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The Mathematical Model of Cyclic Signals in Dynamic Systems as a Cyclically Correlated Random Process

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Abstract: This work is devoted to the procedure for constructing of a cyclically correlated random process of a continuous argument as a mathematical model of cyclic signals in dynamic systems, which makes it possible to consistently describe cyclic stochastic signals, both with regular and irregular rhythms, not separating them, but complementing them within the framework of a single integrated model. The class of cyclically correlated random processes includes the subclass of cyclostationary (periodically) correlated random processes, which enable the use of a set of powerful methods of analysis and the forecasting of cyclic signals with a stable rhythm. Mathematical structures that model the cyclic, phase and rhythmic structures of a cyclically correlated random process are presented. The sufficient and necessary conditions that the structural function and the rhythm function of the cyclically correlated random process must satisfy have been established. The advantages of the cyclically correlated random process in comparison with other mathematical models of cyclic signals with a variable rhythm are given. The obtained results contribute to the emergence of a more complete and rigorous theory of this class of random processes and increase the validity of the methods of their analysis and computer simulation.

Keywords: cyclic signals; irregular rhythm; dynamic systems; cyclically correlated random process

MSC: 37H05; 60G12

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1. Introduction

Cyclical phenomena and processes occupy one of the main places among phenomena and processes of reality. Many processes have a cyclical structure in physical, chemical, technical, biological, economic and social dynamic systems. The development of modern information systems and technologies for the processing and simulation of cyclic signals of various natures enables us to automate and substantially intensify the procedure for the analysis, diagnosis and forecasting of the state of the systems in which they occur, opening the possibility of conducting computer simulation experiments. Typical examples of such systems include: computer systems of cardiac diagnostics; information systems for the analysis and forecasting of cyclic economic processes; information systems of person's authentication from their biometric dynamic data; automated systems for the analysis and forecasting of electric, gas, water and oil consumption; hardware generation and simulation of cyclic signals in modern telecommunication systems.

The first decisive stage in the design of information systems for the processing and simulation of cyclic signals is to create their mathematical models that adequately reflect the important aspects of cyclic signals in terms of research tasks. Mathematical models of cyclic phenomena and signals are devoted to a large number of scientific works. The classical theory of oscillations has become significant in developing and spreading in relation to the simulation and analysis of cyclic phenomena of different physical natures. The basic mathematical apparatus of this theory is the apparatus of differential equations: linear and nonlinear, deterministic and stochastic differential equations, ordinary differential equations and differential equations in partial derivatives [1–4]. This approach to the analysis of oscillatory processes in dynamic systems is constructive, since it is modeling the mechanism (algorithm) of the generation of the oscillations of the system under investigation and justifies the conditions under which the resulting process will have an oscillatory, cyclic structure. However, in the processing of real cyclic signals in information systems, for example, for a number of biological, economic and astrophysical cyclic signals, it is not possible or too difficult to write a differential equation, the solution of which would describe the investigated signals. It is also to note that in a number of applied problems in the analysis of cyclic signals in information systems, there is no need to consider the mechanism of their generation, since this greatly complicates the solution of these problems, and therefore it is quite sufficient to analyze only the spatio-temporal structure of the cyclic signal itself.

Actually, a number of other approaches deal with the modeling, methods of analysis and forecasting of the spatio-temporal structure of cyclic signals in dynamic systems. Among such approaches, it is possible to specify the spectral analysis of signals, in particular, based on the mathematical apparatus of the Fourier series and the Fourier transformation, which use harmonic, periodical and almost-periodical deterministic functions [5–7]. Another approach is a stochastic approach, based on the theory of random processes and sequences. The stochastic approach to the modeling and processing of cyclic signals is based on the spectral correlation theory of stationary random processes [8–10]; the theory of cyclostationary and almost-cyclostationary random processes [11–24]; the theory of periodic Markov random processes and chains [25–30]; the theory of periodic solutions of stochastic difference and differential equations [31–33]; and the theory of linear periodic random processes and fields [34–36].

The most significant results were obtained within the framework of the theory of cyclostationary and almost-cyclostationary random processes, which has been successfully developing for about 60 years. This theory covers a number of classes of random processes of continuous and discrete arguments. These include: cyclostationary correlated stochastic processes (wide-sense cyclostationary processes or periodically correlated random processes); cyclostationary stochastic processes in the strict sense (periodically distributed random processes); poly-periodic cyclostationary correlated stochastic processes (second-order poly-periodic cyclostationary processes); almost-cyclostationary correlated stochastic processes (second-order almost-cyclostationary processes, almost-cyclostationary processes in the wide sense, almost-periodically correlated processes); and almost-cyclostationary stochastic processes in the strict sense (almost-periodically distributed random processes). These random processes will be fruitfully applied for the modeling and analysis of a wide class of cyclic signals that occur in telecommunication, energy, astrophysical, mechanical and biological dynamic systems [37–50].

Despite the presence of generally different classes of random processes in the theory of cyclostationary random processes, all of them are based on the concepts of periodicity, poly-periodicity and almost-periodicity of probabilistic characteristics of random processes in both the wide and strict sense. The concept of periodicity (stochastic or deterministic) does not cover in full the content of the concept of signal cyclicity in full, but only partially, since the property of periodicity postulates the repetition of the values or the probabilistic characteristics of the cyclic signal through a strictly prescribed number, which is called a period, but, in fact, such strict repeatability in the time or/and space of the structure of many cyclic signals in dynamic systems is absent. Deterministic periodic and stochastic periodic functional dependences only partially adequately describe the cyclic structure of the investigated signals, namely, the cyclic signals with a stable (regular) rhythm. However, in cases in which the rhythm (tempo) of the oscillatory process changes significantly, due to the lack of adaptation to changes in the rhythm of oscillation, these mathematical models are inadequate, which leads to low informativeness of diagnostic, authentication and prognostic features in information systems for diagnostics, authentication and prediction by cyclic signals with variable (irregular) rhythm. Typical examples of cyclic signals with a variable rhythm are the signals of the cardiovascular dynamic system of the human body (electrocardio signals, magnetocardio signals, phonocardio signals, etc.), cyclic economic processes (indexes of business activity of all sectors of the economy, gross national product of countries, seasonal indices of enterprise incomes, etc.) and self-organization processes on the surface of materials.

Almost-cyclostationary stochastic processes are a broad class of random processes, since they are based on the concept of almost-periodicity, which is not entirely equivalent to the concept of cyclicity. Almost-cyclostationary stochastic processes do not reflect the cyclical structure of the investigated signals, since the strategy of generalizing periodic processes, when constructing almost periodic random processes, is based on class-forming properties different from those of cyclicity. Namely, the definition of almost periodic functions uses the approach of generalizing of periodic functions based on the possibility of representing almost periodic functions in the form of an analogue of the Fourier series, in which the fact of the multiplicity of frequencies and periods of the components of this series, which is characteristic of periodic functions, is no longer required. In particular, it is not quite adequate for the direct modeling and analyzing of cyclical signals with a variable rhythm on the basis of almost-cyclostationary stochastic processes.

2. Related Work

There are several approaches for considering the variability of the rhythm of cyclic signals, which take into account various deviations from the periodic model in both deterministic and stochastic constructions. One of the approaches to modeling and forming cyclic signals with variable rhythm is the use of modulation technologies for periodic signals. In particular, in radiotransmission systems, by modulation, an information message is inserted into the parameters of the high frequency periodic signal (carrying signal). Carrying periodic signals, basically, are harmonic and pulsed high-frequency signals. If the carrier is a harmonic signal, then amplitude, angular (frequency, phase) modulation and combinations of these modulations are used, for example, in amplitude-frequency modulation. The mathematical model of the resulting modulated signal is a quasi-harmonic function, in which the variability of the rhythm is taken into account in such concepts as the instantaneous angular frequency [51,52] and the instantaneous period [53]. In the case in which a carrier is a pulsed periodic signal (meander), amplitude-pulse, frequency pulse, phase-pulse and pulse-width modulation types, as well as some of their combinations, are used. In this case, the variability of the rhythm is reflected in such concepts as the frequency [54] and the variable period [55].

In the English-language scientific literature, the rhythm variability of the cyclic signal in mathematical models within the framework of the correlation theory of random processes has been studied in works [56–60]. In particular, in the works [56–58], mathematical models of cyclic signals with irregular cyclicity (irregular cyclostationary process, time-warped almost-cyclostationary process) have been built, and methods of analyzing such signals have been developed. In the paper [59], such a mathematical model was applied to electrocardio signals. In the work [60], a mathematical model of a cyclic signals was developed, which was formed by using the scale transformation operator (time warping) of a periodically correlated random process and multiplying the resulting (scaled) random process by a deterministic function. Such a resulting random process is called the cyclostationary process with evolving periods and amplitudes. All these scientific works, when defining new classes of random processes, move away from the classical definition of cyclostationary (almost-cyclostationary) correlated stochastic process and apply its time-scale transformation (time warping). The main task of the methods for processing such processes developed within the framework of these works is to reduce the new random processes to a cyclostationary correlated stochastic processes or to an almost-cyclostationary correlated stochastic processes.

Methods of reducing the cyclic signals with a variable rhythm to classic models and processing methods of cyclic signals with regular cyclicity within the framework of the theory of cyclostationary stochastic process in the strict sense (more precisely within the framework of the theory of linear periodic random functions) were first performed in 2000 and 2001 in Ukrainian-language scientific papers [61–63], which concerned the study of a wide class of cardiac signals in computer diagnostic systems. In these works, the concept of the zone time structure of cyclic signals was introduced, which made it possible to create mathematical models and methods for processing cyclic cardiac signals that took into account the variability of time intervals between single-phase values in different cycles of the cardiac signal and, consequently, made it possible to reduce the effect of "blurring" of the statistical characteristics of heart signals by applying a sequence of static scaling and shifting operators to the corresponding zones of the cardiac signal in all its cycles.

More general approaches and ideas regarding the consideration of variable rhythm in mathematical models of cyclic signals that extend the approach of works [61–63] were published in 2005–2008 [64–74] and were included in the dissertation [75], in which a new theoretical foundation of mathematical modeling, methods of computer simulation, sampling, statistical evaluation and the spectral analysis of cyclic signals in automated information dynamic systems, was created. The results obtained in these works take into account a wide range of possible attributes of cyclicity within the framework of deterministic, stochastic, fuzzy and interval modeling paradigms, and the significant structural diversity of patterns of variability and commonality of the rhythm of cyclic signals. They also have the means of adaptation to changes in rhythm of cyclic signals, which, in a practical aspect, has increased the accuracy, reliability and level of informativeness of the processing and simulation of signals with a cyclic space-time structure in intellectualized information systems. Special attention in the works [64–75] was paid to mathematical modeling and the processing of cyclic signals with a variable (irregular) rhythm. In particular, analytical dependencies between the rhythm functions of cyclic random processes, which are connected through the time-scale transformation operator, were established, which made it possible to analytically study the transformations of the rhythm of cyclic stochastic signals. Methods of statistical evaluation of the probabilistic characteristics of a cyclic random process and a vector of cyclic rhythmically connected random processes have also been developed, which, due to adaptation to changes in the rhythm of cyclic signals, significantly weaken the negative effect of "blurring" their statistical characteristics, and due to taking into account the commonality of the rhythm of a set of interconnected cyclic signals, enable compatible statistical analyses for the needs of complex computer diagnostics and authentication based on a set of interconnected cyclic signals. These models have been widely used in the tasks of modeling cyclic heart signals of different physical nature (electrical, magnetic, acoustic) [76–78], economic cyclical processes [79], dynamic biometric authentication signals [80], surface processes in materials science [81,82], and processes in energetics [83].

However, despite the fruitful use of cyclic random processes (in a wide and strict sense), there are almost no works that contain a complete procedure for constructing such processes as well as the proofs of the necessary properties of cyclic random processes. In particular, this also applies to the cyclically correlated random process, which takes into account the cyclicity and stochasticity of cyclic signals within the framework of the correlation theory of random processes and has effective means of taking into account both the regularity and irregularity of the rhythm of cyclic signals in dynamic systems. This article is devoted to the procedure of constructing of the cyclically correlated random process of a continuous argument and to the establishment of its fundamental properties.

3. Results

3.1. The Main Requirements for the Model and Basic Concepts that Reflect the Cyclic, Phase, and Rhythm Structures of the Signals

The main requirements for the model of cyclic signals are the following:

(1) the model must be fully interpretable, namely, to consistently reflect at the formal level the intuitive basic concepts that underlie the understanding of cyclical motion (cyclic signal) and such constituent concepts as cycle, phase and rhythm;
(2) the class of models should, as a special case, include a class of periodically correlated random processes (cyclostationary correlated stochastic process);
(3) the model must correctly display the variability of the rhythm of the simulated cyclic signals;
(4) the model must enable the development of effective methods of statistical analysis, forecasting and computer simulations of cyclic stochastic signals both with regular and irregular rhythms.

According to the dissertation [75], in order to take into account the results of analyses of the terminology and conceptual apparatus of the theory of the modeling and processing of cyclic signals, as well as the significant homonymy and synonymy of the given terminology, let's give a non formal definitions of following basic concepts: "cyclic signal", "cycle", "phase" and "rhythm", which will be used within the framework of this article and in their totality, in fact, constitute a conceptual (verbal) model of the cyclic signals.

1. The cyclic structure of signals exists only in relation to a certain set (in the partial case of a one-element set) of attributes (characteristics, properties) of a signals (attributes of signals cyclicity), for example, in relation to a set of their certain probabilistic characteristics under a stochastic approach or in relation to its values under a deterministic approach to signal modeling. All attributes of the cyclicity of the signal must be consistent with each other regarding the phase and cyclic structures of the signal.
2. A cyclic by attributes signal is an ordered set of cycles.
3. Cycles are the smallest segments into which a cyclic by attributes signal can be divided, and there is a similarity between these attributes and the same type of phase order in them.
4. Phases: these are stages of deployment in the time or space of a cyclic signal. A cyclic signal can be divided into sets of phases of the same type. The set of phases of the same type contains all the phases that are contained only in different cycles of the signal and have the same value of the cyclicity attribute. Each signal's cycle consists of an ordered set of phases of different types.
5. Any cyclic signal is characterized by its rhythm, which is its property, which sets the values of time (spatial) intervals (distances) between the phases of the same type of the cyclic signal for all its cycles and phases.

We note that in this paper, since the modeling of cyclic signals is within the framework of the correlation theory of random processes, we understand the mathematical expectation and correlation function of the investigated signals to be its attributes of cyclicity.

3.2. Development of Mathematical Model of the Cyclic Signal within the Framework of the Correlation Theory of Random Processes

The procedure for building the model is the correct sequential formalization of all components of the conceptual (informal) model of the cyclic signal. Putting correctly the abstract mathematical objects and structures into conformity with the basic notions of the conceptual model mentioned above, we will develop the mathematical foundations of the theory of the modeling and processing of cyclic signals within the framework of the correlation theory of random processes.

In the general, the mathematical model of the cyclic signal is some random process $\xi(\omega, t), \omega \in \boldsymbol{\Omega}, t \in \boldsymbol{R}$ ($\xi : \boldsymbol{R} \to \boldsymbol{L}_2(\boldsymbol{\Omega}, \boldsymbol{P})$), given on the some probability space $(\boldsymbol{\Omega}, \boldsymbol{F}, \boldsymbol{P})$ and on the set $\boldsymbol{R}$ of real numbers. The argument t can have a physical interpretation of the spatial or time coordinate, and the set of values is a space of random variables (e.g., Hilbert space $\boldsymbol{L}_2(\boldsymbol{\Omega}, \boldsymbol{P})$), given in the same probability space $(\boldsymbol{\Omega}, \boldsymbol{F}, \boldsymbol{P})$. The structure of the random process $\xi(\omega, t), \omega \in \boldsymbol{\Omega}, t \in \boldsymbol{R}$ should reflect the basic properties of cyclic signals, its cyclic, phase and rhythm structures which are reflected in the conceptual model presented above. To attain this, we will give a series of such preliminary definitions that will be used in the mathematical modelling of cyclic signals within the framework of the correlation theory of random processes.

