.\end{aligned}$$

By applying the same treatment as above, we get

$$\begin{aligned}||\varsigma(\mathcal{S}\mathcal{Q}\eta,\mathcal{Q}\eta)|| \leq & \mathfrak{f}(\mathcal{Q}\eta,\eta,\mathfrak{x})||\varsigma(\mathcal{Q}\eta,\eta)|| \\ & +\mathfrak{z}(\mathcal{Q}\eta,\eta,\mathfrak{x})||\varsigma(\mathcal{Q}\eta,\mathcal{S}\mathcal{Q}\eta)|| \\ & +\gamma(\mathcal{Q}\eta,\eta,\mathfrak{x})||\varsigma(\eta,\mathcal{Q}\eta)||.\end{aligned} \tag{5}$$

Let $\tau_0 \in \Re$ and $\{\tau_\alpha\}$ be defined by Equation (1). We claim $\{\tau_\alpha\}$ is Cauchy.

From Proposition 1 and the inequalities in Equations (4) and (5), for all $\iota = 0,1,2,\ldots$ we obtain

$$\begin{aligned}||\varsigma(\tau_{2\iota+1},\tau_{2\iota})|| = & ||\varsigma(\mathcal{S}\mathcal{Q}\tau_{2\iota-1},\mathcal{Q}\tau_{2\iota-1})|| \\ \leq & \mathfrak{f}(\mathcal{Q}\tau_{2\iota-1},\tau_{2\iota-1},\mathfrak{x})||\varsigma(\mathcal{Q}\tau_{2\iota-1},\tau_{2\iota-1})|| \\ & +\mathfrak{z}(\mathcal{Q}\tau_{2\iota-1},\tau_{2\iota-1},\mathfrak{x})||\varsigma(\mathcal{Q}\tau_{2\iota-1},\mathcal{S}\mathcal{Q}\tau_{2\iota-1})|| \\ & +\gamma(\mathcal{Q}\tau_{2\iota-1},\tau_{2\iota-1},\mathfrak{x})||\varsigma(\mathcal{Q}\tau_{2\iota-1},\tau_{2\iota-1})|| \\ = & \mathfrak{f}(\tau_{2\iota},\tau_{2\iota-1},\mathfrak{x}||\varsigma(\tau_{2\iota-1},\tau_{2\iota})|| \\ & +\mathfrak{z}(\tau_{2\iota},\tau_{2\iota-1},\mathfrak{x})||\varsigma(\tau_{2\iota},\tau_{2\iota+1})|| \\ & +\gamma(\tau_{2\iota},\tau_{2\iota-1},\mathfrak{x})||\varsigma(\tau_{2\iota-1},\tau_{2\iota})|| \\ \leq & \mathfrak{f}(\tau_0,\tau_{2\iota-1},\mathfrak{x})||\varsigma(\tau_{2\iota-1},\tau_{2\iota})|| \\ & +\mathfrak{z}(\tau_0,\tau_{2\iota-1},\mathfrak{x})||\varsigma(\tau_{2\iota+1},\tau_{2\iota})|| \\ & +\gamma(\tau_0,\tau_{2\iota-1},\mathfrak{x})||\varsigma(\tau_{2\iota-1},\tau_{2\iota})|| \\ \leq & \mathfrak{f}(\tau_0,\tau_1,\mathfrak{x})||\varsigma(\tau_{2\iota-1},\tau_{2\iota})||+\mathfrak{z}(\tau_0,\tau_1,\mathfrak{x})||\varsigma(\tau_{2\iota+1},\tau_{2\iota})|| \\ & +\gamma(\tau_0,\tau_1,\mathfrak{x})||\varsigma(\tau_{2\iota-1},\tau_{2\iota})||,\end{aligned}$$

which yields that

$$||\varsigma(\tau_{2\iota+1},\tau_{2\iota})|| \leq \frac{\{\mathfrak{f}(\tau_0,\tau_1,\mathfrak{x})+\gamma(\tau_0,\tau_1,\mathfrak{x})\}}{1-\mathfrak{z}(\tau_0,\tau_1,\mathfrak{x})}||\varsigma(\tau_{2\iota-1},\tau_{2\iota})||.$$

Similarly, one can obtain

$$||\varsigma(\tau_{2\iota+2},\tau_{2\iota+1})|| \leq \frac{\{\mathfrak{f}(\tau_0,\tau_1,\mathfrak{x})+\gamma(\tau_0,\tau_1,\mathfrak{x})\}}{1-\mathfrak{z}(\tau_0,\tau_1,\mathfrak{x})}||\varsigma(\tau_{2\iota},\tau_{2\iota+1})||.$$

Let $\mathcal{P} = \frac{\mathfrak{f}(\tau_0,\tau_1,\mathfrak{x})+\gamma(\tau_0,\tau_1,\mathfrak{x})}{1-\mathfrak{z}(\tau_0,\tau_1,\mathfrak{x})} < 1$.

Since $\mathfrak{f}(\tau_0,\tau_1,\mathfrak{x})+\mathfrak{z}(\tau_0,\tau_1,\mathfrak{x})+\gamma(\tau_0,\tau_1,\mathfrak{x})+\beta(\tau_0,\tau_1,\mathfrak{x}) < 1$,
we have, $||\varsigma(\tau_{2\iota+2},\tau_{2\iota+1})|| \leq \mathcal{P}||\varsigma(\tau_{2\iota},\tau_{2\iota+1})||$
or, in fact, $||\varsigma(\tau_{\alpha+1},\tau_\alpha)|| \leq \mathcal{P}||\varsigma(\tau_{\alpha-1},\tau_\alpha)||\forall\alpha \in \mathbb{N}$.

Clearly, $\{\tau_\alpha\}$ is a Cauchy sequence in $(\Re,\varsigma)$ by Lemma 3.

Since $\Re$ is complete, there exists $\varrho \in \Re$ such that $\tau_\alpha \to \varrho$ as $\alpha \to +\infty$.

We now claim that ϱ is a fixed point of $\mathcal{S}$.

