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On Some Properties of Addition Signed Cayley Graph $\Sigma_n^{\wedge}$

Obaidullah Wardak [1], Ayushi Dhama [2] and Deepa Sinha [1,*]

[1] Department of Mathematics, South Asian University, New Delhi 110 021, India
[2] Centre for Mathematical Sciences, Banasthali University, Rajasthan 304 022, India
* Correspondence: deepasinha@sau.ac.in

Abstract: We define an addition signed Cayley graph on a unitary addition Cayley graph G_n represented by $\Sigma_n^{\wedge}$, and study several properties such as balancing, clusterability and sign compatibility of the addition signed Cayley graph $\Sigma_n^{\wedge}$. We also study the characterization of canonical consistency of $\Sigma_n^{\wedge}$, for some n.

Keywords: addition signed Cayley graph $\Sigma_n^{\wedge}$

MSC: 05C 22; 05C 75

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1. Introduction

We refer to standard books of Harary [1] and West [2] for graph theory. For the signed graphs, we refer to Zaslavsky [3,4]. All the signed graphs considered in this paper are simple, finite and loopless.

For the preliminaries, definition and notation of signed graph S, underlying graph S^u, its negation $\eta(S)$, signed isomorphism and its positive (negative) section, we refer to [5,6].

Some Basic Lemma and Theorems which are used in this paper are stated below as a reference.

Lemma 1 ([7])**.** *A signed graph in which every chordless cycle is positive is balanced.*

Theorem 1 ([8])**.** *A signed graph S is clusterable if—and only if—S does not contains a cycle with exactly one negatively charged edge.*

For balancing, clusterability, marking, canonical marking ($\mathcal{C}$*-marking*), consistency, $\mathcal{C}$*-consistency*, S consistency, sign compatibility, line signed graph $L(S)$, line signed root graph, $\times$-line signed graph, $\times$-line signed root graph and the common-edge signed graph $C_E(S)$ of signed graph, S we refer to [6,9–16].

Addition Signed Cayley Graph $\Sigma_n^{\wedge}$

A *unitary addition Cayley graph* G_n, where $n \in I^+$, I^+ is set of positive integers, is a graph in which the vertex set is a ring of integers modulo n, Z_n. Any two vertices x_1 and x_2 are adjacent in G_n if—and only if—$(x_1 + x_2) \in U_n$, where U_n denotes the unit set.

Unitary addition Cayley graphs for n = 2, 3, 4, 5, 6 and 7 are shown in Figure 1.

The study of unitary Cayley graphs began in order to gain some insight into the graph representation problem (see [17]), and we can extend it to the signed graphs (see [18]). Now, we introduce the definition of an *addition signed Cayley graph* $\Sigma_n^{\wedge}$ as follows:

The *addition signed Cayley graph* $\Sigma_n^{\wedge} = (G_n, \sigma^{\wedge})$ is a signed graph whose underlying graph is a unitary addition Cayley graph G_n, where $n \in I^+$ and for an edge ab of $\Sigma_n^{\wedge}$,

$$\sigma^{\wedge}(ab) = \begin{cases} + & \text{if } a, b \in U_n, \\ - & \text{otherwise.} \end{cases}$$

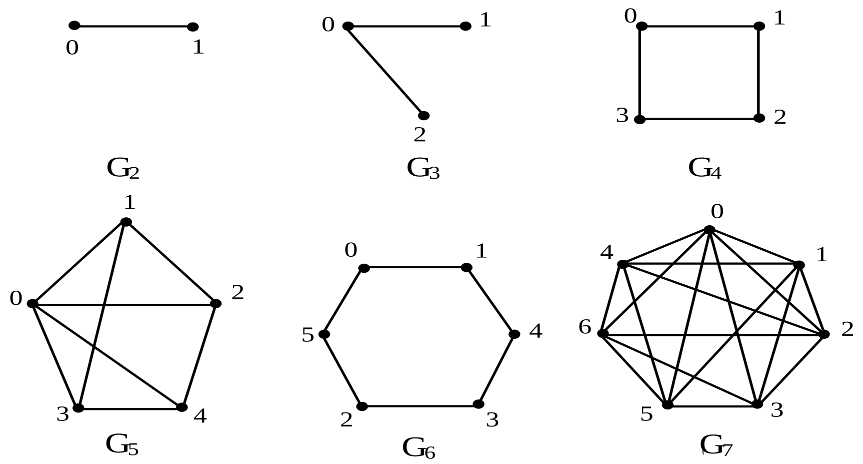

Figure 1. Examples of unitary addition Cayley graphs.

