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Equivalence of Competitive Equilibria, Fuzzy Cores, and Fuzzy Bargaining Sets in Finite Production Economies

Jiuqiang Liu [1,2]

[1] College of Big Data Statistics, Guizhou University of Finance and Economics, Guiyang 550025, China; jiuqiang68@126.com
[2] Department of Mathematics, Eastern Michigan University, Ypsilanti, MI 48197, USA

Abstract: For an exchange economy with a continuum of traders and a finite-dimensional commodity space under some standard assumptions, Aumann showed that the core and the set of competitive allocations (two most important solutions) coincide and Mas-Colell proved that the bargaining set and the set of competitive allocations coincide. However, in the case of exchange economies with a finite number of traders, it is well-known that the set of competitive allocations could be a strict subset of the core which can also be a strict subset of the bargaining set. In this paper, we establish the equivalence of the fuzzy core, the fuzzy bargaining set (or Aubin bargaining set), and the set of competitive allocations in a finite coalition production economy with an infinite-dimensional commodity space under some standard assumptions. We first derive a continuous equivalence theorem and then discretize it to obtain the desired equivalence in finite economies.

Keywords: coalition production economy; competitive equilibrium; core; fuzzy core; fuzzy bargaining set

MSC: 91A86; 91B50

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1. Introduction

Bargaining sets, cores, and competitive equilibria are important solutions for economies and games. In 1954, Arrow and Debreu [1] established the celebrated existence theorem of competitive (Walrasian) equilibria in finite exchange economies and some special finite production economies under a set of assumptions. Since then, the existence of competitive equilibrium for different economies under various assumptions has been studied extensively in the literature, including those for finite economies by Bewley [2], Mas-Colell [3], and Podczeck and Yannelis [4], and those for continuum economies by Aumann [5], Hildenbrand [6], and Sondermann [7].

While competitive equilibrium outcomes are in the core, in general, the core is larger than the set of equilibrium allocations in a finite economy. The question of the relationship of the core of an economy to the set of competitive equilibrium allocations originates with Edgeworth's conjecture that, if an economy were replicated, the core of an economy would shrink to the set of competitive equilibrium allocations. In 1963, Debreu and Scarf [8] gave a rigorous treatment of Edgeworth's conjecture. Aumann [9] proved a remarkable result that the core and the set of competitive allocations coincide in an exchange economy with a continuum (infinite many) of traders under some standard assumptions, and Rustichini and Yannelis [10] extended Aumann's result to the setting where the commodity space is an ordered separable Banach space. In 1989, Mas-Colell [11] extended Aumann's result by proving the following well-known fact: The bargaining set and the set of competitive allocations coincide in an exchange economy with a continuum of traders under some standard assumptions. This result is extended to coalition production economies with a continuum of traders and finite-dimensional Euclidean commodity spaces by Liu and Zhang [12].

For a finite economy, it is well-known that the set of competitive allocations could be a strict subset of the core which can also be a strict subset of the bargaining set. In 1979, Aubin [13] introduced the notion of fuzzy core (see also [14,15]) for an exchange economy by using fuzzy coalitions which allow agents to participate in a coalition with any level between 0 and 100 percentage, and proved that the fuzzy core and the set of competitive equilibrium allocations coincide in a finite exchange economy. Using Aubin's veto mechanism [16] through fuzzy coalitions, Hervés-Estévez and Moreno-García [17] and Liu [18] proved that the fuzzy (Aubin) bargaining set coincides with the set of competitive allocations in a finite economy with a finite-dimensional Euclidean commodity space under standard assumptions.

In this paper, we establish the equivalence of the fuzzy core, the fuzzy bargaining set, and the set of competitive allocations in a finite coalition production economy with an infinite-dimensional commodity space under some standard assumptions. We first derive a continuous equivalence theorem and then discretize it to obtain the desired equivalence in finite economies. Our equivalence theorem is built on a general model—coalition production economies with separable Banach spaces as commodity spaces and thus includes many of the corresponding existing equivalence theorems in the literature. Studying the equivalence of the set of competitive allocations, the fuzzy core, and the fuzzy bargaining set is useful as the existence of any of them implies the existence of the others if they are equivalent, and Liu [19] established the non-emptiness (existence) of the fuzzy core in a finite production economy with infinite-dimensional commodity space.

We conclude this section with the following remark: Equilibrium analysis in the infinite-dimensional setting differs in important ways from equilibrium analysis in the finite dimensional setting as noted by Zame [20]. For example, while Aumann proved that the core and the set of competitive allocations coincide in exchange economies with a continuum of traders and a finite-dimensional Euclidean commodity space, Podczeck [21] and Tourky and Yannelis [22] showed that the core and the set of competitive allocations are not equivalent in an atomless exchange economy with a continuum of traders and a non-separable Banach space as commodity space.

2. The Equivalence between the Set of Competitive Allocations and the Bargaining Set in a Continuum Economy

A *Banach space* is a complete normed vector space and note that a separable Banach space is a locally convex Hausdorff space. Given a Banach space $\mathcal{H}$, its dual $\mathcal{H}^*$ is the set of continuous linear functionals $p : \mathcal{H} \longrightarrow R$. Throughout this paper, we shall let $\mathcal{H}$ be a separable Banach space such that the positive cone $\mathcal{H}_+$ is closed and convex and has an interior point. We assume that $\mathcal{H}$ is equipped with a reflexive, transitive, anti-symmetric order relation $\leq$ such that (1) if $x \leq y$ and $z \geq 0$, then $x + z \leq y + z$ (2) if $x \leq y$ and $c \in R_+$, then $cx \leq cy$ (3) $x \leq y$ implies $||x|| \leq ||y||$. By $x < y$, we mean that $x \leq y$ and $x \neq y$. We define the *positive cone* $\mathcal{H}_+ = \{x \in \mathcal{H} : x \geq 0\}$ and the dual cone of $\mathcal{H}_+$ to be $\mathcal{H}^*_+ = \{p \in \mathcal{H}^* : p(u) \geq 0 \text{ for any } u \in \mathcal{H}_+\}$.

The following concepts for continuum coalition production economies with infinite dimensional commodity spaces are natural extensions of those corresponding concepts for continuum economies with finite-dimensional commodity space R^l_+ (see [6,9,12]). For a continuum economy, the set of agents is the closed interval $[0, 1]$, denoted by T, such that (T, τ, μ) forms an atomless measure space, where μ is Lebesgue measure. Let

$$X : T \mapsto \mathcal{H}_+$$

be a measurable consumption correspondence, where $X(t)$ is interpreted as the consumption set of agent $t \in T$. We use $\mathcal{F}$ to denote the set of all coalitions which are all nonnull Lebesgue measurable subsets of T, all integrals are Bochner integrals (see [10] for the definition) taken with respect to Lebesgue measure μ, and we use notation $\int_S f(t, \cdot)dt$ instead of $\int_S f(t, \cdot)d\mu$ for convenience.

A *coalition production economy with a continuum of agents and an infinite-dimensional commodity space* $\mathcal{H}$ is

$$\mathcal{E} = (\mathcal{H}, (X, \succ_t, \mathbf{w})_{t\in T}, (Y^S)_{S\in\mathcal{F}}, \beta(t,p)),$$

where each $Y^S \subseteq \mathcal{H}$ is the production set of the firm (coalition) $S \in \mathcal{F}$ with $Y^T = Y$ being the total production set, $\mathbf{w}(t) \in \mathcal{H}_+$ is player t's endowment vector which is Bochner integrable, and $\beta(t,p) : T \times \mathcal{H}_+^* \to [0,\infty)$ is the profit distribution function for the total production set Y such that $\int_T \beta(t,p)dt = 1$ and $\beta(t,p)$ is continuous with respect to p, where each agent t receives profit share $(p\cdot y)\beta(t,p)$ from the total profit $p \cdot y$ at production $y \in Y$ and price vector p.

Definition 1. *A* (feasible) allocation *(or "trade") for a continuum economy is a Bochner integrable assignment* $\mathbf{x}$ *for which*

$$\int_T [\mathbf{x}(t) - \mathbf{w}(t)]dt = \mathbf{y} \in Y. \tag{1}$$

An allocation $\mathbf{y}$ *blocks an allocation* $\mathbf{x}$ *via a coalition* S *if* $\mathbf{y}(t) \succ_t \mathbf{x}(t)$ *for each* $t \in S$ *and*

$$\int_S [\mathbf{y}(t) - \mathbf{w}(t)]dt \in Y^S. \tag{2}$$

Definition 2. *The* core *of a production economy* $\mathcal{E}$ *with a continuum of agents is the set of all (feasible) allocations that are not blocked via any nonnull coalition.*

For a continuum coalition production economy, the following assumptions are standard.