According to the second statement of the conceptual model, any cyclic signal is a signal that evolves in cycles, consists of cycles, and therefore it can be divided into segment-cycles that do not intersect and, as a result, they are specified in the areas of definition, that are not intersect. Moreover, in the general case, the set of such cycles should be a countable set. Proceeding from such considerations about cyclic structure, we introduce mathematical objects that formalize these representations.

Given a certain ordered (ordered by m) countable partition $\boldsymbol{D}_{\boldsymbol{R}}^{c} = \{\boldsymbol{W}_{c_m},\ m \in \boldsymbol{Z}\}$ of definition domain $\boldsymbol{R}$ that is for the elements of partition $\boldsymbol{D}_{\boldsymbol{R}}^{c}$, the following relations are performed:

$$\cup_{m\in \boldsymbol{Z}} \boldsymbol{W}_{c_m} = \boldsymbol{R}, \boldsymbol{W}_{c_m} \neq \varnothing,\ \ \boldsymbol{W}_{c_{m_1}} \cap \boldsymbol{W}_{c_{m_2}} = \varnothing,\ \ m_1 \neq m_2, m, m_1, m_2 \in \boldsymbol{Z}, \quad (1)$$

where $\boldsymbol{W}_{c_m} = \left[\widetilde{t}_m, \widetilde{t}_{m+1}\right),\ m \in \boldsymbol{Z}\ \left(0 < \widetilde{t}_{m+1} - \widetilde{t}_m < \infty\right)$. Set $\boldsymbol{D}_c = \left\{\widetilde{t}_m,\ m \in \boldsymbol{Z}\right\}$ is a subset of $\boldsymbol{R}$, the elements of which correspond to the moments of the beginning of cycles of a cyclic signal. Let us assume that the elements of the partition $\boldsymbol{D}_{\boldsymbol{R}}^{c} = \{\boldsymbol{W}_{c_m} \subset \boldsymbol{W},\ m \in \boldsymbol{Z}\}$ are mathematical objects that define the definition domains for cycles of a cyclic signal.

Due to the linear ordering of the numerical set $\boldsymbol{R}$, elements $\boldsymbol{W}_{c_m}$ of the partition $\boldsymbol{D}_{\boldsymbol{R}}^{c}$ are also linearly ordered numerical sets. Let us consider the elements $\boldsymbol{W}_{c_m}$ of partition $\boldsymbol{D}_{\boldsymbol{R}}^{c}$ as carriers of relational systems $\langle \boldsymbol{W}_{c_m}, \leq \rangle$ with a binary relation of linear order $\leq$ (reflexive, antisymmetric, transitive, consistent binary relation). Thus, the partition $\boldsymbol{D}_{\boldsymbol{R}}^{c}$ generates ordered (ordered by m) countable family $\boldsymbol{RS}_{\boldsymbol{R}}^{c} = \{\langle \boldsymbol{W}_{c_m}, \leq \rangle,\ m \in \boldsymbol{Z}\}$ of subrelational systems of a relational system $\langle \boldsymbol{R}, \leq \rangle$, between which there is an isomorphism with respect to linear order $\leq$, namely:

(a) there is a bijection (we will denote its sign " $\Leftrightarrow$ ") between $\boldsymbol{W}_{c_{m_1}}$ and $\boldsymbol{W}_{c_{m_2}}$ $(m_1, m_2 \in \boldsymbol{Z})$, namely: any one $t \in \boldsymbol{W}_{c_{m_1}}$ corresponds to only one $t' \in \boldsymbol{W}_{c_{m_2}}$ $(t \to t')$ and any one $t' \in \boldsymbol{W}_{c_{m_2}}$ corresponds to only one $t \in \boldsymbol{W}_{c_{m_1}}$ $(t' \to t)$, and for any different $t_1, t_2 \in \boldsymbol{W}_{c_{m_1}}$ $(t_1 \neq t_2)$, their images $t_1', t_2' \in \boldsymbol{W}_{c_{m_2}}$ are different $(t_1' \neq t_2')$, and vice versa (we will say that the elements t and t' are bijective connected (or are in a bijective connection) regarding bijection (bijective mapping) $\Leftrightarrow$ and is denoted as follows: $t \leftrightarrow t'$);

(b) the same type of linear ordering of sets $\boldsymbol{W}_{c_{m_1}}$ and $\boldsymbol{W}_{c_{m_2}}$ and takes place, namely $\forall t_1, t_2 \in \boldsymbol{W}_{c_{m_1}}\ \exists t_1', t_2' \in \boldsymbol{W}_{c_{m_2}}$ where $t_1' \leftrightarrow t_1, t_2' \leftrightarrow t_2$ and where such relation takes place $t_1' \leq t_2'$ if $t_1 \leq t_2$, and vice versa.

We carry out a series of important steps in the formalization of the cycles of the cyclic signal itself as some functional relations with certain common properties. According to the conceptual model, the ordering of the phases and cycles of the cyclic signal is postulated, and it is argued that all the cycles have the same type of phase order. In order to reflect these statements, we introduce the linear order in the random process $\xi(\omega, t), \omega \in \boldsymbol{\Omega}, t \in \boldsymbol{R}$ itself and we will show an isomorphism with respect to the linear order of its segments, which describe the cycles of a cyclic signal.

In this case, we will write the random process $\xi(\omega, t), \omega \in \boldsymbol{\Omega}, t \in \boldsymbol{R}$ as a set of pairs (argument t, value $\xi(\omega, t)$) $\xi = \{(t, \xi(\omega, t)) :\ t \in \boldsymbol{R}\}$ and consider the bijective mapping $\boldsymbol{R} \Leftrightarrow \xi$ of the definition domain $\boldsymbol{R}$ of random process ξ on the random process ξ itself. Such a bijective mapping ($\boldsymbol{R} \Leftrightarrow \xi$) can always be constructed, because any element $t \in \boldsymbol{R}$ corresponds to one and only one ordered pair (vector) $(t,\ \xi(\omega, t))$ from ξ and vice versa, and for the two different $t_1, t_2 \in \boldsymbol{R}$, the corresponding ordered pairs $(t_1, \xi(\omega, t_1)), (t_2, \xi(\omega, t_2))$ are also different and vice versa.

The bijective mapping $\boldsymbol{R} \Leftrightarrow \xi$ induces (generates) a linear order into the random process $\xi = \{(t, \xi(\omega, t)) :\ t \in \boldsymbol{R}\}$ itself, which in this case can be considered as a carrier of the relational system $\langle \xi, \leq_2 \rangle$ with a binary relation of linear order $\leq_2$. The ordinal type of ξ coincides with the ordinal type of the set $\boldsymbol{R}$. Namely, for any two ordered pairs $(t_1, \xi(\omega, t_1)) \in \xi, (t_2, \xi(\omega, t_2)) \in \xi$, it is always possible to specify their order: $(t_1, \xi(\omega, t_1)) \leq_2 (t_2, \xi(\omega, t_2))$ if $t_1 \leq t_2$ or $(t_2, \xi(\omega, t_2)) \leq_2 (t_1, \xi(\omega, t_1))$ if $t_2 \leq t_1$. In other words, the bijective mapping $\boldsymbol{R} \Leftrightarrow \xi$ is an isomorphism between the relational system $\langle \boldsymbol{R}, \leq \rangle$ and the relational system $\langle \xi, \leq_2 \rangle$ with respect to the binary relations of linear order $\leq$ and $\leq_2$ ($\langle \boldsymbol{R}, \leq \rangle \Leftrightarrow \langle \xi, \leq_2 \rangle$). Therefore, we will talk about $\xi = \{(t, \xi(\omega, t)) : t \in \boldsymbol{R}\}$ as about a linear ordered random process by the type of ordering of its domain $\boldsymbol{R}$.

Due to the bijective mapping between $\boldsymbol{R}$ and ξ, the ordered countable partition $\boldsymbol{D}^c_{\boldsymbol{R}} = \{\boldsymbol{W}_{c_m}, m \in \boldsymbol{Z}\}$ of domain $\boldsymbol{R}$ generates an ordered countable partition $\boldsymbol{D}^c_{\xi} = \{\xi_{c_m} \subset \xi, m \in \boldsymbol{Z}\}$ of random process $\xi = \{(t, \xi(\omega, t)) : t \in \boldsymbol{R}\}$, where every random process $\xi_{c_m} = \{(t, \xi(\omega, t)) \in \xi : t \in \boldsymbol{W}_{c_m}\}$ is the truncation of the random process $\xi = \{(t, \xi(\omega, t)) : t \in \boldsymbol{R}\}$ to the set $\boldsymbol{W}_{c_m}$. Namely, each set $\boldsymbol{W}_{c_m}$ matches the random process $\xi_{c_m} = \{(t, \xi(\omega, t)) \in \xi : t \in \boldsymbol{W}_{c_m}\} \subset \xi$, which is its image concerning bijective mapping $\boldsymbol{R} \Leftrightarrow \xi$. That is, every random process $\xi_{c_m} = \{(t, \xi(\omega, t)) \in \xi : t \in \boldsymbol{W}_{c_m}\}$ is the set of those ordered pairs $\{(t, \xi(\omega, t)) : t \in \boldsymbol{W}_{c_m}\}$ of the random process ξ, the argument t of which belongs to $\boldsymbol{W}_{c_m} (t \leftrightarrow (t, \xi(\omega, t)), t \in \boldsymbol{W}_{c_m})$.

Since the random process ξ is the carrier of the relational system $\langle \xi, \leq_2 \rangle$, with its partition $\boldsymbol{D}^c_{\xi}$, it can always form the countable family $\boldsymbol{RS}^c_{\xi} = \{\langle \xi_{c_m}, \leq_2 \rangle, m \in \boldsymbol{Z}\}$ of the subrelational systems of system $\langle \xi, \leq_2 \rangle$. From the isomorphism between the subrelational systems $\{\langle \boldsymbol{W}_{c_m}, \leq \rangle, m \in \boldsymbol{Z}\}$ with respect to the binary relation of linear order $\leq$ due to the isomorphism $\langle \boldsymbol{R}, \leq \rangle \Leftrightarrow \langle \xi, \leq_2 \rangle$ follows the isomorphism between the subrelational systems $\{\langle \xi_{c_m}, \leq_2 \rangle, m \in \boldsymbol{Z}\}$ with respect to binary relation of linear order $\leq_2$. Namely, for any $m_1, m_2 \in \boldsymbol{Z}$, arbitrary subrelational systems $\langle \xi_{c_{m_1}}, \leq_2 \rangle$ and $\langle \xi_{c_{m_2}}, \leq_2 \rangle$ from $\boldsymbol{RS}^c_{\xi}$ are isomorphic with respect to binary relation of linear order $\leq_2$, and for any $m \in \boldsymbol{Z}$, the random process ξ_{c_m} is the linear ordered random process by the type of ordering of its domain $\boldsymbol{W}_{c_m}$. More precisely, the isomorphism between the subrelational systems $\{\langle \xi_{c_m}, \leq_2 \rangle, m \in \boldsymbol{Z}\}$ means that:

(a) there is a bijective mapping between $\xi_{c_{m_1}}$ and $\xi_{c_{m_2}}$ ($\xi_{c_{m_1}} \Leftrightarrow \xi_{c_{m_2}}, m_1, m_2 \in \boldsymbol{Z}$), namely: any one $(t, \xi(\omega, t)) \in \xi_{c_{m_1}}$ corresponds to only one $(t', \xi(\omega, t')) \in \xi_{c_{m_2}}$ and any one $(t', \xi(\omega, t')) \in \xi_{c_{m_2}}$ corresponds to only one $(t, \xi(\omega, t)) \in \xi_{c_{m_1}}$, and for any different $(t_1, \xi(\omega, t_1)), (t_2, \xi(\omega, t_2)) \in \xi_{c_{m_1}} (t_1 \neq t_2)$, their images $(t'_1, \xi(\omega, t'_1)), (t'_2, \xi(\omega, t'_2)) \in \xi_{c_{m_2}}$ are different ($t'_1 \neq t'_2$), and vice versa;

(b) due to identity of the types of ordering of the sets $\boldsymbol{W}_{c_{m_1}}$ and $\boldsymbol{W}_{c_{m_2}}$, the types of ordering of random processes $\xi_{c_{m_1}}$ and $\xi_{c_{m_2}}$ are identical, that is, for any different ordered pairs $(t_1, \xi(\omega, t_1)), (t_2, \xi(\omega, t_2)) \in \xi_{c_{m_1}}$ there exist such ordered pairs $(t'_1, \xi(\omega, t'_1)), (t'_2, \xi(\omega, t'_2)) \in \xi_{c_{m_2}}$ that are in bijective connection with ordered pairs $(t_1, \xi(\omega, t_1)), (t_2, \xi(\omega, t_2)) \in \xi_{c_{m_1}}$ $((t_1, \xi(\omega, t_1)) \leftrightarrow (t'_1, \xi(\omega, t'_1)), (t_2, \xi(\omega, t_2)) \leftrightarrow (t'_2, \xi(\omega, t'_2)))$, and the following relations take place: $(t_1, \xi(\omega, t_1)) \leq_2 (t_2, \xi(\omega, t_2))$ and $(t'_1, \xi(\omega, t'_1)) \leq_2 (t'_2, \xi(\omega, t'_2))$, if $t'_1 \leq t'_2$, $t_1 \leq t_2$, $(t'_1 \leftrightarrow t_1, t'_2 \leftrightarrow t_2)$, and vice versa.

So, taking into account mentioned above, it can be argued that there are: (1) an isomorphism with respect to binary relations of linear order $\leq$ and $\leq_2$ between relational systems $\langle \boldsymbol{R}, \leq \rangle$ and $\langle \xi, \leq_2 \rangle$; (2) an isomorphism with respect to binary relation of linear order $\leq$ between elements of the countable family $\boldsymbol{RS}^c_{\boldsymbol{R}} = \{\langle \boldsymbol{W}_{c_m}, \leq \rangle, m \in \boldsymbol{Z}\}$ of subrelational systems of relational system $\langle \boldsymbol{R}, \leq \rangle$; (3) an isomorphism with respect to binary relation of linear order $\leq_2$ between elements of the countable family $\boldsymbol{RS}^c_{\xi} = \{\langle \xi_{c_m}, \leq_2 \rangle, m \in \boldsymbol{Z}\}$ of subrelational systems of relational system $\langle \xi, \leq_2 \rangle$; and (4) an isomorphism with respect to binary relations of linear order $\leq$ and $\leq_2$ between arbitrary pair $\boldsymbol{W}_{c_{m_2}}$ and $\xi_{c_{m_1}}$, $m_1, m_2 \in \boldsymbol{Z}$, taken from the countable partition $\boldsymbol{D}^c_{\boldsymbol{R}} = \{\boldsymbol{W}_{c_m}, m \in \boldsymbol{Z}\}$ of domain $\boldsymbol{R}$ and from the countable partition $\boldsymbol{D}^c_{\xi} = \{\xi_{c_m} \subset \xi, m \in \boldsymbol{Z}\}$ of random process $\xi = \{(t, \xi(\omega, t)) : t \in \boldsymbol{R}\}$.

Let us consider the ordered Cartesian square.

$\xi^2 = \{((t_1, \xi(\omega, t_1)), (t_2, \xi(\omega, t_2))) : t_1, t_2 \in \boldsymbol{R}\}$ of the random process ξ, and the bijective mapping $\boldsymbol{R}^2 \Leftrightarrow \xi^2$, which can always be constructed, because any ordered pair $(t_1, t_2) \in \boldsymbol{R}^2$ corresponds to one and only one ordered pair $((t_1, \xi(\omega, t_1)), (t_2, \xi(\omega, t_2))) \in \xi^2$ and vice versa, and for the two different ordered pairs $(t_1, t_2) \in \boldsymbol{R}^2$ and $(t_3, t_4) \in \boldsymbol{R}^2$ the corresponding ordered pairs $((t_1, \xi(\omega, t_1)), (t_2, \xi(\omega, t_2))), ((t_3, \xi(\omega, t_3)), (t_4, \xi(\omega, t_4))) \in \xi^2$ are also different, and vice versa.

