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A New Parameter Choice Strategy for Lavrentiev Regularization Method for Nonlinear Ill-Posed Equations

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Abstract: In this paper, we introduced a new source condition and a new parameter-choice strategy which also gives the known best error estimate. To obtain the results we used the assumptions used in earlier studies. Further, we studied the proposed new parameter-choice strategy and applied it to the method (in the finite-dimensional setting) considered in George and Nair (2017).

Keywords: nonlinear ill-posed equations; finite dimension; iterative method; Lavrentiev regularization; a new parameter-choice strategy

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1. Introduction

Let $\mathcal{H} : D(\mathcal{H}) \subseteq \mathcal{U} \to \mathcal{U}$ be a nonlinear monotone operator, i.e.,

$$\langle \mathcal{H}(v) - \mathcal{H}(w), v - w \rangle \geq 0, \quad \forall v, w \in D(\mathcal{H}),$$

defined on the real Hilbert space $\mathcal{U}$. Here and below $\langle ., . \rangle$ and $\|.\|$, respectively, denote the inner product and corresponding norm in $\mathcal{U}$; $B(u, r)$ and $\overline{B(u, r)}$, respectively, denote open and closed ball in $\mathcal{U}$ with center $u \in \mathcal{U}$ and radius $r > 0$. We are concerned with finite dimensional approximation of the ill-posed equation

$$\mathcal{H}(u) = y, \tag{1}$$

which has a solution $\hat{u}$ for exact data y. However, we have $y^\delta \in \mathcal{U}$ for some $\delta > 0$, are the available data, such that

$$\|y - y^\delta\| \leq \delta. \tag{2}$$

Due to the ill-posedness of (1), one has to apply regularization method to obtain an approximation for $\hat{u}$. For (1) with monotone $\mathcal{H}$, Lavrentiev regularization (LR) method is widely used (see [1–6]). In (LR) method the solution u_α^δ of the equation

$$\mathcal{H}(u) + \alpha(u - u_0) = y^\delta, \tag{3}$$

is used as an approximation for $\hat{u}$. Here (and below) u_0 is an initial approximation of $\hat{u}$ with $\|u_0 - \hat{u}\| \leq r_0$ for some $r_0 > 0$. The solution of (3), with y in place of y^δ is denoted by u_α, i.e., (cf. [5])

$$\mathcal{H}(u_\alpha) + \alpha(u_\alpha - u_0) = y. \tag{4}$$

Let u_α^δ and u_α be as in Equations (3) and (4) , respectively. Then, we have the following inequalities (cf. [5]).

$$\begin{aligned} \|u_\alpha - \hat{u}\|^2 &\leq \langle u_0 - \hat{u}, u_\alpha - \hat{u} \rangle, \\ \|u_\alpha^\delta - u_\alpha\| &\leq \frac{\delta}{\alpha}, \end{aligned} \tag{5}$$

and hence,

$$\|\hat{u} - u_\alpha^\delta\| \leq \|\hat{u} - u_\alpha\| + \frac{\delta}{\alpha} \tag{6}$$

and

$$\|\hat{u} - u_\alpha\| \leq \|\hat{u} - u_0\|. \tag{7}$$

For proving our result, we assume that, either $\mathcal{H}'(u)$ is self-adjoint or $\mathcal{H}'(u)$ is positive type, i.e.,

$$\sigma(\mathcal{H}'(u)) \subseteq [0, \infty)$$

and

$$\|(\mathcal{H}'(u) + sI)^{-1}\| \leq \frac{c}{s}, \ s > 0, \ \text{for some constant } c > 0, \ u \in \overline{B(u_0, r)}$$

(see [7]). Here and below $\mathcal{H}'(u)$ is the Fréchet derivative of $\mathcal{H}(u)$ (if $\mathcal{H}'(u)$ is self-adjoint, then $c = 1$).

Remark 1. *If $\mathcal{H}'(u)$ is positive type, then*

$$\|(\mathcal{H}'(u) + sI)^{-1}\mathcal{H}'(u)\| = \|I - s(\mathcal{H}'(u) + sI)^{-1}\| \leq 1 + c.$$

Further as in [8] (Lemma 2.2) one can prove

$$\|(\mathcal{H}'(u) + sI)^{-1}\mathcal{H}'(u)^\mu\| = O(s^\mu), \ 0 \leq \mu < 1.$$

So, the results in this paper hold for positive type operator $\mathcal{H}'(u)$ up to a constant. Therefore, for convenience, hereafter we assume $\mathcal{H}'(.)$ is self-adjoint.

In earlier studies such as [4–6,9,10], the following source condition:

$$u_0 - \hat{u} = \mathcal{H}'(\hat{u})^{\mu_1} z, \ \|z\| \leq \rho, \quad 0 < \mu_1 \leq 1. \tag{8}$$

or

$$u_0 - \hat{u} = \mathcal{H}'(u_0)^{\mu_2} z, \ \|z\| \leq \rho, \quad 0 < \mu_2 \leq 1 \tag{9}$$

was used to obtain an estimate for $\|\hat{u} - u_\alpha\|$. In fact, if the source condition (8) is satisfied, then, we have [5]

$$\|\hat{u} - u_\alpha\| = O(\alpha^{\mu_1})$$

and if (9) is satisfied, then, we have [2]

$$\|\hat{u} - u_\alpha\| = O(\alpha^{\mu_2}).$$

In this study, we introduce a new source condition,

$$u_0 - \hat{u} = A^\nu z, \ \|z\| \leq \rho, \quad 0 < \nu \leq 1, \tag{10}$$

where $\rho > 0$ and $A = \int_0^1 \mathcal{H}'(\hat{u} + t(u_0 - \hat{u}))dt$. We shall use this source condition (10) to obtain a convergence rate for $\|\hat{u} - u_\alpha\|$ and to introduce a new parameter-choice strategy.

Remark 2. *(a) Note that in a posteriori parameter-choice strategy, the regularization parameter* α *(depending on* δ *and* y^δ*) is chosen at the time of computing* u_α^δ *(see [11]). The new source condition (10) is used to choose the parameter* α *(depending on* δ *and* y^δ*) and independent of* ν*, before computing* u_α^δ *(see Section 2) and also it gives the best known convergence order (see Remark 4). This is the innovation of our approach.*
(b) Notice that, the operator A *and* A^ν *are used to obtain an estimate for* $\|\hat{u} - u_\alpha\|$*. In actual computation of the approximation* $u_{n+1,\alpha}^{h,\delta}$ *(see Equation (38)) and* α *(see Section 4) we do not require the operator* A *or* A^ν*.*

The following formula ([12], p. 287) for fractional power of positive type operators $\mathcal{B}$ is used in our analysis.

$$\begin{aligned}\mathcal{B}^z x &= \frac{\sin \pi z}{\pi}\int_0^\infty \tau^z\left[(\mathcal{B}+\tau I)^{-1}x - \frac{\Theta(\tau)}{\tau}x + \ldots + (-1)^n\frac{\Theta(\tau)}{\tau^n}\mathcal{B}^{n-1}x\right]d\tau \\ &+\frac{\sin \pi z}{\pi}\left[\frac{x}{z} - \frac{\mathcal{B}x}{z-1} + \ldots + (-1)^{n-1}\frac{\mathcal{B}^{n-1}x}{z-n+1}\right],\ x \in \mathcal{U},\end{aligned}$$

where

$$\Theta(\varsigma) = \begin{cases} 0 & \text{if} \quad 0 \le \varsigma \le 1 \\ 1 & \text{if} \quad 1 < \varsigma < \infty \end{cases}$$

and z is a complex number such that $0 < Rez < n$.

Let $z = \nu$, and $\mathcal{B} = \mathcal{H}'(.)$. Then, we have

$$\mathcal{H}'(.)^\nu x = \frac{\sin \pi(\nu)}{\pi}\left[\frac{x}{\nu} + \int_0^\infty \tau^\nu(\mathcal{H}'(.)+\tau I)^{-1}x d\tau - \int_1^\infty \frac{x}{\tau^{1-\nu}}d\tau\right]. \quad (11)$$

Note that, if $\mathcal{H}'(.)$ is self-adjoint, then, A is self-adjoint. Further, suppose $\mathcal{H}'(.)$ is positive type, then we have

$$\begin{aligned}\|(A+sI)^{-1}\| &= \|(\int_0^1 \mathcal{H}'(\hat{u}+t(u_0-\hat{u}))dt + sI)^{-1}\| \\ &= \|(\int_0^1 (\mathcal{H}'(\hat{u}+t(u_0-\hat{u})) + sI)dt)^{-1}\| \\ &\le \int_0^1 \|(\mathcal{H}'(\hat{u}+t(u_0-\hat{u})) + sI)^{-1}\|dt \\ &\le \frac{c}{s},\end{aligned}$$

i.e., A is positive type.

Next, we shall prove that (10) implies

$$u_0 - \hat{u} = \begin{cases} \mathcal{H}'(u_0)^{\nu_1}\xi_z,\ \|\xi_z\| \le \rho_0 \text{ for} & 0 < \nu_1 < \nu < 1 \\ \mathcal{H}'(u_0)\xi_{z_1},\ \|\xi_{z_1}\| \le \rho_1 \text{ for} & \nu = 1, \end{cases} \quad (12)$$

for some constants ρ_0 and ρ_1. For this, we use the standard non-linear assumptions in the literature (cf. [4,13]).

Assumption 1. For every $u, v \in \overline{B(u_0, r)}$ and $w \in \mathcal{U}$, there exists $k_0 > 0$ and an element $\Phi(u, v, w) \in \mathcal{U}$ with

$$[\mathcal{H}'(u) - \mathcal{H}'(v)]w = \mathcal{H}'(v)\Phi(u, v, w)$$

and

$$\|\Phi(u, v, w)\| \le k_0\|w\|\|u - v\|.$$

Suppose (10) holds for $\nu < 1$, then

$$\begin{aligned} u_0 - \hat{u} &= A^{\nu} z \\ &= [A^{\nu} - \mathcal{H}'(u_0)^{\nu}]z + \mathcal{H}'(u_0)^{\nu} z \\ &= -\frac{\sin \pi(\nu)}{\pi} \int_0^{\infty} \tau^{\nu} (\mathcal{H}'(u_0) + \tau I)^{-1} (A - \mathcal{H}'(u_0))(A + \tau I)^{-1} z d\tau \\ &\quad + \mathcal{H}'(u_0)^{\nu} z, \end{aligned}$$

so by the definition of A and Assumption 1, we have

$$\begin{aligned} u_0 - \hat{u} &= [A^{\nu} - \mathcal{H}'(u_0)^{\nu}]z + \mathcal{H}'(u_0)^{\nu} z \\ &= -\frac{\sin \pi(\nu)}{\pi} \int_0^{\infty} \tau^{\nu} (\mathcal{H}'(u_0) + \tau I)^{-1} \\ &\quad \times \int_0^1 (\mathcal{H}'(\hat{u} + t(u_0 - \hat{u})) - \mathcal{H}'(u_0)) dt (A + \tau I)^{-1} z d\tau \\ &\quad + \mathcal{H}'(u_0)^{\nu} z \\ &= -\frac{\sin \pi(\nu)}{\pi} \int_0^{\infty} \tau^{\nu} (\mathcal{H}'(u_0) + \tau I)^{-1} \mathcal{H}'(u_0) \\ &\quad \times \int_0^1 \Phi(\hat{u} + t(u_0 - \hat{u}), u_0, (A + \tau I)^{-1} z) dt d\tau + \mathcal{H}'(u_0)^{\nu} z \\ &= \mathcal{H}'(u_0)[-\frac{\sin \pi(\nu)}{\pi} \int_0^{\infty} \tau^{\nu} (\mathcal{H}'(u_0) + \tau I)^{-1} \\ &\quad \times \int_0^1 \Phi(\hat{u} + t(u_0 - \hat{u}), u_0, (A + \tau I)^{-1} z) dt d\tau] + \mathcal{H}'(u_0)^{\nu} z \\ &= \mathcal{H}'(u_0)^{\nu_1} \xi_z, \ \nu_1 < \nu, \end{aligned}$$