(II.1) For every $t \in T$, $X(t) \subseteq \mathcal{H}_+$ is convex and closed containing 0, weakly compact, integrably bounded, and $\mathbf{w}(t) \in int(X(t))$ is Bochner integrable.
(II.2) *Desirability* $(of\ the\ commodities)$: $x > y$ implies $x \succ_t y$.
(II.3) *Continuity* $(of\ the\ commodities)$: For each $y \in X(t)$, the sets $\{x : x \succ_t y\}$ and $\{x : y \succ_t x\}$ are open (relative to $X(t)$) for a.e. $t \in T$.
(II.4) *Measurability*: If $\mathbf{x}$ and $\mathbf{y}$ are assignments, then the set $\{t : \mathbf{x}(t) \succ_t \mathbf{y}(t)\}$ is Lebesgue measurable in T.
(II.5) $\succ_t$ is irreflexive and transitive for all $t \in T$.

We remark here that assumptions (II.2)–(II.5) are standard assumptions as in Aumann [9], and assumption (II.1) similar to that in [23].

For convenience, denote $\sup\{p \cdot \mathbf{y}' : \mathbf{y}' \in Y\}$ by $\sup p \cdot Y$. The following concept of competitive equilibrium for a continuum production economy is given in [6,12], with commodity space being a separable Banach space and the price system being the unit ball

$$\mathcal{B}_+^* = \{f \in \mathcal{H}_+^* : ||f|| \leq 1\} \tag{3}$$

which is compact with respect to weak* topology on $\mathcal{H}^*$ by the well-known Banach—Alaoglu Theorem.

Definition 3. *A* competitive equilibrium *(Walrasian equilibrium) of a continuum economy* $\mathcal{E}$ *consists of a price system* $p \in \mathcal{B}_+^*$*, a feasible allocation* $\mathbf{x}(t)$*, and a production* $\mathbf{y} \in Y$ *such that*
(i) $\int_T [\mathbf{x}(t) - \mathbf{w}(t)]dt = \mathbf{y} \in Y$;
(ii) $p \cdot \mathbf{y} = \sup p \cdot Y$ *and for any coalition* $S \in \mathcal{F}$, $\sup p \cdot Y^S \leq (\sup p \cdot Y) \int_S \beta(t,p)dt$;
(iii) for almost every (a.e.) trader t, $\mathbf{x}(t)$ *is maximal with respect to* $\succ_t$ *in* t*'s budget set*

$$B_p(t) = \{\mathbf{x} \in X(t) : p \cdot \mathbf{x} \leq p \cdot \mathbf{w}(t) + (\sup p \cdot Y)\beta(t,p)\},$$

that is, for almost every $t \in T$,

$$p \cdot \mathbf{x}(t) \leq p \cdot \mathbf{w}(t) + (\sup p \cdot Y)\beta(t,p) \text{ and}$$

$$v \succ_t \mathbf{x}(t) \text{ implies } p \cdot v > p \cdot \mathbf{w}(t) + (\sup p \cdot Y)\beta(t,p).$$

We make the following assumptions on the production sets and price systems:

(P.1′) Y^S is a closed, convex, and weakly compact subset of $\mathcal{H}$ containing 0 and $Y^S \subseteq Y$ for each $S \in \mathcal{F}$.

(P.2′) For each $p \in \mathcal{B}_+^*$ and any coalition $S \in \mathcal{F}$,

$$\sup p \cdot Y^S = (\sup p \cdot Y) \int_S \beta(t,p)dt.$$

(P.3′) The wealth map $(x,p) \longrightarrow p \cdot x$ is joint continuous (i.e., continuous with respect to $x \in \mathcal{H}$ and $p \in \mathcal{B}_+^*$).

Remark 1. *We remark that assumption (P.2′) holds for a coalition production economy with a continuum of agents satisfying that Y^S is a closed convex cone of $\mathcal{H}$ with vertex 0 for each $S \in \mathcal{F}$: In fact, we must have $p \cdot y \leq 0$ for any $y \in Y^S$ and for any price vector $p \in \mathcal{B}_+^*$. Otherwise, if $p \cdot y > 0$ for some $y \in Y^S$, then $cy \in Y^S$ for any $c \geq 0$ as the production set Y^S is a convex cone and the profit $p \cdot (cy) = c(p \cdot y)$ approaches positive infinity as c approaches the positive infinity, which is impossible. For each $S \in \mathcal{F}$, since $0 \in Y^S$, it follows that $\sup p \cdot Y^S = 0$ for any $p \in \mathcal{H}^*$. Thus, assumption (P.2′) holds.*

The next fact for a coalition production economy with a continuum of agents and an infinite-dimensional commodity space can be proved easily similar to Theorem 2.3 in [24].

Lemma 1. *Any competitive allocation belongs to the core in a coalition production economy $\mathcal{E}$ with a continuum of agents and an infinite-dimensional commodity space.*

The following concepts of objections, counterobjections, and bargaining sets are natural extensions of the corresponding concepts for exchange economies given by Mas-Colell [11].

Definition 4. *An objection to the allocation $\mathbf{x}$ is a pair $(S, \mathbf{y})$, where $S \in \mathcal{F}$ and $\mathbf{y}$ is an allocation such that*

(a) $\int_S [\mathbf{y}(t) - \mathbf{w}(t)]dt \in Y^S$;

(b) $\mathbf{y}(t) \succeq_t \mathbf{x}(t)$ for a.e. $t \in S$ and $\mu\{t \in S : \mathbf{y}(t) \succ_t \mathbf{x}(t)\} > 0$.

Definition 5. *Let $(S, \mathbf{y})$ be an objection to the allocation $\mathbf{x}$. A counterobjection to $(S, \mathbf{y})$ is a pair $(Q, \mathbf{z})$, where $Q \in \mathcal{F}$ and $\mathbf{z}$ is an allocation such that*

(a) $\int_Q [\mathbf{y}(t) - \mathbf{w}(t)]dt \in Y^Q$,

(b) $\mu(Q) > 0$,

(c) $\mathbf{z}(t) \succ_t \mathbf{y}(t)$ for a.e. $t \in S \cap Q$ and $\mathbf{z}(t) \succ_t \mathbf{x}(t)$ for a.e. $t \in Q \setminus S$.

Definition 6. *An objection $(S, \mathbf{y})$ is said to be justified if there is no counterobjection to it. The bargaining set $\mathcal{B}(\mathcal{E})$ of the economy $\mathcal{E}$ is the set of all allocations which have no justified objection.*

Clearly, the core of the economy $\mathcal{E}$ is contained in the bargaining set $\mathcal{B}(\mathcal{E})$. In 1989, Mas-Colell [11] proved that following well-known result.

Theorem 1 (Mas-Colell, 1989)**.** *For an exchange economy with a continuum of traders and a commodity space R_+^l satisfying assumptions (II.1)–(II.4), the bargaining set coincides with the set of competitive allocations.*

The following theorem generalizes Theorem 1 in two main aspects: exchange economies are extended to coalition production economies and commodity spaces are extended from Euclidean spaces to separable Banach spaces.

Theorem 2. *Let $\mathcal{E}$ be a coalition production economy with a continuum of agents and commodity space $\mathcal{H}_+$, where $\mathcal{H}$ is an ordered separable Banach space. If $\mathcal{E}$ satisfies assumptions (II.1)–(II.5) and (P.1′)–(P.3′), then its bargaining set coincides with its set of competitive allocations.*

We postpone the proof for Theorem 2 to Section 4.

3. Fuzzy Cores, Fuzzy Bargaining Sets, and Competitive Equilibria of Finite Economies

Throughout this paper, we let $\mathcal{H}$ be an ordered separable Banach space such that the positive cone $\mathcal{H}_+$ is closed convex and has an interior point. In this section, we will establish the equivalence of the set of competitive allocations, the fuzzy core, and the fuzzy bargaining set in a finite coalition production economy with infinite-dimensional commodity space.

We first recall the following concept of a finite coalition production economy and some necessary preliminaries from the literature. For simplicity, we assume that the preference orderings are representable by real valued concave continuous utility functions u^i, which can be used to approximate rather general preference relations according to Section 4.6 in Debreu [25]. Let $N = \{1, 2, \ldots, n\}$ be the set of n agents and denote by $\mathcal{N}$ the set of all nonempty subsets (coalitions) of N.