Note that the set $\boldsymbol{R}^2$ will be considered as the carrier of the relational system $\langle \boldsymbol{R}^2, \leq_3 \rangle$, with a binary relation of linear order $\leq_3$. Namely, for any two ordered pairs $(t_1, t_2), (t_3, t_4) \in \boldsymbol{R}^2$, it is always possible to specify their order: $(t_1, t_2) \leq_3 (t_3, t_4)$ if $t_1 \leq t_3$ or $(t_3, t_4) \leq_3 (t_1, t_2)$ if $t_3 \leq t_1$. In the case when $t_1 = t_3$, we will have the following order: $(t_1, t_2) \leq_3 (t_3, t_4)$ if $t_2 \leq t_4$ or $(t_3, t_4) \leq_3 (t_1, t_2)$ if $t_4 \leq t_2$.The bijective mapping $\boldsymbol{R}^2 \Leftrightarrow \xi^2$ induces (generates) a linear order in the Cartesian square ξ^2 itself, which in this case can be considered as a carrier of the relational system $\langle \xi^2, \leq_4 \rangle$ with a binary relation of linear order $\leq_4$. The ordinal type of ξ^2 coincides with the ordinal type of the set $\boldsymbol{R}^2$. Namely, for any two ordered pairs $((t_1, \xi(\omega, t_1)), (t_2, \xi(\omega, t_2))), ((t_3, \xi(\omega, t_3)), (t_4, \xi(\omega, t_4))) \in \xi^2$ it is always possible to specify their order: $((t_1, \xi(\omega, t_1)), (t_2, \xi(\omega, t_2))) \leq_4 ((t_3, \xi(\omega, t_3)), (t_4, \xi(\omega, t_4)))$ if $t_1 \leq t_3$ or $((t_3, \xi(\omega, t_3)), (t_4, \xi(\omega, t_4))) \leq_4 ((t_1, \xi(\omega, t_1)), (t_2, \xi(\omega, t_2)))$ if $t_3 \leq t_1$. In the case when $t_1 = t_3$, we will have such an order: $((t_1, \xi(\omega, t_1)), (t_2, \xi(\omega, t_2))) \leq_4 ((t_3, \xi(\omega, t_3)), (t_4, \xi(\omega, t_4)))$ if $t_2 \leq t_4$ or $((t_3, \xi(\omega, t_3)), (t_4, \xi(\omega, t_4))) \leq_4 ((t_1, \xi(\omega, t_1)), (t_2, \xi(\omega, t_2)))$ if $t_4 \leq t_2$.

In other words, the bijective mapping $\boldsymbol{R}^2 \Leftrightarrow \xi^2$ is an isomorphism between the relational system $\langle \boldsymbol{R}^2, \leq_3 \rangle$ and the relational system $\langle \xi^2, \leq_4 \rangle$ with respect to the binary relations of linear order $\leq_3$ and $\leq_4$ ($\langle \boldsymbol{R}^2, \leq_3 \rangle \Leftrightarrow \langle \xi^2, \leq_4 \rangle$). That is, we will talk about ξ^2 as if about a linear ordered one by the type of ordering of the $\boldsymbol{R}^2$ Cartesian square of the random process ξ.

Let us form an ordered by m countable partition $\boldsymbol{D}^c_{\boldsymbol{R}^2} = \{\boldsymbol{W}_{c_m} \times \boldsymbol{R},\ m \in \boldsymbol{Z}\}$ of $\boldsymbol{R}^2$ based on the ordered countable partition $\boldsymbol{D}^c_{\boldsymbol{R}} = \{\boldsymbol{W}_{c_m},\ m \in \boldsymbol{Z}\}$ of domain $\boldsymbol{R}$. Due to the linear ordering $\leq_3$ of the set $\boldsymbol{R}^2$, the elements $\boldsymbol{W}_{c_m} \times \boldsymbol{R}$ of the partition $\boldsymbol{D}^c_{\boldsymbol{R}^2}$ are also linearly ordered numerical sets. Let us consider the elements $\boldsymbol{W}_{c_m} \times \boldsymbol{R}$ of partition $\boldsymbol{D}^c_{\boldsymbol{R}^2}$ as the carriers of relational systems $\langle \boldsymbol{W}_{c_m} \times \boldsymbol{R}, \leq_3 \rangle$ with a binary relation of linear order $\leq_3$. Thus, the partition $\boldsymbol{D}^c_{\boldsymbol{R}^2}$ generates ordered (ordered by m) countable family $\boldsymbol{RS}^c_{\boldsymbol{R}^2} = \{\langle \boldsymbol{W}_{c_m} \times \boldsymbol{R}, \leq_3 \rangle,\ m \in \boldsymbol{Z}\}$ of subrelational systems of relational system $\langle \boldsymbol{R}^2, \leq_3 \rangle$, between which there is an isomorphism with respect to linear order $\leq_3$, namely:

(a) there is a bijection between $\boldsymbol{W}_{c_{m_1}} \times \boldsymbol{R}$ and $\boldsymbol{W}_{c_{m_2}} \times \boldsymbol{R}$ ($m_1, m_2 \in \boldsymbol{Z}$), namely: any one $(t_1, t_2) \in \boldsymbol{W}_{c_{m_1}} \times \boldsymbol{R}$ corresponds to only one $(t'_1, t'_2) \in \boldsymbol{W}_{c_{m_2}} \times \boldsymbol{R}$ $((t_1, t_2) \to (t'_1, t'_2))$ and any one $(t'_1, t'_2) \in \boldsymbol{W}_{c_{m_2}} \times \boldsymbol{R}$ corresponds to only one $(t_1, t_2) \in \boldsymbol{W}_{c_{m_1}} \times \boldsymbol{R}$ $((t'_1, t'_2) \to (t_1, t_2))$, and for any different $(t_1, t_2), (t_3, t_4) \in \boldsymbol{W}_{c_{m_1}} \times \boldsymbol{R}$ their images $(t'_1, t'_2), (t'_3, t'_4) \in \boldsymbol{W}_{c_{m_2}} \times \boldsymbol{R}$ are different, and vice versa (we will say that the elements (t_1, t_2) and (t'_1, t'_2) are bijective connected (or are in a bijective connection) regarding bijection (bijective mapping) $\Leftrightarrow$ and denote this as follows: $(t_1, t_2) \to (t'_1, t'_2)$);

(b) the same type of linear ordering of sets $\boldsymbol{W}_{c_{m_1}} \times \boldsymbol{R}$ and $\boldsymbol{W}_{c_{m_2}} \times \boldsymbol{R}$ takes place, namely, $\forall (t_1, t_2),\ (t_3, t_4) \in \boldsymbol{W}_{c_{m_1}} \times \boldsymbol{R}\ \exists (t'_1, t'_2), (t'_3, t'_4) \in \boldsymbol{W}_{c_{m_2}} \times \boldsymbol{R}$ where $(t'_1, t'_2) \leftrightarrow (t_1, t_2), (t'_3, t'_4) \leftrightarrow (t_3, t_4)$ and such relation take place $(t'_1, t'_2) \leq_3 (t'_3, t'_4)$ if $(t_1, t_2) \leq_3 (t_3, t_4)$, and vice versa.

Due to the bijective mapping $\boldsymbol{R}^2 \Leftrightarrow \xi^2$, the partition $\boldsymbol{D}^c_{\boldsymbol{R}^2} = \{\boldsymbol{W}_{c_m} \times \boldsymbol{R},\ m \in \boldsymbol{Z}\}$ of $\boldsymbol{R}^2$ generates an ordered countable partition $\boldsymbol{D}^c_{\xi^2} = \{\xi_{c_m} \times \xi \subset \xi^2,\ m \in \boldsymbol{Z}\}$ of Cartesian square ξ^2, where every $\xi_{c_m} \times \xi$ is the truncation of the ξ^2 to the set $\boldsymbol{W}_{c_m} \times \boldsymbol{R}$. Namely, each set $\boldsymbol{W}_{c_m} \times \boldsymbol{R}$ matches the $\xi_{c_m} \times \xi$, which is its image according to bijective mapping $\boldsymbol{R}^2 \Leftrightarrow \xi^2$. That is, every $\xi_{c_m} \times \xi$ is the set of those ordered pairs $\{((t_1, \xi(\omega, t_1)), (t_2, \xi(\omega, t_2))) : t_1 \in \boldsymbol{W}_{c_m}, t_2 \in \boldsymbol{R}\}$ of the ξ^2, the argument t_1 of which belongs to $\boldsymbol{W}_{c_m}$ and the argument t_2 belongs to $\boldsymbol{R}$.

Since the Cartesian square ξ^2 is the carrier of the relational system $\langle \xi^2, \leq_4 \rangle$, then with its partition $\boldsymbol{D}^c_{\xi^2}$, it can always form the countable family $\boldsymbol{RS}^c_{\xi^2} = \{\langle \xi_{c_m} \times \xi, \leq_4 \rangle,\ m \in \boldsymbol{Z}\}$ of the subrelational systems of system $\langle \xi^2, \leq_4 \rangle$. From the isomorphism between the subrelational systems $\{\langle \boldsymbol{W}_{c_m} \times \boldsymbol{R}, \leq_3 \rangle,\ m \in \boldsymbol{Z}\}$ with respect to the binary relation of linear order $\leq_3$ due to the isomorphism $\langle \boldsymbol{R}^2, \leq_3 \rangle \Leftrightarrow \langle \xi^2, \leq_4 \rangle$ follows the isomorphism between the subrelational systems $\{\langle \xi_{c_m} \times \xi, \leq_4 \rangle,\ m \in \boldsymbol{Z}\}$ with respect to binary relation of linear order $\leq_4$. Namely, for any $m_1, m_2 \in \boldsymbol{Z}$, arbitrary subrelational systems $\langle \xi_{c_{m_1}} \times \xi, \leq_4 \rangle$ and $\langle \xi_{c_{m_2}} \times \xi, \leq_4 \rangle$ from $\boldsymbol{RS}^c_{\xi^2}$ are isomorphic with respect to the binary relation of linear order

$\leq_4$, and for any $m \in \boldsymbol{Z}$ Cartesian product, $\xi_{c_m} \times \xi$ is linearly ordered by the type of ordering of its domain $\boldsymbol{W}_{c_m} \times \boldsymbol{R}$.

So, taking into account mentioned above, it can be argued that there are: (1) an isomorphism with respect to binary relations of linear order $\leq_3$ and $\leq_4$ between relational systems $\langle \boldsymbol{R}^2, \leq_3 \rangle$ and $\langle \xi^2, \leq_4 \rangle$; (2) an isomorphism with respect to the binary relation of linear order $\leq_3$ between the elements of the countable family $\boldsymbol{RS}^c_{\boldsymbol{R}^2} = \{ \langle \boldsymbol{W}_{c_m} \times \boldsymbol{R}, \leq_3 \rangle,\ m \in \boldsymbol{Z} \}$ of subrelational systems of relational system $\langle \boldsymbol{R}^2, \leq_3 \rangle$; (3) an isomorphism with respect to the binary relation of linear order $\leq_4$ between elements of the countable family $\boldsymbol{RS}^c_{\xi^2} = \{ \langle \xi_{c_m} \times \xi, \leq_4 \rangle,\ m \in \boldsymbol{Z} \}$ of subrelational systems of the relational system $\langle \xi^2, \leq_4 \rangle$; (4) an isomorphism with respect to the binary relations of linear order $\leq_3$ and $\leq_4$ between arbitrary pair $\boldsymbol{W}_{c_{m_2}} \times \boldsymbol{R}$ and $\xi_{c_{m_1}} \times \xi,\ m_1, m_2 \in \boldsymbol{Z}$, taken from the countable partition $\boldsymbol{D}^c_{\boldsymbol{R}^2} = \{ \boldsymbol{W}_{c_m} \times \boldsymbol{R},\ m \in \boldsymbol{Z} \}$ of set $\boldsymbol{R}^2$ and from the countable partition $\boldsymbol{D}^c_{\xi^2} = \{ \xi_{c_m} \times \xi \subset \xi^2,\ m \in \boldsymbol{Z} \}$ of the Cartesian square ξ^2.

The mathematical objects introduced above, namely, the Cartesian square ξ^2 and binary relation of linear order $\leq_4$, allow us to supplement the relational system $\langle \xi, \leq_2 \rangle$ with a new carrier ξ^2 and relation $\leq_4$. As a result, we form a new relational system $\langle \xi, \xi^2, \{\leq_2, \leq_4\} \rangle$, which is the basis for taking into account not only the one-dimensional cyclic structure of the investigated signal, but also its two-dimensional cyclic structure.

The mathematical objects constructed above formally reflect the segmental structure of the cyclic signals, the same type of phases ordering in all segments-cycles of the cyclic signals, which are necessary but not sufficient conditions of full-fledged cyclic signal modeling within the correlation theory of random processes. To build an adequate mathematical model of the cyclic signals within the correlation theory of random processes, it is necessary to take into account the similarities of the cycles of a cyclic signal, not only regarding their type of phases ordering, but also regarding their mathematical expectation $m_\xi(t)$ and correlation function $r_{2\xi}(t_1, t_2)$. For this purpose, the previously developed mathematical structures will be supplemented with new objects that formally take into account the similarity between the cycles of the cyclic signal in relation to the mathematical expectation and the correlation function of the random process ξ.

First of all, let us supplement the relational system $\langle \xi, \xi^2, \{\leq_2, \leq_4\} \rangle$ with a new carrier $\boldsymbol{R}$ and two new functional relations $p_1 : \xi \to \boldsymbol{R}$ and $p_2 : \xi^2 \to \boldsymbol{R}$. As a result, we form a new relational system $\langle \xi, \xi^2, \boldsymbol{R}, \{\leq_2, \leq_4, p_1 : \xi \to \boldsymbol{R}, p_2 : \xi^2 \to \boldsymbol{R}\} \rangle$ with three carriers $\xi, \xi^2, \boldsymbol{R}$ and four relations $\leq_2, \leq_4, p_1 : \xi \to \boldsymbol{R},\ p_2 : \xi^2 \to \boldsymbol{R}$, where $p_1 : \xi \to \boldsymbol{R}$ is a functional relation, which represents the mathematical expectation operator $\boldsymbol{E}\{\cdot\}$, acting on the values of the random process ξ, the result of which is the mathematical expectation $m_\xi(t)$ of this random process, namely:

$$p_1((t, \xi(\omega, t))) = p_1(\xi(\omega, t)) = \boldsymbol{E}\{\xi(\omega, t)\} = m_\xi(t) \in \boldsymbol{R},\ t \in \boldsymbol{R}; \tag{2}$$

$p_2 : \xi^2 \to \boldsymbol{R}$ is a functional relation, which represents the correlation operator $\boldsymbol{Cor}\{\cdot,\cdot\}$, acting on the values of the random process ξ, the result of which is the correlation function $r_{2\xi}(t_1, t_2)$ of a random process ξ:

$$\begin{aligned} &p_2((t_1, \xi(\omega, t_1)), (t_2, \xi(\omega, t_2))\ ) = p_2(\xi(\omega, t_1), \xi(\omega, t_2)) = \\ &= \boldsymbol{Cor}\{\xi(\omega, t_1), \xi(\omega, t_2)\} = \boldsymbol{E}\{ (\xi(\omega, t_1) - m_\xi(t_1)) \cdot (\xi(\omega, t_2) - m_\xi(t_2)) \} = \\ &= r_{2\xi}(t_1, t_2) \in \boldsymbol{R},\ t_1, t_2 \in \boldsymbol{R},\ \omega \in \boldsymbol{\Omega}. \end{aligned} \tag{3}$$

In order to exclude non-cyclic processes (namely, stationary in the wide sense random processes), we will consider only such functional relations $p_1((t, \xi(\omega, t)))$ and $p_2((t_1, \xi(\omega, t_1)), (t_2, \xi(\omega, t_2))\ )$, for which exist such numbers $T \in \boldsymbol{R}$, that there are the following inequalities:

$$p_1((t, \xi(\omega, t))\ ) \neq p_1((t + T, \xi(\omega, t + T))\ ),\ t \in \boldsymbol{R},$$

$$p_2((t_1, \xi(\omega, t_1)), (t_2, \xi(\omega, t_2))\ ) \neq p_2((t_1 + T, \xi(\omega, t_1 + T)), (t_2 + T, \xi(\omega, t_2 + T))\ ),$$
$$t_1, t_2 \in \boldsymbol{R}.$$

The partition $\boldsymbol{D}^c_{\xi} = \{\xi_{c_m} \subset \xi,\ m \in \boldsymbol{Z}\}$ of the random process ξ and the partition $\boldsymbol{D}^c_{\xi^2} = \{\xi_{c_m} \times \xi \subset \xi^2,\ m \in \boldsymbol{Z}\}$ of Cartesian square ξ^2 generate the family of subrelational systems $\boldsymbol{RS}^c_{\xi,\xi^2} = \{\langle \xi_{c_m}, \xi_{c_m} \times \xi, \boldsymbol{R}, \{\leq_2, \leq_4, p_1 : \xi \to \boldsymbol{R}, p_2 : \xi^2 \to \boldsymbol{R}\}\rangle,\ m \in \boldsymbol{Z}\}$ of the relational system $\langle \xi, \xi^2, \boldsymbol{R}, \{\leq_2, \leq_4, p_1 : \xi \to \boldsymbol{R}, p_2 : \xi^2 \to \boldsymbol{R}\}\rangle$, where $\xi_{c_m}, \xi_{c_m} \times \xi, \boldsymbol{R}$ are carriers of the subrelational system $\langle \xi_{c_m}, \xi_{c_m} \times \xi, \boldsymbol{R}, \{\leq_2, \leq_4, p_1 : \xi \to \boldsymbol{R}, p_2 : \xi^2 \to \boldsymbol{R}\}\rangle$.

