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Asymptotics of Solutions to a Differential Equation with Delay and Nonlinearity Having Simple Behaviour at Infinity

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Abstract: In this paper, we study nonlocal dynamics of a nonlinear delay differential equation. This equation with different types of nonlinearities appears in medical, physical, biological, and ecological applications. The type of nonlinearity in this paper is a generalization of two important for applications types of nonlinearities: piecewise constant and compactly supported functions. We study asymptotics of solutions under the condition that nonlinearity is multiplied by a large parameter. We construct all solutions of the equation with initial conditions from a wide subset of the phase space and find conditions on the parameters of equations for having periodic solutions.

Keywords: delay; asymptotics; large parameter; relaxation oscillations; periodic solutions

MSC: 34K25; 34K13

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1. Introduction

Differential equations with delay

$$\dot{x} = G(x, x(t-\tau)), \tag{1}$$

where x is from $\mathbb{R}^n$, G is some continuous function, and $\tau > 0$ is a delay time, arise as mathematical models in different areas of science (see [1,2] and references therein). Many studies are devoted to the construction of solutions or the analysis of the stability of solutions to differential equations with delay [3–11].

Consider differential equation with delay

$$\dot{u} + \nu u = \lambda F(u(t-T)). \tag{2}$$

Here, u is a scalar real function, and parameters ν and λ and delay time T are positive. This equation plays an important role in mathematical modelling and is of great interest for fundamental research.

This equation simulates a process of production and destruction where the single state variable u decays with a rate ν proportional to u at the present and is produced with a rate dependent on the value of u some time in the past. Such processes arise in many biological applications, for example, in normal and pathological behaviour of control systems in the physiology of blood cell production and respiration and periodic or irregular activity in neural networks (see Table 1 in [12], paper [13] and references therein).

Equation (2) with compactly supported nonlinearity simulates an oscillator with nonlinear delayed feedback with an RC low-pass filter of the first order [14,15]. Additionally, Equation (2) with another nonlinear functions F occurs in laser optics [1,16] and in mathematical ecology [2,17].

There are many studies on the dynamics of this equation: its dynamics were studied in the case of piecewise constant [13], monotone [18,19], or compactly supported nonlinearity [20] in the case of positive and negative feedback [21]. Asymptotics of solutions [22] and the existence of periodic solutions [23] were studied in the case of a singularly perturbed equation:

$$\varepsilon\dot{u} + u = F(u(t-T)), \quad (0 < \varepsilon \ll 1).$$

In [24], the authors determined how dynamics of this differential equation when ε is small related with dynamics of this equation in the case of $\varepsilon = 0$. In [25,26], the authors proposed methods to reconstruct Equation (2) from time series.

For systems of two [27], three [28] and $N > 3$ [29]-coupled oscillators (2) with compactly supported nonlinearity F and $\lambda \gg 1$, asymptotics of relaxation solutions were constructed.

Using simple renormalizations, we can obtain that the coefficient ν in (2) is equal to one. Therefore, without limiting generality, below, we consider case $\nu = 1$.

In the present work, we analytically study behaviour at $t \to +\infty$ of solutions to Equation (2) with initial conditions from a wide subset of the phase space $C[-T, 0]$ under conditions

$$\lambda \gg 1$$

and

$$F(x) = \begin{cases} b, & x \le p_L, \\ f(x), & p_L < x < p_R, \\ d, & x \ge p_R, \end{cases} \tag{3}$$

where $p_L < 0 < p_R$.

We assume that nonlinear function $f(x)$ is bounded and piecewise-smooth. We consider positive, negative, and zero values of parameters b and d (but we assume that at least one of the parameters b or d is nonzero, because if $b = d = 0$, then F is a compactly supported function, and this case has been studied in [20]).

This type of function, $F(u)$, is a generalization of two important applications [12,15] regarding types of nonlinearity: compactly supported and piecewise constant nonlinearities. The class of nonlinearity $F(u)$ is broad because constants $p_L < 0$, $p_R > 0$, b, and d are arbitrary, and conditions on function f are quite general. Therefore, this type of nonlinearity may occur in many applied problems, and the results obtained in this paper can be directly applied to study dynamics of the mathematical models with certain nonlinear functions F (if function F satisfies conditions (3)).

We analytically draw a conclusion about qualitative and quantitative properties of solutions to Equation (2) with arbitrary function F (satisfying conditions (3)) with initial conditions from a wide subset of the phase space and give numerical illustrations of the obtained results. It is important to mention that it is impossible to obtain such a result using only numerical methods because it is impossible to iterate through all functions F from the considered class and through all the considered initial conditions.

The method of investigation in this paper is the following.

1. We select two sets of initial conditions: S_- and S_+. The set S_- consists of continuous functions $u(s)$, ($s \in [-T, 0]$), such that $u(s) \le p_L$ on $s \in [-T, 0)$, and $u(0) = p_L$. The set S_+ consists of continuous functions $u(s)$, ($s \in [-T, 0]$), such that $u(s) \ge p_R$ on $s \in [-T, 0)$, and $u(0) = p_R$.

2. We take initial conditions from sets S_- and S_+ and construct asymptotics at $\lambda \to +\infty$ of all solutions to Equation (2) using the method of steps [30].

3. By the asymptotics of solutions, we draw conclusions about the behaviour of solutions at $t \to +\infty$.

In this paper we conclude that two types of behaviour at $t \to +\infty$ of solutions to Equation (2) with initial conditions from the set S_+ or S_- are possible: (1) the solution tends to a constant at $t \to +\infty$, or (2) after the pre-period, the solution becomes a cycle.

The idea of the proof that after the pre-period, the solution becomes a cycle is the following: 1. it follows from the form of sets S_- and S_+ and properties of function $F(u)$, that on the first step ($t \in [0, T]$) all solutions from the set S_- (S_+) coincide with each other. Thus, all solutions with initial conditions from S_- (S_+) coincide with each other for all $t \ge 0$; 2. if we take initial conditions from one of these sets (S_- or S_+) and if there exists a time moment t_* such that $u(t_* + s)$ (where $s \in [-T, 0]$) belongs to the chosen set, then there exists a periodic solution to Equation (2).

4. We generalize obtained results to the wide sets of initial conditions $u(s) \geq p_R$ (or $u(s) \leq p_L$) at $s \in [-T, 0]$.

The paper has the following structure: in Sections 2–5, we construct asymptotics of solutions to Equation (2) considering cases of different signs of b and d under condition $bd \neq 0$; in Sections 6 and 7 we consider cases $b \neq 0$ and $d = 0$; and in Sections 8 and 9, we consider cases $b = 0$ and $d \neq 0$. In Section 10, we generalize results of Sections 2–9 to wide sets of initial conditions $u(s) \geq p_R$ (or $u(s) \leq p_L$) at $s \in [-T, 0]$.

2. Asymptotics of Solutions in the Case $b > 0$ and $d > 0$

Firstly, we consider asymptotics of the solution to Equation (2) with initial conditions from S_+. We solve our equation using the method of steps.