where $\xi_z = \mathcal{H}'(u_0)^{1-\nu_1}(-\frac{\sin \pi(\nu)}{\pi} \int_0^{\infty} \tau^{\nu} (\mathcal{H}'(u_0) + \tau I)^{-1} \int_0^1 \Phi(\hat{u} + t(u_0 - \hat{u}), u_0, (A + \tau I)^{-1} z) dt) d\tau + \mathcal{H}'(u_0)^{\nu - \nu_1} z$. Further note that

$$\begin{aligned} \|\xi_z\| &\leq \frac{1}{\pi} \| \left(\int_0^{\infty} \mathcal{H}'(u_0)^{1-\nu_1} \tau^{\nu} (\mathcal{H}'(u_0) + \tau I)^{-1} \right. \\ &\quad \left. \times \int_0^1 \Phi(\hat{u} + t(u_0 - \hat{u}), u_0, (A + \tau I)^{-1} z) dt \right) d\tau \| + \|\mathcal{H}'(u_0)^{\nu - \nu_1} z\| \\ &\leq \frac{1}{\pi} [\int_0^1 \tau^{\nu} \|\mathcal{H}'(u_0)^{1-\nu_1} (\mathcal{H}'(u_0) + \tau I)^{-1}\| k_0 \frac{\|u_0 - \hat{u}\|}{2} \|(A + \tau I)^{-1} z\| d\tau \\ &\quad + \int_1^{\infty} \tau^{\nu} \|\mathcal{H}'(u_0)^{1-\nu_1} (\mathcal{H}'(u_0) + \tau I)^{-1}\| k_0 \frac{\|u_0 - \hat{u}\|}{2} \|(A + \tau I)^{-1} z\| d\tau] \\ &\quad + \|\mathcal{H}'(u_0)^{\nu - \nu_1}\| \rho \\ &\leq \frac{1}{\pi} [\int_0^1 \tau^{\nu - \nu_1 - 1} d\tau k_0 \frac{\|u_0 - \hat{u}\|}{2} \|z\| \\ &\quad + \|\mathcal{H}'(u_0)^{1-\nu_1}\| \int_1^{\infty} \tau^{\nu - 2} d\tau k_0 \frac{\|u_0 - \hat{u}\|}{2} \|z\|] + \|\mathcal{H}'(u_0)^{\nu - \nu_1}\| \rho \\ &\leq \frac{1}{\pi} \left[\frac{1}{\nu - \nu_1} + \frac{\|\mathcal{H}'(u_0)^{1-\nu_1}\|}{1 - \nu} \right] k_0 \frac{r_0}{2} \rho + \|\mathcal{H}'(u_0)^{\nu - \nu_1}\| \rho := \rho_0. \end{aligned}$$

Suppose

$$\begin{aligned} u_0 - \hat{u} &= Az \\ &= [A - \mathcal{H}'(u_0) + \mathcal{H}'(u_0)]z \\ &= [\int_0^1 (\mathcal{H}'(\hat{u} + t(u_0 - \hat{u})) - \mathcal{H}'(u_0)) dt + \mathcal{H}'(u_0)]z \end{aligned}$$

$$\begin{aligned} &= \mathcal{H}'(u_0)[\int_0^1 \Phi(\hat{u} + t(u_0 - \hat{u}), u_0, z)dt + z] \\ &= \mathcal{H}'(u_0)\xi_{z_1}, \end{aligned}$$

where $\xi_{z_1} = \int_0^1 \Phi(\hat{u} + t(u_0 - \hat{u}), u_0, z)dt + z$. Observe that

$$\|\xi_{z_1}\| \leq (k_0 \frac{\|\hat{u} - u_0\|}{2} + 1)\|z\| \leq (\frac{k_0 r_0}{2} + 1)\rho = \rho_1.$$

So $u_0 - \hat{u} = Az$ implies $u_0 - \hat{u} = \mathcal{H}'(u_0)\xi_{z_1}$, $\|\xi_{z_1}\| \leq \rho_1$ i.e., (10) implies (12). Similarly one can show that (10) implies

$$u_0 - \hat{u} = \begin{cases} \mathcal{H}'(\hat{u})^{\nu_1}\xi_z, \ \|\xi_z\| \leq \rho_2 \text{ for} & 0 < \nu_1 < \nu < 1 \\ \mathcal{H}'(\hat{u})\xi_{z_1}, \ \|\xi_{z_1}\| \leq \rho_1 \text{ for} & \nu = 1, \end{cases}$$

for some constant ρ_2. Throughout the paper, we use the relation (Fundamental Theorem of Integration),

$$\mathcal{H}(u) - \mathcal{H}(x) = \left[\int_0^1 \mathcal{H}'(x + t(u - x))\, dt\right](u - x)$$

for all x and u in a ball contained in $D(\mathcal{H})$.

Remark 3. *In general, it is believed that (see [5]) a priori parameter-choice strategy is not a good strategy to choose α since the choice is depending on the unknown ν. In this study, we introduce a new parameter-choice strategy which is not depending on unknown ν and gives the best known convergence order $O(\delta^{\frac{\nu}{\nu+1}})$.*

In some recent papers, the first author and his collaborators considered iterative methods [14,15] for obtaining stable approximate solutions for (3) (see [8,16]). In most of the iterative methods Fréchet derivative of the operator involved is used. In [10], Semenova considered the iterative method defined for fixed α, δ, by

$$u^\delta_{n+1,\alpha} = u^\delta_{n,\alpha} - \gamma[\mathcal{H}(u^\delta_{n,\alpha}) + \alpha(u^\delta_{n,\alpha} - u_0) - y^\delta]. \tag{13}$$

Note that, the above iterative method is derivative-free. Convergence analysis in [10] is based on the assumption that $\mathcal{H}$ is Lipschitz continuous and the Lipschitz constant R satisfies

$$0 < \gamma < \min\left\{\frac{1}{\alpha}, \frac{2\alpha}{\alpha^2 + R^2}\right\}, \tag{14}$$

where γ is a constant. Contraction mapping arguments are used to prove the convergence in [10].

In [16], George and Nair considered the method (13), but with β independent on the regularization parameter α and the Lipschitz constant R, instead of γ. The source condition on $u_0 - \hat{u}$ in [16] depends on the known u_0 and the analysis in [16] is not based on the contraction mapping arguments as in [10].

The purpose of this paper is threefold: (1) introduce a new source condition, (2) introduce a new parameter-choice strategy, and (3) apply the parameter-choice strategy to the (finite–dimensional setting of the) method in [16].

The remainder of the paper is organized as follows. In Section 2, we present the error bounds under the source condition (10) and a new parameter-choice strategy. In Section 3, we present the finite dimensional realization of method (13). In Section 4, we present the finite dimensional realization of (10). Section 5 contains the numerical example and the conclusion is given in Section 6.

2. Error Bounds under (10) and a New Parameter Choice Strategy

First we obtain an estimate for $\|\hat{u} - u_\alpha\|$ using (10).

Theorem 1. *Let $\frac{3}{2}k_0 r_0 < 1$, Assumption 1 and (10) be satisfied. Then,*

$$\|\hat{u} - u_\alpha\| \leq \frac{2 + k_0 r_0}{3 - 2k_0 r_0} \alpha^\nu \|z\|.$$

Proof. Since $\mathcal{H}(\hat{u}) = y$ and $\mathcal{H}(u_\alpha) + \alpha\,(u_\alpha - u_0) = y$, we have

$$\mathcal{H}(u_\alpha) - \mathcal{H}(\hat{u}) + \alpha\,(u_\alpha - u_0) = 0,$$

i.e.,

$$\mathcal{H}(u_\alpha) - \mathcal{H}(\hat{u}) + \alpha\,(u_\alpha - \hat{u}) = \alpha\,(u_0 - \hat{u}), \tag{15}$$

or

$$(M_\alpha + \alpha\, I)\,(u_\alpha - \hat{u}) = \alpha\,(u_0 - \hat{u}), \tag{16}$$

where

$$M_\alpha = \int_0^1 \mathcal{H}'(\hat{u} + t(u_\alpha - \hat{u}))\,dt.$$

Again (16) can be written as

$$(A_0 + \alpha I)(u_\alpha - \hat{u}) = (A_0 - M_\alpha)(u_\alpha - \hat{u}) + \alpha(u_0 - \hat{u}),$$

where $A_0 = \mathcal{H}'(u_0)$. Thus, we have

$$\begin{aligned}
u_\alpha - \hat{u} &= -(A_0 + \alpha\, I)^{-1}\,(M_\alpha - A_0)\,(u_\alpha - \hat{u}) + \alpha\,(A_0 + \alpha\, I)^{-1}\,(u_0 - \hat{u}) \\
&= -(A_0 + \alpha\, I)^{-1}\left[\int_0^1 [\mathcal{H}'(\hat{u} + t\,(u_\alpha - \hat{u})) - \mathcal{H}'(u_0)]\right](u_\alpha - \hat{u})\,dt \\
&\quad +\alpha\,(A_0 + \alpha\, I)^{-1}\,(u_0 - \hat{u}) \\
&= -(A_0 + \alpha\, I)^{-1} A_0 \int_0^1 \Phi(\hat{u} + t\,(u_\alpha - \hat{u}), u_0, u_\alpha - \hat{u})\,dt \\
&\quad +\alpha\,(A_0 + \alpha\, I)^{-1}\,(u_0 - \hat{u})
\end{aligned}$$

and hence

$$\begin{aligned}
\|u_\alpha - \hat{u}\| &\leq k_0[\frac{\|u_\alpha - \hat{u}\|}{2} + \|\hat{u} - u_0\|]\|u_\alpha - \hat{u}\| \\
&\quad +\|\alpha\,(A_0 + \alpha\, I)^{-1}\,(u_0 - \hat{u})\| \\
&\leq \frac{3}{2}k_0 r_0\,\|u_\alpha - \hat{u}\| + \alpha\|(A + \alpha I)^{-1}(u_0 - \hat{u})\| \text{ by (7)} \\
&\quad +\alpha\|[(A_0 + \alpha I)^{-1} - (A + \alpha I)^{-1}](u_0 - \hat{u})\| \\
&\leq \frac{3}{2}k_0 r_0\|u_\alpha - \hat{u}\| + \alpha\|(A + \alpha I)^{-1}(u_0 - \hat{u})\| \\
&\quad +\|(A_0 + \alpha I)^{-1}(A - A_0)\alpha(A + \alpha I)^{-1}(u_0 - \hat{u})\| \\
&\leq \frac{3}{2}k_0 r_0\|u_\alpha - \hat{u}\| + \alpha\|(A + \alpha I)^{-1}(u_0 - \hat{u})\| \\
&\quad +\|A_0(A_0 + \alpha I)^{-1}\int_0^1 \Phi(\hat{u} + t(u_0 - \hat{u}), u_0, \alpha(A + \alpha I)^{-1}(u_0 - \hat{u}))dt\| \\
&\leq \frac{3}{2}k_0 r_0\|u_\alpha - \hat{u}\| + \alpha\|(A + \alpha I)^{-1}(u_0 - \hat{u})\| \\
&\quad +\frac{k_0 r_0}{2}\|\alpha(A + \alpha I)^{-1}(u_0 - \hat{u})\|
\end{aligned}$$

i.e.,

$$
\begin{aligned}
\left(1-\frac{3}{2}k_0r_0\right)\|u_\alpha-\hat{u}\| &\leq (1+\frac{k_0r_0}{2})\|\alpha\,(A+\alpha\, I)^{-1}\,(u_0-\hat{u})\| \\
\frac{2-3k_0r_0}{2+k_0r_0}\|\hat{u}-u_\alpha\| &\leq \|\alpha\,(A+\alpha\, I)^{-1}\,A^\nu\, z\| \text{ by (10)} \\
&\leq \sup_{\lambda\in\sigma(A)}\left|\frac{\alpha\lambda^\nu}{\lambda+\alpha}\right|\|z\| \\
&\leq \alpha^\nu\|z\|.
\end{aligned} \tag{17}
$$

□

Theorem 2. *Suppose Assumption 1 and (10) hold. Then,*

$$\|u_\alpha^\delta-\hat{u}\| \leq \max\{1,\frac{2+k_0r_0}{3-2k_0r_0}\|z\|\}(\frac{\delta}{\alpha}+\alpha^\nu).$$

In particular, if $\alpha=\delta^{\frac{1}{\nu+1}}$, then

$$\|u_\alpha^\delta-\hat{u}\| = O\Big(\delta^{\frac{\nu}{\nu+1}}\Big).$$

Proof. Follows from (6) and Theorem 1. □

Remark 4. *Note that the best value for $\frac{\delta}{\alpha}+\alpha^\nu$ is attained when $\frac{\delta}{\alpha}=\alpha^\nu$, i.e., $\alpha=\delta^{\frac{1}{\nu+1}}$, and in this case the optimal order is $O\Big(\delta^{\frac{\nu}{\nu+1}}\Big)$. However, the above choice of α is depending on the unknown ν. In view of this, our aim is to choose α (not depending on ν), so that we obtain $\|u_\alpha^\delta-\hat{u}\| = O\Big(\delta^{\frac{\nu}{\nu+1}}\Big)$.*

A New Parameter Choice Strategy

For $u\in\mathcal{U}$, define

$$\phi(\alpha,u) := \|\alpha^2(A_0+\alpha I)^{-2}\,(\mathcal{H}(u_0)-u)\|, \tag{18}$$

where $A_0=\mathcal{H}'(u_0)$.