Using Equation (2) and Proposition 1, we have

$$
\begin{aligned}
\varsigma(\varrho, \mathcal{S}\varrho) &\preceq_{i_3} \varsigma(\varrho, \mathcal{Q}\tau_{2\alpha+1}) + \varsigma(\mathcal{Q}\tau_{2\alpha+1}, \mathcal{S}\varrho) \\
&= \varsigma(\varrho, \tau_{2\alpha+2}) + \varsigma(\mathcal{S}\varrho, \mathcal{Q}\tau_{2\alpha+1}) \\
&\preceq_{i_3} \varsigma(\varrho, \tau_{2\alpha+2}) + \mathfrak{f}(\varrho, \tau_{2\alpha+1}, \mathfrak{x})\varsigma(\varrho, \tau_{2\alpha+1}) + \mathfrak{z}(\varrho, \tau_{2\alpha+1}, \mathfrak{x}) \\
&\times \frac{\varsigma(\varrho, \mathcal{S}\varrho)\varsigma(\tau_{2\alpha+1}, \mathcal{Q}\tau_{2\alpha+1})}{1 + \varsigma(\varrho, \tau_{2\alpha+1})} + \beta(\varrho, \tau_{2\alpha+1}, \mathfrak{x}) \\
&\times \frac{\varsigma(\tau_{2\alpha+1}, \mathcal{S}\varrho)\varsigma(\varrho, \mathcal{Q}\tau_{2\alpha+1})}{1 + \varsigma(\varrho, \tau_{2\alpha-1})} + \gamma(\varrho, \tau_{2\alpha+1}, \mathfrak{x}) \\
&\times \frac{\{\varsigma(\varrho, \mathcal{S}\varrho)\varsigma(\varrho, \mathcal{Q}\tau_{2\alpha+1}) + \varsigma(\tau_{2\alpha+1}, \mathcal{Q}\tau_{2\alpha+1})\varsigma(\tau_{2\alpha+1}, \mathcal{S}\varrho)\}}{1 + \varsigma(\varrho, \mathcal{Q}\tau_{2\alpha+1}) + \varsigma(\tau_{2\alpha+1}, \mathcal{S}\varrho)} \\
&\preceq_{i_3} \varsigma(\varrho, \tau_{2\alpha+2}) + \mathfrak{f}(\varrho, \tau_1, \mathfrak{x})\varsigma(\varrho, \tau_{2\alpha+1}) + \mathfrak{z}(\varrho, \tau_1, \mathfrak{x}) \\
&\times \frac{\varsigma(\varrho, \mathcal{S}\varrho)\varsigma(\tau_{2\alpha+1}, \tau_{2\alpha+2})}{1 + \varsigma(\varrho, \tau_{2\alpha+1})} \\
&+ \beta(\varrho, \tau_1, \mathfrak{x}) \frac{\varsigma(\tau_{2\alpha+1}, \mathcal{S}\varrho)\varsigma(\varrho, \tau_{2\alpha+2})}{1 + \varsigma(\varrho, \tau_{2\alpha+2})} + \gamma(\varrho, \tau_1, \mathfrak{x}) \\
&\times \frac{\{\varsigma(\varrho, \mathcal{S}\varrho)\varsigma(\varrho, \tau_{2\alpha+2}) + \varsigma(\tau_{2\alpha+1}, \tau_{2\alpha+2})\varsigma(\tau_{2\alpha+1}, \mathcal{S}\varrho)\}}{1 + \varsigma(\varrho, \tau_{2\alpha+2}) + \varsigma(\tau_{2\alpha+1}, \mathcal{S}\varrho)}
\end{aligned}
$$

which upon letting $\alpha \to +\infty$ gives $\varsigma(\varrho, \mathcal{S}\varrho) = 0 \Rightarrow \mathcal{S}\varrho = \varrho$.

We now prove that ϱ is a fixed point of $\mathcal{Q}$.

From Equation (2), we have

$$
\begin{aligned}
\varsigma(\varrho, \mathcal{Q}\varrho) &\preceq_{i_3} \varsigma(\varrho, \mathcal{S}\tau_{2\alpha}) + \varsigma(\mathcal{S}\tau_{2\alpha}, \mathcal{Q}\varrho) \\
&\preceq_{i_3} (\varrho, \tau_{2\alpha+1}) + \mathfrak{f}(\tau_{2\alpha}, \varrho, \mathfrak{x})\varsigma(\tau_{2\alpha}, \varrho) + \mathfrak{z}(\tau_{2\alpha}, \varrho, \mathfrak{x}) \\
&\times \frac{\varsigma(\tau_{2\alpha}, \mathcal{S}\tau_{2\alpha})\varsigma(\varrho, \mathcal{Q}\varrho)}{1 + \varsigma(\tau_{2\alpha}, \varrho)} + \beta(\tau_{2\alpha}, \varrho, \mathfrak{x}) \frac{\varsigma(\varrho, \mathcal{S}\tau_{2\alpha})\varsigma(\tau_{2\alpha}, \mathcal{Q}\varrho)}{1 + \varsigma(\tau_{2\alpha}, \varrho)} \\
&+ \gamma(\tau_{2\alpha}, \varrho, \mathfrak{x}) \left\{ \frac{\varsigma(\tau_{2\alpha}, \mathcal{S}\tau_{2\alpha})\varsigma(\tau_{2\alpha}, \mathcal{Q}\varrho) + \varsigma(\varrho, \mathcal{Q}\varrho)\varsigma(\varrho, \mathcal{S}\tau_{2\alpha})}{1 + \varsigma(\tau_{2\alpha}, \mathcal{Q}\varrho) + \varsigma(\varrho, \mathcal{S}\tau_{2\alpha})} \right\} \\
&\preceq_{i_3} \varsigma(\varrho, \tau_{2\alpha+1}) + \mathfrak{f}(\tau_0, \varrho, \mathfrak{x})\varsigma(\tau_{2\alpha}, \varrho) + \mathfrak{z}(\tau_0, \varrho, \mathfrak{x}) \\
&\times \frac{\varsigma(\tau_{2\alpha}, \tau_{2\alpha+1})\varsigma(\varrho, \mathcal{Q}\varrho)}{1 + \varsigma(\tau_{2\alpha}, \varrho)} + \beta(\tau_0, \varrho, \mathfrak{x}) \frac{\varsigma(\varrho, \tau_{2\alpha+1})\varsigma(\tau_{2\alpha}, \mathcal{Q}\varrho)}{1 + \varsigma(\tau_{2\alpha}, \varrho)} \\
&+ \gamma(\tau_0, \varrho, \mathfrak{x}) \left\{ \frac{\varsigma(\tau_{2\alpha}, \tau_{2\alpha+1})\varsigma(\tau_{2\alpha}, \mathcal{Q}\varrho) + \varsigma(\varrho, \mathcal{Q}\varrho)\varsigma(\varrho, \tau_{2\alpha+1})}{1 + \varsigma(\tau_{2\alpha}, \mathcal{Q}\varrho) + \varsigma(\varrho, \tau_{2\alpha+1})} \right\},
\end{aligned}
$$

which upon making $\alpha \to +\infty$, we get $\varsigma(\varrho, \mathcal{Q}\varrho) = 0$.