On the first step (on the segment $t \in [0, T]$), the function $u(t - T)$ is greater than or equal to p_R, which is why on this segment, Equation (2) has the form

$$\dot{u} + u = \lambda d. \tag{4}$$

Hence, on this time segment, the solution to Equation (2) has the form

$$u(t) = p_R e^{-t} + \lambda d(1 - e^{-t}). \tag{5}$$

Because $d > 0$ and λ is sufficiently large, we obtain $u(t) > p_R$ on $t \in [0, T]$. Therefore, Equation (2) has the form of (4) on the next step $t \in [T, 2T]$ and so on (Equation (2) has the form of (4); until then $u(t) < p_R$). However, at $\lambda \gg 1$, the condition $u(t) = p_R e^{-t} + \lambda d(1 - e^{-t}) < p_R$ is not true for all $t \geq 0$, so Equation (2) has the form of (4) for all $t \geq 0$, and the solution has the form of (5) for all $t \geq 0$ (see Figure 1).

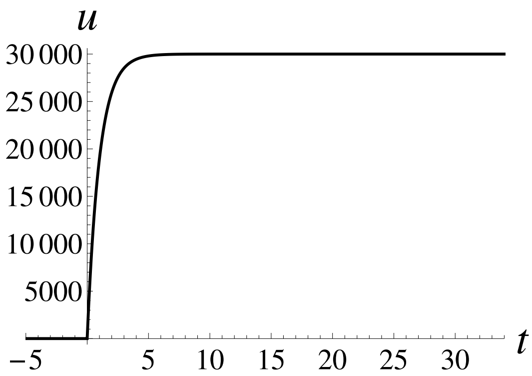


Figure 1. Solution to Equation (2) with initial conditions from S_+ in the case $b > 0$ and $d > 0$. Values of parameters: $\lambda = 10^4$, $T = 5$, $p_L = -1$, $p_R = 2$, $b = 2$, $d = 3$.

Secondly, we take initial conditions from S_- and construct asymptotics for these initial conditions.

Then, on the first step (on the segment $t \in [0, T]$), the function $u(t - T)$ is less than or equal to p_L, which is why on this segment, Equation (2) has the form

$$\dot{u} + u = \lambda b. \tag{6}$$

It follows from (6) that the solution has the form

$$u(t) = p_L e^{-t} + \lambda b(1 - e^{-t}). \tag{7}$$

Therefore,

$$u(T) = p_L e^{-T} + \lambda b(1 - e^{-T}). \tag{8}$$

Lemma 1. *The leading part of the asymptotics of the solution to Equation (2) on the segment $t \in [T, 2T]$ coincides with the leading part of the asymptotics of the solution to the Cauchy problem (4) and (8). The solution to Equation (2) in this interval has the form*

$$u(t) = \lambda b(1 - e^{-T})e^{-(t-T)} + \lambda d(1 - e^{-(t-T)}) + o(\lambda). \tag{9}$$

Proof. On the segment $t \in [0, T]$, the solution to Equation (2) has the form of (7). This expression is an increasing function of t because $\lambda b > 0$ and $p_L < 0$. Therefore, (7) is greater than p_L for all $t \in [0, T]$. It is easy to see that expression (7) is less than p_R for all $t \in [0, \delta)$, where

$$\delta = \ln\Big(1 + \frac{p_R - p_L}{\lambda b - p_R}\Big), \tag{10}$$

and is greater than p_R for all $t \in (\delta, T]$. Note that δ is asymptotically small by λ at $\lambda \to +\infty$ (it has order $O(\lambda^{-1})$).

It follows from the estimation of the expression (7) that on the segment $t \in [T, T + \delta]$, Equation (2) has the form

$$\dot{u} + u = \lambda f(u(t - T)), \tag{11}$$

and on the interval $t \in (T + \delta, 2T]$, it has the form of (4).

At the segment $t \in [T, T + \delta]$, the exact solution to Equation (2) (which is Equation (11) in this interval) has the form

$$u(t) = (p_L e^{-T} + \lambda b(1 - e^{-T}))e^{-(t-T)} + \lambda \int_T^t e^{s-t} f(u(s - T))ds. \tag{12}$$

Function f is bounded; therefore there exists a constant M such that $|f(u(s - T))| < M$ for all $s \in [T, T + \delta]$. Thus,

$$\left|\lambda \int_T^t e^{s-t} f(u(s-T))ds\right| \le \lambda \int_T^t |e^{s-t} f(u(s-T))|ds \le \lambda \int_T^t M ds \le \lambda \int_T^{T+\delta} M ds = \lambda\delta M \le M_1, \tag{13}$$

where M_1 is some constant. The last inequality is true because δ has order $O(\lambda^{-1})$ at $\lambda \to +\infty$.

Note that on the interval $t \in [T, T + \delta]$

$$0 \le \lambda d(1 - e^{-(t-T)}) \le \lambda d(1 - e^{-(T+\delta-T)}) \le M_2, \tag{14}$$

(where M_2 is some constant), δ has order $O(\lambda^{-1})$ at $\lambda \to +\infty$. It follows from inequalities (13) and (14) that on the segment $t \in [T, T + \delta]$, the leading terms of asymptotics at $\lambda \to +\infty$ of expressions (12) and (9) coincide.

On the segment $t \in [T + \delta, 2T]$, the exact solution to Equation (2) (which is Equation (4) in this interval) has the form

$$u(t) = \lambda b(1 - e^{-T})e^{-(t-T)} + \lambda d(1 - e^{-(t-T)}e^{\delta}) + \\ (p_L e^{-(T+\delta)} + \lambda \int_T^{T+\delta} e^{s-(T+\delta)} f(u(s-T))ds)e^{-(t-(T+\delta))}. \tag{15}$$

Since $\delta = O(\lambda^{-1})$ at $\lambda \to +\infty$, then on the segment $t \in [T + \delta, 2T]$, the leading terms of asymptotics at $\lambda \to +\infty$ of expressions (15) and (9) coincide. Thus, the solution to Equation (2) has the form of (9) on the whole segment $t \in [T, 2T]$.

The exact solution to the Cauchy problem (4), (8) has the form

$$u(t) = \lambda b(1 - e^{-T})e^{-(t-T)} + \lambda d(1 - e^{-(t-T)}) + p_L e^{-t}. \tag{16}$$

It is easy to see that the leading terms of asymptotics at $\lambda \to +\infty$ of expressions (16) and (9) coincide on the whole segment $t \in [T, 2T]$.

Thus, on this segment, the leading part of asymptotics of solution to Equation (2) coincides with the leading part of asymptotics of the solution to the Cauchy problem (4) and (8). □

Expression (9) is greater than p_R for all $t \in [T, +\infty)$. Therefore, Equation (2) has the form of (4) for all $t \geq T + \delta$, and the solution of Equation (2) has the form of (9) for all $t \geq T$ (see Figure 2).