Theorem 3. *For each $u\in\mathcal{U}$, and $\alpha>0$ the function $\alpha\to\phi(\alpha,u)$ is continuous, monotonically increasing and*

$$\lim_{\alpha\to 0}\phi(\alpha,u)=0 \quad\text{and}\quad \lim_{\alpha\to\infty}\phi(\alpha,u)=\|\mathcal{H}(u_0)-u\|.$$

Proof. Note that

$$\phi(\alpha,u)^2=\int_0^{\|A_0\|}\left(\frac{\alpha}{\lambda+\alpha}\right)^4 d\|E_\lambda(\mathcal{H}(u_0)-u)\|^2,$$

where E_λ is the spectral family of A_0. Note that for each $\lambda>0$,

$$\alpha\longrightarrow\left(\frac{\alpha}{\lambda+\alpha}\right)^4$$

is strictly increasing and satisfies $\lim_{\alpha\longrightarrow 0}\left(\frac{\alpha}{\lambda+\alpha}\right)^4=0$ and $\lim_{\alpha\longrightarrow\infty}\left(\frac{\alpha}{\lambda+\alpha}\right)^4=1$. Hence, by Dominated Convergence Theorem $\phi(\alpha,u)$ is strictly increasing, continuous, $\lim_{\alpha\to 0}\phi(\alpha,u)=0$ and $\lim_{\alpha\to\infty}\phi(\alpha,u)=\|\mathcal{H}(u_0)-u\|$. □

In addition to (2), we assume that

$$c\delta \le \|\mathcal{H}(u_0) - y^\delta\|, \tag{19}$$

for some $c > 1$. The following theorem is a consequence of the intermediate value theorem.

Theorem 4. *Let y^δ satisfies (2) and (19). Then,*

$$\phi(\alpha, y^\delta) = c\delta \tag{20}$$

has a unique solution α.

Next, we shall show that if $\alpha = \alpha(\delta, u_0)$ satisfies (10) and (20) hold, then $\|\hat{u} - u_\alpha\| = O(\delta^{\frac{\nu}{\nu+1}})$. Our proof is based on the following moment inequality for positive type operator B (see [12], p. 290)

$$\|B^u x\| \le \|B^v x\|^{\frac{u}{v}} \|x\|^{1-\frac{u}{v}}, \quad 0 \le u \le v. \tag{21}$$

Theorem 5. *Let $\frac{3}{2}k_0 r_0 < 1$, Assumption 1 and (10) be satisfied. Let $\alpha = \alpha(\delta, u_0)$ be the solution of (20). Then,*

$$\|\hat{u} - u_\alpha\| \le O(\delta^{\frac{\nu}{\nu+1}}).$$

Proof. By taking $B = \alpha\,(A + \alpha\, I)^{-1} A$ and $x = \alpha^{1-\nu}\,(A + \alpha\, I)^{-(1-\nu)}\, z$ in (17) and then using (21) with $u = \nu$, $v = 1 + \nu$, we have

$$\begin{aligned}
\frac{2 - 3k_0 r_0}{2 + k_0 r_0}\|\hat{u} - u_\alpha\| &\le \|B^\nu x\| \\
&\le \|B^{1+\nu} x\|^{\frac{\nu}{1+\nu}} \|x\|^{\frac{1}{1+\nu}} \\
&= \|\alpha^2\,(A + \alpha\, I)^{-2} A^{1+\nu} z\|^{\frac{\nu}{1+\nu}} \|z\|^{\frac{1}{1+\nu}} \\
&= \|\alpha^2\,(A + \alpha\, I)^{-2} A\,(u_0 - \hat{u})\|^{\frac{\nu}{1+\nu}} \|z\|^{\frac{1}{1+\nu}} \\
&= \|\alpha^2\,(A + \alpha\, I)^{-2}(\mathcal{H}(u_0) - y)\|^{\frac{\nu}{1+\nu}} \|z\|^{\frac{1}{1+\nu}} \\
&\le \Big(\|\alpha^2\,(A + \alpha\, I)^{-2}\,(\mathcal{H}(u_0) - y^\delta)\| \\
&\quad + \|\alpha^2\,(A + \alpha\, I)^{-2}\,(y^\delta - y)\|\Big)^{\frac{\nu}{1+\nu}} \|z\|^{\frac{1}{1+\nu}} \\
&= (\mathcal{B}_1 + \delta)^{\frac{\nu}{1+\nu}} \|z\|^{\frac{1}{1+\nu}}
\end{aligned} \tag{22}$$

where $\mathcal{B}_1 = \|\alpha^2\,(A + \alpha\, I)^{-2}\,(\mathcal{H}(u_0) - y^\delta)\|$ and we used the inequality,

$$\|\alpha^2\,(A + \alpha\, I)^{-2}\,(y^\delta - y)\| \le \delta.$$

We have,

$$\begin{aligned}
\mathcal{B}_1 &= \|\alpha^2\,(A + \alpha\, I)^{-2}\,(\mathcal{H}(u_0) - y^\delta)\| \\
&= \|\alpha^2[(A + \alpha I)^{-2} - (A_0 + \alpha I)^{-2}](\mathcal{H}(u_0) - y^\delta) \\
&\quad + \alpha^2(A_0 + \alpha I)^{-2}(\mathcal{H}(u_0) - y^\delta)\| \\
&\le \|\alpha^2[(A + \alpha I)^{-2} - (A_0 + \alpha I)^{-2}](\mathcal{H}(u_0) - y^\delta)\| \\
&\quad + \|\alpha^2(A_0 + \alpha I)^{-2}(\mathcal{H}(u_0) - y^\delta)\| \\
&=: \mathcal{D}_1 + \phi(\alpha, y^\delta)
\end{aligned} \tag{23}$$

where $\mathcal{D}_1 = \|\alpha^2[(A+\alpha I)^{-2} - (A_0+\alpha I)^{-2}](\mathcal{H}(u_0) - y^\delta)\|$. Let $w = \alpha^2(A_0+\alpha I)^{-2}(\mathcal{H}(u_0)$ $-y^\delta)$. Note that,

$$\begin{aligned}
\mathcal{D}_1 &= \|\alpha^2[(A+\alpha I)^{-2} - (A_0+\alpha I)^{-2}](\mathcal{H}(u_0) - y^\delta)\| \\
&= \|(A+\alpha I)^{-2}[A_0^2 - A^2 + 2\alpha(A_0 - A)]w\| \\
&= \|(A+\alpha I)^{-2}[(A+A_0) + 2\alpha I](A_0 - A)w\| \\
&= \|(A+\alpha I)^{-2}[A_0 - A + 2A + 2\alpha I](A_0 - A)w\| \\
&= \|[(A+\alpha I)^{-1}(A_0 - A)]^2 w + 2(A+\alpha I)^{-1}(A_0 - A)w\| \\
&\leq (\|\Gamma\|^2 + 2\|\Gamma\|)\|w\| = (\|\Gamma\|^2 + 2\|\Gamma\|)\phi(\alpha, y^\delta), \qquad (24)
\end{aligned}$$

where $\Gamma = (A+\alpha I)^{-1}(A_0 - A)$. By Assumption 1, we obtain

$$\begin{aligned}
\|\Gamma x\| &\leq \|[(A+\alpha I)^{-1} - (A_0+\alpha I)^{-1}](A_0 - A)x\| \\
&\quad + \|(A_0+\alpha I)^{-1}(A_0 - A)x\| \\
&= \|(A_0+\alpha I)^{-1}[A_0 - A](A+\alpha I)^{-1}(A_0 - A)x\| \\
&\quad + \|(A_0+\alpha I)^{-1}(A_0 - A)x\| \\
&\leq \|(A_0+\alpha I)^{-1}A_0 \\
&\quad \times \int_0^1 \Phi(\hat{u} + t(u_0 - \hat{u}), u_0, (A+\alpha I)^{-1}(A_0 - A)x)dt\| \\
&\quad + \|(A_0+\alpha I)^{-1}A_0 \int_0^1 \Phi((\hat{u} + t(u_0 - \hat{u}), u_0, x)dt\| \\
&\leq \frac{k_0 r_0}{2}\|\Gamma x\| + \frac{k_0 r_0}{2}\|x\|,
\end{aligned}$$

i.e.,

$$(1 - \frac{k_0 r_0}{2})\|\Gamma x\| \leq k_0 r_0 \|x\|, \qquad (25)$$

and hence

$$\mathcal{B}_1 \leq [\frac{2k_0 r_0}{2 - k_0 r_0}(\frac{2k_0 r_0}{2 - k_0 r_0} + 2) + 1]\phi(\alpha, y^\delta) = O(\delta). \qquad (26)$$

The result now follows from (23)–(26). □

Theorem 6. *Suppose Assumption 1 and (10) hold and if $\alpha = \alpha(\delta, u_0)$ is chosen as a solution of (20). Then,*

$$\frac{\delta}{\alpha} = O\Big(\delta^{\frac{\nu}{\nu+1}}\Big).$$

Proof. By (20), we have

$$\begin{aligned}
c\delta &= \|\alpha^2 (A_0 + \alpha I)^{-2}(\mathcal{H}(u_0) - y^\delta)\| \\
&\leq \|\alpha^2 (A_0 + \alpha I)^{-2}(\mathcal{H}(u_0) - y)\| \\
&\quad + \|\alpha^2 (A_0 + \alpha I)^{-2}(y - y^\delta))\| \\
&\leq \|\alpha^2 (A_0 + \alpha I)^{-2}(\mathcal{H}(u_0) - y)\| + \delta,
\end{aligned}$$

so

$$\begin{aligned}
(c-1)\delta &\leq \|\alpha^2 \Big[(A_0 + \alpha I)^{-2} - (A + \alpha I)^{-2}\Big](\mathcal{H}(u_0) - y)\| \\
&\quad + \|\alpha^2 (A + \alpha I)^{-2}(\mathcal{H}(u_0) - y)\| \\
&= \|(A_0 + \alpha I)^{-2} \Big[(A + \alpha I)^2 - (A_0 + \alpha I)^2\Big] \\
&\quad \times \alpha^2 (A + \alpha I)^{-2} (\mathcal{H}(u_0) - y)\| \\
&\quad + \|\alpha^2 (A + \alpha I)^{-2}(\mathcal{H}(u_0) - y)\|. \qquad (27)
\end{aligned}$$