A *coalition production economy* with n agents is

$$\mathcal{E} = (\mathcal{H}, (X^i, u^i, \mathbf{w}^i)_{i \in N}, (Y^S)_{S \in \mathcal{N}})$$

which consists of a collection of the commodity space $\mathcal{H}$, agents' characteristics $(X^i, u^i, \mathbf{w}^i)_{i \in N}$, and coalitions' production sets $(Y^S)_{S \in \mathcal{N}}$. The triple $(X^i, u^i, \mathbf{w}^i)$ is agent i's characteristics as a consumer: $X^i \subseteq \mathcal{H}_+$ is his consumption set, $u^i : X^i \to R$ is his utility function, and $\mathbf{w}^i \in \mathcal{H}$ is his endowment vector. The set $Y^S \subseteq \mathcal{H}$ is the production set of the firm (coalition) S for which every agent $i \in S$ works and Y^S consists of all production plans that can be achieved through a joint action by the members of S. We use $Y = Y^N$ for the total production possibility set of the economy. For convenience, we simply call $\mathcal{E}$ an economy.

An exchange economy is a special coalition production economy with $Y^S = \{0\}$ for all $S \in \mathcal{N}$.

For each $S \in \mathcal{N}$, the set $F_{\mathcal{E}}(S)$ of S-allocations is

$$F_S(\mathcal{E}) = \{(\mathbf{x}^i)_{i \in S} : \mathbf{x}^i \in X^i \text{ for each } i \in S \text{ and } \sum_{i \in S} (\mathbf{x}^i - \mathbf{w}^i) \in Y^S\}.$$

The set of all (feasible) allocations of the economy $\mathcal{E}$ is

$$F(\mathcal{E}) = F_N(\mathcal{E}) = \{(\mathbf{x}^i)_{i \in N} : \mathbf{x}^i \in X^i \text{ for each } i \in N \text{ and } \sum_{i \in N} (\mathbf{x}^i - \mathbf{w}^i) \in Y^N = Y\}$$

which is assumed to be nonempty and compact.

We make the following assumptions on consumption sets, utility functions, and the sets of allocations:

(A.1) For every agent $i \in N$, X^i is a closed convex subset of $\mathcal{H}_+$ containing 0 and $\mathbf{w}^i \in int(X^i)$ (where $int(A)$ stands for the interior of A).
(A.2) *Desirability (of the commodities)*: $\mathbf{x} > \mathbf{y}$ implies $u^i(\mathbf{x}) > u^i(\mathbf{y})$.
(A.3) For each $i \in N$, $u^i : X^i \to R$ is continuous and strongly convex (i.e., for all x^i and $\overline{x}^i$ such that $x^i \neq \overline{x}^i$ and $u^i(x^i) \geq u^i(\overline{x}^i)$, and for all α with $0 < \alpha < 1$,

$$u^i(\alpha x^i + (1 - \alpha)\overline{x}^i) > u^i(\overline{x}^i).$$

(A.4) The set $F(\mathcal{E})$ of all (feasible) allocations is nonempty and compact.

Note that, for each $S \in \mathcal{N}$, $F_S(\mathcal{E}) \neq \emptyset$ if and only if $(\sum_{i \in S} X^i) \cap (\sum_{i \in S} \mathbf{w}^i + Y^S) \neq \emptyset$, and $0 \in Y^S$ implies that $(\mathbf{w}^i)_{i \in S} \in F_S(\mathcal{E})$.

Definition 7. *By a* price *(or* price system*), we shall mean a continuous linear functional $p \in \mathcal{B}^*_+$ (see (3)), and we denote the value of p at the vector x by $p \cdot x$.*

One can think $p \cdot x$ as the profit at price p and production x and view the map $(x, p) \longrightarrow p \cdot x$ as the wealth map.

We make the following assumption on the production sets and the wealth map:

(P.1) Y^S is a closed convex cone with the vertex at the origin in $\mathcal{H}$ and $Y^S \subseteq Y$ for each $S \in \mathcal{N}$.

(P.2) The wealth map $(x, p) \longrightarrow p \cdot x$ is joint continuous (i.e., continuous with respect to $x \in \mathcal{H}$ and $p \in \mathcal{H}^*_+$).

Note that assumption (P.1) implies the following common assumptions: (1) $Y^S = \{0\}$ for all $S \in \mathcal{N}$ (exchange economy, see [9,11,26]); (2) Y is a closed convex cone with vertex at the origin and $Y^S = Y$ for each $S \in \mathcal{N}$ (see [8]).

Remark 2. *Similar to the remarked by [8], under assumption (P.1), we have $p \cdot y \leq 0$ for any $y \in Y^S$, any $S \in \mathcal{N}$ and for any price vector $p \in \mathcal{H}^*_+$ provided $\sup p \cdot Y^S < \infty$. For otherwise, if $p \cdot y > 0$ for some $y \in Y^S$, then $cy \in Y^S$ for any $c \geq 0$ as the production set Y^S is a convex cone and the profit $p \cdot (cy) = c(p \cdot y)$ approaches the positive infinity as c approaches to the positive infinity, which is impossible. For each $S \in \mathcal{N}$, since $0 \in Y^S$, it follows that $\sup p \cdot Y^S = 0$ for any $p \in \mathcal{H}^*_+$.*

The following concept of competitive equilibrium for an economy satisfying assumption (P.1) is a natural extension of the corresponding concept given in [8,18].

Definition 8. *For an allocation $\mathbf{x} \in X$ and a price vector $p \in \mathcal{B}^*_+$, the couple $(\mathbf{x}, p)$ is a* competitive equilibrium *of an economy $\mathcal{E}$ if the profit is maximized on Y and for each $i \in N$, $\mathbf{x}^i$ satisfies the preferences of the i-th consumer under the constraint $p \cdot \mathbf{x}^i \leq p \cdot \mathbf{w}^i$ that is, for each $i \in N$,*

$$p \cdot \mathbf{x}^i = p \cdot \mathbf{w}^i \text{ and } u^i(\mathbf{v}^i) > u^i(\mathbf{x}^i) \text{ imply } p \cdot \mathbf{v}^i > p \cdot \mathbf{x}^i.$$

We say that an allocation $x = (x^i)_{i \in N}$ in a coalition production economy $\mathcal{E}$ is blocked by a coalition S if there is an attainable S-allocation $(y^i)_{i \in S}$ such that $u^i(y^i) \geq u^i(x^i)$ for all $i \in S$ with at least one of the inequalities being strict (see [8]). The *core* $C(\mathcal{E})$ of an economy $\mathcal{E}$ is the set of all attainable allocations that cannot be blocked by any coalition. Recall from Aubin [16] that a fuzzy coalition is a vector $s = (s_1, s_2, \ldots, s_n)$ with $0 \leq s_i \leq 1$ for each $1 \leq i \leq n$ (where s_i is the participation level of agent i). A crisp coalition $S \subset N$ corresponds to a special fuzzy coalition $s = (s_1, s_2, \ldots, s_n)$ with $s_i = 1$ if $i \in S$ and $s_i = 0$ if $i \notin S$. The following fuzzy core concept is a refinement of the core for an economy (see Florenzano [14]).

Definition 9. *The* fuzzy core *$C_F(\mathcal{E})$ of an economy $\mathcal{E}$ is the set of all allocations which can not be blocked by any fuzzy coalition, where an allocation $\mathbf{x}$ is blocked by a fuzzy coalition s, which means that there exists $\mathbf{y} \in X = \prod_{i \in N} X^i$ such that $\sum_{i \in N} s_i(\mathbf{y}^i - \mathbf{w}^i) \in Y^S$ with $S = car(s) = \{i \in N : s_i > 0\}$, and $u^i(\mathbf{y}^i) \geq u^i(\mathbf{x}^i)$ for each $i \in S$ with at least one of the inequalities being strict.*

Clearly, the fuzzy core $C_F(\mathcal{E})$ is a subset of the core $C(\mathcal{E})$ in an economy $\mathcal{E}$. The next standard fact is given in [18] for coalition production economies with Euclidean commodity spaces, which works exactly the same for Banach spaces as commodity spaces.

Lemma 2. *For an economy $\mathcal{E}$ satisfying assumptions (A.2), (A.3), and (P.1), any competitive allocation of $\mathcal{E}$ is in the fuzzy core of $\mathcal{E}$.*

In 1989, Mas-Colell [11] introduced the (Mas-Colell) bargaining set for exchange economies. Motivated by Aubin's veto mechanism [16], Hervés-Estévez and Moreno-García [17] and Liu [18] gave the following fuzzy extension of the corresponding concept for economies with Euclidean commodity spaces, which is stated in the setting of Banach spaces here.