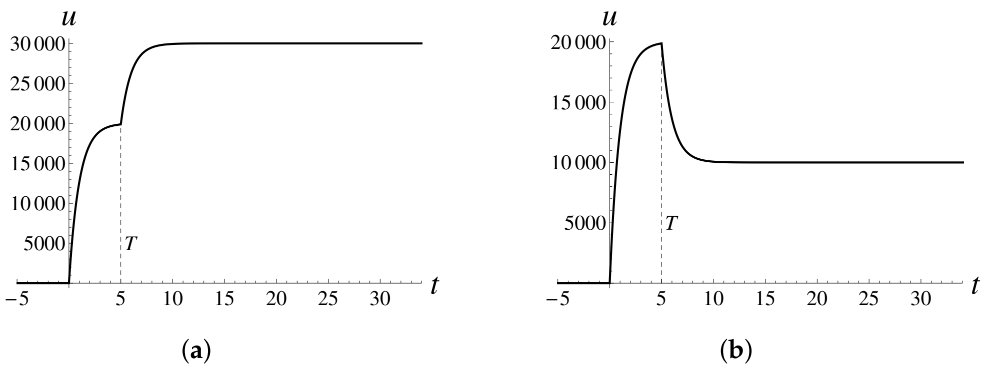


Figure 2. Typical graphs of solutions to Equation (2) with initial conditions from S_- in the case $b > 0$ and $d > 0$ if (**a**) $d > b > 0$ and (**b**) $b > d > 0$. Values of parameters: $\lambda = 10^4$, $T = 5$, $p_L = -1$, $p_R = 2$, $b = 2$, (**a**) $d = 3$, and (**b**) $d = 1$.

Therefore, in the case $b > 0$ and $d > 0$, all solutions with initial conditions from sets S_+ and S_- tend to the constant λd at $t \to +\infty$.

3. Asymptotics of Solutions in the Case $b < 0$ and $d < 0$

Initially, we consider asymptotics of solution to Equation (2) with initial conditions from S_+. In the first step (on the segment $t \in [0, T]$), function $u(t - T)$ is greater or equal than p_R, which is why in this segment Equation (2) has the form of (4). Therefore, for $t \in [0, T]$, the solution of Equation (2) has the form of (5).

In this case, $d < 0$, so we obtain $u(t) < p_L$ for $t \in [\delta, T + \delta]$, where $\delta = O(\lambda^{-1})$ at $\lambda \to +\infty$; therefore, Equation (2) has the form of (6) in the segment $t \in [T + \delta, 2T + \delta]$. As in the previous case, in the time segment $t \in [T, T + \delta]$, the solution $u(t)$ depends on the values of the function f, but this dependence is smaller than the leading term of the asymptotics of the solution, and this leading term of the asymptotics of the solution coincides with the leading term of the asymptotics of the solution to the Cauchy problem (6),

$$u(T) = p_R e^{-T} + \lambda d(1 - e^{-T}).$$

Hence, it follows that the solution of Equation (2) with initial conditions from set S_+ has the form

$$u(t) = \lambda d(1 - e^{-T})e^{-(t-T)} + \lambda b(1 - e^{-(t-T)}) + o(\lambda). \tag{17}$$

Since $b < 0$ and $d < 0$, expression (17) is less then p_L for all $t \in [T, +\infty)$. This is why Equation (2) has the form of (6) for all $t \in [T + \delta, +\infty)$, and Formula (17) holds for all $t \geq T$.

Now, we study asymptotics of the solution to Equation (2) with the initial conditions from S_-.

On the segment $t \in [0, T]$, the function $u(t - T)$ is less than or equal to p_L, which is why on this segment Equation (2) has the form of (6), and its solution has the form of (7).

As $b < 0$, we obtain $u(t) < p_L$, and Equation (2) has the form of (6); until then, $u(t) > p_L$. However, expression (7) is less than p_L for all $t > 0$. This is why solution has the form of (7) for all $t \in [0, +\infty)$.

Therefore, in the case that $b < 0$ and $d < 0$, all solutions with initial conditions from sets S_+ and S_- tend to the constant λb at $t \to +\infty$.

4. Asymptotics of Solutions in the Case $b < 0$ and $d > 0$

Firstly, we consider the asymptotics of the solution to Equation (2) with initial conditions from S_+.

In the first step (in the segment $t \in [0, T]$), the function $u(t - T)$ is greater than or equal to p_R, which is why on this segment, Equation (2) has the form of (4), and the solution to Equation (2) has the form of (5). As in Section 2, we identify that Expression (5) is greater than p_R for all $t > 0$; therefore, Equation (2) has the form of (4) for all $t > 0$. This is why the solution of (2) with initial conditions from S_+ does not depend on the values of f and b and has the form of (5) for all $t > 0$.

Similarly, the solution of Equation (2) with initial conditions from S_- does not depend on the values of f and d and for all $t > 0$ has the form of (7).

Therefore, in the case that $b < 0$ and $d > 0$, the solutions with initial conditions from S_+ tend to the constant λd, and solutions with initial conditions from S_- tend to the constant λb at $t \to +\infty$.

5. Asymptotics of Solutions in the Case $b > 0$ and $d < 0$

In this section, we study behaviour of solutions with initial conditions from sets S_+ and S_- under the assumption that $b > 0$ and $d < 0$.

Firstly, we take initial conditions from S_+ and begin to construct asymptotics of solutions. Then, on the first step $t \in [0, T]$, Equation (2) has the form of (4) and solution has the form of (5) and

$$u(T) = \lambda d(1 - e^{-T} + o(1)). \tag{18}$$

Since $d < 0$, there exists an asymptotically small λ value $\delta_1 > 0$ such that $p_L < u(t) < p_R$ for $t \in (0, \delta_1)$ and $u(t) < p_L$ for all $t \in (\delta_1, T]$. Therefore, on the segment $t \in [T + \delta_1, 2T]$, Equation (2) has the form of (6). In the segment $t \in [T, T + \delta_1]$, the solution to Equation (2) depends on the values of the function f, but the leading term of the asymptotics of the solution to Equation (2) coincides with the leading term of the asymptotics of the solution to Equation (6) with initial conditions from (18). This is why in the whole segment $t \in [T, 2T]$, the solution has the form of (17). Note that in the case that $b > 0$ and $d < 0$, Expression (17) is an increasing function.