Examples of addition signed Cayley graph for $n = 5, 6$ and 10 can be seen in Figure 2a–c. Throughout the paper, we consider $n \geq 2$.

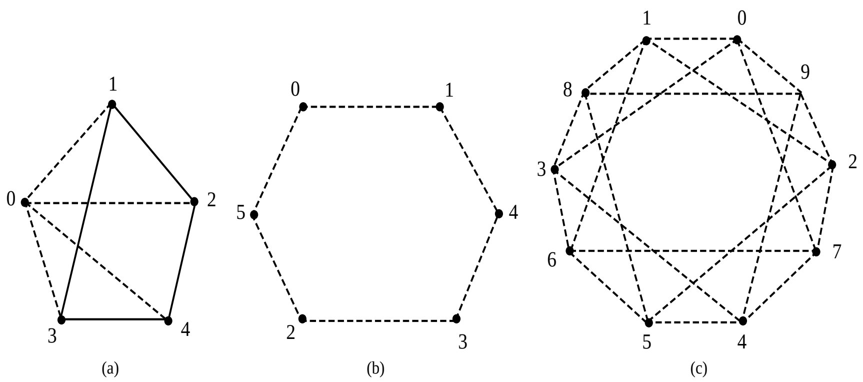

Figure 2. Examples of addition signed Cayley graph $\Sigma_n^\wedge$.

2. Some Properties of $\Sigma_n^\wedge$

2.1. Balancing in $\Sigma_n^\wedge$

The balancing of some derived signed Cayley graphs has been studied in the literature (see [19]). Here, we find out the property of balancing for the addition signed Cayley graph $\Sigma_n^\wedge$, for which the following well-known result can be used as a tool.

Theorem 2 ([20])**.** *G_n, $n \geq 2$, is bipartite if—and only if—either $n = 3$ or n is even.*

Lemma 2. *$i \in U_n \Rightarrow (n-i) \in U_n$ and $i \notin U_n \Rightarrow (n-i) \notin U_n$.*

Lemma 3. *Addition signed Cayley graph $\Sigma_n^\wedge = (G_n, \sigma^\wedge)$, for n even, is an all-negative signed graph.*

Proof. Given an addition signed Cayley graph $\Sigma_n^\wedge = (G_n, \sigma^\wedge)$, where n is even. Suppose the conclusion is false. Let there be a positive edge, say ij, in $\Sigma_n^\wedge$. By the definition of $\Sigma_n^\wedge$, $i, j \in U_n$. Since n is even, U_n consists only of odd numbers. Thus, i and j are odd numbers and their addition $i + j$ is an even number. This shows that $i + j \notin U_n$, i.e., i and j, are not adjacent in $\Sigma_n^\wedge$. Thus, we have a contradiction. Hence, if n is even, then $\Sigma_n^\wedge$ is all-negative signed graph. □

Sampathkumar [21] gave the famous characterization to prove the balancing in a signed graph, which is as follows:

Theorem 3 (Marking Criterion [21]). *A signed graph $S = (G, \sigma)$ is balanced if—and only if—there exists a marking μ of its vertices such that each edge uv in S satisfies $\sigma(uv) = \mu(u)\mu(v)$.*

Lemma 4. *For the addition signed Cayley graph $\Sigma_n^\wedge = (G_n, \sigma^\wedge)$, $\Sigma_n^\wedge$ is a balanced signed graph, if for any prime p, $n = p^a$.*

Proof. $n = p^a$, where p is a prime number. Now, we assign a marking μ to the vertices of $\Sigma_n^\wedge$ in such a manner that if $u \in U_n$, then $\mu(u) = +$ and if $u \notin U_n$, then $\mu(u) = -$, $\forall$ $u \in V(\Sigma_n^\wedge)$. Suppose there is an edge, say ij, in $\Sigma_n^\wedge$.

Case I: Let $\sigma^\wedge(ij) = +$. Then, $i, j \in U_n$ and according to the marking $\mu(i) = \mu(j) = +$. Thus, $\sigma^\wedge(ij) = \mu(i)\mu(j) = +$.