Let $w_1 = \alpha^2 (A + \alpha I)^{-2}(\mathcal{H}(u_0) - y)$. Then, similar to (24), we have

$$(c_1 - 1)\delta \quad \leq \quad (\|\Gamma_1\|^2 + 2\|\Gamma_1\| + 1)\|w_1\|, \tag{28}$$

where $\Gamma_1 = (A_0 + \alpha I)^{-1}(A - A_0)$. Note that,

$$\begin{aligned} \|\Gamma_1 x\| &= \|(A_0 + \alpha I)^{-1}(A - A_0)x\| \\ &= \|(A_0 + \alpha I)^{-1} A_0 \int_0^1 \Phi(\hat{u} + t(u_0 - \hat{u}), u_0, x)dt\| \\ &\leq \frac{k_0}{2}\|u_0 - \hat{u}\|\|x\| \\ &\leq \frac{k_0 r_0}{2}\|x\|, \end{aligned}$$

so

$$\|\Gamma_1\| \leq \frac{k_0 r_0}{2}. \tag{29}$$

Therefore, by (10), (28) and (29), we have

$$\begin{aligned} (c-1)\delta &\leq \left[(\frac{k_0 r_0}{2})^2 + k_0 r_0 + 1\right]\|w_1\| \\ &= \left[(\frac{k_0 r_0}{2})^2 + k_0 r_0 + 1\right]\|\alpha^2 (A + \alpha I)^{-2} A (u_0 - \hat{u})\| \\ &= \left[(\frac{k_0 r_0}{2})^2 + k_0 r_0 + 1\right]\|\alpha^2 (A + \alpha I)^{-2} A^{1+\nu} z\| \\ &\leq \left[(\frac{k_0 r_0}{2})^2 + k_0 r_0 + 1\right]\|\alpha^2 (A + \alpha I)^{-1} A^{\nu} z\| \\ &\leq \left[(\frac{k_0 r_0}{2})^2 + k_0 r_0 + 1\right]\alpha^{1+\nu}\|z\|, \end{aligned}$$

or

$$\alpha^{1+\nu} \geq \frac{c-1}{\left[(\frac{k_0 r_0}{2})^2 + k_0 r_0 + 1\right]\|z\|}\delta. \tag{30}$$

Thus,

$$\frac{\delta}{\alpha} = \delta^{\frac{\nu}{\nu+1}}\left(\frac{\delta}{\alpha^{\nu+1}}\right)^{\frac{1}{\nu+1}} = O(\delta^{\frac{\nu}{\nu+1}}).$$

□

Combining Theorems 5 and 6, we obtain:

Theorem 7. *Let Assumption 1 and (10) be satisfied and let $\alpha = \alpha(\delta, u_0)$ be the solution of (20). Then,*

$$\|u_\alpha^\delta - \hat{u}\| = O(\delta^{\frac{\nu}{\nu+1}}).$$

In [16], the following estimates was given (see [16], Theorem 2.3)

$$\|u_\alpha^\delta - u_{n,\alpha}^\delta\| \leq k q_\alpha^n, \tag{31}$$

where $q_\alpha = 1 - \beta\alpha$ and $k \geq r_0 + 1$ with $\beta = \frac{1}{\beta_0 + \alpha}$, $\beta_0 \geq \|\mathcal{H}'(u)\|$, $\forall u \in \overline{B(u, 2(r_0 + 1))}$. Suppose

$$n_{\alpha,\delta} := \min\{n \in \mathbb{N} : \alpha q_\alpha^n \leq \delta\}.$$

Theorem 8. *Let Assumption 1 and (10) be satisfied and let $\alpha = \alpha(\delta, u_0)$ be the solution of (20). Then,*

$$\|u_{n_{\alpha,\delta},\alpha}^\delta - \hat{u}\| = O(\delta^{\frac{\nu}{\nu+1}}).$$

Proof. Follows from the inequality

$$\|u^{\delta}_{n_{\alpha,\delta},\alpha} - \hat{u}\| \le \|u^{\delta}_{n_{\alpha,\delta},\alpha} - u^{\delta}_{\alpha}\| + \|u^{\delta}_{\alpha} - \hat{u}\|,$$

Equation (31), Theorems 6 and 7. □

3. Finite Dimensional Realization of (13)

Consider a family $\{P_h\}_{h>0}$ of orthogonal projections of $\mathcal{U}$ onto the range $R(P_h)$ of P_h. Let there exists $b_0 > 0$ such that

$$\|(I - P_h)\hat{u}\| := b_h \le b_0,$$

and let

$$r \ge 2(2r_0 + \max\{\|\hat{u}\|, 1\} + b_h) \quad \text{with} \quad r_0 := \|\hat{u} - u_0\|.$$

We assume that;

(i) $\overline{B(P_h u_0, r)} \subseteq D(\mathcal{H})$,

(ii) there exists $\beta_0 > 0$ such that

$$\|P_h \mathcal{H}'(u) P_h\| \le \beta_0 \ \forall\ u \in \overline{B(P_h u_0, r)}. \tag{32}$$

(iii) there exists $\varepsilon_0 > 0$ such that

$$\|\mathcal{H}'(u)(I - P_h)\| := \varepsilon_h(u) \le \varepsilon_h \le \varepsilon_0 \ \forall\ u \in \overline{B(P_h u_0, r)}. \tag{33}$$

Remark 5. *(a) Suppose $\mathcal{H}'(u)$ is self-adjoint for $u \in \overline{B(P_h u_0, r)}$. Then, $\|\mathcal{H}'(u)(I - P_h)\| = \|(I - P_h)\mathcal{H}'(u)\|$, and by Assumption 1, we have $\mathcal{H}'(u)v = \mathcal{H}'(P_h u_0)(v + \varphi(u, P_h u_0, v))$. Hence,*

$$\begin{aligned} \|\mathcal{H}'(u)(I - P_h)v\| &= \|(I - P_h)\mathcal{H}'(P_h u_0)(v + \varphi(u, P_h u_0, v))\| \\ &\le \|(I - P_h)\mathcal{H}'(P_h u_0)\|[\|v\| + k_0\|u - P_h u_0\|\|v\|] \\ &\le (1 + k_0 r)\|(I - P_h)\mathcal{H}'(P_h u_0)\|\|v\|, \end{aligned}$$

so, $\|\mathcal{H}'(u)(I - P_h)\| \le (1 + k_0 r)\|(I - P_h)\mathcal{H}'(P_h u_0)\|$.
Therefore, in this case, we can take, $\varepsilon_h = (1 + k_0 r)\|(I - P_h)\mathcal{H}'(P_h u_0)\|$.

(b) Suppose, $\mathcal{H}'(u)$ is not self-adjoint for $u \in \overline{B(P_h u_0, r)}$. In this case, under the additional assumption (see [17])

$$\mathcal{H}'(u) = R_u \mathcal{H}'(P_h u_0),\ u \in \overline{B(P_h u_0, r)}$$

with $\|I - R_u\| \le C_R\|u - P_h u_0\|$, we have

$$\begin{aligned} \|\mathcal{H}'(u)(I - P_h)\| &= \|R_u \mathcal{H}'(P_h u_0)(I - P_h)\| \\ &\le \|R_u\|\|\mathcal{H}'(P_h u_0)(I - P_h)\| \\ &\le (1 + C_R r)\|\mathcal{H}'(P_h u_0)(I - P_h)\|. \end{aligned}$$

Therefore, in this case, we can take, $\varepsilon_h = (1 + C_R r)\|\mathcal{H}'(P_h u_0)(I - P_h)\|$.

From now on, we assume $\delta \in (0, d]$ and $\alpha \in [\delta + \varepsilon_h, a)$ with $a > d + \varepsilon_0$.
First we shall prove that

$$(P_h \mathcal{H} P_h)(u) + \alpha P_h(u - u_0) = P_h y^{\delta} \tag{34}$$

has a unique solution $u_\alpha^{h,\delta} \in R(P_h)$, under the assumption

$$R(P_h) \subseteq D(\mathcal{H}). \tag{35}$$

Proposition 1. *Suppose (35) holds. Then (34) has a unique solution $u_\alpha^{h,\delta}$ in $B(P_h u_0, r)$ for all $u_0 \in \mathcal{U}$ and $y^\delta \in \mathcal{U}$.*

Proof. Since $\mathcal{H}$ is monotone, we have

$$\langle (P_h \mathcal{H} P_h)(u) - (P_h \mathcal{H} P_h)(v), u - v \rangle = \langle \mathcal{H}(P_h(u)) - \mathcal{H}(P_h(v)), P_h(u) - P_h(v) \rangle \geq 0,$$

so $P_h \mathcal{H} P_h$ is monotone and $D(P_h \mathcal{H} P_h) = \mathcal{U}$. Hence by Minty–Browder Theorem(see [18,19]), Equation (34) has a unique solution $u_\alpha^{h,\delta}$ for all $u_0 \in \mathcal{U}$ and $y^\delta \in \mathcal{U}$.

Next, we shall prove that $u_\alpha^{h,\delta} \in B(P_h u_0, r)$. Note that by (34), we have

$$P_h \mathcal{H}(P_h u_\alpha^{h,\delta}) + \alpha P_h (u_\alpha^{h,\delta} - \hat{u}) - P_h \mathcal{H}(\hat{u}) = P_h (y^\delta - y) + \alpha P_h (u_0 - \hat{u}). \tag{36}$$

Let $M = \int_0^1 \mathcal{H}'(\hat{u} + t(P_h u_\alpha^{h,\delta} - \hat{u}))dt$. Then by (36), we have

$$P_h M (P_h u_\alpha^{h,\delta} - \hat{u}) + \alpha P_h (u_\alpha^{h,\delta} - \hat{u}) = P_h (y^\delta - y) + \alpha P_h (u_0 - \hat{u}).$$

or

$$(P_h M P_h + \alpha I)(u_\alpha^{h,\delta} - P_h \hat{u}) = P_h (y^\delta - y) + \alpha P_h (u_0 - \hat{u}) + P_h M (I - P_h) \hat{u}.$$

So, we have

$$\begin{aligned} \|u_\alpha^{h,\delta} - P_h \hat{u}\| &= \|(P_h M P_h + \alpha I)^{-1} \\ &\quad \times [\alpha P_h (u_0 - \hat{u}) + P_h (y^\delta - y) + (P_h M (I - P_h))(\hat{u})]\| \\ &\leq \|P_h (u_0 - \hat{u})\| + \frac{\|P_h (y^\delta - y)\|}{\alpha} + \frac{\|P_h M (I - P_h)\| \|\hat{u}\|}{\alpha} \\ &\leq r_0 + \frac{\delta}{\alpha} + \frac{\varepsilon_h \|\hat{u}\|}{\alpha} \end{aligned}$$

and hence

$$\begin{aligned} \|u_\alpha^{h,\delta} - P_h u_0\| &\leq \|u_\alpha^{h,\delta} - P_h \hat{u}\| + \|P_h (\hat{u} - u_0)\| \\ &\leq 2r_0 + \max\{\|\hat{u}\|, 1\} \frac{\delta + \varepsilon_h}{\alpha} \\ &\leq 2r_0 + \max\{\|\hat{u}\|, 1\} < r, \end{aligned} \tag{37}$$

i.e., $u_\alpha^{h,\delta} \in B(P_h u_0, r)$. □

The method: The rest of this section, $\mathcal{H}'(u)$, $u \in \overline{B(P_h u_0, r)}$ is assumed to be positive self-adjoint operator. We consider the sequence $\{u_{n,\alpha}^{h,\delta}\}$ defined iteratively by

$$u_{n+1,\alpha}^{h,\delta} = u_{n,\alpha}^{h,\delta} - \beta P_h [F P_h (u_{n,\alpha}^{h,\delta}) + \alpha (u_{n,\alpha}^{h,\delta} - u_0) - y^\delta] \tag{38}$$

where

$$u_{0,\alpha}^{h,\delta} = P_h u_0 \quad \text{and} \quad \beta := \frac{1}{\beta_0 + a}.$$

Note that if $\lim_{n \longrightarrow \infty} \{u_{n,\alpha}^{h,\delta}\}$ exists, then the limit is the solution $u_\alpha^{h,\delta}$ of (34).