Since $u(T) < 0$ and (17) increases to the positive value, there exists an asymptotically small by λ value $\delta_2 < 0$ and value $t_1 > T + \delta_1$, such that $u(t_1) = 0$ and $u(t_1 + \delta_2) = p_L$. It follows from the definition of t_1 and δ_2 that Equation (2) has the form of (6) on the segment $t \in [T + \delta_1, t_1 + \delta_2 + T]$. It easily follows from (17) that

$$e^{-(t_1 - T)} = b/(b - d(1 - e^{-T})), \tag{19}$$

and, consequently,

$$u(t_1 + T) = \lambda b(1 - e^{-T} + o(1)). \tag{20}$$

Note that expression (20) is greater than p_R when $\lambda \gg 1$. Thus, for $t > t_1 + T + \delta_3$ (where $\delta_3 > 0$ denotes an asymptotically small by λ value such that $u(t_1 + \delta_3) = p_R$), until then, the $u(t) < p_R$ solution Equation (2) has the form of (4). Therefore, for $t > t_1 + T + \delta_3$, until then, the $u(t) < p_R$ solution to Equation (2) has the form of

$$u(t) = (\lambda b(1 - e^{-T}) - \lambda d)e^{-(t - (t_1 + T))} + \lambda d + o(\lambda). \tag{21}$$

Expression (21) is a decreasing function, and there exists $t = t_2$ such that (21) is equal to zero. Additionally, for $t_* = t_2 + o(1)$, it is true that $u(t_*) = p_R$ and $u(t_* + s) > p_R$ for all $s \in [-T, 0)$. Thus, at the point $t = t_*$, we return to the initial situation (the function $u(t_* + s)$ belongs to the set S_+). This is why if we take this function as the initial conditions for Equation (2), we obtain a periodic solution to this equation with an amplitude of the order $O(\lambda)$ (see Formulas (18) and (20)) and period

$$t_* = 2T + \ln\left(\frac{(b(1-e^{-T}) - d)(b - d(1-e^{-T}))}{-bd}\right) + o(1). \tag{22}$$

We mention that the logarithm in Formula (22) is positive because its argument is greater than 1 for all $b > 0$, $d < 0$, and $T > 0$.

Note that there exists a point $t_L = t_1 + \delta_2$ such that $u(t_L) = p_L$ and $u(s + t_L) < p_L$ on the segment $s \in [-T, 0)$. Additionally, we stress that if we take an initial function such that it is less than or equal to p_L in some segment of the length T: $s \in [\tilde{t} - T, \tilde{t})$ and is equal to p_L at the point $\tilde{t}$, then the solution to Equation (2) on the next interval $t \in [\tilde{t}, \tilde{t} + T]$ does not depend on the "history" values of $u(s + \tilde{t})$ on $s \in [-T, 0)$. This is why if we consider the initial conditions from the set S_- and construct the asymptotics of the solution to Equation (2), we obtain the periodic solution obtained earlier in this section, but this solution will be shifted.

From the results of Sections 2–5, we derive the following theorem.

Theorem 1. *Let $bd \neq 0$. Then, Equation (2) with sufficiently large $\lambda > 0$ has a cycle with initial conditions from S_+ or S_- if and only if $b > 0$ and $d < 0$. This sign-changing cycle $u_*(t)$ has asymptotics*

$$\begin{aligned} u_*(t) &= p_R e^{-(t-nt_*)} + \lambda d(1 - e^{-(t-nt_*)}), & t \in [nt_*, nt_* + T], \\ u_*(t) &= \lambda d(1-e^{-T}) e^{-(t-(T+nt_*))} + \lambda b(1 - e^{-(t-(T+nt_*))}) + o(\lambda), & \\ & & t \in [nt_* + T, nt_* + t_1 + T], \\ u_*(t) &= (\lambda b(1-e^{-T}) - \lambda d) e^{-(t-(t_1+T+nt_*))} + \lambda d + o(\lambda), & \\ & & t \in [nt_* + t_1 + T, (n+1)t_*]. \end{aligned} \tag{23}$$

at $\lambda \to +\infty$ (where $n = 0, 1, 2, \ldots$ represents the number of periods of the cycle), and the period of this cycle t_* is given in (22).

Note that Formula (23) was obtained from Formulas (5), (17), and (21) using a shift in the time variable t by n periods t_* of solution $u_*(t)$.

It should also be noted that all shifts of cycle $u_*(t + C)$ where $C \in \mathbb{R}$ are solutions to Equation (2), but we consider them as a single object.

A cycle of Equation (2) in the case that $b > 0$ and $d < 0$ is shown in Figure 3.

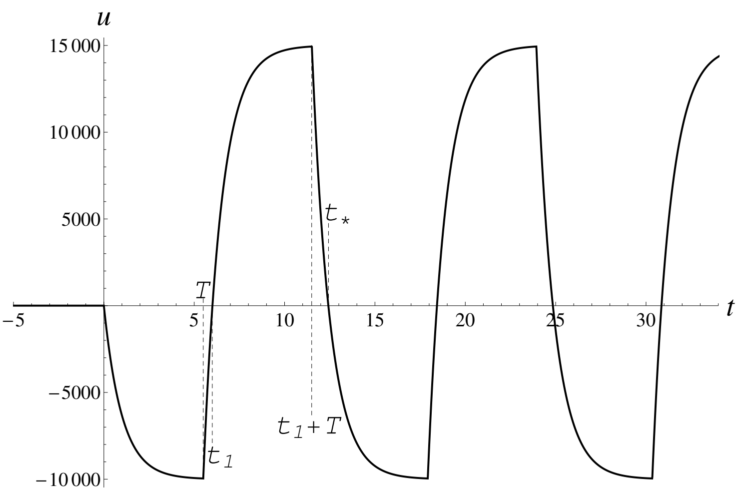


Figure 3. Cycle of Equation (2) in the case that $b > 0$ and $d < 0$. Values of parameters: $\lambda = 10^4$, $T = 5.5$, $p_L = -2$, $p_R = 3$, $b = 1.5$, and $d = -1$.

6. Asymptotics of Solutions in the Case $b > 0$ and $d = 0$

Firstly, we consider initial conditions from S_+. Then, on the first step $t \in [0, T]$, Equation (2) has the form

$$\dot{u} + u = 0. \tag{24}$$

Therefore, the solution has the form

$$u(t) = p_R e^{-t}. \tag{25}$$

Since the solution to (25) belongs to the interval $u \in (0, p_R)$ in the interval $t \in (0, T]$, then in the segment $t \in [T, 2T]$, it depends on the values of function f. It has the form

$$u(t) = p_R e^{-t} + \lambda \int_T^t e^{s-t} f(p_R e^{T-s}) ds. \tag{26}$$

In this segment, the asymptotics of the solution to (2) crucially depend on the values of the integral in Formula (26). Below, we assume that this integral preserves its sign on the segment $t \in (T, 2T]$ (if the integral changes its sign, then we cannot construct the asymptotics of the solution at the segment $t \in [2T, 3T]$ for an arbitrary unknown function f). Consider the first case:

$$\int_T^t e^{s-t} f(p_R e^{T-s}) ds > 0 \text{ for all } t \in (T, 2T]. \tag{27}$$