Thus, $\mathcal{Q}\varrho = \varrho$.

From the above, it is clear that ϱ is a common fixed point of $\mathcal{S}$ and $\mathcal{Q}$.

Uniqueness follows from Equation (2) based on Equation (3).

The proof is complete. □

Example 4. *Let $\Re = [0, 1]$ and $\varsigma\colon \Re \times \Re \to \mathcal{C}_3$ be defined by $\varsigma(\tau, \eta) = |\tau - \eta| \mathfrak{e}^{i_3 \frac{\pi}{6}}$. Then, $(\Re, \varsigma)$ is a CTVMS. Let $\mathcal{S}, \mathcal{Q}\colon \Re \to \Re$ be self-maps given by $\mathcal{S}(\tau) = \frac{\tau}{5}$ and $\mathcal{Q}(\eta) = \frac{\eta}{5}$. Furthermore, for all $\tau, \eta \in \Re$ and for fixed $\mathfrak{x} = \frac{1}{4} \in \Re$, we define the functions $\mathfrak{f}, \mathfrak{z}, \beta, \gamma\colon \Re \times \Re \times \Re \to [0, 1)$ by*

$$
\begin{aligned}
\mathfrak{f}(\tau, \eta, \mathfrak{x}) &= \left(\frac{\tau}{5} + \frac{\eta}{6} + \mathfrak{x} \right), \quad \mathfrak{z}(\tau, \eta, \mathfrak{x}) = \frac{\tau\eta\mathfrak{x}}{11}, \\
\beta(\tau, \eta, \mathfrak{x}) &= \frac{\tau^2\eta^2\mathfrak{x}^2}{11}, \quad \gamma(\tau, \eta, \mathfrak{x}) = \frac{\tau^3\eta^3\mathfrak{x}^3}{11}.
\end{aligned}
$$

Clearly, $\mathfrak{f}(\tau,\eta,\mathfrak{x}) + \mathfrak{z}(\tau,\eta,\mathfrak{x}) + \beta(\tau,\eta,\mathfrak{x}) + \gamma(\tau,\eta,\mathfrak{x}) < 1$ *for all* $\tau,\eta \in \Re$ *and for a fixed* $\mathfrak{x} = \frac{1}{4} \in \Re$.

Now consider

$$\mathfrak{f}(\mathcal{QS}\tau,\eta,\mathfrak{x}) = \mathfrak{f}\left(\mathcal{Q}\left(\frac{\tau}{5}\right),\eta,\mathfrak{x}\right) = \mathfrak{f}\left(\frac{\tau}{25},\eta,\mathfrak{x}\right)$$
$$= \frac{\tau}{125} + \frac{\eta}{6} + \mathfrak{x} \leq \frac{\tau}{5} + \frac{\eta}{6} + \mathfrak{x} = \mathfrak{f}(\tau,\eta,\mathfrak{x}).$$

That is, $\mathfrak{f}(\mathcal{QS}\tau,\eta,\mathfrak{x}) \leq \mathfrak{f}(\tau,\eta,\mathfrak{x})$ *for all* $\tau,\eta \in \Re$ *and for a fixed* $\mathfrak{x} = \frac{1}{4} \in \Re$.

In addition, consider

$$\mathfrak{f}(\tau,\mathcal{SQ}\eta,\mathfrak{x}) = \mathfrak{f}\left(\tau,\mathcal{S}\left(\frac{\eta}{5}\right),\mathfrak{x}\right) = \mathfrak{f}\left(\tau,\frac{\eta}{25},\mathfrak{x}\right)$$
$$= \frac{\tau}{5} + \frac{\eta}{210} + \mathfrak{x} \leq \frac{\tau}{5} + \frac{\eta}{6} + \mathfrak{x} = \mathfrak{f}(\tau,\eta,\mathfrak{x}).$$

That is, $\mathfrak{f}(\tau,\mathcal{SQ}\eta,\mathfrak{x}) \leq \mathfrak{f}(\tau,\eta,\mathfrak{x})$ *for all* $\tau,\eta \in \Re$ *and for a fixed* $\mathfrak{x} = \frac{1}{4} \in \Re$.

Similarly, we can show that

$$\mathfrak{z}(\mathcal{QS}\tau,\eta,\mathfrak{x}) \leq \mathfrak{z}(\tau,\eta,\mathfrak{x}) \quad \text{and} \quad \mathfrak{z}(\tau,\mathcal{SQ}\eta,\mathfrak{x}) \leq \mathfrak{z}(\tau,\eta,\mathfrak{x})$$
$$\beta(\mathcal{QS}\tau,\eta,\mathfrak{x}) \leq \beta(\tau,\eta,\mathfrak{x}) \quad \text{and} \quad \beta(\tau,\mathcal{SQ}\eta,\mathfrak{x}) \leq \beta(\tau,\eta,\mathfrak{x})$$
$$\gamma(\mathcal{QS}\tau,\eta,\mathfrak{x}) \leq \gamma(\tau,\eta,\mathfrak{x}) \quad \text{and} \quad \gamma(\tau,\mathcal{SQ}\eta,\mathfrak{x}) \leq \gamma(\tau,\eta,\mathfrak{x}).$$

Finally, we assert that Equation (2) also holds.