Definition 10. *A fuzzy objection to an allocation* $\mathbf{x}$ *is a pair* $(S, \mathbf{y})$*, where* $S \in \mathcal{N}$ *and* $\mathbf{y}$ *is defined on* S*, for which there exist* $0 < \alpha_i \leq 1$ *for each* $i \in S$*, such that*
(a) $\sum_{i \in S} \alpha_i(\mathbf{y}^i - \mathbf{w}^i) \in Y^S$*,*
(b) $u^i(\mathbf{y}^i) \geq u^i(\mathbf{x}^i)$ *for each* $i \in S$ *with at least one of these inequalities being strict.*

Definition 11. *Let* $(S, \mathbf{y})$ *be a fuzzy objection to an allocation* $\mathbf{x}$*. A fuzzy counterobjection to* $(S, \mathbf{y})$ *is a pair* $(Q, \mathbf{z})$*, where* $Q \in \mathcal{N}$ *and* $\mathbf{z}$ *is defined on* Q*, for which there exist* $0 < \lambda_i \leq 1$ *for each* $i \in Q$*, such that*
(a) $\sum_{i \in Q} \lambda_i(\mathbf{z}^i - \mathbf{w}^i) \in Y^Q$*,*
(b) $u^i(\mathbf{z}^i) > u^i(\mathbf{y}^i)$ *for each* $i \in S \cap Q$ *and* $u^i(\mathbf{z}^i) > u^i(\mathbf{x}^i)$ *for each* $i \in Q \setminus S$*.*

Definition 12. *A fuzzy objection* $(S, \mathbf{y})$ *is said to be justified if there is no fuzzy counterobjection to it. The* fuzzy (Aubin) bargaining set $\mathcal{B}_F(\mathcal{E})$ *of an economy* $\mathcal{E}$ *is the set of all allocations which have no justified Aubin objection.*

When restricted to the crisp case, that is, $\alpha_i = 1$ for all $i \in S$ and $\lambda_i = 1$ for all $i \in Q$, we obtain the (Mas-Colell) bargaining set $\mathcal{B}(\mathcal{E})$ for the economy $\mathcal{E}$. Clearly, the fuzzy bargaining set $\mathcal{B}_F(\mathcal{E})$ is a subset of the bargaining set $\mathcal{B}(\mathcal{E})$ in an economy $\mathcal{E}$.

Note that it follows from the definitions that, in an economy $\mathcal{E}$, the core $C(\mathcal{E})$ is a subset of the bargaining set $\mathcal{B}(\mathcal{E})$ and the fuzzy core $C_F(\mathcal{E})$ is a subset of the fuzzy bargaining set $\mathcal{B}_F(\mathcal{E})$. The fuzzy core of an economy can be viewed as a refinement to the core of the economy by allowing agents to cooperate at a different participation level (from 0 percent to 100 percent), thereby with more blocking power, the fuzzy bargaining set does the same to the bargaining set for an economy.

Given a finite economy

$$\mathcal{E} = (\mathcal{H}, (X^i, u^i, \mathbf{w}^i)_{i \in N}, (Y^S)_{S \in \mathcal{N}})$$

with n agents satisfying assumption (P.1), we construct a special continuum economy $\mathcal{E}_C$ with n types of distinct agents as follows: We divide the set $T = [0, 1]$ of agents into n subintervals $I_i = [\frac{i-1}{n}, \frac{i}{n})$ for $1 \leq i \leq n-1$ and $I_n = [\frac{n-1}{n}, 1]$, where all agents in I_i are identical to agent i in the economy $\mathcal{E}$, that is,

$$\mathcal{E}_C = (\mathcal{H}, (X(t), \succ_t, \mathbf{w}(t))_{t \in T}, (Y^S)_{S \in \mathcal{F}}, \beta(t, p)_{t \in T, p \in P}),$$

where $X(t) = X^i$ and $\mathbf{w}(t) = \mathbf{w}^i$ for all $t \in I_i$; for any $t \in I_i$ and $\mathbf{x}, \mathbf{y} \in X(t)$, $\mathbf{x} \succ_t \mathbf{y}$ if and only if $u^i(\mathbf{x}) > u^i(\mathbf{y})$; for each coalition $S \in \mathcal{F}$ (where $\mathcal{F}$ is the set of all measurable subsets of T), define $Y^S = Y^{S'}$, where $S' = \{i \in N : \mu(S \cap I_i) > 0\}$ (thus, $Y^T = Y^N = Y$) with μ being Lebesgue measure and $\beta(t, p) = 1$ for any $t \in T$ and any $p \in P$.

For each allocation $\mathbf{x} = (\mathbf{x}^i)_{i \in N} \in F(\mathcal{E})$, define the step function $f_{\mathbf{x}}(t)$ by

$$f_{\mathbf{x}}(t) = \mathbf{x}^i \text{ if } t \in I_i. \tag{4}$$

Then, $f_{\mathbf{x}}(t)$ is an allocation in $\mathcal{E}_C$. In fact, $\mathbf{x} \in F(\mathcal{E})$ implies that $\sum_{i \in N}(\mathbf{x}^i - \mathbf{w}^i) \in Y$. By assumption (P.1),

$$\int_T [f_{\mathbf{x}}(t) - \mathbf{w}(t)]dt = \sum_{i \in N} \frac{1}{n}(\mathbf{x}^i - \mathbf{w}^i) = \frac{1}{n} \sum_{i \in N}(\mathbf{x}^i - \mathbf{w}^i) \in Y,$$

which implies that $f_{\mathbf{x}}(t)$ is an allocation in $\mathcal{E}_C$ by (4).

To prove our main theorem, we need the following lemma which is the Lemma together with its remark by García-Cutrín and Hervés-Beloso [26].

Lemma 3 (García-Cutrín and Hervés-Beloso, [26])**.** *Let $\succeq$ be a convex and continuous preference relation and $\mathcal{H}$ be a Banach space. If $S \subseteq T$ has positive measure, $g : S \to \mathcal{H}_+$ is an integrable function and $\mathbf{x} \in \mathcal{H}_+$ is such that $g(t) \succ \mathbf{x}$ (or $g(t) \succeq \mathbf{x}$) for all $t \in S$, then*

$$\frac{1}{\mu(S)} \int_S g(t)dt \succ \mathbf{x} \text{ (or } \frac{1}{\mu(S)} \int_S g(t)dt \succeq \mathbf{x} \quad \text{resp.).}$$

In the next theorem, an allocation $\mathbf{f}(t)$ in $\mathcal{E}_C$ yields an allocation $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^n)$ in $\mathcal{E}$ with $\mathbf{x}^i = n \int_{I_i} \mathbf{f}(t)dt$ for $1 \le i \le n$; and an allocation $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^n)$ in $\mathcal{E}$ gives rise to an allocation $\mathbf{f}(t)$ in $\mathcal{E}_C$ with $\mathbf{f}(t) = \mathbf{x}^i$ for all $t \in I_i$ and $i \in N$. The following proof is motivated by the proof of Theorem 1 from [26].

Theorem 3. *Let $\mathcal{E}$ be a finite economy satisfying assumptions (A.3) and (P.1). Then, $(p, \mathbf{f}, \mathbf{y})$ is a competitive equilibrium for $\mathcal{E}_C$ if and only if $(p, \mathbf{x}, n\mathbf{y})$ is a competitive equilibrium for $\mathcal{E}$.*

Proof. Let $(p, \mathbf{x}, n\mathbf{y})$ is a competitive equilibrium for $\mathcal{E}$. Then, we have $\mathbf{f}(t)$ in $\mathcal{E}_C$ defined by $\mathbf{f}(t) = \mathbf{x}^i$ for all $t \in I_i$ and $i \in N$. Since Y is a cone by assumption (P.1), $\sum_{i \in N}(\mathbf{x}^i - \mathbf{w}^i) = n\mathbf{y} \in Y$ implies

$$\int_T [\mathbf{f}(t) - \mathbf{w}(t)]dt = \sum_{i \in N} \int_{I_i} [\mathbf{f}(t) - \mathbf{w}(t)]dt$$

$$= \frac{1}{n} \sum_{i \in N} (\mathbf{x}^i - \mathbf{w}^i) = \mathbf{y} \in Y$$

and so $\mathbf{f}(t)$ is an allocation in $\mathcal{E}_C$. By Remark 1, we have $\sup p \cdot Y = p \cdot (n\mathbf{y}) = 0 = p \cdot \mathbf{y}$. Moreover, we have $p \cdot \mathbf{x}^i \le p \cdot \mathbf{w}^i$, and $u^i(\mathbf{v}^i) > u^i(\mathbf{x}^i)$ implies $p \cdot \mathbf{v}^i > p \cdot \mathbf{w}^i$ for each $i \in N$. It follows that, for each $1 \le i \le n$ and for all $t \in I_i$,

$$p \cdot \mathbf{f}(t) = p \cdot \mathbf{x}^i \le p \cdot \mathbf{w}^i = p \cdot \mathbf{w}(t)$$

and $\mathbf{v}(t) \succ_t \mathbf{x}(t)$ implies $u^i(\mathbf{v}^i) > u^i(\mathbf{x}^i)$, where $\mathbf{v}^i = \mathbf{v}(t)$ for all $t \in I_i$, and so

$$p \cdot \mathbf{v}(t) = p \cdot \mathbf{v}^i > p \cdot \mathbf{w}^i = p \cdot \mathbf{w}(t).$$

Thus, $(p, \mathbf{f}, \mathbf{y})$ is a competitive equilibrium for $\mathcal{E}_C$.