Let us amplify the isomorphism between the relational systems of the family $\boldsymbol{RS}^c_{\xi,\xi^2}$ by adding the requirements of equality of values of mathematical expectations for bijective connected ordered pairs $(t, \xi(\omega, t)) \in \xi_{c_{m_1}}$ and $(t', \xi(\omega, t')) \in \xi_{c_{m_2}}$ from two different arbitrary random processes $\xi_{c_{m_1}}$ and $\xi_{c_{m_2}}$, and the requirements of equality of values of correlation functions for bijective connected ordered pairs $((t_1, \xi(\omega, t_1)), (t_2, \xi(\omega, t_2))) \in \xi_{c_{m_1}} \times \xi$ and $((t'_1, \xi(\omega, t'_1)), (t'_2, \xi(\omega, t'_2))) \in \xi_{c_{m_2}} \times \xi$ from two different arbitrary Cartesian products $\xi_{c_{m_1}} \times \xi$ and $\xi_{c_{m_2}} \times \xi$. Namely, isomorphism with respect to the binary relations of linear order $\leq_2$ and $\leq_4$ for two arbitrary relational systems $\langle \xi_{c_{m_1}}, \xi_{c_{m_1}} \times \xi, \boldsymbol{R}, \{\leq_2, \leq_4, p_1 : \xi \to \boldsymbol{R}, p_2 : \xi^2 \to \boldsymbol{R}\}\rangle$ and $\langle \xi_{c_{m_2}}, \xi_{c_{m_2}} \times \xi, \boldsymbol{R}, \{\leq_2, \leq_4, p_1 : \xi \to \boldsymbol{R}, p_2 : \xi^2 \to \boldsymbol{R}\}\rangle$ must be supplemented by an isomorphism with respect to functional relations $p_1 : \xi \to \boldsymbol{R}$ and $p_2 : \xi^2 \to \boldsymbol{R}$.

This kind of isomorphism between relational systems $\langle \xi_{c_{m_1}}, \xi_{c_{m_1}} \times \xi, \boldsymbol{R}, \{\leq_2, \leq_4, p_1 : \xi \to \boldsymbol{R}, p_2 : \xi^2 \to \boldsymbol{R}\}\rangle$ and $\langle \xi_{c_{m_2}}, \xi_{c_{m_2}} \times \xi, \boldsymbol{R}, \{\leq_2, \leq_4, p_1 : \xi \to \boldsymbol{R}, p_2 : \xi^2 \to \boldsymbol{R}\}\rangle$ will be called an isomorphism with respect to the linear order, mathematical expectation and correlation function. Let us assign a strict definition to this type of isomorphism between the relational systems $\langle \xi_{c_{m_1}}, \xi_{c_{m_1}} \times \xi, \boldsymbol{R}, \{\leq_2, \leq_4, p_1 : \xi \to \boldsymbol{R}, p_2 : \xi^2 \to \boldsymbol{R}\}\rangle$ and $\langle \xi_{c_{m_2}}, \xi_{c_{m_2}} \times \xi, \boldsymbol{R}, \{\leq_2, \leq_4, p_1 : \xi \to \boldsymbol{R}, p_2 : \xi^2 \to \boldsymbol{R}\}\rangle$ for any $m_1, m_2 \in \boldsymbol{Z}$.

Definition 1. *The bijective mapping $\xi_{c_{m_1}} \Leftrightarrow \xi_{c_{m_2}}$ between random processes $\xi_{c_{m_1}}$ and $\xi_{c_{m_2}}$ and the bijective mapping $\xi_{c_{m_1}} \times \xi \Leftrightarrow \xi_{c_{m_2}} \times \xi$ between Cartesian products $\xi_{c_{m_1}} \times \xi$ and $\xi_{c_{m_2}} \times \xi$, which are carriers of relational systems $\langle \xi_{c_{m_1}}, \xi_{c_{m_1}} \times \xi, \boldsymbol{R}, \{\leq_2, \leq_4, p_1 : \xi \to \boldsymbol{R}, p_2 : \xi^2 \to \boldsymbol{R}\}\rangle$ and $\langle \xi_{c_{m_2}}, \xi_{c_{m_2}} \times \xi, \boldsymbol{R}, \{\leq_2, \leq_4, p_1 : \xi \to \boldsymbol{R}, p_2 : \xi^2 \to \boldsymbol{R}\}\rangle$ will be called* ***the isomorphism with respect to binary relations of linear order $\leq_2$ and $\leq_4$, mathematical expectation $m_\xi(t)$ and correlation function $r_{2\xi}(t_1, t_2)$*** *between relational systems $\langle \xi_{c_{m_1}}, \xi_{c_{m_1}} \times \xi, \boldsymbol{R}, \{\leq_2, \leq_4, p_1 : \xi \to \boldsymbol{R}, p_2 : \xi^2 \to \boldsymbol{R}\}\rangle$ and $\langle \xi_{c_{m_2}}, \xi_{c_{m_2}} \times \xi, \boldsymbol{R}, \{\leq_2, \leq_4, p_1 : \xi \to \boldsymbol{R}, p_2 : \xi^2 \to \boldsymbol{R}\}\rangle$ if the following statements are made:*

1. *There is an isomorphism between relational systems $\langle \xi_{c_{m_1}}, \xi_{c_{m_1}} \times \xi, \boldsymbol{R}, \{\leq_2, \leq_4, p_1 : \xi \to \boldsymbol{R}, p_2 : \xi^2 \to \boldsymbol{R}\}\rangle$ and $\langle \xi_{c_{m_2}}, \xi_{c_{m_2}} \times \xi, \boldsymbol{R}, \{\leq_2, \leq_4, p_1 : \xi \to \boldsymbol{R}, p_2 : \xi^2 \to \boldsymbol{R}\}\rangle$ with respect to the linear order $\leq_2$; namely, the types of the ordering of random processes $\xi_{c_{m_1}}$ and $\xi_{c_{m_2}}$ are identical.*
2. *There is an isomorphism between relational systems $\langle \xi_{c_{m_1}}, \xi_{c_{m_1}} \times \xi, \boldsymbol{R}, \{\leq_2, \leq_4, p_1 : \xi \to \boldsymbol{R}, p_2 : \xi^2 \to \boldsymbol{R}\}\rangle$ and $\langle \xi_{c_{m_2}}, \xi_{c_{m_2}} \times \xi, \boldsymbol{R}, \{\leq_2, \leq_4, p_1 : \xi \to \boldsymbol{R}, p_2 : \xi^2 \to \boldsymbol{R}\}\rangle$ with respect to the linear order $\leq_4$; namely, the types of ordering of Cartesian products $\xi_{c_{m_1}} \times \xi$ and $\xi_{c_{m_2}} \times \xi$ are identical.*
3. *There is an isomorphism between relational systems $\langle \xi_{c_{m_1}}, \xi_{c_{m_1}} \times \xi, \boldsymbol{R}, \{\leq_2, \leq_4, p_1 : \xi \to \boldsymbol{R}, p_2 : \xi^2 \to \boldsymbol{R}\}\rangle$ and $\langle \xi_{c_{m_2}}, \xi_{c_{m_2}} \times \xi, \boldsymbol{R}, \{\leq_2, \leq_4, p_1 : \xi \to \boldsymbol{R}, p_2 : \xi^2 \to \boldsymbol{R}\}\rangle$ with respect to mathematical expectation $m_\xi(t)$ of a random process ξ; namely, for all the bijective connected ordered pairs $(t, \xi(\omega, t)) \in \xi_{c_{m_1}}$ and $(t', \xi(\omega, t')) \in \xi_{c_{m_2}}((t, \xi(\omega, t)) \leftrightarrow (t', \xi(\omega, t'))\ )$, there is equality of mathematical expectations, namely:*

$$m_\xi(t) = m_\xi(t'),\ t \in \boldsymbol{W}_{c_{m_1}},\ t' \in \boldsymbol{W}_{c_{m_2}},\ \ t \leftrightarrow t',\ \ m_1, m_2 \in \boldsymbol{Z} \quad (4)$$

4. *There is an isomorphism between relational systems $\langle \xi_{c_{m_1}}, \xi_{c_{m_1}} \times \xi, \boldsymbol{R}, \{\leq_2, \leq_4, p_1 : \xi \to \boldsymbol{R}, p_2 : \xi^2 \to \boldsymbol{R}\}\rangle$ and $\langle \xi_{c_{m_2}}, \xi_{c_{m_2}} \times \xi, \boldsymbol{R}, \{\leq_2, \leq_4, p_1 : \xi \to \boldsymbol{R}, p_2 : \xi^2 \to \boldsymbol{R}\}\rangle$ with respect to correlation function $r_{2\xi}(t_1, t_2)$ of a random process ξ, namely, for all the bijective con-*

nected ordered pairs $((t_1, \xi(\omega, t_1)), (t_2, \xi(\omega, t_2))) \in \xi_{c_{m_1}} \times \xi$ *there exist ordered pairs* $((t_1', \xi(\omega, t_1')), (t_2', \xi(\omega, t_2'))) \in \xi_{c_{m_2}} \times \xi$ *bijective connected to them* $((t_1, \xi(\omega, t_1)) \leftrightarrow (t_1', \xi(\omega, t_1')), (t_2, \xi(\omega, t_2)) \leftrightarrow (t_2', \xi(\omega, t_2')))$, *and there is an equality of correlation functions, namely:*

$$r_{2\xi}(t_1, t_2) = r_{2\xi}(t_1', t_2'), t_1 \in \boldsymbol{W}_{c_{m_1}}, \ t_1' \in \boldsymbol{W}_{c_{m_2}}, \ t_2, t_2' \in \boldsymbol{R}, t_1' \leftrightarrow t_1, t_2' \leftrightarrow t_2, \ m_1, m_2 \in \boldsymbol{Z}. \tag{5}$$

Definition 2. *The Cartesian products* $\xi_{c_{m_1}} \times \xi$ *and* $\xi_{c_{m_2}} \times \xi$*, which are carriers of isomorphic relational systems* $\langle \xi_{c_{m_1}}, \xi_{c_{m_1}} \times \xi, \boldsymbol{R}, \{\leq_2, \leq_4, p_1 : \xi \to \boldsymbol{R}, p_2 : \xi^2 \to \boldsymbol{R}\}\rangle$ *and* $\langle \xi_{c_{m_2}}, \xi_{c_{m_2}} \times \xi, \boldsymbol{R}, \{\leq_2, \leq_4, p_1 : \xi \to \boldsymbol{R}, p_2 : \xi^2 \to \boldsymbol{R}\}\rangle$*, will be called* ***the isomorphic Cartesian products with respect to the binary relation of the linear order*** $\leq_4$ ***and correlation function*** $r_{2\xi}(t_1, t_2)$ *or for short,* ***the isomorphic Cartesian products****.*

Definition 3. *The random processes* $\xi_{c_{m_1}}$ *and* $\xi_{c_{m_2}}$*, which are carriers of isomorphic relational systems* $\langle \xi_{c_{m_1}}, \xi_{c_{m_1}} \times \xi, \boldsymbol{R}, \{\leq_2, \leq_4, p_1 : \xi \to \boldsymbol{R}, p_2 : \xi^2 \to \boldsymbol{R}\}\rangle$ *and* $\langle \xi_{c_{m_2}}, \xi_{c_{m_2}} \times \xi, \boldsymbol{R}, \{\leq_2, \leq_4, p_1 : \xi \to \boldsymbol{R}, p_2 : \xi^2 \to \boldsymbol{R}\}\rangle$*, will be called* ***the isomorphic random processes with respect to binary relation of linear order*** $\leq_2$ ***and mathematical expectation*** $m_\xi(t)$ *(or for short,* ***isomorphic random processes****).*

The family $\boldsymbol{RS}^c_{\xi,\xi^2} = \{\langle \xi_{c_m}, \xi_{c_m} \times \xi, \boldsymbol{R}, \{\leq_2, \leq_4, p_1 : \xi \to \boldsymbol{R}, p_2 : \xi^2 \to \boldsymbol{R}\}\rangle, \ m \in \boldsymbol{Z}\}$ *of isomorphic subrelational systems, the carriers of which are the elements of the ordered countable partitions* $\boldsymbol{D}^c_\xi = \{\xi_{c_m} \subset \xi, \ m \in \boldsymbol{Z}\}$ *and* $\boldsymbol{D}^c_{\xi^2} = \{\xi_{c_m} \times \xi \subset \xi^2, \ m \in \boldsymbol{Z}\}$*, constructed above, makes it possible to give the definition of a cyclically correlated random process, which generalizes the periodically (cyclostationary) correlated random process.*

Definition 4. *The random process* $\xi(\omega, t), \omega \in \boldsymbol{\Omega}, t \in \boldsymbol{R}$ *(*$\xi : \boldsymbol{R} \to L_2(\boldsymbol{\Omega}, \boldsymbol{P})$*), given on the probability space* $(\boldsymbol{\Omega}, \boldsymbol{F}, \boldsymbol{P})$ *and on the set* $\boldsymbol{R}$ *of real numbers, will be called* ***the cyclically correlated random process****, if its ordered countable partition* $\boldsymbol{D}^c_\xi = \{\xi_{c_m} \subset \xi, \ m \in \boldsymbol{Z}\}$ *and the ordered countable partition* $\boldsymbol{D}^c_{\xi^2} = \{\xi_{c_m} \times \xi \subset \xi^2, \ m \in \boldsymbol{Z}\}$ *of its Cartesian square* ξ^2 *exist, the elements of which are carriers of isomorphic relational systems* $\boldsymbol{RS}^c_{\xi,\xi^2} = \{\langle \xi_{c_m}, \xi_{c_m} \times \xi, \boldsymbol{R}, \{\leq_2, \leq_4, p_1 : \xi \to \boldsymbol{R}, p_2 : \xi^2 \to \boldsymbol{R}\}\rangle, \ m \in \boldsymbol{Z}\}$ *with respect to the binary relations of linear order* $\leq_2$ *and* $\leq_4$*, the mathematical expectation* $m_\xi(t)$ *and the correlation function* $r_{2\xi}(t_1, t_2)$ *of the random process* $\xi(\omega, t)$*.*

3.3. The Cycle Structure of the Cyclically Correlated Random Process

Proceeding from the definition of the cyclically correlated random process, we will give a mathematical definitions of the cycle and the set of cycles of the cyclic signal, whose model is a cyclically correlated random process. First of all, we note that not every ordered countable partition $\boldsymbol{D}^c_\xi = \{\xi_{c_m} \subset \xi, \ m \in \boldsymbol{Z}\}$ into the isomorphic random processes of a cyclically correlated random process ξ is its partition into cycles. Since each element ξ_{c_m} of such partition can include two or more cycles of a cyclically correlated random process, and according to the conceptual model of the cyclic signal, the cycle is understood as its smallest segment, which includes all types of phases of the signal only once. To account for this requirement of the conceptual model of formalizing the cycle and the partitioning into cycles of the cyclic signal within a cyclically correlated random process, only a minimal partition needs to be considered.