Before proceeding further, it may be noted that for all $\tau,\eta \in \Re$*, we have*

$$0 \preceq_{i_3} \varsigma(\tau,\eta), \varsigma(\mathcal{S}\tau,\mathcal{Q}\eta), \frac{\varsigma(\tau,\mathcal{S}\tau),\varsigma(\eta,\mathcal{Q}\eta)}{1+\varsigma(\tau,\eta)}, \frac{\varsigma(\eta,\mathcal{S}\tau),\varsigma(\tau,\mathcal{Q}\eta)}{1+\varsigma(\tau,\eta)},$$
$$\frac{\varsigma(\tau,\mathcal{S}\tau),\varsigma(\tau,\mathcal{Q}\eta)+\varsigma(\eta,\mathcal{Q}\eta),\varsigma(\eta,\mathcal{S}\tau)}{1+\varsigma(\tau,\mathcal{Q}\eta)+\varsigma(\eta,\mathcal{S}\tau)}$$

It is sufficient to show that

$$\varsigma(\mathcal{S}\tau,\mathcal{Q}\eta) \preceq_{i_3} \mathfrak{f}(\tau,\eta,\mathfrak{x})\varsigma(\tau,\eta).$$

Consider

$$\varsigma(\mathcal{S}\tau,\mathcal{Q}\eta) = \varsigma\left(\frac{\tau}{5},\frac{\eta}{5}\right) = \left|\frac{\tau}{5} - \frac{\eta}{5}\right| \mathrm{e}^{i_3\frac{\pi}{6}} = \frac{1}{5}\left|\tau - \eta\right| \mathrm{e}^{i_3\frac{\pi}{6}}$$
$$\preceq_{i_3} \frac{1}{4}\left|\tau-\eta\right| \mathrm{e}^{i_3\frac{\pi}{6}} \preceq_{i_3} \left(\frac{\tau}{5} + \frac{\eta}{6} + \frac{1}{4}\right)\left|\tau-\eta\right| \mathrm{e}^{i_3\frac{\pi}{6}} \text{ for all } \tau,\eta \in \Re.$$
$$= \mathfrak{f}(\tau,\eta,\mathfrak{x})\varsigma(\tau,\eta), \quad \text{for all} \quad \tau,\eta \in \Re \quad \text{and for} \quad \mathfrak{x} = \frac{1}{4} \in \Re.$$

That is, $\varsigma(\mathcal{S}\tau,\mathcal{Q}\eta) \preceq_{i_3} \mathfrak{f}(\tau,\eta,\mathfrak{x})\varsigma(\tau,\eta)$ *for all* $\tau,\eta \in \Re$ *and for* $\mathfrak{x} = \frac{1}{4} \in \Re$.

By choosing $\mathfrak{z} = 0, \beta = 0, \gamma = 0$ *in Theorem 1, we have the following:*

Corollary 1. *Let* $(\Re,\varsigma)$ *be a CTVMS and* $\mathcal{S},\mathcal{Q}\colon \Re \to \Re$. *If there exist maps* $\mathfrak{f}\colon \Re\times\Re\times\Re \to [0,1)$ *such that*

$$\mathfrak{f}(\mathcal{QS}\tau,\eta,\mathfrak{x}) \leq \mathfrak{f}(\tau,\eta,\mathfrak{x}) \text{ and } \mathfrak{f}(\tau,\mathcal{SQ}\eta,\mathfrak{x}) \leq \mathfrak{f}(\tau,\eta,\mathfrak{x}),$$

satisfying

$$\varsigma(\mathcal{S}\tau,\mathcal{Q}\eta) \preceq_{i_3} \mathfrak{f}(\tau,\eta,\mathfrak{x})\varsigma(\tau,\eta),$$

for all $\tau, \eta \in \Re$ and for a fixed $\mathfrak{x} \in \Re$,
then $\mathcal{S}$ and $\mathcal{Q}$ have a unique common fixed point.

The following corollary is obtained by setting $\mathfrak{z} = \beta = \gamma = 0$ in Theorem 1:

Corollary 2. *Let $(\Re, \varsigma)$ be a CTVMS and $\mathcal{S}, \mathcal{Q} \colon \Re \to \Re$. If there exist maps $\mathfrak{f}, \gamma \colon \Re \times \Re \times \Re \to [0,1)$ such that for all $\tau, \eta \in \Re$ and for a fixed $\mathfrak{x} \in \Re$, we have*

$$\mathfrak{f}(\mathcal{Q}\mathcal{S}\tau, \eta, \mathfrak{x}) \leq \mathfrak{f}(\tau, \eta, \mathfrak{x}) \text{ and } \mathfrak{f}(\tau, \mathcal{S}\mathcal{Q}\eta, \mathfrak{x}) \leq \mathfrak{f}(\tau, \eta, \mathfrak{x}),$$
$$\gamma(\mathcal{Q}\mathcal{S}\tau, \eta, \mathfrak{x}) \leq \gamma(\tau, \eta, \mathfrak{x}) \text{ and } \gamma(\tau, \mathcal{S}\mathcal{Q}\eta, \mathfrak{x}) \leq \gamma(\tau, \eta, \mathfrak{x}),$$

and

$$\mathfrak{f}(\tau, \eta, \mathfrak{x}) + \gamma(\tau, \eta, \mathfrak{x}) < 1,$$

also satisfying

$$\varsigma(\mathcal{S}\tau, \mathcal{Q}\eta) \preceq_{i_3} \mathfrak{f}(\tau, \eta, \mathfrak{x})\varsigma(\tau, \eta) + \gamma(\tau, \eta, \mathfrak{x}) \times \frac{\varsigma(\tau, \mathcal{S}\tau)\varsigma(\tau, \mathcal{Q}\eta) + \varsigma(\eta, \mathcal{Q}\eta)\varsigma(\eta, \mathcal{S}\tau)}{1 + \varsigma(\tau, \mathcal{Q}\eta) + \varsigma(\eta, \mathcal{S}\tau)}$$

then $\mathcal{S}$ and $\mathcal{Q}$ have a unique common fixed point.