Conversely, let $(p, \mathbf{f}, \mathbf{y})$ be a competitive equilibrium for $\mathcal{E}_C$. Then, $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^n)$ satisfies $\mathbf{x}^i = n \int_{I_i} \mathbf{f}(t)dt = \frac{1}{|I_i|} \int_{I_i} \mathbf{f}(t)dt$ for $1 \le i \le n$. Since Y is a cone by assumption (P.1), $\int_T [\mathbf{f}(t) - \mathbf{w}(t)]dt = \mathbf{y} \in Y$ implies that

$$\sum_{i \in N} (\mathbf{x}^i - \mathbf{w}^i) = n \left[\sum_{i \in N} \frac{1}{n} (\mathbf{x}^i - \mathbf{w}^i) \right]$$

$$= n \int_T \left[\mathbf{f}(t) - \mathbf{w}(t) \right] dt = n\mathbf{y} \in Y.$$

Thus, $\mathbf{x}$ is an allocation for $\mathcal{E}$. Since $p \cdot \mathbf{y} = \sup p \cdot Y = 0$ by Remark 2, $p \cdot n\mathbf{y} = 0$. Thus, the profit is maximized on Y at $n\mathbf{y}$ for the economy $\mathcal{E}$.

Next, we show that each $\mathbf{x}^i$ is in the budget set $B_p(i) = \{x \in X^i : p \cdot x \le p \cdot \mathbf{w}^i\}$. Since $\mathbf{f}(t)$ is in the budget set $B_p(t) = \{x \in X(t) : p \cdot x \le p \cdot \mathbf{w}(t)\}$ for all $t \in I_i$, we have for each $1 \le i \le n$, $p \cdot \mathbf{f}(t) \le p \cdot \mathbf{w}(t)$ and

$$p \cdot \mathbf{x}^i = p \cdot n \int_{I_i} \mathbf{f}(t)dt \le p \cdot n \int_{I_i} \mathbf{w}(t)dt = p \cdot \mathbf{w}^i.$$

To show that $(p, \mathbf{x}, n\mathbf{y})$ is a competitive equilibrium for $\mathcal{E}$, it suffices to show that, for $1 \leq i \leq n$, $u^i(\mathbf{v}^i) > u^i(\mathbf{x}^i)$ implies $p \cdot \mathbf{v}^i > p \cdot \mathbf{w}^i$. Let $u^i(\mathbf{v}^i) > u^i(\mathbf{x}^i)$ and define $\mathbf{v}(t)$ by setting $\mathbf{v}(t) = \mathbf{v}^i$ for all $t \in I_i$. We claim that there exists $t' \in I_i$ such that $\mathbf{v}(t') \succ_{t'} \mathbf{f}(t')$. Suppose, otherwise, that $\mathbf{f}(t) \succeq_t \mathbf{v}(t) = \mathbf{v}^i$ for all $t \in I_i$. By assumption (A.3) and Lemma 3, we have

$$\mathbf{x}^i = \frac{1}{|I_i|} \int_{I_i} \mathbf{f}(t) dt \succeq \mathbf{v}^i$$

which implies $u^i(\mathbf{v}^i) \leq u^i(\mathbf{x}^i)$ by the construction of $\mathcal{E}_C$, contradicting $u^i(\mathbf{v}^i) > u^i(\mathbf{x}^i)$. Thus, the claim holds. Since $(p, \mathbf{f}, \mathbf{y})$ is a competitive equilibrium for $\mathcal{E}_C$, $\mathbf{v}(t') \succ_{t'} \mathbf{f}(t')$ implies that $p \cdot \mathbf{v}(t') > p \cdot \mathbf{w}(t')$. Since $\mathbf{v}(t') = \mathbf{v}^i$ and $\mathbf{w}(t') = \mathbf{w}^i$, it follows that $p \cdot \mathbf{v}^i > p \cdot \mathbf{w}^i$. Thus, $(p, \mathbf{x}, n\mathbf{y})$ is a competitive equilibrium for $\mathcal{E}$. □

It is well-known that, in a finite exchange economy, which is a special production economy with $Y^S = \{0\}$ for all $S \in \mathcal{N}$, the set of competitive allocations could be a proper subset of the core which can also be a proper subset of the bargaining set. Here, we will prove the following equivalence between the set of competitive allocations and the fuzzy bargaining set in a finite coalition production economy with infinite-dimensional commodity space which generalizes the equivalence theorems in [17,18]. The proof is similar to the proof of Theorem 4.5 in [18], with major difficulties and complications caused by infinite-dimensional commodity spaces involved, much of the difficulties occurred in the proof of Theorem 2 given in Section 4.

Theorem 4. *Let $\mathcal{E}$ be a finite economy satisfying assumptions (A.1)–(A.4), (P.1), and (P.2). Then, the set of competitive allocations and the fuzzy bargaining set coincide in $\mathcal{E}$.*

Proof. First, by Remarks 1 and 2, it is easy to check that the assumptions (A.1)–(A.4), (P.1) and (P.2) in Theorem 4 imply assumptions (II.1)–(II.5) and (P.1′)–(P.3′) in Theorem 2. By Lemma 2 and the fact that the fuzzy core $C_F(\mathcal{E})$ is a subset of the fuzzy bargaining set $\mathcal{B}_F(\mathcal{E})$, we conclude that the set of competitive allocations of $\mathcal{E}$ is a subset of the fuzzy bargaining set $\mathcal{B}_F(\mathcal{E})$.

Similar to the proof of Theorem 4.5 in [18], applying Lemma 3 and Theorems 2 and 3, one can show that the fuzzy bargaining set $\mathcal{B}_F(\mathcal{E})$ is a subset of the set of competitive allocations of $\mathcal{E}$. □

Since the set of competitive allocations is a subset of the fuzzy core which is a subset of the fuzzy bargaining set, Theorem 4 implies the next equivalence theorem immediately.

Theorem 5. *Let $\mathcal{E}$ be a finite economy satisfying assumptions (A.1)–(A.4), (P.1), and (P.2). Then, the set of competitive allocations and the fuzzy core coincide in $\mathcal{E}$.*

Proof. By Lemma 2, the set of competitive allocations is a subset of the fuzzy core in $\mathcal{E}$. Recall that the fuzzy core is a subset of the fuzzy bargaining set in $\mathcal{E}$. It follows from Theorem 4 that we must have the set of competitive allocations, the fuzzy core, and the fuzzy bargaining set coincide in $\mathcal{E}$. □

The following example provides an economy which satisfies the assumptions in Theorems 4 and 5.

Example 1. *Let $\mathcal{E} = (\mathcal{H}, (X^i, u^i, \mathbf{w}^i)_{i \in N}, (Y^S)_{S \in \mathcal{N}})$ be the exchange economy: $N = \{1, 2\}$, $X^1 = X^2 = [0, 1]$, $\mathbf{w}^1 = \mathbf{w}^2 = \frac{1}{2}$, $Y^S = \{0\}$ for any $S \subseteq N$, $u^1(x^1) = x^1$ for all $x^1 \in X^1$ and $u^2(x^2) = x^2$ for all $x^2 \in X^2$. Then, it is easy to check that $\mathcal{E}$ satisfies assumptions (A.1)–(A.4), (P.1), and (P.2). Thus, Theorems 4 and 5 can be applied here.*

4. Cores, Bargaining Sets, Competitive Allocations in Continuum Economies

In this section, we will prove Theorem 2 along the same line as the proof of Theorem 1 by Mas-Colell [11] through competitive objections defined below, using an approach similar to the proof of the extension of Theorem 1 to continuum coalition production economies with finite-dimensional Euclidean commodity space R^l_+ given in [12]. Note that there are significant differences due to important structural differences between finite-dimensional Euclidean space R^l and infinite-dimensional Banach spaces.

Definition 13. *The objection $(S, \mathbf{y})$ to the allocation $\mathbf{x}$ is competitive if there is a price system $p \in \mathcal{B}^*_+$ such that for a.e. $t \in T$:*
(i) $p \cdot v \geq p \cdot \mathbf{w}(t) + (\sup p \cdot Y)\beta(t, p)$ for $v \in X(t)$ satisfying $v \succeq_t \mathbf{y}(t)$, $t \in S$, with strict inequality if $v \succ_t \mathbf{y}(t)$;
(ii) $p \cdot v \geq p \cdot \mathbf{w}(t) + (\sup p \cdot Y)\beta(t, p)$ for $v \in X(t)$ satisfying $v \succeq_t \mathbf{x}(t)$, $t \in T \setminus S$, with strict inequality if $v \succ_t \mathbf{x}(t)$.