Case II: Let $\sigma^\wedge(ij) = -$. Then, there are three possibilities:

(a) $i \in U_n, j \notin U_n$.
(b) $i \notin U_n, j \in U_n$.
(c) $i, j \notin U_n$.

Now, for (a) and (b), by marking μ, we get $\mu(j) = -$ and $\mu(i) = +$ or vice versa. Therefore, $\sigma^\wedge(ij) = \mu(i)\mu(j) = -$. Now, if $i, j \notin U_n$. Then, i and j are both multiples of p, and then $i + j = kp$, where k is some positive integer and $i + j \notin U_n$. So $ij \notin E(\Sigma_n^\wedge)$. Thus, condition (c) is not possible. So in every condition we get $\sigma^\wedge(ij) = \mu(i)\mu(j)$. Since ij is an arbitrary edge, using Theorem 3, $\Sigma_n^\wedge$ is balanced. □

Theorem 4. *The addition signed Cayley graph $\Sigma_n^\wedge$ is balanced if—and only if—either n is even or if n has exactly one prime factor, then n is odd.*

Proof. ***Necessity***: First, suppose $\sum_n^\wedge$ is balanced. Now, let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m}$; $p_1, p_2, \dots, p_m$ being distinct primes, $p_1 \neq 2$, $p_1 \leq p_2 \leq \dots \leq p_m$.

In the unitary addition Cayley graph G_n, $p_1 + 1 \neq k_1 p_i$ for $i = 1, 2, \dots, m$ and k_1 are some positive integers i.e., $p_1 + 1 \in U_n$, so p_1 is adjacent with one. Now, we claim that p_1 and p_2 are adjacent in G_n. On the contrary, suppose $p_1 p_2$ is not an edge in G_n. Then, $p_1 + p_2 \notin U_n$. Thus, $p_1 + p_2 = k_2 p_i$ for some $i = 1, 2, \dots, m$ and k_2 are some positive integers. Let $p_1 + p_2$ be a multiple of p_1.

$$\begin{aligned} p_1 + p_2 &= \alpha p_1 \\ p_2 &= \alpha p_1 - p_1 \\ &= (\alpha - 1) p_1 \end{aligned}$$

for the positive integer α, a contradiction. With the same argument, we can show that $p_1 + p_2$ is not a multiple of p_2. Now, let $p_1 + p_2 = \alpha p_i$, for $i = 3, 4, \dots, m$. As we know, the addition of two prime factors is always even; $p_1 + p_2$ is even. So, α is even and is at least 2. However, as $p_1 < p_2 < p_i$, $p_1 + p_2$ is always less than any multiple of p_i for $i = 3, 4, \dots, m$. Thus, $p_1 + p_2 \in U_n$ and $p_1 p_2$ is an edge in G_n. Next, if p_2 is adjacent to 1 in G_n, we get a cycle

$$C = (p_1, p_2, 1, p_1)$$

in $\Sigma_n^\wedge$. Clearly, p_1 and p_2 are not in U_n, then by definition of $\Sigma_n^\wedge$, C is a negative cycle. Thus, we have a negative cycle in $\Sigma_n^\wedge$, implying that $\Sigma_n^\wedge$ is not balanced. Now, suppose $p_2 + 1 \notin E(G_n)$, since $p_2 + 1 \notin U_n$. Then, $p_2 + 1 = c p_i$; $i = 1, 2, \dots, m$, c are positive integers. Clearly,

$$p_2 + 1 = \alpha p_1 \tag{1}$$

α is a positive integer.

Since $p_2 \notin U_n$, according to Lemma 2, $n - p_2 \notin U_n$. Next, we claim that $n - p_2$ is adjacent to 1 or $n - p_2 + 1 = n - (p_2 - 1) \in U_n$. If $p_2 - 1 \in U_n$, then according to Lemma 2, $n - p_2 + 1 = n - (p_2 - 1) \in U_n$. Suppose $p_2 - 1 \notin U_n$. Then, $p_2 - 1 = \beta p_i$; $i = 1, 2, \dots, m$,

β are positive integers. Let $p_2 - 1 = \beta p_1$. However, from Equation (1), $p_2 = \alpha p_1 - 1$. This implies

$$\begin{aligned} p_2 - 1 &= \beta p_1 \\ \alpha p_1 - 1 - 1 &= \beta p_1 \\ \alpha p_1 - 2 &= \beta p_1 \\ \alpha p_1 - \beta p_1 &= 2 \\ (\alpha - \beta) p_1 &= 2. \end{aligned}$$