Then, expression (26) is greater than p_R on the segment $t \in [T + \delta_1, 2T]$ (here, $\delta_1 \geq 0$ is some asymptotically small by λ value; it has order $O(\lambda^{-1})$ at $\lambda \to +\infty$). In the segment $t \in [2T, 2T + \delta_1]$, the leading term of the asymptotics of the solution to Equation (2) coincides with the leading term of the asymptotics of the solution to Equation (24) with the initial conditions

$$u(2T) = p_R e^{-2T} + \lambda \int_T^{2T} e^{s-2T} f(p_R e^{T-s}) ds,$$

and in the segment $t \in [2T + \delta_1, 3T]$, Equation (2) has the form of (24). This is why in the whole segment $t \in [2T, 3T]$, the solution to (2) has the form of

$$u(t) = \lambda \Big(\int_T^{2T} e^{s-2T} f(p_R e^{T-s}) ds + o(1) \Big) e^{-(t-2T)}. \tag{28}$$

Note that Expression (28) is greater than p_R in the segment with length $O(\ln \lambda)$ at $\lambda \to +\infty$ (and this is why Equation (2) has the form of (24) in this segment), and this expression decreases and tends to zero at $t \to +\infty$. Therefore, there exists a time moment $t_* = 2T + (1 + o(1)) \ln \lambda > 3T$ such that $u(t_*) = p_R$ and $u(t_* + s) > p_R$ on the interval $s \in [-T, 0)$. Thus, we come to the initial situation (the function $u(t_* + s)$ belongs to the set S_+), and if we take this function as the initial conditions to our equation, then we obtain a positive relaxation cycle of Equation (2) with the amplitude $O(\lambda)$ and period $t_* = 2T + (1 + o(1)) \ln \lambda$.

We obtain the following result.

Theorem 2. *Let $b > 0$ and $d = 0$, and (27) holds. Then, for all sufficiently large $\lambda > 0$, Equation (2) has a positive relaxation cycle with the asymptotics*

$$
\begin{aligned}
&u(t) = p_R e^{-(t-nt_*)}, && t \in [nt_*, nt_* + T],\\
&u(t) = p_R e^{-(t-nt_*)} + \lambda \int\limits_{T+nt_*}^{t} e^{s-t} f(p_R e^{nt_*+T-s})ds, && t \in [nt_* + T, nt_* + 2T],\\
&u(t) = \lambda \Big(\int\limits_{nt_*+T}^{nt_*+2T} e^{s-nt_*-2T} f(p_R e^{nt_*+T-s})ds + o(1) \Big) e^{-(t-nt_*-2T)},\\
& && t \in [nt_* + 2T, (n+1)t_*],
\end{aligned} \tag{29}
$$

(where $n = 0, 1, 2, \dots$ represents number of periods of cycle) and the period $t_* = 2T + (1 + o(1)) \ln \lambda$ at $\lambda \to +\infty$.

Consider the second case:

$$
\int\limits_{T}^{t} e^{s-t} f(p_R e^{T-s})ds < 0 \text{ for all } t \in (T, 2T]. \tag{30}
$$

Then, expression (26) is less than p_L in the segment $t \in [T + \delta_2, 2T]$ (here, $\delta_2 > 0$ denotes an asymptotically small by λ value such that $u(T + \delta_2) = p_L$), and this is why Equation (2) in the segment $t \in [2T + \delta_2, 3T]$ has the form of (6). Thus, in the segment $t \in [2T, 3T]$, the solution has the form

$$
u(t) = \lambda \Big(\int\limits_{T}^{2T} e^{s-2T} f(p_R e^{T-s})ds - b + o(1) \Big) e^{-(t-2T)} + \lambda b. \tag{31}
$$

Note that expression (31) is an increasing function and that there exists a time moment $t_1 > 2T + \delta_2$ such that $u(t_1) = 0$. It is easy to see that

$$
e^{-(t_1-2T)} = \frac{b}{b - \int\limits_{T}^{2T} e^{s-2T} f(p_R e^{T-s})ds}. \tag{32}
$$

On the segment $t \in [2T + \delta_2, t_1 + T + \delta_3]$ equation has the form of (6), and therefore, Formula (31) holds for the solution in this segment (here, $\delta_3 < 0$ is an asymptotically small by λ value that denotes a time moment such that $u(t_1 + \delta_3) = p_L$).

It follows from (32) that

$$
u(t_1 + T + \delta_3) = \lambda b(1 - e^{-T} + o(1)). \tag{33}
$$

Since the value (33) is positive and has order $O(\lambda)$ at $\lambda \to +\infty$, then Equation (2) has the form of (24) in the time interval $t \in [t_1 + \delta_4 + T, t_*)$ (here, $\delta_4 > 0$ denotes an asymptotically small by λ value such that $u(t_1 + \delta_4) = p_R$, and t_* denotes a first time moment such that $t_* > t_1 + T + \delta_4$ and $u(t_*) = p_R$). Therefore, the solution has the form

$$
u(t) = \lambda b(1 - e^{-T} + o(1)) e^{-(t-(t_1+T))}. \tag{34}
$$

Note that $t_* = t_1 + T + (1 + o(1)) \ln \lambda$. This is why $u(t_* + s) > p_R$ for all $s \in [-T, 0)$. Thus, $u(t_* + s)$ belongs to the set S_+, and therefore, if we take this function as the initial condition, we get a sign-changing relaxation cycle with the amplitude of the order $O(\lambda)$ and period $O(\ln \lambda)$ at $\lambda \to +\infty$.

If we consider initial conditions from the set S_-, then on the first step $t \in [0, T]$, the equation has the form of (6) and the solution has the form of (7). Since $b > 0$, there exists an asymptotically small by λ value $\delta > 0$ such that $u(\delta) = p_R$ and $u(t) > p_R$ for all $t \in [\delta, T]$.

Then, for all $t > T + \delta$, until then, the $u(t) = p_R$ equation has the form of (24) and the solution has the form

$$u(t) = \lambda b(1 - e^{-T} + o(1))e^{-(t-T))}. \tag{35}$$

We denote as t_R a time moment such that $t_R > T$ and $u(t_R) = p_R$. This value exists because Expression (35) decreases and tends to zero at $t \to +\infty$. Note that $t_R = O(\ln \lambda)$ at $\lambda \to +\infty$. Therefore, function $u(t_R + s)$ ($s \in [-T, 0]$) belongs to the set S_+, and we return to a problem considered earlier in this section.

From the results of this section we obtain the following statement.

Theorem 3. *Let $b > 0$, $d = 0$, and condition (30) holds. Then, for all sufficiently large $\lambda > 0$, Equation (2) has a sign-changing relaxation cycle with the asymptotics*

$$\begin{aligned}
&u(t) = p_R e^{-(t-nt_*)}, && t \in [nt_*, nt_* + T],\\
&u(t) = p_R e^{-(t-nt_*)} + \lambda \int\limits_{T+nt_*}^{t} e^{s-t} f(p_R e^{nt_*+T-s})ds, && t \in [nt_* + T, nt_* + 2T],\\
&u(t) = \lambda\Big(\int\limits_{nt_*+T}^{nt_*+2T} e^{s-nt_*-2T} f(p_R e^{nt_*+T-s})ds - b + o(1)\Big)e^{-(t-nt_*-2T)} + \lambda b, && \\
& && t \in [nt_* + 2T, nt_* + t_1 + T],\\
&u(t) = \lambda b(1 - e^{-T} + o(1))e^{-(t-(nt_*+t_1+T))}, && t \in [nt_* + t_1 + T, (n+1)t_*],
\end{aligned}$$

(where $n = 0, 1, 2, \ldots$ represents the number of periods of a cycle) and period $t_* = t_1 + T + (1 + o(1)) \ln \lambda$ at $\lambda \to +\infty$.