Letting $\mathfrak{z} = \gamma = 0$ in Theorem 1 results in the following corollary:

Corollary 3. *Let $(\Re, \varsigma)$ be a CTVMS and $\mathcal{S}, \mathcal{Q} \colon \Re \to \Re$. If there exist maps $\mathfrak{f}, \beta \colon \Re \times \Re \times \Re \to [0,1)$ such that for all $\tau, \eta \in \Re$ and for a fixed $\mathfrak{x} \in \Re$, we have*

$$\mathfrak{f}(\tau, \eta, \mathfrak{x}) + \beta(\tau, \eta, \mathfrak{x}) < 1,$$

and

$$\mathfrak{f}(\mathcal{Q}\mathcal{S}\tau, \eta, \mathfrak{x}) \leq \mathfrak{f}(\tau, \eta, \mathfrak{x}) \text{ and } \mathfrak{f}(\tau, \mathcal{S}\mathcal{Q}\eta, \mathfrak{x}) \leq \mathfrak{f}(\tau, \eta, \mathfrak{x}),$$
$$\beta(\mathcal{Q}\mathcal{S}\tau, \eta, \mathfrak{x}) \leq \beta(\tau, \eta, \mathfrak{x}) \text{ and } \beta(\tau, \mathcal{S}\mathcal{Q}\eta, \mathfrak{x}) \leq \beta(\tau, \eta, \mathfrak{x}),$$

also satisfying

$$\varsigma(\mathcal{S}\tau, \mathcal{Q}\eta) \preceq_{i_3} \mathfrak{f}(\tau, \eta, \mathfrak{x})\varsigma(\tau, \eta) + \beta(\tau, \eta, \mathfrak{x}) \frac{\varsigma(\eta, \mathcal{S}\tau)\varsigma(\tau, \mathcal{Q}\eta)}{1 + \varsigma(\tau, \eta)}$$

then $\mathcal{S}$ and $\mathcal{Q}$ have a unique common fixed point.

Corollary 4. *Let $(\Re, \varsigma)$ be a CTVMS and $\mathcal{S}, \mathcal{Q} \colon \Re \to \Re$. If there exists a map $\mathfrak{f}, \mathfrak{z} \colon \Re \times \Re \times \Re \to [0,1)$ such that for all $\tau, \eta \in \Re$ and for a fixed $\mathfrak{x} \in \Re$, we have*

$$\mathfrak{f}(\tau, \eta, \mathfrak{x}) + \mathfrak{z}(\tau, \eta, \mathfrak{x}) < 1,$$

and

$$\mathfrak{f}(\mathcal{Q}\mathcal{S}\tau, \eta, \mathfrak{x}) \leq \mathfrak{f}(\tau, \eta, \mathfrak{x}) \text{ and } \mathfrak{f}(\tau, \mathcal{S}\mathcal{Q}\eta, \mathfrak{x}) \leq \mathfrak{f}(\tau, \eta, \mathfrak{x}),$$
$$\mathfrak{z}(\mathcal{Q}\mathcal{S}\tau, \eta, \mathfrak{x}) \leq \mathfrak{z}(\tau, \eta, \mathfrak{x}) \text{ and } \mathfrak{z}(\tau, \mathcal{S}\mathcal{Q}\eta, \mathfrak{x}) \leq \mathfrak{z}(\tau, \eta, \mathfrak{x}),$$

also satisfying

$$\varsigma(\mathcal{S}\tau, \mathcal{Q}\eta) \preceq_{i_3} \mathfrak{f}(\tau, \eta, \mathfrak{x})\varsigma(\tau, \eta) + \mathfrak{z}(\tau, \eta, \mathfrak{x}) \frac{\varsigma(\tau, \mathcal{S}\tau)\varsigma(\tau, \mathcal{Q}\eta)}{1 + \varsigma(\tau, \eta)}$$

then $\mathcal{S}$ and $\mathcal{Q}$ have a unique common fixed point.

Theorem 2. *Let $(\Re, \varsigma)$ be a CTVMS and $\mathcal{Q}\colon \Re \to \Re$. If there exist mappings $\mathfrak{f}, \mathfrak{z}\colon \Re \times \Re \times \Re \to [0,1)$ such that for all $\tau, \eta \in \Re$ and for a fixed $\mathfrak{x} \in \Re$, we have*

(a)

$$\mathfrak{f}(\mathcal{Q}\tau, \eta, \mathfrak{x}) \leq \mathfrak{f}(\tau, \eta, \mathfrak{x}) \quad \text{and} \quad \mathfrak{f}(\tau, \mathcal{Q}\eta, \mathfrak{x}) \leq \mathfrak{f}(\tau, \eta, \mathfrak{x}),$$
$$\mathfrak{z}(\mathcal{Q}\tau, \eta, \mathfrak{x}) \leq \mathfrak{z}(\tau, \eta, \mathfrak{x}) \quad \text{and} \quad \mathfrak{z}(\tau, \mathcal{Q}\eta, \mathfrak{x}) \leq \mathfrak{z}(\tau, \eta, \mathfrak{x});$$

(b)

$$\varsigma(\mathcal{Q}\tau, \mathcal{Q}\eta) \preceq_{i_3} \mathfrak{f}(\tau, \eta, \mathfrak{x})\varsigma(\tau, \eta) + \mathfrak{z}(\tau, \eta, \mathfrak{x})\frac{\varsigma(\eta, \mathcal{Q}\eta)[1 + \varsigma(\tau, \mathcal{Q}\tau)]}{1 + \varsigma(\tau, \eta)}; \tag{6}$$

(c)

$$\mathfrak{f}(\tau, \eta, \mathfrak{x}) + \mathfrak{z}(\tau, \eta, \mathfrak{x}) < 1.$$

then $\mathcal{Q}$ has a unique fixed point.

Proof. Let $\tau_0 \in \Re$ and $\{\tau_\alpha\}_{\alpha=0}^{+\infty}$ by $\tau_{\alpha+1} = \mathcal{Q}\tau_\alpha$.

From the condition in Equation (6), we have

$$\begin{aligned}
\varsigma(\tau_{\alpha+1}, \tau_{\alpha+2}) &= \varsigma(\mathcal{Q}\tau_\alpha, \mathcal{Q}\tau_{\alpha+1}) \\
&\preceq_{i_3} \mathfrak{f}(\tau_\alpha, \tau_{\alpha+1}, \mathfrak{x})\varsigma(\tau_\alpha, \tau_{\alpha+1}) + \mathfrak{z}(\tau_\alpha, \tau_{\alpha+1}, \mathfrak{x}) \\
&\times \frac{\varsigma(\tau_{\alpha+1}, \mathcal{Q}\tau_{\alpha+1})[1 + \varsigma(\tau_\alpha, \mathcal{Q}\tau_\alpha)]}{1 + \varsigma(\tau_\alpha, \tau_{\alpha+1})} \\
&= \mathfrak{f}(\tau_\alpha, \tau_{\alpha+1}, \mathfrak{x})\varsigma(\tau_\alpha, \tau_{\alpha+1}) + \mathfrak{z}(\tau_\alpha, \tau_{\alpha+1}, \mathfrak{x}) \\
&\times \frac{\varsigma(\tau_{\alpha+1}, \tau_{\alpha+1})[1 + \varsigma(\tau_\alpha, \tau_\alpha)]}{1 + \varsigma(\tau_\alpha, \tau_{\alpha+1})}
\end{aligned} \tag{7}$$