Under a ***minimal ordered countable partition*** into isomorphic random processes with respect to binary relation of linear order $\leq_2$ and the mathematical expectation $m_\xi(t)$ of cyclically correlated random process $\xi = \{(t, \xi(\omega, t)) : \ t \in \boldsymbol{R}\}$, we understand the following partition $\boldsymbol{D}^c_\xi = \{\xi_{c_m} \subset \xi, \ m \in \boldsymbol{Z}\}$, when the arbitrary partitioning of its elements ξ_{c_m} form a new one smaller partition $\{\xi_n \subset \xi, n \in \boldsymbol{Z}\}$, between the all elements ξ_n of which simultaneously there is no isomorphism with respect to the binary relation of linear order $\leq_2$ and the mathematical expectation $m_\xi(t)$.

Definition 5. *The minimal ordered countable partition* $\boldsymbol{D}_\xi^c = \{\xi_{c_m} \subset \xi,\ m \in \boldsymbol{Z}\}$ *into isomorphic random processes with respect to binary relation of linear order* $\leq_2$ *and the mathematical expectation* $m_\xi(t)$ *of cyclically correlated random process* $\xi = \{(t, \xi(\omega, t)) :\ t \in \boldsymbol{R}\}$ *will be called* ***the partition into cycles of cyclically correlated random process*** ξ, *and random process* ξ_{c_m} *will be called the* m-***th cycle of the cyclically correlated random process*** ξ.

That is, the m-th cycle is a random process ξ_{c_m} which is the truncation of the random process ξ to the area $\boldsymbol{W}_{c_m}$. In this case, we will give the following definition:

Definition 6. *The set* $\boldsymbol{W}_{c_m}$ *will be called* ***the definition domain of*** m-***th cycle*** ξ_{c_m} ***of the cyclically correlated random process*** ξ.

Given the fact that a cyclically correlated random process ξ, in addition to the one-dimensional probability structure determined by its mathematical expectation $m_\xi(t)$, has a two-dimensional probability structure given by its correlation function $r_{2\xi}(t_1, t_2)$, then, in addition to the partition $\boldsymbol{D}_\xi^c = \{\xi_{c_m} \subset \xi,\ m \in \boldsymbol{Z}\}$ into one-dimensional cycles ξ_{c_m}, it is possible to give a definition of partition $\boldsymbol{D}_{\xi^2}^c = \{\xi_{c_m} \times \xi \subset \xi^2,\ m \in \boldsymbol{Z}\}$ Cartesian square ξ^2 into two-dimensional cycles $\xi_{c_m} \times \xi$ of the random process ξ.

Similar to the concept of the minimal ordered countable partition of a cyclic random process into isomorphic random processes, under the ***minimal ordered countable*** partition of Cartesian square ξ^2 of cyclic random process $\xi = \{(t, \xi(\omega, t)) :\ t \in \boldsymbol{R}\}$ into isomorphic Cartesian products $\xi_{c_m} \times \xi$ with respect to binary relation of linear order $\leq_4$ and correlation function $r_{2\xi}(t_1, t_2)$ of random process ξ, we understand the following partition $\boldsymbol{D}_{\xi^2}^c = \{\xi_{c_m} \times \xi \subset \xi^2,\ m \in \boldsymbol{Z}\}$, when the arbitrary partitioning of its elements $\xi_{c_m} \times \xi$ form a new one smaller partition $\{\xi_n \times \xi \subset \xi^2, n \in \boldsymbol{Z}\}$, between the all elements $\xi_n \times \xi$ of which simultaneously there are no isomorphisms with respect to the binary relation of the linear order $\leq_4$ and correlation function $r_{2\xi}(t_1, t_2)$ of the random process ξ.

Definition 7. *The minimal ordered countable partition* $\boldsymbol{D}_{\xi^2}^c = \{\xi_{c_m} \times \xi \subset \xi^2,\ m \in \boldsymbol{Z}\}$ *of Cartesian square* ξ^2 *of the cyclically correlated random process* $\xi = \{(t, \xi(\omega, t)) :\ t \in \boldsymbol{R}\}$ *into isomorphic Cartesian products* $\xi_{c_m} \times \xi$ *with respect to the binary relation of linear order* $\leq_4$ *and correlation function* $r_{2\xi}(t_1, t_2)$ *of the random process* ξ *will be called* ***the partition into two-dimensional cycles of cyclically correlated random process*** ξ, *and the Cartesian product* $\xi_{c_m} \times \xi$ *will be called* ***the*** m-***th two-dimensional cycle of the cyclically correlated random process*** ξ.

That is, the m-th two-dimensional cycle is a Cartesian product $\xi_{c_m} \times \xi$, which is the truncation of the Cartesian square ξ^2 to the area $\boldsymbol{W}_{c_m} \times \boldsymbol{R}$. In this case we will give the following definition:

Definition 8. *The set* $\boldsymbol{W}_{c_m} \times \boldsymbol{R}$ *will be called* ***the definition domain of two-dimensional*** m-***th cycle*** $\xi_{c_m} \times \xi$. ***of the cyclically correlated random process*** ξ.

3.4. The Phase Structure of the Cyclically Correlated Random Process

Another basic concept that characterizes a stage in the development of oscillating, cyclic motion, is the concept of phase. The mathematical structures introduced above, formalizing the concept of a cycle and a cyclic signal, are sufficient to correctly define the concept of a phase as a mathematical object. We will carry out such a formalization.

With partitions $\boldsymbol{D}_{\boldsymbol{R}}^c$ and $\boldsymbol{D}_\xi^c$ two more partitions $\boldsymbol{D}_{\boldsymbol{R}}^{ph}$ and $\boldsymbol{D}_\xi^{ph}$ are connected which are used in the formalization of the concept of phase of a cyclic signal within its model in the form of the cyclically correlated random process. Let us show how the partitions $\boldsymbol{D}_{\boldsymbol{R}}^{ph}$ and $\boldsymbol{D}_{\xi}^{ph}$ are formed.

Let us have the definition domain $\boldsymbol{W}_{c_{m_1}}$ of m_1-th cycle $\xi_{c_{m_1}}$. Due to the isomorphism between relational systems $\langle \boldsymbol{W}_{c_{m_1}}, \leq \rangle$ and $\langle \boldsymbol{W}_{c_{m_2}}, \leq \rangle$ ($m_1, m_2 \in \boldsymbol{Z}$, $m_1 \neq m_2$), for any $t_{m_1}^{\varphi} \in \boldsymbol{W}_{c_{m_1}}$ in the definition domain $\boldsymbol{W}_{c_{m_2}}$ of arbitrary m_2-th cycle $\xi_{c_{m_2}}$, there is only one element $t_{m_2}^{\varphi} \in \boldsymbol{W}_{c_{m_2}}$, which is bijective connected with $t_{m_1}^{\varphi} \left(t_{m_1}^{\varphi} \leftrightarrow t_{m_2}^{\varphi} \right)$. Since for a cyclically correlated random process ξ we have a countable set $\boldsymbol{D}_{\xi}^{c}$ of cycles, then for every $t_{m_1}^{\varphi} \in \boldsymbol{W}_{c_{m_1}}$ we will have a countable set $\boldsymbol{W}_{\varphi}$ of elements t_m^{φ}, which are bijective connected with it. Set $\boldsymbol{W}_{\varphi}$ of all bijective connected elements with an element $t_{m_1}^{\varphi}$ is defined as follows:

$$\boldsymbol{W}_{\varphi} = \left\{ t_m^{\varphi} : t_m^{\varphi} \in \boldsymbol{W}_{c_m}, t_m^{\varphi} \leftrightarrow t_{m_1}^{\varphi},\ m \in \boldsymbol{Z}, m_1 = const \right\}. \tag{6}$$

For each fixed $t_{m_1}^{\varphi} \in \boldsymbol{W}_{c_{m_1}}$ we will have specific set $\boldsymbol{W}_{\varphi}$. If $t_{m_1}^{\varphi}$ runs the ordered set $\boldsymbol{W}_{c_{m_1}}$ then we obtain the ordered in the index φ uncountable partition $\boldsymbol{D}_{\boldsymbol{R}}^{ph} = \left\{ \boldsymbol{W}_{\varphi}, \varphi \in \boldsymbol{I} \subset \boldsymbol{R} \right\}$ (***I***—index numerical set) of the definition domain $\boldsymbol{R}$ of cyclically correlated random process ξ. Index numeric set $\boldsymbol{I}$. can be taken as equal to any of the sets $\boldsymbol{W}_{c_m}$. Let us accept that $\boldsymbol{I} = \boldsymbol{W}_{c_0} = \left[\widetilde{t}_0, \widetilde{t}_1 \right)$.

Let us create an ordered in the index φ uncountable partition $\boldsymbol{D}_{\xi}^{ph} = \left\{ \xi_{\varphi},\ \varphi \in \boldsymbol{I} \right\}$ of cyclically correlated random process ξ by bijective mapping elements $\boldsymbol{W}_{\varphi}$ from partition $\boldsymbol{D}_{\boldsymbol{R}}^{ph} = \left\{ \boldsymbol{W}_{\varphi}, \varphi \in \boldsymbol{I} \right\}$ into subsets ξ_{φ} of random process ξ ($\boldsymbol{W}_{\varphi} \Leftrightarrow \xi_{\varphi}$); that is, every $\boldsymbol{W}_{\varphi}$ is matched by the subset $\xi_{\varphi} = \left\{ (t, \xi(\omega, t)), t \in \boldsymbol{W}_{\varphi} \right\} \subset \xi$ of those pairs $(t, \xi(\omega, t))$ of the cyclically correlated random process ξ, the first elements t of which belongs to $\boldsymbol{W}_{\varphi} \left(t \leftrightarrow (t, \xi(\omega, t)), t \in \boldsymbol{W}_{\varphi} \right)$. Since $\boldsymbol{W}_{\varphi}$ is a countable set, then ξ_{φ} is also a countable set, defined as:

$$\xi_{\varphi} = \left\{ \left(t_m^{\varphi}, \xi\left(\omega, t_m^{\varphi} \right) \right) : t_m^{\varphi} \in \boldsymbol{W}_{c_m}, t_m^{\varphi} \leftrightarrow t_{m_1}^{\varphi},\ m \in \boldsymbol{Z}, m_1 = const \right\}. \tag{7}$$

As it can be seen from Expression (7), any set $\xi_{\varphi} = \left\{ \left(t_m^{\varphi}, \xi\left(\omega, t_m^{\varphi} \right) \right), m \in \boldsymbol{Z} \right\}$ is ordered by m countable set.

Because the index numerical set $\boldsymbol{I}$ is isomorphic with respect to the linear order $\leq$ for any set $\boldsymbol{W}_{c_m}$, then between the partition $\boldsymbol{D}_{\xi}^{ph} = \left\{ \xi_{\varphi},\ \varphi \in \boldsymbol{I} \right\}$ and the arbitrary sets $\boldsymbol{W}_{c_m}$ there is an isomorphism with respect to linear order, or rather, there is an isomorphism between the relational system $\langle \boldsymbol{D}_{\xi}^{ph}, \leq^{ph} \rangle$ and arbitrary relational system $\langle \boldsymbol{W}_{c_m}, \leq \rangle$ with respect to the binary relations of linear order $\leq^{ph}$ and $\leq$ ($\langle \boldsymbol{D}_{\xi}^{ph}, \leq^{ph} \rangle \Leftrightarrow \langle \boldsymbol{W}_{c_m}, \leq \rangle$). Moreover, the ordering type of partition $\boldsymbol{D}_{\xi}^{ph}$ is determined by the type of ordering of the set $\boldsymbol{W}_{c_m}$, in particular, determined by the type of ordering $\boldsymbol{W}_{c_0}$, since it is accepted that $\boldsymbol{I} = \boldsymbol{W}_{c_0}$, that is: $\forall\ \varphi_1 = t_1,\ \varphi_2 = t_2 \in \boldsymbol{W}_{c_0}$, where $t_1 \leq t_2$, $\exists \xi_{\ \varphi_1}, \xi_{\ \varphi_2} \in \boldsymbol{D}_{\xi}^{ph}$, where $t_1 \leftrightarrow \xi_{\ \varphi_1}$, $t_2 \leftrightarrow \xi_{\ \varphi_2}$ and there is the order $\xi_{\ \varphi_1} \leq^{ph} \xi_{\ \varphi_2}$, and vice versa. This result reflects the fact of ordering and the fact of the same type of phase ordering in all cycles of the cyclic signal, as postulated in the conceptual model.

Let us note that ξ_{φ} is the countable set of such ordered pairs of the cyclically correlated random process ξ, among which there are no two pairs belonging to the same cycle; that is, among the elements of ξ_{φ} there are no two pairs $\left(t_{m_1}^{\varphi}, \xi\left(\omega, t_{m_1}^{\varphi} \right) \right)$ where $t_{m_1}^{\varphi} \in \boldsymbol{W}_{c_{m_1}}$ and $\left(t_{m_2}^{\varphi}, \xi\left(\omega, t_{m_2}^{\varphi} \right) \right)$ where $t_{m_2}^{\varphi} \in \boldsymbol{W}_{c_{m_2}}$ for which $\boldsymbol{W}_{c_{m_1}} = \boldsymbol{W}_{c_{m_2}}$. This fact is also fully consistent with the relevant requirement of the conceptual model.

Because ξ_{φ} is a set of bijective connected pairs of random process ξ, then according to equality (2) for all of them, there is equality of mathematical expectations, namely:

$$m_{\xi_{c_{m_1}}}\left(t_{m_1}^{\varphi} \right) = m_{\xi_{c_{m_2}}}\left(t_{m_2}^{\varphi} \right) \in \boldsymbol{R}, t_{m_1}^{\varphi} \in \boldsymbol{W}_{c_{m_1}},\ t_{m_2}^{\varphi} \in \boldsymbol{W}_{c_{m_2}}, t_{m_1}^{\varphi} \leftrightarrow t_{m_2}^{\varphi},\ \ m_1, m_2 \in \boldsymbol{Z}. \tag{8}$$

For two different ξ_{φ_1} and ξ_{φ_2} of random process ξ, according to equality (3), there is an equality of correlation functions, namely:

$$r_{2\xi}\left(t_{m_1}^{\varphi_1}, t_{m_2}^{\varphi_2}\right) = r_{2\xi}\left(t_{m_3}^{\varphi_1}, t_{m_4}^{\varphi_2}\right), t_{m_1}^{\varphi_1} \in \boldsymbol{W}_{c_{m_1}},\ t_{m_2}^{\varphi_2} \in \boldsymbol{W}_{c_{m_2}},\ t_{m_3}^{\varphi_1} \in \boldsymbol{W}_{c_{m_3}},\ t_{m_4}^{\varphi_2} \in \boldsymbol{W}_{c_{m_4}},$$
$$t_{m_1}^{\varphi_1} \leftrightarrow t_{m_3}^{\varphi_1}, t_{m_2}^{\varphi_2} \leftrightarrow t_{m_4}^{\varphi_2},\ \ m_1, m_2, m_3, m_4 \in \boldsymbol{Z}. \tag{9}$$

Based on the results obtained above, we give a mathematical definition of the phase of the cyclically correlated random process ξ, which adequately reflects the concept of phase (stage) of the cyclic signal (oscillatory phenomenon).