The cycles of Equation (2) in the case that $b > 0$ and $d = 0$ are shown in Figure 4.

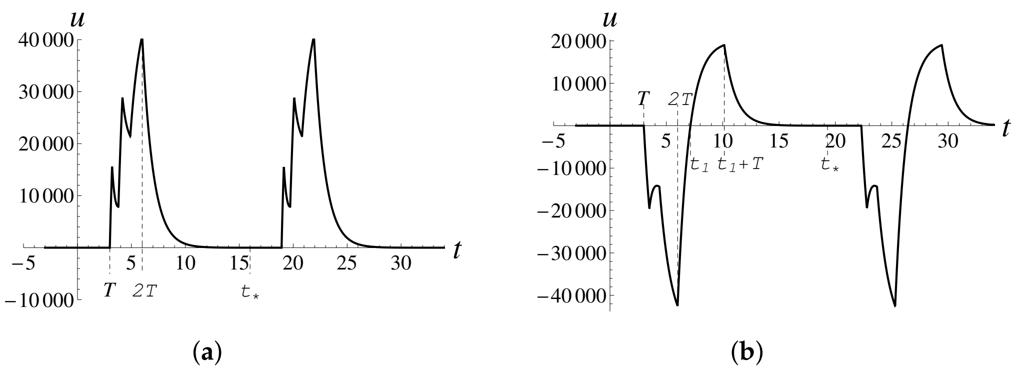


Figure 4. Relaxation cycles of Equation (2) in the case that $b > 0$ and $d = 0$; function $f(u)$ satisfies the condition (**a**) (27) (**b**) (30). Values of parameters: $\lambda = 10^4$, $T = 3$, $p_L = -1$, $p_R = 2$, $b = 2$, and $d = 0$.

7. Asymptotics of Solutions in the Case $b < 0$ and $d = 0$

Firstly, consider initial conditions from S_+. As in the previous section, in the interval $t \in [0, T]$, the solution has the form of (25), and in the interval $t \in [T, 2T]$, it has the form of (26).

If condition (27) holds, then this case is absolutely similar to the case in Section 6, and we obtain the following result.

Theorem 4. *Let $b < 0$, $d = 0$, and (27) holds. Then for all sufficiently large $\lambda > 0$, Equation (2) has a positive relaxation cycle with the asymptotics (29) and period $t_* = 2T + (1 + o(1)) \ln \lambda$ at $\lambda \to +\infty$.*

If condition (30) is true, then there exists an asymptotically small by λ value $\delta > 0$ such that $u(T + \delta) = p_L$ and $u(t) < p_L$ in the interval $t \in (T + \delta, 2T]$. That is why in the segment $t \in [2T + \delta, 3T]$, the equation has the form of (6), and the solution has the form of (31) in the segment $t \in [2T, 3T]$. One can easily see that under conditions $b < 0$ and (30), Expression (31) is less than p_L for all $t > 2T$. This is why Equation (2) has the form of (6), and the solution has the asymptotics of (31) for all $t > 3T$.

Therefore, in the case that $b < 0$ and $d = 0$, if Condition (30) is true, then all solutions with initial conditions from S_+ tend to a constant λb at $t \to +\infty$.

Now, consider initial conditions from S_-. Then, absolutely similarly as in Section 3, the solution has the asymptotics of (7) for all $t > 0$.

Thus, in the case that $b < 0$ and $d = 0$, all solutions with initial conditions from S_- tend to a constant λb at $t \to +\infty$.

8. Asymptotics of Solutions in the Case $b = 0$ and $d < 0$

Firstly, consider initial conditions from S_-. Then, on the first step, $t \in [0, T]$, Equation (2) has the form of (24) and solution has the form

$$u(t) = p_L e^{-t}. \tag{36}$$

It follows from (36) that in the segment $t \in [0, T]$, function $u(t)$ satisfies the inequality $p_L < u(t) < 0$, which is why in the second step, $t \in [T, 2T]$, the solution has the form

$$u(t) = p_L e^{-t} + \lambda \int_T^t e^{s-t} f(p_L e^{T-s}) ds. \tag{37}$$

If function $f(u)$ satisfies the condition

$$\int_T^t e^{s-t} f(p_L e^{T-s}) ds < 0 \text{ for all } t \in (T, 2T], \tag{38}$$

then there exists an asymptotically small by λ value $\delta > 0$ such that Expression (37) is less than p_L on the interval $t \in (T + \delta, 2T]$. It is easy to see that on the segment $t \in [2T, 3T]$, the leading term of the asymptotics of the solution to Equation (2) coincides with the leading term of the asymptotics of the solution to Equation (24) with the initial condition

$$u(2T) = p_L e^{-2T} + \lambda \int_T^{2T} e^{s-2T} f(p_L e^{T-s}) ds. \tag{39}$$

This is why, in the segment $t \in [2T, 3T]$, the solution to Equation (2) has the form

$$u(t) = \lambda \Big(\int_T^{2T} e^{s-2T} f(p_L e^{T-s}) ds + o(1) \Big) e^{-(t-2T)}. \tag{40}$$

Note that Expression (40) is less than p_L in the segment $t \in [2T, 3T]$, which is why Equation (2) has the form of (24) until the Function (40) becomes greater than p_L. There exists a value $t_* > 3T$ such that $u(t_*) = p_L$ and $u(t) < p_L$ for all $t \in (2T, t_*)$. This is why at the point $t = t_*$, we return to the initial situation: the function $u(t_* + s)$ $(s \in [-T, 0))$ belongs to the set S_-. Therefore, if we consider the function $u(t_* + s)$ as the initial conditions of Equation (2), then we get a negative relaxation cycle. Note that it follows from (40) that $t_* - 2T = (1 + o(1)) \ln \lambda$ at $\lambda \to +\infty$.

We obtain the following statement.