In other words, we have

$$\varsigma(\tau_{\alpha+1}, \tau_{\alpha+2}) \preceq_{i_3} \mathfrak{f}(\tau_\alpha, \tau_{\alpha+1}, \mathfrak{x})\varsigma(\tau_\alpha, \tau_{\alpha+1}) + \mathfrak{z}(\tau_\alpha, \tau_{\alpha+1}, \mathfrak{x})\varsigma(\tau_{\alpha+1}, \tau_{\alpha+2}). \tag{8}$$

Now, we have

$$\begin{aligned}
\mathfrak{f}(\tau_\alpha, \tau_{\alpha+1}, \mathfrak{x}) &= \mathfrak{f}(\mathcal{Q}\tau_{\alpha-1}, \tau_{\alpha+1}, \mathfrak{x}) \\
&\leq \mathfrak{f}(\tau_{\alpha-1}, \tau_{\alpha+1}, \mathfrak{x}) = \mathfrak{f}(\mathcal{Q}\tau_{\alpha-2}, \tau_{\alpha-1}, \mathfrak{x}) \\
&\leq \mathfrak{f}(\tau_{\alpha-2}, \tau_{\alpha+1}, \mathfrak{x}) = \mathfrak{f}(\tau_{\alpha-3}, \tau_{\alpha+1}, \mathfrak{x}) \\
&\cdots\cdots\cdots \\
&\leq \mathfrak{f}(\tau_0, \tau_{\alpha+1}, \mathfrak{x}),
\end{aligned}$$

and similarly

$$\mathfrak{z}(\tau_\alpha, \tau_{\alpha+1}, \mathfrak{x}) \leq \mathfrak{z}(\tau_0, \tau_{\alpha+1}, \mathfrak{x}).$$

Then, from Equation (8), we have

$$\varsigma(\tau_{\alpha+1}, \tau_{\alpha+2}) \preceq_{i_3} \mathfrak{f}(\tau_0, \tau_{\alpha+1}, \mathfrak{x})\varsigma(\tau_\alpha, \tau_{\alpha+1}) + \mathfrak{z}(\tau_0, \tau_{\alpha+1}, \mathfrak{x})\varsigma(\tau_{\alpha+1}, \tau_{\alpha+2}).$$

Arguing the same as above, we obtain

$$\varsigma(\tau_{\alpha+1}, \tau_{\alpha+2}) \preceq_{i_3} \mathfrak{f}(\tau_0, \tau_0, \mathfrak{x})\varsigma(\tau_\alpha, \tau_{\alpha+1}) + \mathfrak{z}(\tau_0, \tau_0, \mathfrak{x})\varsigma(\tau_{\alpha+1}, \tau_{\alpha+2}).$$

Therefore, we have

$$||\varsigma(\tau_{\alpha+1},\tau_{\alpha+2})|| \leq \mathfrak{f}(\tau_0,\tau_0,\mathfrak{x})||\varsigma(\tau_\alpha,\tau_{\alpha+1})|| + \mathfrak{z}(\tau_0,\tau_0,\mathfrak{x})||\varsigma(\tau_{\alpha+1},\tau_{\alpha+2})||,$$

which implies

$$||\varsigma(\tau_{\alpha+1},\tau_{\alpha+2})|| \leq \frac{\mathfrak{f}(\tau_0,\tau_0,\mathfrak{x})}{1-\mathfrak{z}(\tau_0,\tau_0,\mathfrak{x})}||\varsigma(\tau_{\alpha+1},\tau_{\alpha+2})||,$$

for all $\alpha = 0,1,2,\ldots$ By letting $\iota = \frac{\mathfrak{f}(\tau_0,\tau_0,\mathfrak{x})}{1-\mathfrak{z}(\tau_0,\tau_0,\mathfrak{x})} < 1$, then

$$||\varsigma(\tau_{\alpha+1},\tau_{\alpha+2})|| \leq \iota||\varsigma(\tau_\alpha,\tau_{\alpha+1})||, forall \alpha = 0,1,2,\ldots$$

Using Lemma 3, $\{\tau_\alpha\}$ is Cauchy in $(\Re,\varsigma)$. Since $\Re$ is complete, there exists $\varrho \in \Re$ s.t. $\tau_\alpha \to \varrho$ as $\alpha \to +\infty$. We claim that $\varrho \in \mathcal{Q}$ is a fixed point. From Equation (6), we have

$$\begin{aligned}
\varsigma(\varrho,\mathcal{Q}\varrho) &\preceq_{i_3} \varsigma(\varrho,\mathcal{Q}\tau_\alpha) + \varsigma(\mathcal{Q}\tau_\alpha,\mathcal{Q}\varrho) \\
&\preceq_{i_3} \varsigma(\varrho,\mathcal{Q}\tau_\alpha) + \mathfrak{f}(\tau_\alpha,\varrho,\mathfrak{x})\varsigma(\tau_\alpha,\varrho) \\
&+ \mathfrak{z}(\tau_\alpha,\varrho,\mathfrak{x})\frac{\varsigma(\varrho,\mathcal{Q}\varrho)[1+\varsigma(\tau_\alpha,\mathcal{Q}\tau_\alpha)]}{1+\varsigma(\varrho,\tau_\alpha)} \\
&\preceq_{i_3} \varsigma(\varrho,\tau_{\alpha+1}) + \mathfrak{f}(\tau_0,\varrho,\mathfrak{x})\varsigma(\tau_\alpha,\varrho) \\
&+ \mathfrak{z}(\tau_0,\varrho,\mathfrak{x})\frac{\varsigma(\varrho,\mathcal{Q}\varrho)[1+\varsigma(\tau_\alpha,\mathcal{Q}\tau_\alpha)]}{1+\varsigma(\varrho,\tau_\alpha)}.
\end{aligned}$$

which, on making $\alpha \to +\infty$, reduces to

$$\varsigma(\varrho,\mathcal{Q}\varrho) \preceq_{i_3} \mathfrak{z}(\tau_0,\varrho,\mathfrak{x})\varsigma(\varrho,\mathcal{Q}\varrho).$$

so that

$$||\varsigma(\varrho,\mathcal{Q}\varrho)|| \leq \mathfrak{z}(\tau_0,\varrho,\mathfrak{x})||\varsigma(\varrho,\mathcal{Q}\varrho||,$$

which is a contradiction since $\mathfrak{z}(\tau_0,\varrho,\mathfrak{x}) < 1$.