Theorem 9. *Let $\delta \in (0, d]$, $\alpha \in [\delta + \varepsilon_h, a)$, $u_\alpha^{h,\delta}$ and u_α^δ are solutions of (3) and (34), respectively. Then*

$$\|u_\alpha^{h,\delta} - u_\alpha^\delta\| \leq \|\hat{u}\| \frac{\varepsilon_h}{\alpha} + b_h + 2\|u_\alpha^\delta - \hat{u}\|.$$

Proof. Note that by (3), we have

$$P_h\mathcal{H}(u_\alpha^\delta) + \alpha P_h(u_\alpha^\delta - u_0) = P_h y^\delta. \tag{39}$$

Therefore, by (34) and (39), we have

$$P_h(\mathcal{H}(u_\alpha^{h,\delta}) - \mathcal{H}(u_\alpha^\delta)) + \alpha P_h(u_\alpha^{h,\delta} - u_\alpha^\delta) = 0. \tag{40}$$

Let $T_h := \int_0^1 \mathcal{H}'(u_\alpha^\delta + t(u_\alpha^{h,\delta} - u_\alpha^\delta))dt$. Then by (40), we have

$$P_h T_h(u_\alpha^{h,\delta} - u_\alpha^\delta) + \alpha P_h(u_\alpha^{h,\delta} - u_\alpha^\delta) = 0$$

or

$$P_h T_h P_h(u_\alpha^{h,\delta} - u_\alpha^\delta) + \alpha P_h(u_\alpha^{h,\delta} - u_\alpha^\delta) = P_h T_h(I - P_h)u_\alpha^\delta. \tag{41}$$

Notice that

$$\begin{aligned}
\|u_\alpha^\delta + t(u_\alpha^{h,\delta} - u_\alpha^\delta) - P_h u_0\| &= \|(1-t)(u_\alpha^\delta - \hat{u} + \hat{u} - P_h u_0) + t(u_\alpha^{h,\delta} - P_h u_0)\| \\
&= \|(1-t)[(u_\alpha^\delta - \hat{u}) + (I - P_h)\hat{u} + P_h(\hat{u} - u_0)] \\
&\quad + t(u_\alpha^{h,\delta} - P_h u_{0)}\| \\
&\leq (1-t)[\|u_\alpha^\delta - \hat{u}\| + \|(I - P_h)\hat{u}\| + \|P_h(\hat{u} - u_0)\|] \\
&\quad + \|t(u_\alpha^{h,\delta} - P_h u_0)\| \\
&\leq (1-t)[\|u_\alpha^\delta - \hat{u}\| + b_h + r_0] + t\|u_\alpha^{h,\delta} - P_h u_0\| \\
&\leq (1-t)[(\frac{\delta}{\alpha} + 2r_0) + b_h] + t(2r_0 + \max\{1, \|\hat{u}\|\}) \\
&\quad \text{by (7) and (37)} \\
&\leq r,
\end{aligned}$$

that is $u_\alpha^\delta + t(u_\alpha^{h,\delta} - u_\alpha^\delta) \in B(P_h u_0, r)$. So, $P_h T_h P_h$ is self-adjoint and hence by (41),

$$\begin{aligned}
\|u_\alpha^{h,\delta} - P_h u_\alpha^\delta\| &= \|(P_h T_h P_h + \alpha I)^{-1} P_h T_h (I - P_h) u_\alpha^\delta\| \\
&\leq \frac{\|P_h T_h (I - P_h) u_\alpha^\delta\|}{\alpha} \\
&\leq \frac{\varepsilon_h}{\alpha}\|u_\alpha^\delta\| \\
&\leq \frac{\varepsilon_h}{\alpha}(\|\hat{u}\| + \|\hat{u} - u_\alpha^\delta\|)
\end{aligned} \tag{42}$$

and

$$\|(I - P_h)u_\alpha^\delta\| \leq \|(I - P_h)\hat{u}\| + \|u_\alpha^\delta - \hat{u}\|. \tag{43}$$

Since $\frac{\varepsilon_h}{\alpha} \leq 1$, by (42) and (43), we have

$$\begin{aligned}
\|u_\alpha^{h,\delta} - u_\alpha^\delta\| &\leq \|u_\alpha^{h,\delta} - P_h u_\alpha^\delta\| + \|(I - P_h)u_\alpha^\delta\| \\
&\leq \|\hat{u}\|\frac{\varepsilon_h}{\alpha} + b_h + 2\|u_\alpha^\delta - \hat{u}\|.
\end{aligned}$$

□

Remark 6. *If $\alpha b_h \leq \delta + \varepsilon_h$ and $\alpha = (\delta + \varepsilon_h)^{\frac{1}{\nu+1}}$, then by Theorems 2 and 9, we have*

$$\|u_\alpha^{h,\delta} - \hat{u}\| = O((\delta + \varepsilon_h)^{\frac{\nu}{\nu+1}}).$$

Theorem 10. *Let $\delta \in (0, d]$ and $\alpha \in [\delta + \varepsilon_h, a)$. Then, $\{u_{n,\alpha}^{h,\delta}\} \in \overline{B(P_h u_0, r)}$ and $\lim_{n\longrightarrow\infty} u_{n,\alpha}^{h,\delta} = u_\alpha^{h,\delta}$. Further*

$$\|u_{n,\alpha}^{h,\delta} - u_\alpha^{h,\delta}\| \leq \kappa q_{\alpha,h}^n,$$

where $q_{\alpha,h} := 1 - \beta\alpha$, $\kappa \geq 2r_0 + \max\{1, \|\hat{u}\|\}$ *and* $\beta := 1/(\beta_0 + a)$.

Proof. We shall show the following using induction;

(1a) $u_{n,\alpha}^{h,\delta} \in \overline{B(P_h u_0, r)}$,

(1b) the operator

$$A_n^h := \int_0^1 \mathcal{H}'(u_\alpha^{h,\delta} + t(u_{n,\alpha}^{h,\delta} - u_\alpha^{h,\delta}))dt$$

is positive self-adjoint, well defined and

(1c) $\|u_{n+1,\alpha}^{h,\delta} - u_\alpha^{h,\delta}\| \leq (1 - \beta\alpha)\|u_{n,\alpha}^{h,\delta} - u_\alpha^{h,\delta}\| \ \forall n = 0, 1, 2, \ldots$

Clearly, $u_{0,\alpha}^{h,\delta} = P_h u_0 \in \overline{B(P_h u_0, r)}$. Furthermore, we have by Proposition 1, $u_\alpha^{h,\delta} \in \overline{B(P_h u_0, r)}$, so by (32), A_0^h is a well defined and positive self-adjoint operator with $\|P_h A_0^h P_h\| \leq \beta_0$. So (1a) and (1b) hold for $n = 0$.

Note that

$$u_{1,\alpha}^{h,\delta} - u_\alpha^{h,\delta} = u_{0,\alpha}^{h,\delta} - u_\alpha^{h,\delta} - \beta P_h[\mathcal{H}(u_{0,\alpha}^{h,\delta}) - \mathcal{H}(u_\alpha^{h,\delta}) + \alpha(u_{0,\alpha}^{h,\delta} - u_\alpha^{h,\delta})].$$

Since,

$$\mathcal{H}(u_{0,\alpha}^{h,\delta}) - \mathcal{H}(u_\alpha^{h,\delta}) = \int_0^1 \mathcal{H}'(u_\alpha^{h,\delta} + t(u_{0,\alpha}^{h,\delta} - u_\alpha^{h,\delta}))(u_{0,\alpha}^{h,\delta} - u_\alpha^{h,\delta})dt = A_0^h(u_{0,\alpha}^{h,\delta} - u_\alpha^{h,\delta})$$

we have

$$u_{1,\alpha}^{h,\delta} - u_\alpha^{h,\delta} = [I - \beta(P_h A_0^h P_h + \alpha I)](u_{0,\alpha}^{h,\delta} - u_\alpha^{h,\delta})]. \tag{44}$$

Since $P_h A_0^h P_h$ is a positive self-adjoint operator (cf. [20]),

$$\begin{aligned} \|I - \beta(P_h A_0^h P_h + \alpha I)\| &= \sup_{\|u\|=1} |\langle[(1 - \beta\alpha)I - \beta P_h A_0^h P_h]u, u\rangle| \\ &= \sup_{\|u\|=1} |(1 - \beta\alpha) - \beta\langle P_h A_0^h P_h u, u\rangle| \end{aligned}$$

and since $\|P_h A_0^h P_h\| \leq \beta_0$ and $\beta = 1/(\beta_0 + a)$, we have

$$0 \leq \beta\langle P_h A_0^h P_h u, u\rangle \leq \beta\|P_h A_0^h P_h\| \leq \beta\beta_0 < 1 - \beta\alpha \ \forall \alpha \in (0, a).$$

Therefore,

$$\|I - \beta(P_h A_0^h P_h + \alpha I)\| \leq 1 - \beta\alpha.$$

Thus, by (44), we have

$$\|u_{1,\alpha}^{h,\delta} - u_\alpha^{h,\delta}\| \leq (1 - \beta\alpha)\|u_{0,\alpha}^{h,\delta} - u_\alpha^{h,\delta}\| \leq q_{\alpha,h}\|P_h u_0 - u_\alpha^{h,\delta}\|.$$

Therefore, we have

$$\|u_{1,\alpha}^{h,\delta} - u_\alpha^{h,\delta}\| \leq q_{\alpha,h}(2r_0 + \max\{1, \|\hat{u}\|\}), \text{ by } (37) = \kappa q_{\alpha,h}.$$

and

$$\begin{aligned} \|u_{1,\alpha}^{h,\delta} - P_h u_0\| &\leq \|u_{1,\alpha}^{h,\delta} - u_\alpha^{h,\delta}\| + \|u_\alpha^{h,\delta} - P_h u_0\| \\ &\leq 2\|P_h u_0 - u_\alpha^{h,\delta}\| \leq 2(2r_0 + \max\{1, \|\hat{u}\|\}) \leq r. \end{aligned}$$

Thus, $u_{1,\alpha}^{h,\delta} \in \overline{B(P_h u_0, r)}$. So, for $n = 0$, (1a)–(1c) hold. The induction for (1a)–(1c) is completed, if we simply replace $u_{1,\alpha}^{h,\delta}, u_{0,\alpha}^{h,\delta}$ in the preceding arguments with $u_{n+1,\alpha}^{h,\delta}, u_{n,\alpha}^{h,\delta}$, respectively. The result now follows from (1c). □

Theorem 11. *Let $\delta \in (0, d]$, $\alpha \in (\delta + \varepsilon_h, a]$ with $d + \varepsilon_0 < a$. Let u_α^δ and u_α be solutions of (3) and (4), respectively. For $\delta \in (0, d]$ and $\alpha \in [\delta + \varepsilon_h, a)$, let $\{u_{n,\alpha}^{h,\delta}\}$ be as in (38). Let*

$$n_{\alpha,\delta} := \min\{m \in \mathbb{N} : \alpha q_{\alpha,h}^m \leq \delta + \varepsilon_h\} \tag{45}$$

and

$$\alpha b_h \leq \delta + \varepsilon_h.$$

Then,

$$\|u_{n_{\alpha,\delta},\alpha}^{h,\delta} - \hat{u}\| = (\kappa + 1 + \max\{\|\hat{u}\|, 3\})\Big(\|\hat{u} - u_\alpha\| + \frac{\delta + \varepsilon_h}{\alpha}\Big). \tag{46}$$

Proof. By Theorems 9 and 10, we have

$$\begin{aligned}
\|u_{n_{\alpha,\delta},\alpha}^{h,\delta} - \hat{u}\| &\leq \|u_{n_{\alpha,\delta},\alpha}^{h,\delta} - u_\alpha^{h,\delta}\| + \|u_\alpha^{h,\delta} - u_\alpha^\delta\| + \|u_\alpha^\delta - \hat{u}\| \\
&\leq \kappa q_{\alpha,h}^n + \frac{\varepsilon_h}{\alpha}\|\hat{u}\| + b_h + 3\|u_\alpha^\delta - \hat{u}\| \qquad (47) \\
&\leq \kappa q_{\alpha,h}^n + \frac{\varepsilon_h}{\alpha}\|\hat{u}\| + b_h + 3(\frac{\delta}{\alpha} + \|u_\alpha - \hat{u}\|) \\
&\leq (\kappa + 1 + \max\{3, \|\hat{u}\|\})\Big(\|\hat{u} - u_\alpha\| + \frac{\delta + \varepsilon_h}{\alpha}\Big). \qquad (48)
\end{aligned}$$

Here, we used the fact that $q_{\alpha,h}^n \leq \frac{\delta+\varepsilon_h}{\alpha}$ for $n = n_{\alpha,\delta}$ and $b_h \leq \frac{\delta+\varepsilon_h}{\alpha}$. Thus, we obtain the required estimate in the theorem. □

Finite dimensional realization of (20) is considered next.