Lemma 4. *For a coalition production economy with a continuum of agents and commodity space $\mathcal{H}_+$ satisfying assumptions (H.1), (II.1)–(II.5) and (P.1′), every competitive objection $(S, \mathbf{y})$ to an allocation $\mathbf{x}$ is justified.*

Proof. Let $p \in \mathcal{B}^*_+$ be the price vector associated with the competitive objection $(S, \mathbf{y})$. Then, we have

$$p \cdot \mathbf{y}(t) \geq p \cdot \mathbf{w}(t) + (\sup p \cdot Y)\beta(t, p) \text{ for a.e. } t \in S.$$

$$p \cdot \mathbf{x}(t) \geq p \cdot \mathbf{w}(t) + (\sup p \cdot Y)\beta(t, p) \text{ for a.e. } t \in T \setminus S.$$

Suppose there is a counterobjection $(Q, \mathbf{z})$ to $(S, \mathbf{y})$. Then, there exists $y' \in Y^Q$ such that $\mu(Q) > 0$,

$$\int_Q [\mathbf{z}(t) - \mathbf{w}(t)]dt = y', \tag{5}$$

$\mathbf{z}(t) \succ_t \mathbf{y}(t)$ for a.e. $t \in S \cap Q$, and $\mathbf{z}(t) \succ_t \mathbf{x}(t)$ for a.e. $t \in Q \setminus S$.

By the definition of competitive objection, we have

$$p \cdot \mathbf{z}(t) > p \cdot \mathbf{w}(t) + (\sup p \cdot Y)\beta(t, p) \text{ for a.e. } t \in S \cap Q, \tag{6}$$

$$p \cdot \mathbf{z}(t) > p \cdot \mathbf{w}(t) + (\sup p \cdot Y)\beta(t, p) \text{ for a.e. } t \in Q \setminus S. \tag{7}$$

It follows from Definition 9 (ii), (6) and (7) that

$$\begin{aligned} & p \cdot \int_Q \mathbf{z}(t)dt - p \cdot \int_Q \mathbf{w}(t)dt \\ & > (\sup p \cdot Y) \int_Q \beta(t, p)dt \geq \sup p \cdot Y^Q. \end{aligned}$$

However, by (5), we have

$$p \cdot \left[\int_Q \mathbf{z}(t)dt - \int_Q \mathbf{w}(t)dt \right] = p \cdot y' \leq \sup p \cdot Y^Q,$$

a contradiction. Therefore, the objection $(S, \mathbf{y})$ is justified. □

The next theorem is Theorem 6.2 in [27].

Theorem 6 (Yannelis, 1991)**.** *Let (T, τ, μ) be a finite atomless measure space, $\mathcal{H}$ be a Banach space and $\phi : T \mapsto 2^{\mathcal{H}}$ be a correspondence. Then, cl $\int_T \phi dt$ is convex.*

The following extension of Fatou's lemma is Theorem 3.1 from [28], where LsF_n is the set of the weak limit superior points of the sequence $\{F_n\}$ of subsets in a Banach space $\mathcal{H}$:

$$LsF_n \equiv w\text{-}\overline{\lim}_{n \longrightarrow \infty} F_n$$

$$= \{y \in \mathcal{H} : y = w\text{-}\lim y_k, y_k \in F_{n_k}, k = 1, 2, \dots\}.$$

Theorem 7 (Yannelis, 1988)**.** *Let (T, τ, μ) be a complete finite atomless measure space and $\mathcal{H}$ be a separable Banach space. Let $\phi_n : T \longrightarrow 2^{\mathcal{H}}$ $(n = 1, 2, \dots$ be a sequence of nonempty closed valued correspondences such that for all n $(n = 1, 2, \dots)$, $\phi_n(t) \subseteq X(t)$ for all $t \in T$, where $X : T \longrightarrow 2^{\mathcal{H}}$ is an integrably bounded, weakly compact, nonempty, convex valued correspondence. Moreover, suppose that $Ls\phi_n(\cdot)$ is closed and convex valued. Then,*

$$Ls \int_T \phi_n(t)d(t) \subseteq \int_T Ls\phi_n(t)d(t).$$

The next theorem extends the corresponding result by Aumann [29] from Euclidean spaces to Banach spaces.

Theorem 8. *Let (T, τ, μ) be a complete finite separable measure space and $\mathcal{H}$ be a separable Banach space. Let $\varphi : \mathcal{B}^* \times T \mapsto 2^{\mathcal{H}}$ be a nonempty, closed, integrably bounded, weakly compact, convex valued correspondence having a measurable graph such that for each $t \in T$, $\varphi(\cdot, t)$ is upper semicontinuous. Then, $\int_T \varphi(p, t)dt$ is upper semicontinuous.*

Proof. To show that $\int_T \varphi(p, t)dt$ is upper semicontinuous, by the Closed Graph Theorem, we need to show that, if $p_n \longrightarrow p$ with all $p_n, p \in \mathcal{B}^*$ and $\int_T z_n(t)dt \longrightarrow \int_T z(t)dt$ with all $z_n(t) \in \varphi(p_n, t)$, then $z(t) \in \varphi(p, t)$. Since φ is closed, weakly compact, and convex valued, it can be proved that $Ls\varphi(p_n, t)$ is closed and convex valued. It follows from Theorem 7 that $\int_T z(t)dt \in Ls \int_T \varphi(p_n, t)dt \subseteq \int_T Ls\varphi(p_n, t)d(t)$, which implies that there exists a subsequence $\{g_{n_k}(t)\}$ with all $g_{n_k}(t) \in \varphi(p_{n_k}, t)$ such that $g_{n_k}(t) \longrightarrow z(t)$ for a.e. $t \in T$. Since $p_{n_k} \longrightarrow p$ and $\varphi(\cdot, t)$ is upper semicontinuous, we have $z(t) \in \varphi(p, t)$. Thus, $\int_T \varphi(p, t)dt$ is upper semicontinuous. □

Note that a separable Banach space $\mathcal{H}$ is a complete metric space, and the following fact is Aumann's Measurable Selection Theorem in [16].

Theorem 9 (Aumann's Measurable Selection Theorem, 1969)**.** *If (T, τ, μ) is a complete finite measure space, $F : T \mapsto 2^{\mathcal{H}}$ has a measurable graph, and $\mathcal{H}$ is separable, then F has a measurable selection. Moreover, if F is also integrably bounded, it admits a Bochner integrable selection.*

The following remark is Remark 1 from [23].

Remark 3. *Let $\mathcal{H}$ be a Banach space and u be an interior element of $\mathcal{H}_+$. Then, for any nonzero $f \in \mathcal{H}_+^*$, $f \cdot u > 0$.*

The proof of the next lemma is along the same line as in [12,30], with major difficulties caused by infinite-dimensional Banach space. Recall that a well-known theorem by James [31] states: A closed and bounded convex subset C of a Banach space $\mathcal{H}$ is weakly compact if and only if every continuous linear functional defined on $\mathcal{H}$ attains its maximum value over C, where weakly compact means compact with respect to the weak topology in a Banach space. The integral of a correspondence F is defined by

$$\int_T F(t)dt$$
$$= \left\{ \int_T f(t)dt : f \in F(t) \text{ and } f \text{ is Bochner integrable} \right\}.$$

The next lemma extends Lemma 3.9 in [12] with a much more complicated proof.

Lemma 5. *Let $\mathcal{E}$ be a coalition production economy with a continuum of agents and commodity space $\mathcal{H}_+$ satisfying assumptions (H.1), (II.1)–(II.5) and (P.1′)–(P.3′). If $\mathbf{x}$ is an allocation which is not competitive in $\mathcal{E}$, then there is a competitive objection $(S, \mathbf{y})$ to $\mathbf{x}$.*

Proof. Assume that $\mathbf{x}$ is an allocation which is not competitive. We will construct a competitive objection to $\mathbf{x}$.

For convenience, we will use a continuous and quasi-concave function $u_t : X \to R$ to represent the preference relation $\succ_t$ (it is a well-known fact that real valued continuous and quasi-concave utility functions can be used to approximate rather general preference relations arbitrarily closely). Recall that the budget set for agent $t \in T$ at each $p \in \mathcal{B}_+^*$ (see (3)) is

$$B_p(t) = \{\mathbf{x}' \in X(t) : p \cdot \mathbf{x}' \leq p \cdot \mathbf{w}(t) + (\sup p \cdot Y)\beta(t, p)\}.$$

Then, it is easy to see that the budget set $B_p(t)$ is closed. It follows that $B_p(t)$ is weakly compact for each $p \in \mathcal{B}_+^*$ as $X(t)$ is weakly compact and $B_p(t) \subseteq X(t)$. By James's Theorem, the continuous function u_t attains maximum on $B_p(t)$ for each $t \in T$ and every $p \in \mathcal{B}_+^*$. For all $p \in \mathcal{B}_+^*$ and all $t \in T$, define

$$D(p,t) = \{\mathbf{z}(t) \in X(t) : \mathbf{z} \text{ maximizes } u_t \text{ on } B_p(t)\}.$$

Then, it is easy to see that $D(p,t)$ is convex and closed, and $D(p,t)$ is nonempty by James's Theorem [31] for any $p \in \mathcal{B}_+^*$ and $t \in T$. Thus, $D(p,t)$ is weakly compact. Since $X(t)$ is integrably bounded by assumption (II.1), $D(p,t)$ is integrably bounded. By Theorem 9, there exists a Bochner integrable selection $f(t) \in D(p,t)$.