Therefore, $||\varsigma(\varrho,\mathcal{Q}\varrho)|| = 0$. Hence, $\varrho = \mathcal{Q}\varrho$, and ϱ is a fixed point of $\mathcal{Q}$.

Uniqueness follows from the condition in Equation (8). This completes the proof. □

The following example validates Theorem 2:

Example 5. *Let $\Re = [0,1]$ and $\varsigma\colon \Re \times \Re \to \mathcal{C}_3$ be defined by $\varsigma(\tau,\eta) = |\tau-\eta|e^{i_3\frac{\pi}{6}}$. Then, $(\Re,\varsigma)$ is a CTVMS. Let $\mathcal{Q}\colon \Re \to \Re$ be defined by $\mathcal{Q}(\tau) = \frac{\tau}{7}$. Functions $\mathfrak{f},\mathfrak{z}\colon \Re\times\Re\times\Re \to [0,1]$ are defined as $\mathfrak{f}(\tau,\eta,\mathfrak{x}) = (\frac{\tau}{4}+\frac{\eta}{5}+\mathfrak{x}), \mathfrak{z}(\tau,\eta,\mathfrak{x}) = \frac{\tau^2\eta^2\mathfrak{x}^2}{60}$ for all $\tau,\eta \in \Re$ and for fixed $\mathfrak{x} = \frac{3}{4} \in \Re$.*

Clearly, $\mathfrak{f}(\tau,\eta,\mathfrak{x}) + \mathfrak{z}(\tau,\eta,\mathfrak{x}) < 1$.

Consider

$$\begin{aligned}
\mathfrak{f}(\mathcal{Q}\tau,\eta,\mathfrak{x}) = \mathfrak{f}\left(\frac{\tau}{7},\eta,\mathfrak{x}\right) &= \frac{\tau}{28}+\frac{\eta}{5}+\frac{3}{4} \leq \frac{\tau}{4}+\frac{\eta}{5}+\frac{3}{4} \\
&= \mathfrak{f}(\tau,\eta,\mathfrak{x})
\end{aligned}$$

and

$$\begin{aligned}
\mathfrak{f}(\tau,\mathcal{Q}\eta,\mathfrak{x}) &= \mathfrak{f}(\tau,\frac{\eta}{7},\frac{3}{4}) \\
&= \frac{\tau}{4}+\frac{\eta}{35}+\frac{3}{4} \leq \frac{\tau}{4}+\frac{\eta}{5}+\frac{3}{4} \leq \mathfrak{f}(\tau,\eta,\mathfrak{x}).
\end{aligned}$$

Similarly, we can show that

$$\mathfrak{z}(\mathcal{Q}\tau,\eta,\mathfrak{x}) \leq \mathfrak{z}(\tau,\eta,\mathfrak{x}) \quad \mathfrak{z}(\tau,\mathcal{Q}\eta,\mathfrak{x}) \leq \mathfrak{z}(\tau,\eta,\mathfrak{x})$$

Now, for verifying Equation (6), one needs to note that

$$0 \preceq_{i_3} \varsigma(\tau,\eta), \varsigma(\mathcal{Q}\tau,\mathcal{Q}\eta), \frac{\varsigma(\eta,\mathcal{Q}\eta)[1+\varsigma(\tau,\mathcal{Q}\tau)]}{1+\varsigma(\tau,\eta)}, forall\tau,\eta \in \Re$$

Now, it is sufficient to show that $\varsigma(\mathcal{Q}\tau,\mathcal{Q}\eta) \preceq_{i_3} \mathfrak{f}(\tau,\eta,\mathfrak{x})\varsigma(\tau,\eta)$.
Consider

$$\begin{aligned}\varsigma(\mathcal{Q}\tau,\mathcal{Q}\eta) = \varsigma\left(\frac{\tau}{7},\frac{\eta}{7}\right) &= \frac{1}{7}\left|\tau-\eta\right|\mathfrak{e}^{i_3\frac{\pi}{6}} \preceq_{i_3} \frac{3}{4}\left|\tau-\eta\right|\mathfrak{e}^{i_3\frac{\pi}{6}} \\ &\preceq_{i_3} \left(\frac{\tau}{4}+\frac{\eta}{5}+\frac{3}{4}\right)\left|\tau-\eta\right|\mathfrak{e}^{i_3\frac{\pi}{6}} = \mathfrak{f}(\tau,\eta,\mathfrak{x})\varsigma(\tau,\eta),\end{aligned}$$

for all $\tau,\eta \in \Re$ *and for* $\mathfrak{x} = \frac{3}{4} \in \Re$.
Thus all axioms of Theorem 2 are satisfied, and $\tau = 0 \in \Re$ *is the unique fixed point of* $\mathcal{Q}$.

4. Application

In this section, we give an application using Theorem 1.

Let $\Re = C[\Phi_1,\Phi_2]$ be a family of all real continuous functions on $[\Phi_1,\Phi_2]$ equipped with the metric $\varsigma(\tau,\eta) = (1+i_3)(|\tau(\mathcal{Q}) - \eta(\mathcal{Q})|)$ for all $\tau,\eta \in C[\Phi_1,\Phi_2]$ and $\mathcal{Q} \in [\Phi_1,\Phi_2]$, where $|.|$ is the usual real modulus. We define the functions $\mathfrak{f}$: $\Re\times\Re\times\Re \to [0,1)$ by

$$\mathfrak{f}(\tau,\eta,\mathfrak{x}) = \left(\frac{\tau}{5}+\frac{\eta}{6}+\mathfrak{x}\right).$$