4. Finite Dimensional Realization of the a New Parameter Choice Strategy (20)

For $u \in \mathcal{U}$, define

$$\phi^h(\alpha, u) := \|\alpha^2 (P_h A_0 P_h + \alpha I)^{-2} P_h(\mathcal{H}(u_0) - u)\|. \tag{49}$$

The proof of the next theorem is similar to that of Theorem 3, so the proof is omitted.

Theorem 12. *For each $u \in \mathcal{U}$, the function $\alpha \to \phi^h(\alpha, u)$ for $\alpha > 0$, defined in (49), is continuous, monotonically increasing and*

$$\lim_{\alpha \to 0} \phi^h(\alpha, u) = 0, \quad \lim_{\alpha \to \infty} \phi^h(\alpha, u) = \|P_h(\mathcal{H}(u_0) - u)\|.$$

In addition to (2), we assume that

$$c_1 \delta + d_1 \varepsilon_h \leq \|P_h(\mathcal{H}(u_0) - y^\delta)\|, \tag{50}$$

for some $c_1 > 1$ and $d_1 > \frac{k_0 r_0^2}{2} + r_0$. The proof of the following theorem follows from the intermediate value theorem.

Theorem 13. *If y^δ satisfies (2) and (50). Then,*

$$\phi^h(\alpha, y^\delta) = c_1 \delta + d_1 \varepsilon_h \tag{51}$$

has a unique solution $\alpha = \alpha(\delta, h, u_0)$.

Next, we shall show that if $\alpha = \alpha(\delta, h, u_0)$ satisfies (51), then $\|\hat{u} - u_\alpha\| = O((\delta + \varepsilon_h)^{\frac{\nu}{\nu+1}})$. Our proof is based on the moment inequality (21).

Theorem 14. *Let Assumption 1 and (10) be satisfied and let $\alpha = \alpha(\delta, h, u_0)$ satisfies (51). Then,*

$$\|\hat{u} - u_\alpha\| \leq O((\delta + \varepsilon_h)^{\frac{\nu}{\nu+1}}).$$

Proof. By (24), the result follows once we prove $\|w\| = O(\delta + \varepsilon_h)$. This can be seen as follows,

$$\begin{aligned}
\|w\| &= \|\alpha^2(A_0+\alpha I)^{-2}(\mathcal{H}(u_0)-y^\delta)\| \\
&\leq \|\alpha^2[(A_0+\alpha I)^{-2}-(P_hA_0P_h+\alpha P_h)^{-2}](\mathcal{H}(u_0)-y^\delta)\| \\
&\quad +\|\alpha^2(P_hA_0P_h+\alpha I)^{-2}P_h(\mathcal{H}(u_0)-y^\delta)\| \\
&\leq \|\alpha^2(A_0+\alpha I)^{-2}[(P_hA_0P_h+\alpha P_h)^2-(A_0+\alpha I)^2] \\
&\quad \times(P_hA_0P_h+\alpha P_h)^{-2}(\mathcal{H}(u_0)-y^\delta)\| + c_1\delta + d_1\varepsilon_h \\
&= \|\alpha^2(A_0+\alpha I)^{-2}[(P_hA_0P_h)^2+2\alpha P_hA_0P_h-(A_0^2+2\alpha A_0)+\alpha^2(P_h-I)] \\
&\quad \times(P_hA_0P_h+\alpha P_h)^{-2}(\mathcal{H}(u_0)-y^\delta)\| + c_1\delta + d_1\varepsilon_h \\
&= \|\alpha^2(A_0+\alpha I)^{-2}[(P_hA_0P_h+A_0)(P_hA_0P_h-A_0)+2\alpha(P_hA_0P_h-A_0)] \\
&\quad \times(P_hA_0P_h+\alpha P_h)^{-2}(\mathcal{H}(u_0)-y^\delta)\| + c_1\delta + d_1\varepsilon_h \\
&= \|\alpha^2(A_0+\alpha I)^{-2}[(P_hA_0P_h-A_0)+2(A_0+\alpha I)] \\
&\quad \times(P_hA_0P_h-A_0)(P_hA_0P_h+\alpha P_h)^{-2}(\mathcal{H}(u_0)-y^\delta)\| + c_1\delta + d_1\varepsilon_h \\
&= \|\Big[[(A_0+\alpha I)^{-1}(P_hA_0P_h-A_0)]^2+2(A_0+\alpha I)^{-1}(P_hA_0P_h-A_0)\Big] \\
&\quad \times\alpha^2(P_hA_0P_h+\alpha P_h)^{-2}(\mathcal{H}(u_0)-y^\delta)\| + c_1\delta + d_1\varepsilon_h \\
&= \|\Big[[(A_0+\alpha I)^{-1}(P_hA_0P_h-A_0P_h+A_0P_h-A_0)]^2 \\
&\quad +2(A_0+\alpha I)^{-1}(P_hA_0P_h-A_0P_h+A_0P_h-A_0)\Big] \\
&\quad \times\alpha^2(P_hA_0P_h+\alpha P_h)^{-2}(\mathcal{H}(u_0)-y^\delta)\| + c_1\delta + d_1\varepsilon_h \\
&= \|\Big[[(A_0+\alpha I)^{-1}(P_hA_0P_h-A_0P_h)]^2+2(A_0+\alpha I)^{-1}(P_hA_0P_h-A_0P_h)\Big] \\
&\quad \times\alpha^2(P_hA_0P_h+\alpha P_h)^{-2}(\mathcal{H}(u_0)-y^\delta)\| + c_1\delta + d_1\varepsilon_h \\
&\leq \|\Big[[(A_0+\alpha I)^{-1}(P_h-I)A_0P_h]^2+2(A_0+\alpha I)^{-1}((P_h-I)A_0P_h)\Big] \\
&\quad \times\alpha^2(P_hA_0P_h+\alpha P_h)^{-2}(\mathcal{H}(u_0)-y^\delta)\| + c_1\delta + d_1\varepsilon_h \\
&\leq \Big(\Big[\frac{\varepsilon_h}{\alpha}+2\Big]\frac{\varepsilon_h}{\alpha}+1\Big)(c_1\delta+d_1\varepsilon_h),
\end{aligned} \tag{52}$$

where, we used $(P_h - I)P_h = 0$. Next, we shall show that $\frac{\varepsilon_h}{\alpha}$ is bounded. Note that,

$$\begin{aligned}
c_1\delta + d_1\varepsilon_h &= \|\alpha^2(P_hA_0P_h+\alpha I)^{-2}\,P_h(\mathcal{H}(u_0)-y^\delta)\| \\
&\leq \|\alpha^2(P_hA_0P_h+\alpha I)^{-2}\,P_h(y-y^\delta)\| \\
&\quad +\|\alpha^2(P_hA_0P_h+\alpha I)^{-2}\,P_h(\mathcal{H}(u_0)-y)\| \\
&\leq \delta+\|\alpha^2(P_hA_0P_h+\alpha I)^{-2}\,P_hA(u_0-\hat{u})\| \\
&\leq \delta+\|\alpha^2(P_hA_0P_h+\alpha I)^{-2}\,P_h(A-A_0)(u_0-\hat{u})\| \\
&\quad +\|\alpha^2(P_hA_0P_h+\alpha I)^{-2}\,P_hA_0(u_0-\hat{u})\|
\end{aligned}$$

$$
\begin{aligned}
&= \delta + \|\alpha^2(P_hA_0P_h+\alpha I)^{-2}\,P_hA_0\int_0^1\Phi(\hat{u}+t(u_0-\hat{u}),u_0,u_0-\hat{u})\| \\
&\quad +\|\alpha^2(P_hA_0P_h+\alpha I)^{-2}\,P_hA_0[P_h+I-P_h])(u_0-\hat{u})\| \\
&\leq \delta + \|\alpha^2(P_hA_0P_h+\alpha)^{-2}\,P_hA_0[P_h+I-P_h]\int_0^1\Phi(\hat{u}+t(u_0-\hat{u}),u_0,u_0-\hat{u})\| \\
&\quad +\|\alpha^2(P_hA_0P_h+\alpha I)^{-2}\,P_hA_0[P_h+I-P_h])(u_0-\hat{u})\| \\
&\leq \delta + (\alpha+\|A_0(I-P_h)\|)\|\int_0^1\Phi(\hat{u}+t(u_0-\hat{u}),u_0,u_0-\hat{u})dt\| \\
&\quad +(\alpha+\|A_0(I-P_h)\|)\|u_0-\hat{u}\| \\
&\leq \delta + (\alpha+\varepsilon_h)\frac{k_0}{2}\|u_0-\hat{u}\|^2 \\
&\quad +(\alpha+\varepsilon_h)\|u_0-\hat{u}\| \\
&\leq \delta + (\frac{k_0r_0^2}{2}+r_0)\varepsilon_h + (\frac{k_0r_0^2}{2}+r_0)\alpha
\end{aligned}
$$

so, we have

$$(d_1-(\frac{k_0r_0^2}{2}+r_0))\varepsilon_h < (c_1-1)\delta + (d_1-(\frac{k_0r_0^2}{2}+r_0))\varepsilon_h \leq (\frac{k_0r_0^2}{2}+r_0)\alpha$$

and hence

$$\frac{\varepsilon_h}{\alpha} \leq \frac{\frac{k_0r_0^2}{2}+r_0}{d_1-(\frac{k_0r_0^2}{2}+r_0)} := C_{r_0}. \tag{53}$$

Now, the result follows from (52) and (53). □

Theorem 15. *Suppose Assumption 1 and (10) hold and if $\alpha=\alpha(\delta,h,u_0)$ is chosen as a solution of (51). Then,*

$$\frac{\delta+\varepsilon_h}{\alpha} = O\Big((\delta+\varepsilon_h)^{\frac{\nu}{\nu+1}}\Big).$$

Proof. By (51), we have

$$
\begin{aligned}
c_1\delta + d_1\varepsilon_h &= \|\alpha^2\,(P_hA_0P_h+\alpha\,I)^{-2}P_h(\mathcal{H}(u_0)-y^\delta)\| \\
&\leq \|\alpha^2\,(P_hA_0P_h+\alpha\,I)^{-2}P_h(\mathcal{H}(u_0)-y)\| \\
&\quad +\|\alpha^2\,(P_hA_0P_h+\alpha\,I)^{-2}P_h(y-y^\delta))\| \\
&\leq \|\alpha^2\,(P_hA_0P_h+\alpha\,I)^{-2}P_h(\mathcal{H}(u_0)-y)\| + \delta,
\end{aligned}
$$

so

$$
\begin{aligned}
(c_1-1)\delta + d_1\varepsilon_h &\leq \|\alpha^2\,(P_hA_0P_h+\alpha\,I)^{-2}P_h(\mathcal{H}(u_0)-y)\| \\
&\leq \|\alpha^2\Big[(P_hA_0P_h+\alpha\,P_h)^{-2}-(A+\alpha\,I)^{-2}\Big](\mathcal{H}(u_0)-y)\| \\
&\quad +\|\alpha^2\,(A+\alpha\,I)^{-2}(\mathcal{H}(u_0)-y)\| \\
&= \|(P_hA_0P_h+\alpha\,P_h)^{-2}\Big[P_h(A+\alpha\,I)^2-(P_hA_0P_h+\alpha\,I)^2\Big] \\
&\quad \times\alpha^2\,(A+\alpha\,I)^{-2}\,(\mathcal{H}(u_0)-y)\| \\
&\quad +\|\alpha^2\,(A+\alpha\,I)^{-2}(\mathcal{H}(u_0)-y)\|.
\end{aligned} \tag{54}
$$