For each $p \in \mathcal{B}_+^*$, define

$$\varphi(p,t) = \begin{cases} D(p,t) & \text{if } u_t(D(p,t)) > u_t(\mathbf{x}(t)) \\ D(p,t) \cup \{\mathbf{w}(t)\} & \text{if } u_t(D(p,t)) = u_t(\mathbf{x}(t)) \\ \{\mathbf{w}(t)\} & \text{if } u_t(D(p,t)) < u_t(\mathbf{x}(t)). \end{cases}$$

Then, $\varphi(p,t)$ is closed and weakly compact as $D(p,t)$ is closed and weakly compact for each $p \in \mathcal{B}_+^*$ and each $t \in T$. Moreover, since $\mathbf{!}(t)$ is integrable by assumption (II.1) and $D(p,t)$ has a Bochner integrable selection $f(t)$, a Bochner integrable function in $\varphi(p,t)$ exists.

Let

$$S = \{t \in T : f(t) \in D(p^*, t) \text{ and } u_t(D(p^*, t)) \geq u_t(\mathbf{x}(t))\}$$

and

$$C(p^*) = \{t \in T : u_t(D(p^*, t)) > u_t(\mathbf{x}(t))\}. \tag{8}$$

Then, $C(p^*) \subseteq S$. By the measurability assumption (II.4), both S and $C(p^*)$ are Lebesgue μ-measurable. Since $\mathbf{x}$ is not competitive, we have $\mu(C(p^*)) > 0$ which implies $\mu(S) > 0$. We claim that, for any $f(t) \in D(p^*, t)$,

$$p^* \cdot f(t) \geq p^* \cdot \mathbf{w}(t) + (\sup p^* \cdot Y)\beta(t, p^*) \text{ for all } t \in S. \tag{9}$$

In fact, suppose that, for $t \in S$,

$$\begin{aligned} p^* \cdot f(t) &< p^* \cdot \mathbf{w}(t) + (\sup p^* \cdot Y)\beta(t, p^*) \\ &= p^* \cdot \mathbf{w}(t) + (p^* \cdot y^*)\beta(t, p^*). \end{aligned}$$

By the continuity assumption (II.3) and the fact that p^* is linear continuous, there exists $g(t) \in X(t) \subseteq \mathcal{H}_+$ such that $u_t(g(t)) > u_t(f(t))$ and

$$p^* \cdot [g(t) - f(t)] < p^* \cdot \mathbf{w}(t) + (p^* \cdot y^*)\beta(t, p^*) - p^* \cdot f(t)$$

which implies that $p^* \cdot g(t) < p^* \cdot \mathbf{w}(t) + (p^* \cdot y^*)\beta(t, p^*)$, contradicting the fact $f(t) \in D(p^*, t)$. Thus, (9) holds.

Since Y^S is weakly compact by assumption (P.1′), it follows from James's Theorem [31] that , for each $p \in \mathcal{B}^*_+$, there exists $y \in Y^S$ such that $p \cdot y = \sup p \cdot Y^S$. For each $p \in \mathcal{B}^*_+$, let

$$Y^S_p = \{y \in Y^S | p \cdot y = \sup p \cdot Y^S\} (\neq \emptyset)$$

and define

$$\begin{aligned}\psi(p) &= \int_T \varphi(p,t)dt - \int_T \mathbf{w}(t)dt - \int_T Y^S_p dt \\ &= \int_T \Big[\varphi(p,t) - \mathbf{w}(t) - Y^S_p\Big]dt.\end{aligned}$$

Then, $\psi(p)$ is nonempty for every $p \in \mathcal{B}^*_+$ as there exists a Bochner integrable function in $\varphi(p,t)$ and $Y^S_p \neq \emptyset$. Since Y^S is closed and $p \cdot x$ is jointly continuous by assumption (P.3′), it is easy to see that Y^S_p is closed. Thus, $\varphi(p,t) - \mathbf{w}(t) - Y^S_p$ is closed as $\varphi(p,t)$ is closed. By Theorem 6, $\psi(p)$ is convex. Moreover, by a standard argument and using assumption (P.3′), one can check that $\varphi(p,t) - \mathbf{w}(t) - Y^S_p$ is upper semicontinuous. It follows from Theorem 8 that ψ is upper semicontinuous on $\mathcal{B}^*_+$.

We claim that $p \cdot v \leq 0$ for any $p \in \mathcal{B}^*_+$ and any $v \in \psi(p)$. Let $p \in \mathcal{B}^*_+$. Since $T = [0,1]$, we have $\int_T y dt = y$ for each $y \in Y^S$. It follows that, for any $v \in \psi(p)$, there exists Bochner integrable $f(t) \in \varphi(p,t)$ and $y \in Y^S_p$ such that

$$\begin{aligned}v &= \int_T f(t)dt - \int_T \mathbf{w}(t)dt - \int_T y dt \\ &= \int_T f(t)dt - \int_T \mathbf{w}(t)dt - y.\end{aligned}$$

By the definition of $\varphi(p,t)$, for any $t \in T$, we have either $f(t) = \mathbf{w}(t)$ or $f(t) \in D(p,t) \subseteq B_p(t)$, which implies that $f(t) = \mathbf{w}(t)$ for $t \in T \setminus S$ and $p \cdot f(t) \leq p \cdot \mathbf{w}(t) + (\sup p \cdot Y)\beta(t,p)$ for a.e. $t \in T$. It follows from assumption (P.2′) that

$$\begin{aligned}p \cdot v = p \cdot \int_T \Big[f(t) - \mathbf{w}(t)\Big]dt - p \cdot y \\ = p \cdot \int_S \Big[f(t) - \mathbf{w}(t)\Big]dt - \sup p \cdot Y^S \\ = \int_S p \cdot \Big[f(t) - \mathbf{w}(t)\Big]dt - (\sup p \cdot Y) \int_S \beta(t,p)dt \\ = \int_S \Big[p \cdot f(t) - p \cdot \mathbf{w}(t) - (\sup p \cdot Y)\beta(t,p)\Big]dt \leq 0.\end{aligned}$$

Thus, the claim holds.

By the Banach extension of the celebrated Gale–Nikaido–Debreu Lemma in Debreu [25] (5.6, (1)), there exist $p^* \in \mathcal{B}^*_+$ and $z^* \in -\mathcal{H}_+$ such that $z^* \in \psi(p^*)$. It follows that there exists a Bochner integrable function $\mathbf{f}(t) \in \varphi(p^*,t)$ for all $t \in T$ and $y^* \in Y^S_{p^*} \subseteq Y^S$ such that

$$z^* = \int_T \mathbf{f}(t)dt - \int_T \mathbf{w}(t)dt - y^*. \tag{10}$$

Recall that $\mathbf{f}(t) = \mathbf{w}(t)$ for $t \in T \setminus S$. It follows from (10) that

$$z^* = \int_S \mathbf{f}(t)dt - \int_S \mathbf{w}(t)dt - y^*. \tag{11}$$

Let

$$\mathbf{y}(t) = \mathbf{f}(t) - \frac{1}{\mu(S)} z^* \text{ for all } t \in S,$$

$$\mathbf{y}(t) = \mathbf{w}(t) \text{ for all } t \in T \setminus S.$$

Since $z^* \leq 0$, we have $-z^* \in \mathcal{H}_+$ and so $p^* \cdot z^* \leq 0$. It follows from (8) that

$$\begin{aligned} p^* \cdot \mathbf{y}(t) &= p^* \cdot \mathbf{f}(t) - \frac{1}{\mu(S)} p^* \cdot z^* \\ &\geq p^* \cdot \mathbf{f}(t) \geq p^* \cdot \mathbf{w}(t) + (\sup p^* \cdot Y)\beta(t, p^*) \text{ for } t \in S. \end{aligned} \tag{12}$$

We now show that $(S, \mathbf{y})$ is a competitive objection. Since $z^* \leq 0$, we have $\mathbf{y}(t) \geq \mathbf{f}(t)$ for all $t \in S$. Note that $\frac{1}{\mu(S)} \int_S z^* dt = z^*$. It follows from (10) that

$$\int_S \mathbf{y}(t)dt - \int_S \mathbf{w}(t)dt = y^* \in Y^S.$$

Moreover, since $\mathbf{y}(t) = \mathbf{w}(t)$ for all $t \in T \setminus S$,

$$\int_T \mathbf{y}(t)dt - \int_T \mathbf{w}(t)dt = \int_S \mathbf{y}(t)dt - \int_S \mathbf{w}(t)dt \in Y^S \subseteq Y$$

and so $\mathbf{y}$ is an allocation satisfying condition (a) in Definition 10. Since $\mathbf{y}(t) \geq \mathbf{f}(t)$ for all $t \in T$, it follows from the desirability assumption (II.2) and the definition for S that

$$u_t(\mathbf{y}(t)) \geq u_t(\mathbf{f}(t)) = u_t(D(p^*, t)) \geq u_t(\mathbf{x}(t))$$

for all $t \in S$. Together with the fact $\mu(C(p^*)) > 0$ (see (8)), we have that $\mathbf{y}$ satisfies condition (b) in Definition 10 and so $(S, \mathbf{y})$ is an objection.