Clearly, $\mathfrak{f}(\tau,\eta,\mathfrak{x}) < 1$ for all $\tau,\eta \in \Re$ and for a fixed $\mathfrak{x} = \frac{1}{4} \in \Re$. Then, $(\Re,\varsigma)$ is a CTVMS. Now, we consider the system of the non-linear Fredholm integral equation

$$\tau(\mathcal{Q}) = \mathfrak{v}(\mathcal{Q}) + \frac{1}{\Phi_2-\Phi_1}\int_{\Phi_1}^{\Phi_2} \iota_1(\mathcal{Q},\mathfrak{s},\tau(\mathfrak{s}))d\mathfrak{s}$$

and

$$\tau(\mathcal{Q}) = \mathfrak{v}(\mathcal{Q}) + \frac{1}{\Phi_2-\Phi_1}\int_{\Phi_1}^{\Phi_2} \iota_2(\mathcal{Q},\mathfrak{s},\tau(\mathfrak{s}))d\mathfrak{s},$$

where $\mathcal{Q},\mathfrak{s} \in [\Phi_1,\Phi_2]$. Assume that $\iota_1,\iota_2 : [\Phi_1,\Phi_2]\times[\Phi_1,\Phi_2]\times\Re \to \mathbb{R}$ and $\mathfrak{v} : [\Phi_1,\Phi_2] \to \mathbb{R}$ are continuous, where $\mathfrak{v}(\mathcal{Q})$ is a given function in $\Re$. We define a partial order $\preceq_{i_3}$ in $\mathbb{C}_3$ as $\tau \preceq_{i_3} \eta$ if $\tau \leq \eta$.

Theorem 3. *Suppose that* $(\Re,\varsigma)$ *is a CTVMS equipped with the metric* $\varsigma(\tau,\eta) = (1+i_3)(|\tau(\mathcal{Q}) - \eta(\mathcal{Q})|)$ *for all* $\tau,\eta \in \Re$, $\mathcal{Q} \in [\Phi_1,\Phi_2]$, *and* $\mathcal{S},\mathcal{Q} : \Re \to \Re$ *is a continuous operator on* $\Re$ *defined by*

$$\mathcal{S}\tau(\mathcal{Q}) = \mathfrak{v}(\mathcal{Q}) + \frac{1}{\Phi_2-\Phi_1}\int_{\Phi_1}^{\Phi_2} \iota_1(\mathcal{Q},\mathfrak{s},\tau(\mathfrak{s}))d\mathfrak{s}. \tag{9}$$

and

$$\mathcal{Q}\tau(\mathcal{Q}) = \mathfrak{v}(\mathcal{Q}) + \frac{1}{\Phi_2-\Phi_1}\int_{\Phi_1}^{\Phi_2} \iota_2(\mathcal{Q},\mathfrak{s},\tau(\mathfrak{s}))d\mathfrak{s}. \tag{10}$$

If there exists $\mathfrak{f} < 1$ *such that for all* $\tau, \eta \in \Re$ *with* $\tau \neq \eta$ *and* $\mathfrak{s}, \mathcal{Q} \in [\Phi_1, \Phi_2]$*, satisfying the following inequality:*

$$|\iota_1(\mathcal{Q}, \mathfrak{s}, \tau(s)) - \iota_2(\mathcal{Q}, \mathfrak{s}, \eta(\mathfrak{s}))| \leq \mathfrak{f}(\tau, \eta, \mathfrak{x})|\tau(\mathcal{Q}) - \eta(\mathcal{Q})|, \tag{11}$$

then the integral operators defined by Equations (9) *and* (10) *have a unique common solution.*

Proof. Consider

$$\begin{aligned}(1+i_3)(|\mathcal{S}\tau(\mathcal{Q}) - \mathcal{Q}\eta(\mathcal{Q})|) &= \frac{(1+i_3)}{|\Phi_2 - \Phi_1|}\left(\left|\int_{\Phi_1}^{\Phi_2} \iota_1(\mathcal{Q}, \mathfrak{s}, \tau(\mathfrak{s}))d\mathfrak{s}\right.\right. \\ &\quad \left.\left. - \int_{\Phi_1}^{\Phi_2} \iota_2(\mathcal{Q}, \mathfrak{s}, \eta(\mathfrak{s}))d\mathfrak{s}\right|\right) \\ &\leq \frac{(1+i_3)}{|\Phi_2 - \Phi_1|}\left(\int_{\Phi_1}^{\Phi_2} |\iota_1(\mathcal{Q}, \mathfrak{s}, \tau(\mathfrak{s}))\right. \\ &\quad \left. - \iota_2(\mathcal{Q}, \mathfrak{s}, \eta(\mathfrak{s}))|d\mathfrak{s}\right) \\ &\leq \frac{(1+i_3)\mathfrak{f}(\tau, \eta, \mathfrak{x})}{|\Phi_2 - \Phi_1|}\left(\int_{\Phi_1}^{\Phi_2} |\tau(\mathcal{Q}) - \eta(\mathcal{Q})|d\mathfrak{s}\right) \\ &\leq \frac{\mathfrak{f}(\tau, \eta, \mathfrak{x})}{|\Phi_2 - \Phi_1|}\int_{\Phi_1}^{\Phi_2} (1+i_3)|\tau(\mathcal{Q}) - \eta(\mathcal{Q})|d\mathfrak{s} \\ &\leq \frac{\mathfrak{f}(\tau, \eta, \mathfrak{x})(1+i_3)|\tau(\mathcal{Q}) - \eta(\mathcal{Q})|}{|\Phi_2 - \Phi_1|}\int_{\Phi_1}^{\Phi_2} d\mathfrak{s}.\end{aligned}$$

Therefore, we have

$$\varsigma(\mathcal{S}\tau, \mathcal{T}\eta)) \leq \mathfrak{f}(\tau, \eta, \mathfrak{x})\varsigma(\tau, \eta).$$

Thus, the axioms of Theorem 2 are satisfied with $\mathfrak{f} < 1$. $\mathfrak{z} = \beta = \gamma = 0$, and so the integral operators $\mathcal{S}$ and $\mathcal{T}$ defined by Equations (9) and (10) have a unique common solution. □

5. Conclusions

In this paper, we established some common fixed-point theorems on a TVMS using control functions. Our results were validated using suitable examples, and we presented an application on the TVMS to find a unique common solution to integral-type contraction. It will be quite interesting to extend our results in the setting of a TVMS using other contractive conditions such as cyclic contractions, multi-valued contractions, etc.

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