Theorem 5. *Let $b = 0$, $d < 0$, and let condition (38) be true. Then, for all sufficiently large $\lambda > 0$, Equation (2) has a negative relaxation cycle with the asymptotics*

$$\begin{aligned}
&u(t) = p_L e^{-(t-nt_*)}, \qquad\qquad t \in [nt_*, nt_* + T], \\
&u(t) = p_L e^{-(t-nt_*)} + \lambda \int\limits_{T+nt_*}^{t} e^{s-t} f(p_L e^{nt_*+T-s}) ds, \quad t \in [nt_* + T, nt_* + 2T], \\
&u(t) = \lambda \Big(\int\limits_{nt_*+T}^{nt_*+2T} e^{s-nt_*-2T} f(p_L e^{nt_*+T-s}) ds + o(1) \Big) e^{-(t-nt_*-2T)}, \\
&\qquad\qquad\qquad\qquad t \in [nt_* + 2T, (n+1)t_*],
\end{aligned} \tag{41}$$

(where $n = 0, 1, 2, \ldots$ represents the number of periods of a cycle) and period $t_* = 2T + (1 + o(1)) \ln \lambda$ at $\lambda \to +\infty$.

If the function $f(u)$ satisfies the condition

$$\int\limits_{T}^{t} e^{s-t} f(p_L e^{T-s}) ds > 0 \text{ for all } t \in (T, 2T], \tag{42}$$

then there exists an asymptotically small by λ value $\delta_1 > 0$ such that $u(T + \delta_1) = p_R$, $u(t) > p_R$ in the interval $(T + \delta_1, 2T]$. Thus, in the segment $t \in [2T, 3T]$, the leading term of the asymptotics of the solution to Equation (2) coincides with the leading term of the asymptotics of the solution to Equation (4) with the initial conditions of (39). This is why this time segment solution has the asymptotics

$$u(t) = \lambda \Big(\int\limits_{T}^{2T} e^{s-2T} f(p_L e^{T-s}) ds - d + o(1) \Big) e^{-(t-2T)} + \lambda d. \tag{43}$$

Since $d < 0$, Expression (43) is decreasing, and there exists a time value t_1 such that $t_1 > 2T$ and Expression (43) is equal to zero at the point t_1 and greater than zero in the interval $t \in (2T, t_1)$. Note that until $u(t) < p_R$, Equation (2) has the form of (4), and the solution has the form of (43). Since λ is sufficiently large, there exists an asymptotically small by λ values $\delta_2 < 0$ and $\delta_3 > 0$ such that $u(t_1 + \delta_2) = p_R$ and $u(t_1 + \delta_3) = p_L$. The length of the interval $(T + \delta_1, t_1 + \delta_2)$ is greater than T, and the solution in this interval is greater than p_R, which is why Equation (2) has the form of (4) in the segment $t \in [t_1 + \delta_2, t_1 + \delta_2 + T]$ and the solution has the form of (43) in this interval.

It is easy to see that

$$u(t_1 + \delta_3 + T) = \lambda d(1 - e^{-T} + o(1)). \tag{44}$$

Since the solution is less than p_L in the interval of the length of delay ($t \in (t_1 + \delta_3, t_1 + \delta_3 + T]$), and $u(t_1 + \delta_3 + T)$ is negative and has the order $O(\lambda)$, Equation (2) has the form of (24) in the segment of the length $(1 + o(1)) \ln \lambda$ (until the solution becomes greater than p_L), and the solution has the form

$$u(t) = \lambda d(1 - e^{-T} + o(1)) e^{-(t-(t_1+T))}. \tag{45}$$

Expression (45) is negative and increases. There exists a time moment $t_* > t_1 + T + \delta_3$ such that Expression (45) is less than p_L for all $t \in [t_1 + T + \delta_3, t_*)$ and is equal to p_L at the point $t = t_*$. Thus, function $u(t_* + s)$ belongs to the set S_-: $u(t_*) = p_L$ and $u(t_* + s) < p_L$ for all $s \in [-T, 0)$. Therefore, if we take this function as the initial conditions of Equation (2), then we get a sign-changing relaxation cycle of this equation with the period $t_* = t_1 + T + (1 + o(1)) \ln \lambda$.

If we take initial functions from S_+, then at the first step, $t \in [0, T]$, the equation has the form of (4) and solution has the form of (5). Then, there exists an asymptotically small by λ value $\delta_4 > 0$ such that $u(t) < p_L$ for all $t \in (\delta_4, T]$. Since, for $t \in [\delta_4, T + \delta_4]$, the solution is less than p_L, Equation (2) has the form of (24), and the solution has the form

$$u(t) = \lambda d(1 - e^{-T} + o(1))e^{-(t-T)} \quad (46)$$

for $t > T + \delta_4$; until then $u(t) > p_L$.

It follows from (46) that there exists a value t_L such that $u(t_L) = p_L$ and $u(s + t_L) < p_L$ in the interval $s \in [-T, 0)$. Therefore, oin the segment $t \in [t_L - T, t_L]$, the solution belongs to the set S_-, which is why we have reduced the problem to the previously studied one.

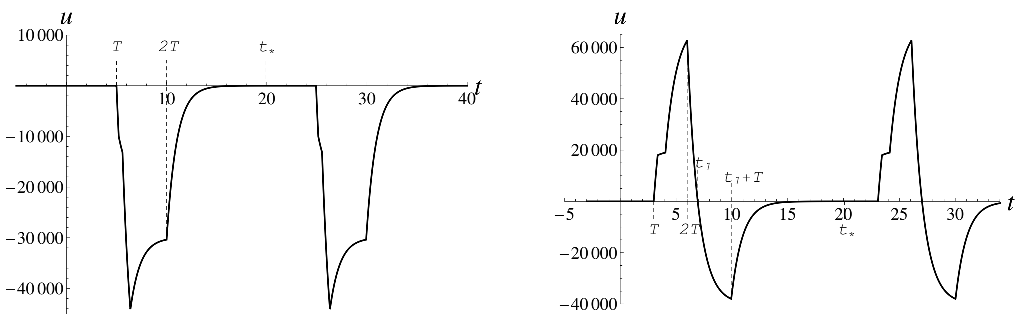
From the above reasoning, we obtain the following statement.

Theorem 6. *Let $b = 0$, $d < 0$, and let condition (42) hold. Then, for all sufficiently large $\lambda > 0$, Equation (2) has a sign-changing relaxation cycle with the asymptotics*

$$\begin{aligned}
&u(t) = p_L e^{-(t-nt_*)}, && t \in [nt_*, nt_* + T],\\
&u(t) = p_L e^{-(t-nt_*)} + \lambda \int\limits_{T+nt_*}^{t} e^{s-t} f(p_L e^{nt_*+T-s})ds, && t \in [nt_* + T, nt_* + 2T],\\
&u(t) = \lambda\Big(\int\limits_{nt_*+T}^{nt_*+2T} e^{s-nt_*-2T} f(p_L e^{nt_*+T-s})ds - d + o(1)\Big)e^{-(t-nt_*-2T)} + \lambda d, &&\\
& && t \in [nt_* + 2T, nt_* + t_1 + T],\\
&u(t) = \lambda d(1 - e^{-T} + o(1))e^{-(t-(nt_*+t_1+T))}, && t \in [nt_* + t_1 + T, (n+1)t_*],
\end{aligned}$$

(where $n = 0, 1, 2, \ldots$ represents the number of periods of cycle) and period $t_* = t_1 + T + (1 + o(1)) \ln \lambda$ at $\lambda \to +\infty$.