Definition 9. *Ordered by index φ partition $\boldsymbol{D}_{\xi}^{ph} = \{\xi_{\varphi},\ \varphi \in \boldsymbol{I}\}$ of cyclically correlated random process ξ whose elements are countable sets formed according to (7) and for which the Equalities (8) and (9) are held, it will be called* ***the partition into phases,*** *and the set ξ_{φ} itself will be called the* ***phase (φ-phase) of the cyclically correlated random process ξ.***

Definition 10. *The m-th element $\left(t_m^{\varphi}, \xi\left(\omega, t_m^{\varphi}\right)\right)$ of the set $\xi_{\varphi} = \left\{\left(t_m^{\varphi}, \xi\left(\omega, t_m^{\varphi}\right)\right), m \in \boldsymbol{Z}\right\}$ will be called* ***the actualization of the phase ξ_{φ} (φ-phase) in m-th cycle of cyclically correlated random process ξ.***

Definition 11. *The set $\boldsymbol{W}_{\varphi}$ which is determined according to Expression (6) will be called* ***the definition domain of phase ξ_{φ} (φ-phase) of the cyclically correlated random process ξ.***

Definition 12. *The set $\boldsymbol{A}_{\varphi}$ which is determined according to expression:*

$$\boldsymbol{A}_{\varphi} = \left\{\xi\left(\omega, t_m^{\varphi}\right) : t_m^{\varphi} \in \boldsymbol{W}_{c_m}, t_m^{\varphi} \leftrightarrow t_{m_1}^{\varphi},\ m \in \boldsymbol{Z}, m_1 = const\right\}, \tag{10}$$

will be called **the φ-set (φ-series) of single-phase values of the cyclically correlated random process ξ.**

The set $\{\boldsymbol{A}_{\varphi}, \varphi \in \boldsymbol{I}\}$ of all sets of single-phase values is ordered by parameter φ.

Note that for the cyclically correlated random process ξ , a whole set $\left\{\boldsymbol{D}_{\xi}^{c_{\varphi}}, \varphi \in \boldsymbol{I}\right\}$ exists of its possible partitions into cycles. In contrast to the different ways of partitioning the cyclically correlated random process ξ into cycles, there is only one its partition $\boldsymbol{D}_{\xi}^{ph} = \{\xi_{\varphi},\ \varphi \in \boldsymbol{I}\}$ into phases. That is, every partition $\boldsymbol{D}_{\xi}^{c_{\varphi}}$ into cycles of the cyclically correlated random process ξ is associated with the certain phase ξ_{φ} of it; namely, it is a phase ξ_{φ} that is observed at the beginning of all the cycles of the cyclically correlated random process ξ, i.e., the following relations take place:

$$t_m^{\varphi} = \widetilde{t}_m,\ m \in \boldsymbol{Z}, \tag{11}$$

$$\xi_{\varphi} = \{(\widetilde{t}_m, \xi(\omega, \widetilde{t}_m)) : \widetilde{t}_m \in \boldsymbol{D}_c,\ \ m \in \boldsymbol{Z}\}. \tag{12}$$

In this case, $\boldsymbol{W}_{\varphi} = \left\{\ t_m^{\varphi},\ m \in \boldsymbol{Z}\right\} = \boldsymbol{D}_c = \{\widetilde{t}_m,\ m \in \boldsymbol{Z}\}$.

With two different phases $\xi_{\ \varphi_1}$ and $\xi_{\ \varphi_2}$ of the random process ξ**,** its two different possible partitions into cycles $\boldsymbol{D}_{\xi}^{c\ \varphi_1}$ and $\boldsymbol{D}_{\xi}^{c\ \varphi_2}$ are connected. In order to reflect the fact that the cycles of the random process ξ and its specific phase ξ_{φ} (φ-phase) are connected, we will use the following notation $\xi_{c_m}^{\varphi} \in \boldsymbol{D}_{\xi}^{c_{\varphi}}$. In this case, the partition $\boldsymbol{D}_{\xi}^{c_{\varphi}}$ into cycles, we denote as follows: $\boldsymbol{D}_{\xi}^{c_{\varphi}} = \{\xi_{c_m}^{\varphi} \subset \xi,\ m \in \boldsymbol{Z}\}$.

Let us assign definitions for the two types of initial phases.

Definition 13. *The specific phase ξ_{φ} (φ-phase) of the cyclically correlated random process ξ, which determines the appropriate partition $\boldsymbol{D}_{\xi}^{c_{\varphi}} = \left\{\xi_{c_m}^{\varphi} \subset \xi,\ m \in \boldsymbol{Z}\right\}$ of the cyclically correlated* *random process ξ, will be called* ***the initial phase for partition $D_\xi^{c_\varphi}$ into cycles of the cyclically correlated random process ξ***.

Definition 14. *The specific phase ξ_{φ_0}, $\varphi_0 \in \boldsymbol{I}$ of the cyclically correlated random process ξ, which includes an ordered pair $(0, \xi(\omega, 0))$, will be called* ***the initial phase of the cyclically correlated random process ξ***.

In general, the initial phase ξ_{φ_0} of the cyclically correlated random process ξ and the initial phase ξ_φ of its partition $\boldsymbol{D}_\xi^{c_\varphi}$ into cycles are not identical ($\xi_{\varphi_0} \neq \xi_\varphi$), which is caused by the inequality $\widetilde{t}_0 \neq 0$. It is often convenient in practical applications to set the identities of these two types of initial phases as requiring the equality $\widetilde{t}_0 = 0$.

In contrast to the countability of the set ξ_φ and the φ-set $\boldsymbol{A}_\varphi$ of single-phase values, the cardinal number of set $\left\{\boldsymbol{D}_\xi^{c_\varphi}, \varphi \in \boldsymbol{I}\right\}$ of possible partitions into cycles and the cardinal number of set $\boldsymbol{D}_\xi^{ph} = \{\xi_\varphi,\ \varphi \in \boldsymbol{I}\}$ of the phases of the cyclically correlated random process ξ are equal to the cardinal number of the index set $\boldsymbol{I}$, which is equal to the continuum c:

$$Card\left(\left\{\boldsymbol{D}_\xi^{c_\varphi}, \varphi \in \boldsymbol{I}\right\}\right) = Card(\{\xi_\varphi,\ \varphi \in \boldsymbol{I}\}) = Card(\boldsymbol{I}) = Card(\boldsymbol{R}) = c. \quad (13)$$

Similarly to the definition of two-dimensional cycles of a cyclically correlated random process, it is possible to define the concept of its two-dimensional phases.

3.5. Different Forms of Representation of a Cyclically Correlated Random Process and Its Mathematical Expectation and Correlation Function

Based on the definition of the cyclically correlated random process $\xi : \boldsymbol{R} \to \boldsymbol{L}_2(\boldsymbol{\Omega}, \boldsymbol{P})$, namely, given that the ordered countable partition $\boldsymbol{D}_\xi^c = \{\xi_{c_m} \subset \xi,\ m \in \boldsymbol{Z}\}$ into cycles always exists, and the ordered countable partition $\boldsymbol{D}_{\xi^2}^c = \{\xi_{c_m} \times \xi \subset \xi^2,\ m \in \boldsymbol{Z}\}$ of the Cartesian square ξ^2 into two-dimensional cycles of cyclically correlated random process, the random process ξ and its Cartesian square ξ^2 can be represented as follows:

$$\xi = \cup_{m\in \boldsymbol{Z}} \xi_{c_m}, \quad (14)$$

$$\xi^2 = \cup_{m\in \boldsymbol{Z}} \xi_{c_m} \times \xi,\ \xi_{c_m} \neq \varnothing,\ \ \xi_{c_{m_1}} \cap \xi_{c_{m_2}} = \varnothing,\ \ m_1 \neq m_2,\ \ m,\ m_1, m_2 \in \boldsymbol{Z}. \quad (15)$$

It is possible in another way to represent the cyclically correlated random process $\xi : \boldsymbol{R} \to \boldsymbol{L}_2(\boldsymbol{\Omega}, \boldsymbol{P})$ if we consider a counted-dimensional vector $\left\{\widetilde{\xi}_m(\omega, t),\ \omega \in \boldsymbol{\Omega}, t \in \boldsymbol{R},\ m \in \boldsymbol{Z}\right\}$ of random processes, which in the areas $\boldsymbol{W}_{c_m}$ coincides with the random processes ξ_{c_m}, but in the areas $\boldsymbol{R} \backslash \boldsymbol{W}_{c_m}$, the random processes $\widetilde{\xi}_m(\omega, t)$ are identically equal to zero $\left(\widetilde{\xi}_m(\omega, t) = 0,\ t \in \boldsymbol{R} \backslash \boldsymbol{W}_{c_m}\right)$. Then the cyclically correlated random process can be given as the sum of the components of the vector $\left\{\widetilde{\xi}_m(\omega, t),\ \omega \in \boldsymbol{\Omega}, t \in \boldsymbol{R},\ m \in \boldsymbol{Z}\right\}$:

$$\xi(\omega, t) = \sum_{m\in \boldsymbol{Z}} \widetilde{\xi}_m(\omega, t),\ \omega \in \boldsymbol{\Omega}, t \in \boldsymbol{R}. \quad (16)$$

Similarly to the representations of the random process ξ and its Cartesian square ξ^2 according to Formulas (14) and (15), representations of the mathematical expectation $m_\xi = \{(t, m_\xi(t)) :\ t \in \boldsymbol{R}\}$ and the correlation function $r_{2\xi} = \{((t_1, t_2), r_{2\xi}(t_1, t_2)) :\ (t_1, t_2) \in \boldsymbol{R}^2\}$ of the cyclically correlated random process can be given as follows:

$$m_\xi = \cup_{m\in \boldsymbol{Z}} m_{\xi_{c_m}},\ m_{\xi_{c_m}} \neq \varnothing,\ \ m_{\xi_{c_{m_1}}} \cap m_{\xi_{c_{m_2}}} = \varnothing,\ \ m_1 \neq m_2,\ \ m,\ m_1, m_2 \in \boldsymbol{Z}, \quad (17)$$

$$r_{2\xi} = \cup_{m\in \boldsymbol{Z}} r_{2\xi_{c_m}},\ r_{2\xi_{c_m}} \neq \varnothing,\ \ r_{2\xi_{c_{m_1}}} \cap r_{2\xi_{c_{m_2}}} = \varnothing,\ \ m_1 \neq m_2, m, m_1, m_2 \in \boldsymbol{Z}, \quad (18)$$

where $m_{\xi_{cm}} = \{(t, m_{\xi_{cm}}(t)) : t \in \boldsymbol{W}_{c_m}\}$ is the mathematical expectation of the m-th cycle of the cyclically correlated random process ξ; and $r_{2\xi_{cm}} = \{((t_1, t_2), r_{2\xi}(t_1, t_2)) : (t_1, t_2) \in \boldsymbol{W}_{c_m} \times \boldsymbol{R}\}$ is the correlation function of the m-th two-dimensional cycle of the cyclically correlated random process ξ.

It is possible in another way to represent the mathematical expectation of the cyclically correlated random process ξ if we consider a counted-dimensional vector $\{\widetilde{m}_{\xi_{cm}}(t),\ t \in \boldsymbol{R},\ m \in \boldsymbol{Z}\}$, whose components $\widetilde{m}_{\xi_{cm}}(t)$ in the areas $\boldsymbol{W}_{c_m}$ coincide with $m_{\xi_{cm}}(t)$, but in the areas $\boldsymbol{R}\backslash\boldsymbol{W}_{c_m}$ $\widetilde{m}_{\xi_{cm}}(t)$ are identically equal to zero $\left(\widetilde{m}_{\xi_{cm}}(t) = 0,\ t \in \boldsymbol{R}\backslash\boldsymbol{W}_{c_m}\right)$. Then the mathematical expectation of the cyclically correlated random process ξ can be given as the sum of the components of the vector $\{\widetilde{m}_{\xi_{cm}}(t),\ t \in \boldsymbol{R},\ m \in \boldsymbol{Z}\}$:

$$m_{\xi}(t) = \sum_{m \in \boldsymbol{Z}} \widetilde{m}_{\xi_{cm}}(t),\ t \in \boldsymbol{R}. \tag{19}$$

Similarly, it is possible to represent the correlation function $r_{2\xi}(t_1, t_2)$ of the cyclically correlated random process ξ if we consider a counted-dimensional vector $\{\widetilde{r}_{2\xi_m}(t_1, t_2),\ (t_1, t_2) \in \boldsymbol{R}^2,\ m \in \boldsymbol{Z}\}$, whose components $\widetilde{r}_{2\xi_m}(t_1, t_2)$ in the areas $\boldsymbol{W}_{c_m} \times \boldsymbol{R}$ coincide with the $r_{2\xi_m}(t_1, t_2)$, but in the areas $\boldsymbol{R}^2\backslash(\boldsymbol{W}_{c_m} \times \boldsymbol{R})$ $\widetilde{r}_{2\xi_m}(t_1, t_2)$ are identically equal to zero $\left(\widetilde{r}_{2\xi_m}(t_1, t_2) = 0,\ (t_1, t_2) \in \boldsymbol{R}^2\backslash(\boldsymbol{W}_{c_m} \times \boldsymbol{R})\right)$. Then the correlation function of the cyclically correlated random process ξ can be given as the sum of the components of the vector $\{\widetilde{r}_{2\xi_m}(t_1, t_2),\ (t_1, t_2) \in \boldsymbol{R}^2,\ m \in \boldsymbol{Z}\}$:

$$r_{2\xi}(t_1, t_2) = \sum_{m \in \boldsymbol{Z}} \widetilde{r}_{2\xi_m}(t_1, t_2), (t_1, t_2) \in \boldsymbol{R}^2. \tag{20}$$

In practice, as a rule, the cyclically correlated random process should not be considered in the entire area $\boldsymbol{R}$ but in some of its subsets $\boldsymbol{V} \subset \boldsymbol{R}$, which, for example, may be:

$$\boldsymbol{V} = \cup_{m=0}^{M} \boldsymbol{W}_{c_m} \ \ or\ \ \boldsymbol{V} = \cup_{m=0}^{\infty} \boldsymbol{W}_{c_m},$$

where M is the integer number. In this case, the mathematical expectation $m_{\xi}(t)$ will also be considered in the set $\boldsymbol{V} \subset \boldsymbol{R}$, and the correlation function $r_{2\xi}(t_1, t_2)$ will be considered in the set $\boldsymbol{V}^2 \subset \boldsymbol{R}^2$.

3.6. Structural Function and Rhythm Function of Cyclically Correlated Random Process

In contrast to the definition of the periodically correlated random process, in the definition of the cyclically correlated random process, the time (or spatial) distance between its actualizations of each phase ξ_{φ} in different neighboring cycles can be different; it is important only to preserve the type of ordering of the phases of the cyclically correlated random process in all its cycles. To characterize the temporal (spatial) patterns and relations that occur between the actualizations of each phase ξ_{φ} in the different cycles of the cyclically correlated random process, we introduce the structural function and rhythm function, which formalize the concept of the rhythm (tempo) of the cyclic signal as provided by the conceptual model.

Based on the definition of the cyclically correlated random process, we formulate the following theorem, which is based on the results of work [72].

Theorem 1. *For the cyclically correlated random process $\xi = \{(t, \xi(\omega, t)) : t \in \boldsymbol{R}\}$, a numerical function $y(t, n),\ t \in \boldsymbol{R},\ n \in \boldsymbol{Z}$ exists, for which the following properties occur:*

$$\begin{gathered}\text{(a) } y(t,n) > t,\ if\ n > 0\ (y(t,1) - t < \infty) \\ \text{(b) } y(t,n) = t,\ if\ n = 0; \\ \text{(c) } y(t,n) < t,\ if\ n < 0,\ \ t \in \boldsymbol{R};\end{gathered} \tag{21}$$

for any $t_1, t_2 \in \boldsymbol{R}$, for which $t_1 < t_2$ for function $y(t, n)$, a strict inequality is present:

$$y(t_1, n) < y(t_2, n),\ \forall n \in \boldsymbol{Z} \tag{22}$$

and for the mathematical expectation $m_\xi(t)$ and correlation function $r_{2\xi}(t_1, t_2)$ of a cyclically correlated random process ξ, there are the following equalities:

$$m_\xi(t) = m_\xi(y(t, n)),\ t \in \boldsymbol{R}, n \in \boldsymbol{Z}; \tag{23}$$

$$r_{2\xi}(t_1, t_2) = r_{2\xi}(y(t_1, n), y(t_2, n)),\ t_1, t_2 \in \boldsymbol{R}, n \in \boldsymbol{Z}. \tag{24}$$

On the contrary, if for a random process ξ, there is a numerical function $y(t, n)$, $t \in \boldsymbol{R}$, $n \in \boldsymbol{Z}$ with all the above-mentioned properties (22) and (21), and if the equalities (23) and (24) are present, then it is a cyclically correlated random process.