Let $w_1=\alpha^2\,(A+\alpha\,I)^{-2}(\mathcal{H}(u_0)-y)$. Then, similar to (24), we have

$$(c_1-1)\delta + d_1\varepsilon_h \leq (\|\Gamma_2\|^2+2\|\Gamma_2\|+1)\|w_1\|, \tag{55}$$

where $\Gamma_2 = (P_h A_0 P_h + \alpha I)^{-1}(P_h A - P_h A_0 P_h)$. Note that,

$$\begin{aligned}
\|\Gamma_2 x\| &= \|(P_h A_0 P_h + \alpha I)^{-1}[P_h(A - A_0) + P_h A_0 (I - P_h)]x\| \\
&\leq \|(P_h A_0 P_h + \alpha I)^{-1}[P_h(A - A_0)x\| + \|(P_h A_0 P_h + \alpha I)^{-1} P_h A_0 (I - P_h)]x\| \\
&= \|(P_h A_0 P_h + \alpha I)^{-1}[P_h A_0 \int_0^1 \Phi(\hat{u} + t(u_0 - \hat{u}), u_0, x) dt\| \\
&\quad + \|(P_h A_0 P_h + \alpha I)^{-1} P_h A_0 (I - P_h)]x\| \\
&= \|(P_h A_0 P_h + \alpha I)^{-1}[P_h A_0 [P_h + I - P_h)] \int_0^1 \Phi(\hat{u} + t(u_0 - \hat{u}), u_0, x) dt\| \\
&\quad + \|(P_h A_0 P_h + \alpha I)^{-1} P_h A_0 (I - P_h)]x\| \\
&\leq \left[(1 + \frac{\varepsilon_h}{\alpha})\frac{k_0}{2}\|u_0 - \hat{u}\| + \frac{\varepsilon_h}{\alpha}\right]\|x\| \\
&\leq \left[(1 + C_{r_0})\frac{k_0}{2} r_0 + C_{r_0}\right]\|x\|,
\end{aligned}$$

so

$$\|\Gamma_2\| \leq \left[(1 + C_{r_0})\frac{k_0}{2} r_0 + C_{r_0}\right] := C_{\Gamma_2}. \tag{56}$$

Therefore, by (10), (55) and (56), we have

$$\begin{aligned}
(c_1 - 1)\delta + d_1 \varepsilon_h &< \left[C_{\Gamma_2}^2 + 2C_{\Gamma_2} + 1\right]\|w_1\| \\
&= \left[C_{\Gamma_2}^2 + 2C_{\Gamma_2} + 1\right]\|\alpha^2 (A + \alpha I)^{-2} A (u_0 - \hat{u})\| \\
&= \left[C_{\Gamma_2}^2 + 2C_{\Gamma_2} + 1\right]\|\alpha^2 (A + \alpha I)^{-2} A^{1+\nu} z\| \\
&\leq \left[C_{\Gamma_2}^2 + 2C_{\Gamma_2} + 1\right]\|\alpha^2 (A + \alpha I)^{-1} A^{\nu} z\| \\
&\leq \left[C_{\Gamma_2}^2 + 2C_{\Gamma_2} + 1\right]\alpha^{1+\nu}\|z\|,
\end{aligned}$$

or

$$\alpha^{1+\nu} \geq \frac{\min\{c_1 - 1, d_1\}}{\left[C_{\Gamma_2}^2 + 2C_{\Gamma_2} + 1\right]\|z\|}(\delta + \varepsilon_h). \tag{57}$$

Thus

$$\frac{\delta + \varepsilon_h}{\alpha} = (\delta + \varepsilon_h)^{\frac{\nu}{\nu+1}}\left(\frac{\delta + \varepsilon_h}{\alpha^{\nu+1}}\right)^{\frac{1}{\nu+1}} = O((\delta + \varepsilon_h)^{\frac{\nu}{\nu+1}}).$$

□

By combining Theorems 11, 14 and 15, we have the following Theorem.

Theorem 16. *Suppose Assumption 1 and (10) hold and if $\alpha = \alpha(\delta, h, u_0)$ is chosen as a solution of (51). Then*

$$\|u_{n_{\alpha,\delta},\alpha}^{h,\delta} - \hat{u}\| = O\Big((\delta + \varepsilon_h)^{\frac{\nu}{\nu+1}}\Big).$$

Remark 7. *Note that in the proposed method a system of equation is solved to obtain the parameter α and used it for computing $u_{n_{\alpha,\delta},\alpha}^{h,\delta}$. Whereas in the classical discrepancy principle one has to compute α and $u_{n_{\alpha,\delta},\alpha}^{h,\delta}$ in each iteration step. This is an advantage of our proposed approach.*

5. Numerical Examples

The following steps are involved in the computation of $u_{n_{\alpha,\delta},\alpha}^{h,\delta}$.
Step I Compute $\alpha = \alpha(\delta, h, u_0) =: \alpha(\delta, \epsilon_h)$ satisfying (51)
Step II Choose n such that $q_{\alpha,h}^n = (1 - \beta\alpha(\delta, \varepsilon_h))^n \leq \frac{\delta + \varepsilon_h}{\alpha(\delta, \varepsilon_h)}$.

Step III Compute $u^{h,\delta}_{n_{\alpha,\delta},\alpha}$ using (38).

To compute $u^{h,\delta}_{n,\alpha}$, consider a sequence (V_m), of finite dimensional subspaces, where $V_m = span\{v_1, v_2, \ldots, v_{m+1}\}$ with $v_i, i = 1, 2, \ldots, m+1$ as the linear splines (in a uniform grid of $m+1$ points in $[0,1]$), so that dimension $V_m = m+1$. Since $u^{h,\delta}_{n,\alpha} \in V_m$, $u^{h,\delta}_{n,\alpha} = \sum_{i=1}^{m+1} \lambda_i^{(n)} v_i$, $\lambda_i, i = 1, 2, \ldots m+1$ are some scalars. Then, from (38), we have

$$\sum_{i=1}^{m+1} \lambda_i^{(n+1)} v_i = \sum_{i=1}^{m+1} \lambda_i^{(n)} v_i - \beta P_m[\mathcal{H}(\sum_{i=1}^{m+1} \lambda_i^{(n)} v_i) + \alpha \sum_{i=1}^{m+1} (\lambda_i^{(n)} - \lambda_i^{(0)}) v_i - y^\delta], \tag{58}$$

where $P_m := P_{h_m}$ is the projection on to V_m with $h_m = \frac{1}{m}$. In this case one can prove as in [21] that $\|\mathcal{H}'(u)(I - P_m)\| = O(\frac{1}{m^2})$. So we have taken $\varepsilon_{h_m} = \frac{1}{m^2}$ in our computation. Since $P_m \mathcal{H}(\sum_{i=1}^{m+1} \lambda_i^{(n)} v_i) \in V_m$, $P_m y^\delta \in V_m$, we approximate

$$P_m \mathcal{H}(\sum_{i=1}^{m+1} \lambda_i^{(n)} v_i) = \sum_{i=1}^{m+1} \mathcal{H}(\sum_{i=1}^{m+1} \lambda_i^{(n)} v_i)(t_i) v_i, \; P_m y^\delta = \sum_{i=1}^{m+1} y^\delta(t_i) v_i,$$

where $t_i, i = 1, 2, \ldots, m+1$ are grid points. So $\lambda^{(n+1)} = (\lambda_1^{(n+1)}, \lambda_2^{(n+1)}, \ldots, \lambda_{m+1}^{(n+1)})^T$ satisfies (58), if $\lambda^{(n+1)}$ satisfies the equation

$$Q[\lambda^{(n+1)} - \lambda^{(n)}] = Q\beta[Y^\delta - (\mathcal{H}_h + \alpha(\lambda^{(n)} - \lambda^{(0)}))],$$

where

$$Q = (\langle v_i, v_j \rangle)_{i,j}, \; i, j = 1, 2, \ldots m+1,$$
$$Y^\delta = (y^\delta(t_1), y^\delta(t_2), \ldots y^\delta(t_{m+1}))^T,$$

and

$$\mathcal{H}_h = (\mathcal{H}(\sum_{i=1}^{m+1} \lambda_i^{(n)} v_i)(t_1), \mathcal{H}(\sum_{i=1}^{m+1} \lambda_i^{(n)} v_i)(t_2), \ldots, \mathcal{H}(\sum_{i=1}^{m+1} \lambda_i^{(n)} v_i)(t_{m+1}))^T.$$

To compute the α satisfying (51), we follow the following steps:

Let $z = (P_m A_0 P_m + \alpha I)^{-2} P_m(\mathcal{H}(u_0) - y^\delta)$, Then $z \in V_m$, so $z = \sum_{i=1}^{m+1} \xi_i v_i$ for some scalars $\xi_i, i = 1, 2, \ldots m+1$. Note that $(P_m A_0 P_m + \alpha I)^2 z = P_m(\mathcal{H}(u_0) - y^\delta)$ or $(P_m A_0 P_m + \alpha I) Z = P_m(\mathcal{H}(u_0) - y^\delta)$, where $Z = (P_m A_0 P_m + \alpha I) z$.

Since $Z \in V_m$, we have $Z = \sum_{i=1}^{m+1} \varsigma_i v_i$. Further $\varsigma = (\varsigma_1, \varsigma_2, \ldots, \varsigma_{m+1})^T$ and $\xi = (\xi_1, \xi_2, \ldots, \xi_{m+1})^T$ satisfies the equations

$$(M + \alpha Q)\varsigma = QB,$$

and

$$(M + \alpha Q)\xi = Q\varsigma,$$

respectively, where

$$M = (\langle A_0 v_i, v_j \rangle)_{i,j}, i, j = 1, 2, \ldots m+1$$

and

$$B = ((\mathcal{H}(u_0) - y^\delta)(t_1), (\mathcal{H}(u_0) - y^\delta)(t_2), \ldots, (\mathcal{H}(u_0) - y^\delta)(t_{m+1}))^T.$$

We compute α in (51), using Newton's method as follows. Let $f(\alpha) = \alpha^4 \|z\|^2 - (c_1\delta + d_1\varepsilon_h)^2$. Then

$$f'(\alpha) = 4\alpha^3 \|z\|^2 + 4\alpha^4 \langle z, ZZ \rangle,$$

where $ZZ = (P_m A_0 P_m + \alpha I)^{-3} P_m(\mathcal{H}(u_0) - y^\delta)$. Let $ZZ = \sum_{i=1}^{m+1} \Theta_i v_i$. The $\Theta = (\Theta_1, \Theta_2, \ldots, \Theta_{m+1})^T$ satisfies the equation

$$(M + \alpha Q)\Theta = Q\xi.$$

So,

$$f(\alpha) = \alpha^4 \xi^T Q \xi - (c_1\delta + d_1\varepsilon_h)^2$$

and

$$f'(\alpha) = 4\alpha^3 \xi^T Q \xi + 4\alpha^4 \xi^T Q \Theta.$$

Then, using Newton's iteration we compute the $(k+1)^{th}$ iterate as; $\alpha_{k+1} = \alpha_k - \frac{f(\alpha)}{f'(\alpha)}$. In our computation, we stop the iterate when $\alpha_{k+1} - \alpha_k \le 10^{-5}$.

We consider a simple one dimensional example studied in [5,7,22,23] to illustrate our results in the previous sections. We also compare our computational results with that adaptive method considered in [16,24]. Let us briefly explain the adaptive method considered in [16]. Choose $\alpha_0 = \delta + \epsilon_h$, $\alpha_j = \varrho^j \alpha_0$. For each j find n_j such that $n_j = \min\{i : q^i_{\alpha,h} \le \frac{1}{\varrho^j}\}$.

Then, find k such that

$$k := \max\{i : \|u^{h,\delta}_{n_i,\delta,\alpha_i} - u^{h,\delta}_{n_j,\delta,\alpha_j}\| \le 4\frac{1}{\varrho^j}, j = 0, 1, \ldots, i-1\}.$$

Choose, $\alpha = \alpha_k$ as the regularization parameter.