Now, we claim that, for each $t \in S$ and $v \in X(t)$,

$$\begin{aligned} &u_t(v) \geq u_t(\mathbf{y}(t)) \text{ implies} \\ &\qquad p^* \cdot v \geq p^* \cdot \mathbf{w}(t) + (\sup p^* \cdot Y)\beta(t, p^*). \end{aligned} \tag{13}$$

For otherwise, suppose that $u_t(v) \geq u_t(\mathbf{y}(t))$ and $p^* \cdot v < p^* \cdot \mathbf{w}(t) + (\sup p^* \cdot Y)\beta(t, p^*)$. Then, it follows from assumption (II.3) that there exists $w \in X(t) \subseteq \mathcal{H}_+$ such that $u_t(w) > u_t(v) \geq u_t(\mathbf{y}(t)) \geq u_t(\mathbf{f}(t)) = u_t(D(p^*, t))$ and $p^* \cdot w \leq p^* \cdot \mathbf{w}(t) + (\sup p^* \cdot Y)\beta(t, p^*)$, contradicting $\mathbf{f}(t) \in D(p^*, t)$. Thus, (13) holds. Next, we show that for each $t \in S$ and $v \in X(t)$, $u_t(v) > u_t(\mathbf{y}(t))$ implies $p^* \cdot v > p^* \cdot \mathbf{w}(t) + (\sup p^* \cdot Y)\beta(t, p^*)$. Suppose that for some $v \in X(t)$, $u_t(v) > u_t(\mathbf{y}(t))$ but $p^* \cdot v \leq p^* \cdot \mathbf{w}(t) + (\sup p^* \cdot Y)\beta(t, p^*)$. Then, $p^* \cdot v = p^* \cdot \mathbf{w}(t) + (\sup p^* \cdot Y)\beta(t, p^*)$ by (14). Since $X(t)$ is convex containing 0, $v \in X(t)$ implies that $\lambda v \in X(t)$ for any $0 \leq \lambda \leq 1$. By assumption (II.3), we can choose $0 < \lambda < 1$ so that $u_t(\lambda v) > u_t(\mathbf{y}(t))$. By (13), we have $p^* \cdot (\lambda v) \geq p^* \cdot \mathbf{w}(t) + (\sup p^* \cdot Y)\beta(t, p^*)$. Since $\mathbf{w}(t)$ is an interior point of $X(t) \subseteq \mathcal{H}_+$, by Remark 3, $p^* \cdot \mathbf{w}(t) > 0$. Since $0 \in Y$ by assumption (P.1′), $\sup p^* \cdot Y \geq 0$ and so $p^* \cdot (\lambda v) > 0$. It follows that $p^* \cdot v > \lambda(p^* \cdot v) = p^* \cdot (\lambda v) \geq p^* \cdot \mathbf{w}(t) + (\sup p^* \cdot Y)\beta(t, p^*)$, a contradiction. Thus, $u_t(v) > u_t(\mathbf{y}(t))$ implies $p^* \cdot v > p^* \cdot \mathbf{w}(t) + (\sup p^* \cdot Y)\beta(t, p^*)$ for each $t \in S$.

Similarly, we claim that, for any $t \in T \setminus S$ and $v \in X(t)$,

$$\begin{aligned} &u_t(v) \geq u_t(\mathbf{x}(t)) \text{ implies} \\ &\qquad p^* \cdot v \geq p^* \cdot \mathbf{w}(t) + (\sup p^* \cdot Y)\beta(t, p^*). \end{aligned} \tag{14}$$

In fact, suppose that $u_t(v) \geq u_t(\mathbf{x}(t))$ but $p^* \cdot v < p^* \cdot \mathbf{w}(t) + (\sup p^* \cdot Y)\beta(t, p^*)$ for some $t \in T \setminus S$ and $v \in X(t)$. By the definition of S, $u_t(D(p^*, t)) \leq u_t(\mathbf{x}(t))$ and $\mathbf{f}(t) = \mathbf{w}(t)$. Thus, $u_t(v) \geq u_t(D(p^*, t))$. Since $p^* \cdot v < p^* \cdot \mathbf{w}(t) + (\sup p^* \cdot Y)\beta(t, p^*)$, by assumptions (II.3) and (P.3′), there exists $v' \in X(t)$ such that $u_t(v') > u_t(v) \geq u_t(D(p^*, t))$ and

$$p^* \cdot v' - p^* \cdot v < p^* \cdot \mathbf{w}(t) + (\sup p^* \cdot Y)\beta(t, p^*) - p^* \cdot v$$

which implies $p^* \cdot v' < p^* \cdot \mathbf{w}(t) + (\sup p^* \cdot Y)\beta(t, p^*)$. It follows from the definition of $D(p^*, t)$ that $u_t(v') \leq u_t(D(p^*, t))$, contradicting the fact $u_t(v') > u_t(v) \geq u_t(D(p^*, t))$. Thus, (4.10) holds. Moreover, similar to the argument in the previous paragraph, we have $u_t(v) > u_t(\mathbf{x}(t))$ implies $p^* \cdot v > p^* \cdot \mathbf{w}(t) + (\sup p^* \cdot Y)\beta(t, p^*)$ for each $t \in T \setminus S$. Therefore, $(S, \mathbf{y})$ is a competitive objection. □

Proof of Theorem 2. By Lemma 1 and the fact that the core of an economy $\mathcal{E}$ is contained in its bargaining set $\mathcal{B}(\mathcal{E})$, we have that the set of competitive allocations is a subset of the bargaining set $\mathcal{B}(\mathcal{E})$. On the other hand, by Lemmas 4 and 5, the bargaining set $\mathcal{B}(\mathcal{E})$ is contained in the set of competitive allocations of $\mathcal{E}$. Thus, the set of competitive allocations coincides with the bargaining set $\mathcal{B}(\mathcal{E})$ in economy $\mathcal{E}$. □

Since, for a continuum coalition production economy $\mathcal{E}$, the set of competitive equilibria is contained in its core by Lemma 2, and the core is contained in its bargaining set, Theorem 2 implies immediately the next fact.

Theorem 10. *Let $\mathcal{E}$ be a coalition production economy with a continuum of agents and commodity space $\mathcal{H}_+$. If $\mathcal{E}$ satisfies assumptions (II.1)–(II.5) and (P.1′)–(P.3′), then its core coincides with its set of competitive allocations.*

Recall that an exchange economy is a special coalition production economy with $Y^S = \{0\}$ for every coalition $S \subseteq T$, Theorem 10 implies the following results for exchange economies with a continuum of traders, which implies the corresponding results by Aumann [9] (for exchange economies with $X(t) = R^l_+$) and by Rustichini and Yannelis [10] (for exchange economies with $X(t) = \mathcal{H}_+$).

Theorem 11. *For an exchange economy with a continuum of traders and commodity space $\mathcal{H}_+$ satisfying assumptions (II.1)–(II.5) and (P.3′), the core coincides with the set of competitive allocations.*

Proof. An exchange economy is a special coalition production economy with $Y^S = \{0\}$ for every coalition $S \subseteq T$, which clearly satisfies assumptions (P.1′) and (P.2′). Theorem 11 follows directly from Theorem 10. □

5. Conclusions

In this paper, we have established the equivalence of the set of competitive allocations, the fuzzy core, and the fuzzy bargaining set in a finite coalition production economy with an ordered separable Banach space being the commodity space through Theorems 4 and 5. In doing so, we obtain an equivalence theorem for the set of competitive allocations and the bargaining set in a continuum coalition production economy through Theorem 2 and then discretize it to obtain the desired equivalence in finite economies.

Based on the equivalence of the set of competitive allocations, the fuzzy core, and the fuzzy bargaining set, the existence of any of them implies the existence of the others.

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