This is not possible, as p_1 is at least 3. Thus, $p_2 - 1$ is not a multiple of any of the p_is, whence $p_2 - 1 \in U_n$. Hence, $n - p_2 + 1 = n - (p_2 - 1) \in U_n$, whence $n - p_2$ is adjacent to 1 in G_n. Now, $n - p_2 + p_1 = n - (p_2 - p_1)$. Since $p_1 < p_2 < \cdots p_m$, $p_2 - p_1 \neq kp_i$; $i = 2, 3, \ldots m.$, k is a positive integer. Additionally, $p_2 - p_1$ is not a multiple of p_1. This shows that $p_2 - p_1 \in U_n$ and by Lemma 2, $n - (p_2 - p_1) \in U_n$. This shows that $n - p_2$ is adjacent to p_1 in Σ_n. Thus, we get a cycle

$$C' = (p_1, n - p_2, 1, p_1)$$

in Σ_n. Clearly, p_1 and $n - p_2$ do not belong to U_n and $1 \in U_n$. Then, by definition $\Sigma_n^{\wedge}$, we have a cycle C' with three negative edges. Thus, a contradiction. So, by contraposition, necessity is true.

Sufficiency: Let n be even. Then, according to Lemma 3, $\Sigma_n^{\wedge}$ is an all-negative signed graph. Additionally, according to Theorem 2, G_n is a bipartite graph. Hence, $\Sigma_n^{\wedge}$, by Lemma 3 and Theorem 2, is balanced.

Now, let n be odd, with exactly one prime factor. Then, according to Lemma 4, $\Sigma_n^{\wedge}$ is balanced, hence the theorem. □

2.2. Clusterability in $\Sigma_n^{\wedge}$

Theorem 5. *The addition signed Cayley graph $\Sigma_n^{\wedge} = (G_n, \sigma^{\wedge})$ is clusterable.*

Proof. Given an addition signed Cayley graph $\Sigma_n^{\wedge} = (G_n, \sigma^{\wedge})$. Suppose $v \in V(\Sigma_n^{\wedge})$. Define $V^* \subseteq V(\Sigma_n^{\wedge})$, such that $V^* = \{u_i : u_i \in V(\Sigma_n^{\wedge})$ and $\sigma^{\wedge}(vu_i) = +\}$. By the definition of $\Sigma_n^{\wedge}$, clearly u_i and v are in U_n.

If, for i and j, $(i \neq j)$, u_i and u_j are adjacent, then $\sigma^{\wedge}(u_i u_j) = +$. Thus, $U_n \subseteq V^*$. Since $|U_n| = \phi(n)$, $n - \phi(n) = k$ (say) vertices are not in U_n. Thus, only negative edges are incident on these k vertices. Put all these vertices in the k partition $V_1, V_2, \ldots, V_k$, such that each partition contains exactly only one vertex. The clearly induced subgraph $< V^* >$ is all positive. Additionally, no positive edge joins the vertex of V^* with the vertex of any of V_i, for $i = 1, 2, \ldots, k$, and there is no edge xy, such that $\sigma^{\wedge}(xy) = -$ and $x, y \in V^*$. Thus, there exists a partition of the $V(\Sigma_n^{\wedge})$, such that every positive edge has end vertices within the same subset and every negative edge has end vertices in a different subset. Hence, the proof. □

2.3. Sign-Compatibility in $\Sigma_n^{\wedge}$

Theorem 6 ([22]). *A signed graph S is sign compatible if—and only if—S does not contain a sub signed graph isomorphic to either of the two signed graphs. S_1 formed by taking the path $P_4 = (x, u, v, y)$ with both edges xu and vy negative and edge uv positive, and S_2 formed by taking S_1 and identifying the vertices x and y* (Figure 3).