The cycles of Equation (2) in the case that $b = 0$ and $d < 0$ are shown in Figure 5.



(**a**) (**b**)

Figure 5. Relaxation cycles of Equation (2) in the case that $b = 0$ and $d < 0$ and function $f(u)$ satisfies the conditions of (**a**) (38) and (**b**) (42). Values of parameters: $\lambda = 10^4$, $p_L = -1.5$, $p_R = 2.5$, $b = 0$, (**a**) $T = 5$, $d = -2$, (**b**) $T = 3$, and $d = -4$.

9. Asymptotics of Solutions in the Case $b = 0$ and $d > 0$

Firstly, consider the initial conditions from S_+. Similar to the case in Section 2, we obtain that for all $t \geq 0$, the solution has the form of (5).

Therefore, in the case that $b = 0$ and $d > 0$, all solutions with initial conditions from the set S_+ tend to a constant λd at $t \to +\infty$.

Now consider the initial conditions from S_-. If function f satisfies Inequality (38), then we obtain that Equation (2) has a negative relaxation cycle (all the reasoning is the same as in Section 8).

The following statement is true.

Theorem 7. *Let $b = 0$, $d > 0$, and let condition (38) be true. Then, for all sufficiently large $\lambda > 0$, Equation (2) has a negative relaxation cycle with the asymptotics of (41) and period $t_* = 2T + (1 + o(1)) \ln \lambda$ at $\lambda \to +\infty$.*

Let us construct the asymptotics of the solution to Equation (2) in the case that function f satisfies the inequality (42). Similarly to Section 8, in the segment $t \in [0, T]$, the solution has the form of (36), in the segment $t \in [T, 2T]$, it has the form of (37), and in the segment $t \in [2T, 3T]$, it has the form of (43). Since Expression (43) is greater than p_R for all $t > 2T$, Equation (2) has the form of (4) for all $t > 3T$, and the solution has the form of (43) for all $t > 2T$.

Thus, in the case that $b = 0$ and $d > 0$, under the condition that function f satisfies Inequality (42), all solutions with initial conditions from the set S_- tend to a constant λd at $t \to +\infty$.

10. Discussion and Conclusions

If we fix the values of b and d, function f, and the set of initial conditions (S_+ or S_-), then for all initial conditions from the chosen set, we obtain an identical behaviour at $t \to +\infty$ (because all solutions with initial conditions from the set S_+ (or S_-) coincide with each other in the segment $t \in [0, T]$, and, therefore, for all $t \geq 0$).

In Sections 2–9 we derived that the behaviour at $t \to +\infty$ of the solutions to Equation (2) with the initial conditions from sets S_+ and S_- may be only of two types: (1) solutions tend to a constant at $t \to +\infty$ or (2) we obtain a cycle.

The following generalization of this result takes place.

Theorem 8. *If we replace the equality $u(0) = p_R$ ($u(0) = p_L$) with the inequality $u(0) \geq p_R$ ($u(0) \leq p_L$, respectively) in the definition of the set S_+ (or S_-, respectively), then the behaviour of the solutions at $t \to +\infty$ does not change.*

Theorem 8 means that if a solution with initial conditions from S_+ tends to a constant at $t \to +\infty$, then a solution with initial conditions satisfying inequality $u(s) \geq p_R$ for all $s \in [-T, 0]$ tends to the same constant at $t \to +\infty$; if we take initial conditions from S_+ and obtain a cycle, then taking initial conditions satisfying inequality $u(s) \geq p_R$, we get the same cycle (but it may be shifted).

The same result is valid for the set S_-.

Proof. Let us prove that if in the definition of S_+, we replace equality $u(0) = p_R$ with inequality $u(0) > p_R$, then the behaviour of the solution does not change.

Denote $u(0)$ as u_0. Since for all $s \in [-T, 0]$, Inequality $u(s) \geq p_R$ holds, then Equation (2) has the form of (4) on the segment $t \in [0, T]$, and the solution has form

$$u_+(t) = u_0 e^{-t} + \lambda d(1 - e^{-t}). \tag{47}$$

Two situations are possible:

(1) There exists a time moment $t_0 > 0$ such that expression (47) is greater than p_R for all $t \in [0, t_0)$ and is equal to p_R at $t = t_0$;

(2) For all $t > 0$, Expression (47) is greater than p_R.

If the first situation occurs, then the function $u_+(t_0 + s)$ ($s \in [-T, 0]$) belongs to the set S_+. All solutions with initial conditions from S_+ for fixed values b and d and function f have the same behaviour, which is why, in this case, for the considered initial conditions, we have the same behaviour of solutions as for the initial conditions from S_+.

The second situation is possible only in the case that $d > 0$ (for all $d \leq 0$ and $u_0 > p_R$, there exists $t_0 > 0$ such that $u_+(t_0) = p_R$). In this situation, for all $t \geq 0$, Equation (2) has the form of (4), and the solution has the form of (47) for all $t \geq 0$. Expression (47) tends to λd at $t \to +\infty$. Since in all cases where $d > 0$, the solutions with initial conditions from S_+

tend to λd at $t \to +\infty$ (see Sections 2, 4 and 9), then in this situation, for the considered initial conditions, we have the same behaviour of solutions as for initial conditions from S_+.

The proof of the Theorem for set S_- is absolutely similar as the proof for the set S_+. $\square$

We have studied the nonlocal dynamics of an equation with delay and nonlinearity having simple behaviour at infinity. This type of nonlinearity is interesting because, on one hand, it is a quite general class of functions, and on the other hand, it is a generalization of two important for application types of nonlinearity: compactly supported and piecewise constant nonlinearities. The key assumption that the nonlinear function F is multiplied by a large parameter λ allows us to construct the asymptotics of all the solutions from the wide sets of initial conditions.

We have studied behaviour at $t \to +\infty$ of the solutions to (2) for wide sets of initial conditions and conclude that two types of behaviour are possible: (1) the solution tends to a constant or (2) after the pre-period, the solution becomes a cycle.

It is important to mention that it is impossible to obtain such general results using numerical simulation because it is impossible to iterate through all the considered functions F and initial conditions. Additionally, even if we take a certain function F and initial conditions, the simulation of this equation is a difficult problem, because the parameter λ is large.

We have found conditions on signs b and d under the condition that $bd \neq 0$ for having a cycle of Equation (2). This cycle has an amplitude of the order $O(\lambda)$ and period of the order $O(1)$ at $\lambda \to +\infty$. We have found conditions on sign b (d) under condition $d = 0$ ($b = 0$, respectively) for having relaxation cycles of Equation (2). Depending on the properties of the function f, this cycle may be sign-changing or sign-preserving.

It is important to mention that most found cycles (see Theorems 1, 3, 6) do not exist in the case of compactly supported nonlinearity [20].

In the future, it will be interesting to study the dynamics of several coupled Equation (2) and to analyse the dependence of the dynamics of the system on the type of coupling.

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