Proof of Theorem 1. According to the definition of a cyclically correlated random process ξ, any of its two cycles $\xi_{c_{m_1}} = \left\{(t, \xi(\omega, t)) \in \xi : \ t \in \boldsymbol{W}_{c_{m_1}}\right\}$ and $\xi_{c_{m_2}} = \left\{(t, \xi(\omega, t)) \in \xi : \ t \in \boldsymbol{W}_{c_{m_2}}\right\}$ are isomorphic with respect to the binary relation of the linear order $\leq_2$, and this isomorphism is due to the isomorphism with respect to binary relation of linear order $\leq$ of the domains $\boldsymbol{W}_{c_{m_1}}$ and $\boldsymbol{W}_{c_{m_2}}$, which are ordinary sets of real numbers. Between isomorphic numerical sets $\boldsymbol{W}_{c_{m_1}}$ and $\boldsymbol{W}_{c_{m_2}}$ some numeric functions $t_{m_2} = y_{m_1m_2}(t_{m_1}) \in \boldsymbol{W}_{c_{m_2}},\ t_{m_1} \in \boldsymbol{W}_{c_{m_1}}$ and $t_{m_1} = y_{m_2m_1}(t_{m_2}) \in \boldsymbol{W}_{c_{m_1}},\ t_{m_{21}} \in \boldsymbol{W}_{c_{m_2}}$ can always be constructed, namely:

(a) there is a bijection between $\boldsymbol{W}_{c_{m_1}}$ and $\boldsymbol{W}_{c_{m_2}}$ $(m_1 m_2 \in \boldsymbol{Z})$, namely: any $t_{m_1} \in \boldsymbol{W}_{c_{m_1}}$ corresponds to only one $t_{m_2} = y_{m_1m_2}(t_{m_1}) \in \boldsymbol{W}_{c_{m_2}}$ and any $t_{m_2} \in \boldsymbol{W}_{c_{m_2}}$ corresponds to only one $t_{m_1} = y_{m_2m_1}(t_{m_2}) \in \boldsymbol{W}_{c_{m_1}}$, and for any different $t_{m_1}, t'_{m_1} \in \boldsymbol{W}_{c_{m_1}}$, their images $t_{m_2}, t'_{m_2} \in \boldsymbol{W}_{c_{m_2}}$ are different ($t_{m_2} \neq t'_{m_2}$), and vice versa: for any different $t_{m_2}, t'_{m_2} \in \boldsymbol{W}_{c_{m_2}}$, their images $t_{m_1}, t'_{m_1} \in \boldsymbol{W}_{c_{m_1}}$ are different ($t_{m_1} \neq t'_{m_1}$);

(b) the same type of linear ordering of sets $\boldsymbol{W}_{c_{m_1}}$ and $\boldsymbol{W}_{c_{m_2}}$ takes place, that is $\forall t_{m_1}, t'_{m_1} \in \boldsymbol{W}_{c_{m_1}}$, $\exists t_{m_2}, t'_{m_2} \in \boldsymbol{W}_{c_{m_2}}$, that $t_{m_2} = y_{m_1m_2}(t_{m_1}), t'_{m_2} = y_{m_1m_2}(t'_{m_1})$ and there is a strong order relation:

$$y_{m_1m_2}(t_{m_1}) < y_{m_1m_2}(t'_{m_1}),\ if\ t_{m_1} < t'_{m_1}, \tag{25}$$

and vice versa: $\forall t_{m_2}, t'_{m_2} \in \boldsymbol{W}_{c_{m_2}}\ \exists t_{m_1}, t'_{m_1} \in \boldsymbol{W}_{c_{m_1}}$, that $t_{m_1} = y_{m_2m_1}(t_{m_2}), t'_{m_1} = y_{m_2m_1}(t'_{m_2})$ and there is a strong order relation:

$$y_{m_2m_1}(t_{m_2}) < y_{m_2m_1}(t'_{m_2}),\ if\ t_{m_2} < t'_{m_2}. \tag{26}$$

Hence, the functions $y_{m_1m_2}(t_{m_1}) \in \boldsymbol{W}_{c_{m_2}}, t_{m_1} \in \boldsymbol{W}_{c_{m_1}}$ and $y_{m_2m_1}(t_{m_2}) \in \boldsymbol{W}_{c_{m_1}}, t_{m_2} \in \boldsymbol{W}_{c_{m_2}}$ are increasing numerical functions.

Taking into account the isomorphism between all possible pairs of cycles of the cyclically correlated random process ξ, we introduce a countable-dimensional matrix of increasing numerical functions which specifies the bijective mapping between the domains of its corresponding cycles, i.e., the following matrix:

$$\left\{y_{m_1m_2}(t_{m_1}) \in \boldsymbol{W}_{c_{m_2}}, t_{m_1} \in \boldsymbol{W}_{c_{m_1}}, \qquad m_1 m_2 \in \boldsymbol{Z}\right\}. \tag{27}$$

Moreover, on the diagonal of the functional matrix (27) when $m_1 = m_2 = m \in \boldsymbol{Z}$, we will have numerical functional relations of identity which are automorphisms with respect to the binary relation of linear order $\leq$ of the domains $\boldsymbol{W}_{c_{m_1}}$ and $\boldsymbol{W}_{c_{m_2}}$, and at the permutation of the places of indices m_1 and m_2 of the function $y_{m_1m_2}(t_{m_1}) \in \boldsymbol{W}_{c_{m_2}}, t_{m_1} \in \boldsymbol{W}_{c_{m_1}}$ we obtain the inverse numerical increasing function $y_{m_2m_1}(t_{m_2}) \in \boldsymbol{W}_{c_{m_1}}, t_{m_2} \in \boldsymbol{W}_{c_{m_2}}$.

Entering the notation $m_1 = m,\ \ m_2 = m + n,\ m, n \in \boldsymbol{Z}$ and taking into account the indices of the elements of the matrix (27), i.e., $t_{m+n} = y_{m,m+n}(t_m) = y(t_m, n)$, from the matrix (27) we obtain the following countable-dimensional vector of increasing numerical functions from the two arguments t_m and n:

$$\{y(t_m, n) \in \boldsymbol{W}_{c_{m+n}},\ t_m \in \boldsymbol{W}_{c_m},\ m, n \in \boldsymbol{Z}\} \tag{28}$$

Each element of the countable-dimensional vector (28) establishes an isomorphism between the domains of the definition of the arbitrary m-th cycle and $m+n$-th cycle, and is remote from it on n cycles. In addition, for all elements of the vector (28), there are the following inequalities:

$$\begin{aligned}&(\mathrm{a})\ y(t_m, n) > t_m,\ if\ n > 0,\ \forall m\ \in \boldsymbol{Z}\,;\\ &(\mathrm{b})\ y(t_m, n) = t_m,\ if\ n = 0,\ \forall m\ \in \boldsymbol{Z}\,;\\ &(\mathrm{c})\ y(t_m, n) < t_m,\ if\ n < 0,\ \forall m\ \in \boldsymbol{Z}.\end{aligned} \tag{29}$$

The first property $(y(t_m, n) > t_m,\ if\ n > 0)$ follows from following facts: $\forall t_m \in \boldsymbol{W}_{c_m}$, $m \in \boldsymbol{Z}$ and $\forall n > 0,\ \exists t_{m+n} = y(t_m, n) \in \boldsymbol{W}_{c_{m+n}}$. Moreover, $t_{m+n} > t_m$, whereas $n > 0$, and therefore, $y(t_m, n) > t_m$.

The second property $(y(t_m, n) = t_m,\ if\ n = 0)$ follows from the fact that $t_{m+0} = t_m \in \boldsymbol{W}_{c_m}$, since $y(t_m, n) = t_m$.

The third property $(y(t_m, n) < t_m,\ if\ n < 0)$ can be proved similarly to the first: $\forall t_m \in \boldsymbol{W}_{c_m}$, $m \in \boldsymbol{Z}$ and $\forall n < 0,\ \exists t_{m+n} = y(t_m, n) \in \boldsymbol{W}_{c_{m+n}}$, and $t_{m+n} < t_m$, whereas $n < 0$ and therefore, $y(t_m, n) < t_m$.

Since for the cyclically correlated random process ξ, there is a continuous set $\left\{\boldsymbol{D}_{\xi}^{c\varphi}, \varphi \in \boldsymbol{I}\right\}$ of its possible partitions into cycles, there exists a set of countable-dimensional vectors (28) corresponding to these partitions. However, since for the cyclically correlated random process ξ, there exists only one of its partition $\boldsymbol{D}_{\xi}^{ph} = \{\xi_\varphi,\ \varphi \in \boldsymbol{I}\}$ into the set of phases, for all possible countable-dimensional vectors (28) corresponding to the partitions from set $\left\{\boldsymbol{D}_{\xi}^{c\varphi}, \varphi \in \boldsymbol{I}\right\}$, there is one and only one numerical function $y(t, n),\ t \in \boldsymbol{R},\ n \in \boldsymbol{Z}$, which is equal to the ordered union (sum) of the elements of the countable-dimensional vector (28) at a fixed n:

$$\{(t, y(t, n)), t \in \boldsymbol{R}\} = \cup_{m \in \boldsymbol{Z}}\{(t_m, y(t, n)), t_m \in \boldsymbol{W}_{c_{m+n}}\}, n \in \boldsymbol{Z}. \tag{30}$$

Due to the order of the union (30), the numerical function $y(t, n),\ t \in \boldsymbol{R}, n \in \boldsymbol{Z}$, similar to the elements of the countable-dimensional vector of functions (28), is an isomorphism with respect to binary relation of linear order $\leq$, and therefore, for it a strict inequality (22) exists, i.e., for any fixed n function, $y(t, n),\ t \in \boldsymbol{R}, n \in \boldsymbol{Z}$ is an increasing numerical function. Properties (21) follow from the properties (29) of the vector components (28), since the numerical function $y(t, n),\ t \in \boldsymbol{R}, n \in \boldsymbol{Z}$, in fact, is "stitched" from these components.

For all the bijective connected ordered pairs $(t_m, \xi(\omega, t_m)) \in \xi_{c_m}$ and $(y(t_m, n), \xi(\omega, y(t_m, n))) \in \xi_{c_{m+n}}$, which belong to the same phase of the cyclically correlated random process ξ, there is an equality of mathematical expectations as follows:

$$m_\xi(t_m) = m_\xi(y(t_m, n)), t_m \in \boldsymbol{W}_{c_m},\ m, n \in \boldsymbol{Z}. \tag{31}$$

If t_m runs through the entire set $\boldsymbol{W}_{c_m}$ and m runs through the entire set $\boldsymbol{Z}$, then Equality (31) will turn into Equalities (23), because $\cup_{m \in \boldsymbol{Z}} \boldsymbol{W}_{c_m} = \boldsymbol{R}$.

For all ordered pairs $(t_{m_1}, \xi(\omega, t_{m_1})) \in \xi_{c_{m_1}}$, $(t_{m_2}, \xi(\omega, t_{m_2})) \in \xi_{c_{m_2}}$, there are bijective connected to them ordered pairs $(y(t_{m_1}, n), \xi(\omega, y(t_{m_1}, n))) \in \xi_{c_{m_1+n}}$, $(y(t_{m_2}, n), \xi(\omega, y(t_{m_2}, n))) \in \xi_{c_{m_2+n}}$, and there is an equality of correlation functions:

$$r_{2\xi}(t_{m_1}, t_{m_2}) = r_{2\xi}(y(t_{m_1}, n), y(t_{m_2}, n)), t_{m_1} \in \boldsymbol{W}_{c_{m_1}}, t_{m_2} \in \boldsymbol{W}_{c_{m_2}},\ m_1, m_2, n \in \boldsymbol{Z}. \tag{32}$$

If the vector (t_{m_1}, t_{m_2}) runs through the entire set $\boldsymbol{W}_{c_{m_1}} \times \boldsymbol{W}_{c_{m_2}}$ and the vector (m_1, m_2) runs through the entire set $\boldsymbol{Z}^2$, then Equality (32) will turn into Equalities (24), because:

$$\cup_{m_1,m_k \in \boldsymbol{Z}} \boldsymbol{W}_{c_{m_1}} \times \boldsymbol{W}_{c_{m_2}} = \boldsymbol{R}^2. \tag{33}$$

The requirement of limited function $y(t,n) - t$ at $n = 1$ $(y(t,1) - t < \infty)$ necessarily follows from the fact that the duration of the cycles is limited, which is formally reflected in the inequalities $0 < \widetilde{t}_{m+1} - \widetilde{t}_m < \infty$ when considering the partition $\boldsymbol{D}^c_{\boldsymbol{R}} = \{\boldsymbol{W}_{c_m} \subset \boldsymbol{R}, m \in \boldsymbol{Z}\}$.

It is easy to see that if for some random process ξ, there is a numerical function $y(t,n)$ that satisfies all the conditions of Theorem 1, and equations (23) and (24) are present, then this random process is a cyclically correlated random process, because in this case, its ordered countable partition $\boldsymbol{D}^c_\xi = \{\xi_{c_m} \subset \xi,\ m \in \boldsymbol{Z}\}$ always exists, and the ordered countable partition $\boldsymbol{D}^c_{\xi^2} = \{\xi_{c_m} \times \xi \subset \xi^2,\ m \in \boldsymbol{Z}\}$ of its Cartesian square ξ^2 always exists, which is the carrier of the isomorphic relational systems $\boldsymbol{RS}^c_{\xi,\xi^2} = \{\langle \xi_{c_m}, \xi_{c_m} \times \xi, \boldsymbol{R}, \{\leq_2, \leq_4, p_1 : \xi \to \boldsymbol{R}, p_2 : \xi^2 \to \boldsymbol{R}\}\rangle,\ m \in \boldsymbol{Z}\}$ with respect to the binary relations of linear order $\leq_2$ and $\leq_4$, the mathematical expectation $m_\xi(t)$ and the correlation function $r_{2\xi}(t_1, t_2)$ of random process $\xi(\omega, t)$. □

Note that for a cyclically correlated random process ξ, the set of functions $y_\gamma(t,n)$, which satisfy the conditions of Theorem 1 is a countable set $\{y_\gamma(t,n), \gamma \in \boldsymbol{N}\}$, since, as noted above, there is a countable set of partitions of a cyclically correlated random process into mutually isomorphic segments which are nested in each other and are determined by initial phase ξ_φ. Actually, every such function $y_\gamma(t,n)$ will generate a separate countable-dimensional partition of a cyclically correlated random process into its isomorphic segments. Since for a cyclically correlated random process ξ, there is always the minimal partition $\boldsymbol{D}^{c\varphi}_\xi = \{\xi^\varphi_{c_m} \subset \xi,\ m \in \boldsymbol{Z}\}$ (which is determined by the initial phase ξ_φ), which is its partition into cycles, then among the functions from the set $\{y_\gamma(t,n), \gamma \in \boldsymbol{N}\}$, the smallest in the modulus $(|y(t,n)| \leq |y_\gamma(t,n)|)$ will always be function $y(t,n) = y_1(t,n)$ $(\gamma = 1)$, which generates (displays) the partition of the cyclically correlated random process ξ into its cycles.