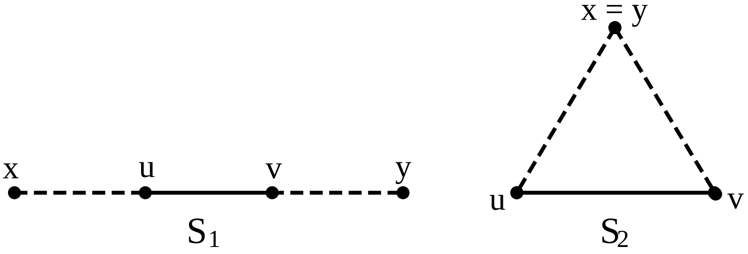

Figure 3. Two forbidden sub signed graphs for a sign-compatible signed graph [13].

Theorem 7. *Addition signed Cayley graph* $\Sigma_n^\wedge$ *is sign compatible if—and only if—n is* 3 *or even.*

Proof. Let addition signed Cayley graph $\Sigma_n^\wedge$ be sign compatible. If possible, suppose the conclusion is not true. Let n be odd but not 3. Now, $01 \in E(\Sigma_n^\wedge)$. As, $n-2+1 = n-1 \in U_n$, $1(n-2) \in E(\Sigma_n^\wedge)$. Additionally, $n-2+0 = n-2 \in U_n$. Thus, we have a triangle $(0, 1, n-2, 0)$ with one positive edge $1(n-2)$ and two negative edges 01 and $(n-2)0$, which again contradict Theorem 6. Hence, the condition is necessary.

Next, let n be even. Thus, according to Lemma 3, $\Sigma_n^\wedge$, which is all-negative, is trivially sign compatible. If $n = 3$, then $\Sigma_n^\wedge$ is P_3, which is trivially sign compatible. □

Acharya and Sinha [23] showed that every line signed graph is sign compatible. Next, we discuss the value of n for which $\Sigma_n^\wedge$ is a line signed graph.

Theorem 8. G_n *is a line graph if—and only if—n is equal to* 2 *or* 3 *or* 4 *or* 6*.*

Proof. ***Necessity***: Let G_n be a line graph. Meanwhile, n is not equal to 2, 3, 4 and 6.

Case I: n is prime. It is clear that $n \geq 5$. Here, n is prime, so by the definition of U_n, there are numbers from 1 to $(n-1)$ in U_n. 0 is connected to every vertex of G_n. The other vertex, $i \neq 0$, in G_n is not connected to only $(n-i)$ by definition. For any $i, j \in V(G_n)$; $i \neq 0$, $j \neq 0$ there is an induced subgraph in G_n (see Figure 4).

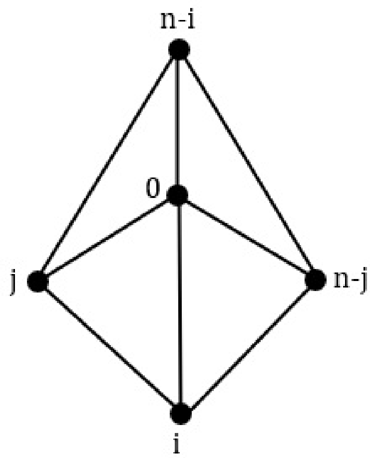

Figure 4. A forbidden subgraph for a line graph in G_n.

Thus, G_n contains forbidden subgraph for a line graph. Thus, G_n is not a line graph.

Case II: n is not prime. 1 is connected to 0 in G_n. Next, 1 is connected to p_1, as $p_1 + 1 \in U_n$, where p_1 is the smallest factor of n. Let $\alpha p_1 = n$, for a positive integer α. Now,

$$\begin{aligned} 1 + (\alpha - 1)p_1 &= 1 + \alpha p_1 - p_1 \\ &= 1 + n - p_1 \\ &= n - (p_1 - 1). \end{aligned}$$

Since $p_1 - 1 \in U_n$, by Lemma 2, $n - (p_1 - 1) \in U_n$. Thus, 1 and $(a-1)p_1$ are adjacent in G_n. Additionally, 0 is not adjacent to p_1 and $(a-1)p_1$, because their sum is a multiple of p_1. In the same way, p_1 and $(a-1)p_1$ are not connected in G_n because their sum is a multiple of p_1. So, we have an induced subgraph in G_n (see Figure 5). Thus, there is a forbidden subgraph $K_{1,3}$ of a line graph. Additionally, G_n is not a line graph.