Example 1. *Let $c > 0$ be a constant. Consider the inverse problem of identifying the distributed growth law $u(t), t \in (0,1)$, in the initial value problem*

$$\frac{dy}{dt} = u(t)y(t), \ \ y(0) = c, \ \ t \in (0,1) \tag{59}$$

from the noisy data $y^\delta(t) \in L^2(0,1)$. One can reformulate the above problem as an (ill-posed) operator equation $\mathcal{H}(u) = y$ with

$$[\mathcal{H}(u)](t) = ce^{\int_0^t u(\theta)d\theta}, \ \ u \in L^2(0,1), \ \ t \in (0,1). \tag{60}$$

Then $\mathcal{H}'$ is given by

$$[\mathcal{H}'(u)h](t) = [\mathcal{H}(u)](t) \int_0^t h(\theta)d\theta. \tag{61}$$

It is proved in [7], that $\mathcal{H}'$ is positive type (sectorial) and spectrum of $\mathcal{H}'(u)$ is the singleton set $\{0\}$. Further it is proved in [5] that $\mathcal{H}'$ satisfies Assumption 1 and that $\hat{u} - u_0 \in R(\mathcal{H}'(\hat{u}))$ provided $u^* := \hat{u} - u_0 \in H^1(0,1)$ and $u^*(0) = 0$. Now since $\hat{u} - u_0 = \mathcal{H}'(\hat{u})w$, we have

$$\begin{aligned}
[\hat{u} - u_0](t) &= [\mathcal{H}(\hat{u})](t) \int_0^t w(\theta)d\theta \\
&= ce^{\int_0^t \hat{u}(\theta)d\theta} \int_0^t w(\theta)d\theta \\
&= \frac{\int_0^1 ce^{\int_0^t [\hat{u}+\tau(u_0-\hat{u})](\theta)d\theta} d\tau \int_0^t w(\theta)d\theta}{\int_0^1 e^{\int_0^t [\tau(u_0-\hat{u})](\theta)d\theta} d\tau} \\
&= [A\bar{w}](t),
\end{aligned}$$

where $\bar{w} = \frac{w}{\int_0^1 e^{\int_0^t [\tau(u_0-\hat{u})](\theta)d\theta} d\tau}$. This shows the source condition (10) is satisfied. For our computation we have taken $\hat{u}(t) = t, u_0(t) = 0$ and $y(t) = e^{\frac{t^2}{2}}$. In Table 1, we present the relative error $E_\alpha = \frac{\|u^{h,\delta}_{n_\alpha,\delta,\alpha} - \hat{u}\|}{\|u^{h,\delta}_{n_\alpha,\delta,\alpha}\|}$, and α values using a new method (51) and adaptive method considered in [16] for different values of δ and n. Furthermore, we provide computational time (CT) for both the methods mentioned above. The relative error obtained for our a new method (51) is lesser than that the adaptive method in [16] for various δ values. As the relative error decreases the accuracy of reconstruction increases.

The solutions obtained for different δ values ($\delta = 0.01, 0.001, 0.0001$) for $n = 500$ are shown in Figures 1–3, respectively, and for $n = 1000$ and $\delta = 0.01, 0.001, 0.0001$ are shown in Figures

4–6, respectively. The exact and noisy data are shown in subfigure (a) of these figures and the computed solution is shown in subfigure(b) (C.S-A priori denotes the figure corresponding to the method (51)). The computed solution for the new method is closer to the actual solution.

Table 1. Relative errors using discrepancy principle.

Method		$n = 500$			$n = 1000$		
		$\delta = 0.01$	$\delta = 0.001$	$\delta = 0.0001$	$\delta = 0.01$	$\delta = 0.001$	$\delta = 0.0001$
(51)	α	4.283954×10^{-3}	4.283969×10^{-3}	4.283972×10^{-3}	3.602506×10^{-3}	3.602505×10^{-3}	3.602536×10^{-3}
	E_α	1.225477×10^{-2}	1.225481×10^{-2}	1.225482×10^{-2}	1.036919×10^{-2}	1.036919×10^{-2}	1.036927×10^{-2}
	CT	3.764950×10^{-1}	3.286400×10^{-1}	3.355110×10^{-1}	1.879650	1.870468	1.802014
Adaptive method	α	1.040604×10^{-4}	1.040604×10^{-6}	1.040604×10^{-8}	1.040604×10^{-4}	1.040604×10^{-6}	1.040604×10^{-8}
	E_α	2.182110×10^{-2}	2.173007×10^{-2}	2.172918×10^{-2}	2.183745×10^{-2}	2.174636×10^{-2}	2.174546×10^{-2}
in [16]	CT	1.246600×10^{-2}	1.159500×10^{-2}	4.501330×10^{-1}	1.352600×10^{-2}	1.191300×10^{-2}	8.252000×10^{-3}

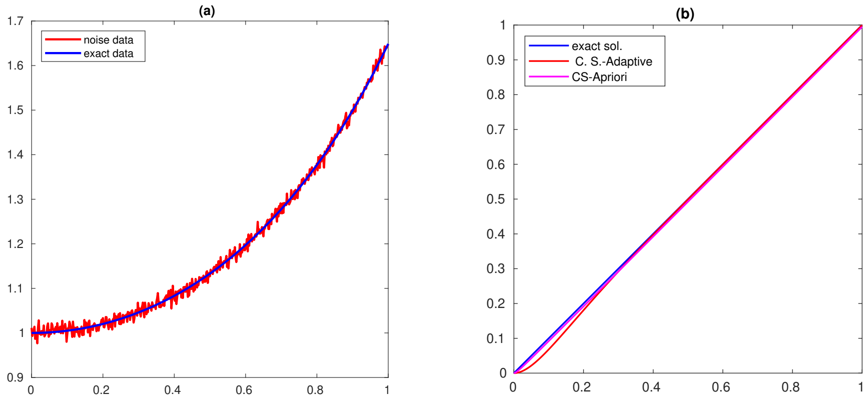

Figure 1. (**a**) data and (**b**) Solution with $\delta = 0.01$ and $n = 500$.

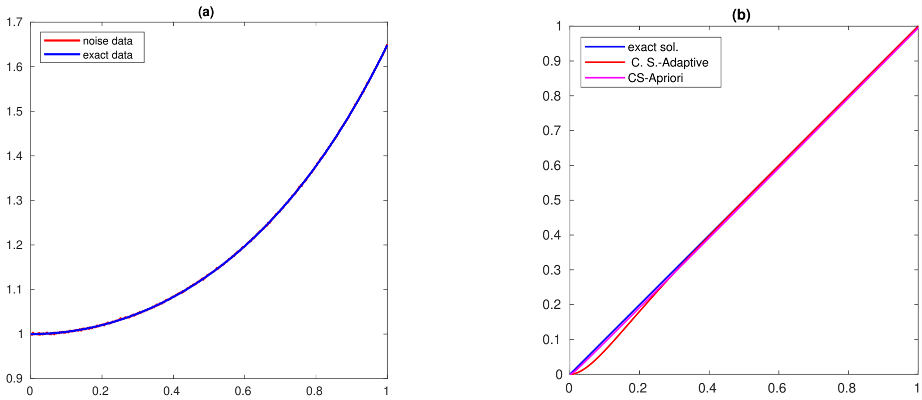

Figure 2. (**a**) data and (**b**) Solution with $\delta = 0.001$ and $n = 500$.

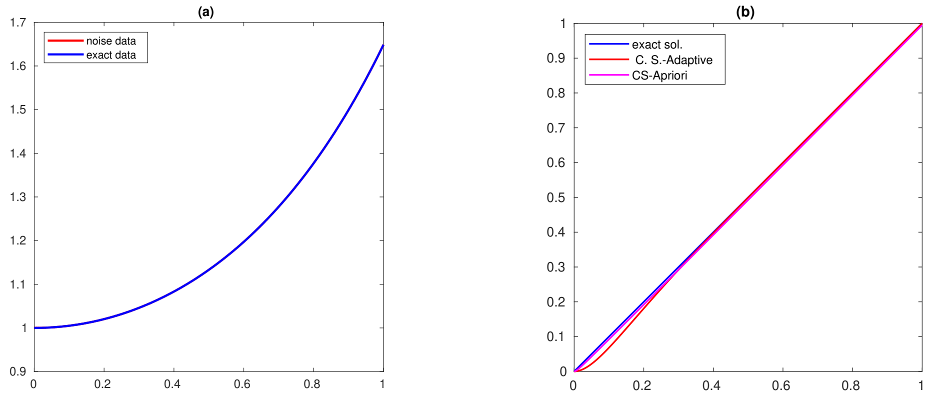

Figure 3. (**a**) data and (**b**) Solution with $\delta = 0.0001$ and $n = 500$.

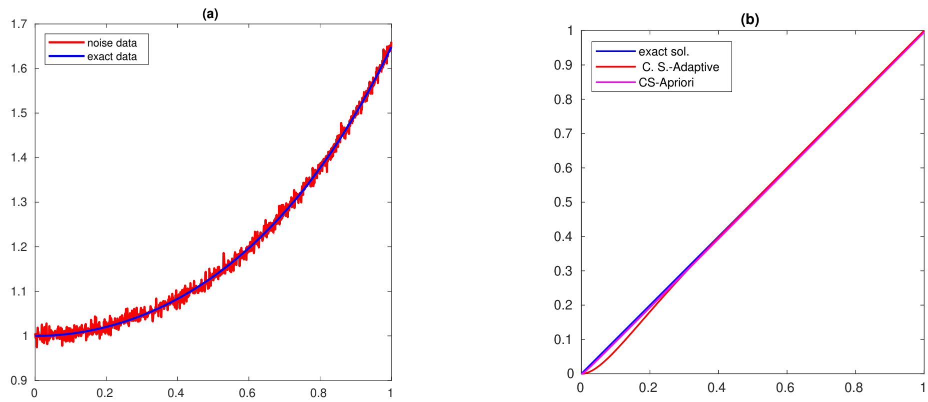

Figure 4. (**a**) data and (**b**) Solution with $\delta = 0.01$ and $n = 1000$.

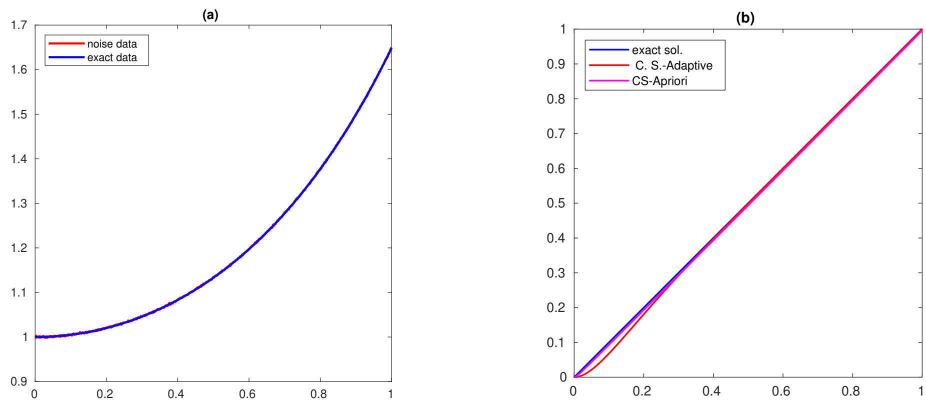

Figure 5. (**a**) data and (**b**) Solution with $\delta = 0.001$ and $n = 1000$.

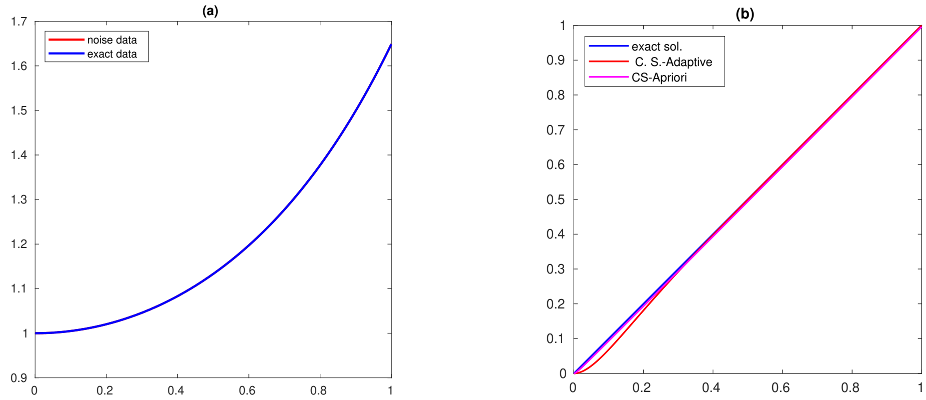

Figure 6. (**a**) data and (**b**) Solution with $\delta = 0.0001$ and $n = 1000$.

6. Conclusions

We introduced a new source condition and a new parameter-choice strategy. The proposed a new parameter-choice strategy is independent of the unknown parameter ν and it provides the optimal order $O(\delta^{\frac{\nu}{\nu+1}})$, for $0 \leq \nu \leq 1$.

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