Let us provide the following definition:

Definition 15. *The function $y(t,n)$, which is the smallest in the modulus $(|y(t,n)| \leq |y_\gamma(t,n)|)$ among all such functions $\{y_\gamma(t,n), \gamma \in \boldsymbol{N}\}$ which satisfy (21)–(24), will be called* ***the structural function of the cyclically correlated random process*** *ξ.*

Let us consider the additive form of the representation of a structural function $y(t,n)$, $t \in \boldsymbol{R}$, $n \in \boldsymbol{Z}$, namely, $y(t,n) = t + T(t,n)$. Function $T(t,n)$ in [72] is called ***the rhythm function of the cyclically correlated random process*** ξ**.** Note, that it is enough to set the rhythm function $T(t,n)$, as well as the structural function $y(t,n)$, not in the whole area of its definition $\boldsymbol{R} \times \boldsymbol{Z}$, but only in the area $\boldsymbol{W}_{c_m} \times \boldsymbol{Z}$. Based on the properties (21) and (22) of the structural function of the cyclically correlated random process ξ, its rhythm function has the following properties:

1. (a) $T(t,n) > 0$ if $n > 0$ $(T(t,1) < \infty)$;

(b) $T(t,n) = 0$, if $n = 0$;

(c) $T(t,n) < 0$ if $n < 0$, $t \in \boldsymbol{R}$. (34)

2. For any $t_1 \in \boldsymbol{R}$ and $t_2 \in \boldsymbol{R}$, for which $t_1 < t_2$, and for function $T(t,n)$, a strict inequality holds:

$$T(t_1,n) + t_1 < T(t_2,n) + t_2, \forall n \in \boldsymbol{Z}. \tag{35}$$

Equations (23) and (24) can be represented by the rhythm function as follows:

$$m_{\xi}(t) = m_{\xi}(t + T(t,n)),\ t \in \boldsymbol{R}, n \in \boldsymbol{Z}; \tag{36}$$

$$r_{2\xi}(t_1, t_2) = r_{2\xi}(t_1 + T(t_1,n), t_2 + T(t_2,n)),\ t_1, t_2 \in \boldsymbol{R}, n \in \boldsymbol{Z}. \tag{37}$$

Function $T(t,n)$ is the smallest in the modulus ($|T(t,n)| \leq |T_{\gamma}(t,n)|$) among all such functions $\{T_{\gamma}(t,n), \gamma \in \}$ which satisfy (34)–(37).

The introduction of the rhythm function $T(t,n)$ and the structural function $y(t,n)$ of the cyclically correlated random process provides a clear mathematical basis for the concept of the rhythm of the cyclic signal, which is what made it possible to quantitatively describe its rhythmic structure. According to the conceptual model, the rhythmic structure of the cyclic signal is a concept that characterizes the law of change in time intervals between single-phase values in different cycles of the cyclic signal, and in fact such a law describes the function of rhythm. Unlike concepts such as variable period or instantaneous signal period, which are sometimes found in scientific publications, the structural function and rhythm function of the cyclically correlated random process are characterized by clear, analytically justified conditions that are determined conditions of Theorem 1, and which are sufficient and necessary with respect to the cyclicity of the cyclically correlated random process.

The rhythm of the cyclic signal in qualitative terms can be regular (stable, unchanging) or irregular (variable, unstable). From the point of view of the introduced concept of the rhythm function, the periodically correlated random process is a ***cyclically correlated random process with a regular (stable) rhythm***, or rather with a rhythm function $T(t,n) = n{\cdot}T, T = const > 0$. A non-rhythmic cyclic signal (variable rhythm signal) is a signal whose model is a cyclically correlated random process with a rhythm function $T(t,n) \neq n{\cdot}T$ ($T(t,1) \neq const$). This random process is called a ***cyclically correlated random process with an irregular (variable) rhythm***.

In the partial case, Inequalities (22) and (35) for the structural function $y(t,n)$ and for the rhythm function $T(t,n)$ can be written through their derivatives by argument t. To attain this, we reduce these inequalities to the following form:

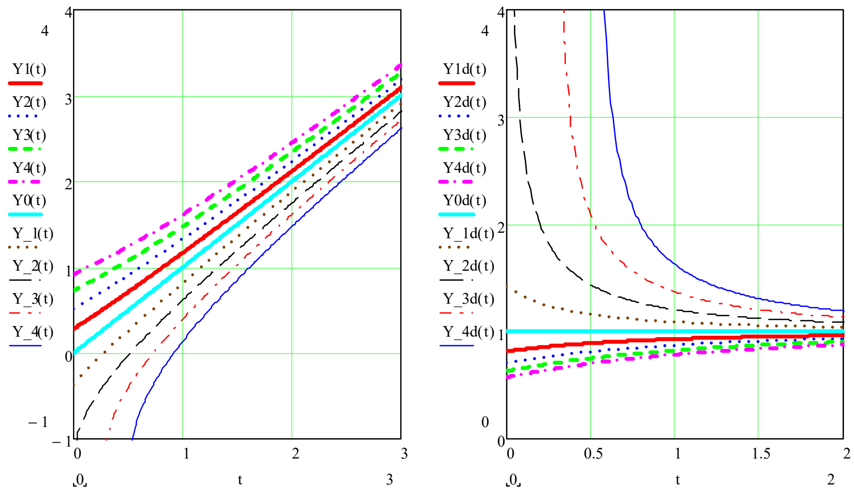
$$\frac{y(t_2,n) - y(t_1,n)}{t_2 - t_1} > 0,\ \frac{T(t_2,n) - T(t_1,n)}{t_2 - t_1} > -1\ \ if\ \ t_1 < t_2. \tag{38}$$

In the case of the existence of the structural function $y(t,n)$ or the rhythm function $T(t,n)$ of their derivatives on t from Inequalities (38), the following strict inequalities follow [66]:

$$\lim_{t_2 - t_1 \to 0} \frac{y(t_2,n) - y(t_1,n)}{t_2 - t_1} = y'(t,n) > 0,\ \ \lim_{t_2 - t_1 \to 0} \frac{T(t_2,n) - T(t_1,n)}{t_2 - t_1} = T'(t,n) > -1. \tag{39}$$

Note, that Inequalities (39) are only partial cases of general conditions (22) and (35), because, for example, for many structural functions $y(t,n)$ and rhythm functions $T(t,n)$, which are continuous by argument t, there is no derivative at some points, which makes it impossible to use Inequalities (39).

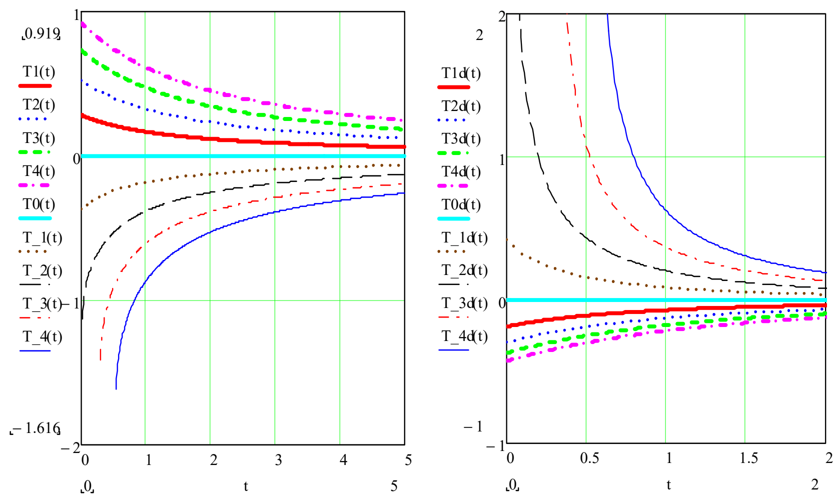
In Figure 1, an example of graphs of cross sections of a structural function $y(t,n) = -\frac{2at+b}{2a} + \sqrt{\frac{4a^2t^2+4abt+b^2}{4a^2} + \frac{\pi n}{a}} + t$ (coefficients $a = 4, b = 10, c = 1$) and its derivative $y'(t,n)$ for various fixed n are given.



(**a**) (**b**)

Figure 1. Example of graphs of cross sections of a structural function (**a**) and its derivative (**b**) for various fixed n.

In Figure 2, an example of graphs of cross sections of a rhythm function $T(t, n) = -\frac{2at+b}{2a} + \sqrt{\frac{4a^2t^2+4abt+b^2}{4a^2} + \frac{\pi n}{a}}$ and its derivative $T'(t, n)$ for various fixed n are given. This rhythm function $T(t, n)$ corresponds to the structural function $y(t, n)$ shown in Figure 1.



(**a**) (**b**)

Figure 2. Example of graphs of cross sections of a rhythm function (**a**) and its derivative (**b**) for various fixed n.

Using the rhythm function $T(t, n)$ of the cyclically correlated random process, similarly to Expressions (7) and (10), expressions for its phase ξ_φ and its φ-set of single-phase values can be written as follows:

$$\xi_\varphi = \left\{ \left(t_0^\varphi + T\left(t_0^\varphi, n\right), \xi\left(\omega, t_0^\varphi + T\left(t_0^\varphi, n\right)\right)\right) : t_0^\varphi = const \in \mathbf{W}_{c_0},\ n \in \mathbf{Z} \right\} \tag{40}$$

$$\mathbf{A}_\varphi = \left\{ \xi\left(\omega, t_0^\varphi + T\left(t_0^\varphi, n\right)\right) : t_0^\varphi = const \in \mathbf{W}_{c_0},\ n \in \mathbf{Z} \right\} \tag{41}$$

The set $\{\mathbf{A}_\varphi, \varphi \in \mathbf{I}\}$ of all sets of single-phase values of the cyclically correlated random process is a set of stationary and stationary connected random sequences in a wide sense, which are the primary statistical materials for the statistical evaluation of the mathematical expectation and the correlation function of the cyclically correlated random process. Corresponding methods of statistical evaluation were developed in the work [67].

4. Discussion

In this paper, when constructing a cyclically correlated random process, a rather unusual approach for the correlation theory of random processes was used. This atypicality is connected with the use of ideas of the category theory, in particular, with the concept of isomorphism between different random processes, which are considered as carriers of certain relational systems. However, the use of such a mathematical apparatus made it possible to clearly mathematically model the cyclic, phase and rhythmic structures of signals within the framework of the correlation theory of random processes.

The main advantages of the class of cyclically correlated random processes include the following:

1. The class of cyclically correlated random processes include the class of cyclostationary (periodically) correlated random processes as its subclass, which enable the use of a set of powerful methods of analysis and the forecasting of cyclic signals with a stable rhythm. These methods were developed during the last 60 years of active research by scientists from different countries of the world within the framework of this stochastic model.
2. The cyclically correlated random process makes it possible to consistently describe cyclic stochastic signals, both with regular and irregular rhythms, not separating them, but complementing them within the framework of a single integrated model.

In contrast to such well-known models as the cyclostationary correlated, poly-cyclostationary correlated and almost-cyclostationary correlated random processes, the cyclically correlated random process in an explicit, direct form describes the cyclic and phase structures of the investigated signals and has formal means of taking into account the variability of the rhythm of the investigated cyclic signals, which significantly expands the scope of the application of the theory of random processes of the second order to solve the problems of the modeling, analysis and forecasting of signals of a cyclic spatio-temporal structure, and especially of cyclic signals with variable rhythm.

In contrast to the works [56–60], in which random processes with irregular cyclicity (irregular cyclostationary process, time-warped almost-cyclostationary process, cyclostationary processes with evolving periods and amplitudes) are defined, the cyclically correlated random process has the following advantages:

1. Unlike the known random processes with irregular cyclicity (with irregular rhythm), which are the results of the time-scale (time-warping) transformation of the cyclostationary or almost-cyclostationary correlated random processes, and for which such constructive representation (and the assumption of the existence of some basic cyclostationary (almost-cyclostationary) correlated random process) in the case of natural (not technical) cyclic signals is artificial and often does not correspond to the real mechanisms of their formation, the cyclically correlated random process, and its cyclic, phase and rhythmic structures directly reflect the time structure and properties of a cyclic signal itself, regardless of the mechanisms of its formation (generation).
2. For a cyclically correlated random process, there is no ambiguity in its representation; however, for the construction of a time-warped cyclostationary correlated process, such ambiguity is always present, since such a construction requires the simultaneous selection of both the time-scale transformation (time-warping) function and the period of the cyclostationary correlated random process.
3. The mathematical tools of the cyclically correlated random process, and in particular, its rhythm function, enable us to research in an explicit form the analytical dependencies between the rhythmic structures of cyclically correlated random processes, which are connected through the time-scale transformation operator; however, the same cannot be said about the known random processes (irregular cyclostationary process, time-warped almost-cyclostationary process, cyclostationary processes with evolving

periods and amplitudes), because in such constructively defined processes, there are no analogues of the rhythm function that would simulate the rhythmic structure of the cyclic signal itself. In addition, in the work [71], within the framework of a cyclic random process, a much wider class of analytical dependencies was investigated between the rhythm functions of cyclic random processes (both with regular and irregular rhythms), which are connected through the time-scale transformation operator, namely, not only the time-scale conversion (time-warping) of cyclic signals with a regular rhythm into cyclic signals with an irregular rhythm and vice versa (time-dewarping), but also the time-scale conversion of cyclic signals with a variable rhythm into cyclic signals with another type of variable rhythm.

4. In contrast to the methods of the processing of cyclic signals within the framework of models of the irregular cyclostationary process, the time-warped almost-cyclostationary process, and cyclostationary processes with evolving periods and amplitudes developed in works [56–60], in which the main task of processing is a reduction to known processing methods of cyclostationary correlated random processes or almost-cyclostationary correlated random processes, the methods of processing (statistical evaluation, discretization, spectral analysis) and the computer simulation of a cyclic random process and cyclically correlated random process, which were developed in the works [64–75], have the means to adapt to changes in the rhythm of the investigated signals, by considering a pre-estimated rhythm function [70,84]. The representation of the rhythm of cyclic signals in the form of the rhythm function of a cyclically correlated random process in an explicit form makes it possible to analytically and statistically describe the rhythmic structure of cyclic signals and as a consequence enables the analysis of the rhythm of cyclic signals as an important informative feature in many natural dynamic systems.

5. Conclusions

In this work, the procedure for constructing the cyclically correlated random process of a continuous argument, which takes into account the cyclicity and stochasticity of cyclic signals within the framework of the correlation theory of random processes and has an effective means of taking into account both the regularity and irregularity of the rhythm of cyclic signals in dynamic systems, for the first time is proposed. Mathematical structures that model the cyclic, phase and rhythmic structures of a cyclically correlated random process are presented. It made it possible to improve the mathematical means of the modeling and the analysis of cyclic signals within the framework of the correlation theory of random processes, in comparison with the possibilities of the classical theory of periodically correlated random processes, in which there are no ways to take into account the variability of the rhythm of cyclic signals, and there are no such expressive formal means of explicitly describing of the cyclic and phase structures of the investigated signals.

By proving the theorem, the sufficient and necessary conditions, which the structural function and the rhythm function of the cyclically correlated random process must satisfy, have been established, which provide a clear mathematical basis for the concept of the rhythm of the cyclic signals and make it possible to quantitatively describe its rhythmic structures.

However, an important area of the study of cyclically correlated random processes is their representation and analysis in the frequency domain. In this direction, it is important to have the means of spectral representation of both the cyclically correlated process itself and its mathematical expectation and correlation function. Additionally, an important direction of further research is the development of mathematical models in the form of integral and differential stochastic equations, the resolution of which is a cyclically correlated random process, which will make it possible to establish the conditions that the parameters of these equations must satisfy so that the mathematical expectation and the correlation function of the output signal of the dynamic system are cyclic. In particular, within the framework of the theory of linear random processes, such conditions were established in [85].

The developed approach to the construction of a cyclically correlated random process can be extended to the construction of other types of mathematical models of signals which are cyclic according to certain attributes of cyclicity, in particular, to cyclic random processes in the strict sense and cyclic random processes relative to its higher-order moment functions. The obtained results contribute to the emergence of a more complete and rigorous theory of this class of random processes and increase the validity of the methods of analysis, the forecasting, and the computer simulation of cyclic signals in dynamic systems.

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