Sufficiency: Let $n = 2$ or $n = 3$ or $n = 4$ or $n = 6$. Then, $G_2 \cong L(P_3)$, $G_3 \cong L(P_4)$, $G_4 \cong L(C_4)$ and $G_6 \cong L(C_6)$ (see Figure 6). Hence, the result. □

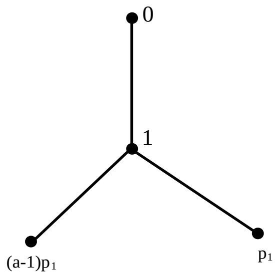

Figure 5. A forbidden subgraph for a line graph in G_n.

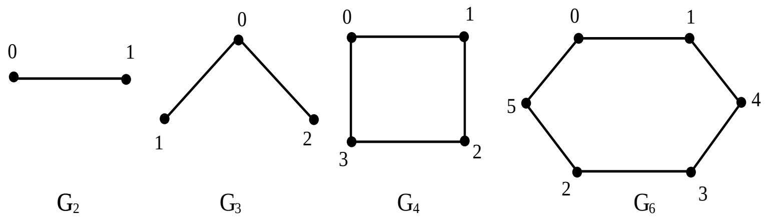

Figure 6. Showing G_2, G_3, G_4 and G_6.

Theorem 9. *$\Sigma_n^\wedge$ is a line signed graph if—and only if—$n = 2$ or $n = 3$ or $n = 4$ or $n = 6$.*

Proof. ***Necessity***: Let, if possible, n be unequal to 2, 3, 4 and 6. Theorem 8 shows that $G_n \ncong L(G)$, for any graph G. Thus, a contradiction and the condition are necessary.

Sufficiency: Now, suppose $n = 2$ or $n = 3$ or $n = 4$ or $n = 6$. Line signed graphs of an addition signed Cayley graph, for these values of n, are displayed in Figure 7, hence the sufficiency. □

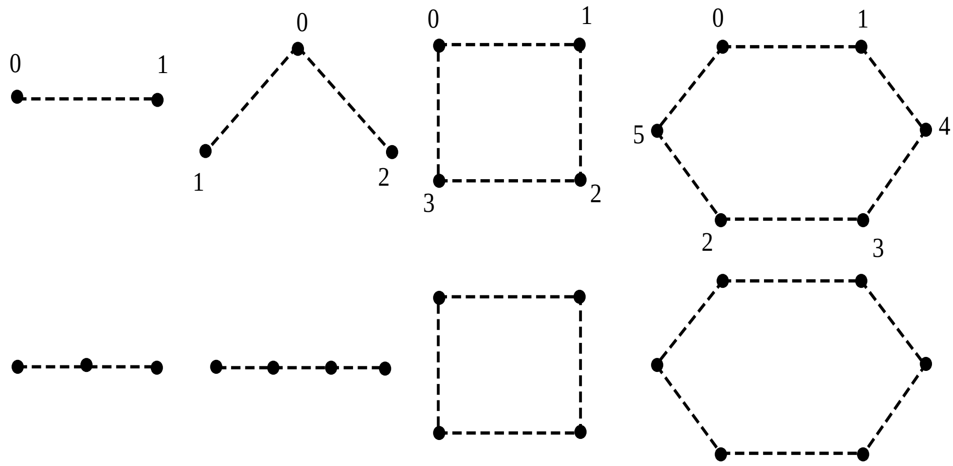

Figure 7. Showing $\Sigma_2^\wedge$, $\Sigma_3^\wedge$, $\Sigma_4^\wedge$ and $\Sigma_6^\wedge$ and its line signed root graphs.

Remark 1. *$\Sigma_n^\wedge$ is a $\times$-line signed graph if—and only if—$n = 2$ or $n = 3$ or $n = 4$ or $n = 6$.*

Proof. Let $\Sigma_n^\wedge$ be a $\times-$line signed graph. We know that the underlying structure for line signed graphs and $\times-$line signed graphs is the same. Thus, the condition comes from Theorem 8.

Next, let $n \in \{2,3,4,6\}$. $\Sigma_2^\wedge$, $\Sigma_3^\wedge$, $\Sigma_4^\wedge$ and $\Sigma_6^\wedge$ and its $\times-$line signed root graphs are displayed in Figure 8. From Theorem 4, it is clear that for these values of n, an addition signed Cayley graph is balanced. Additionally, $L_\times(S)$ of any signed graph is always balanced, and its underlying graph is a line graph (see [24]). This result comes from Theorems 4 and 8. □

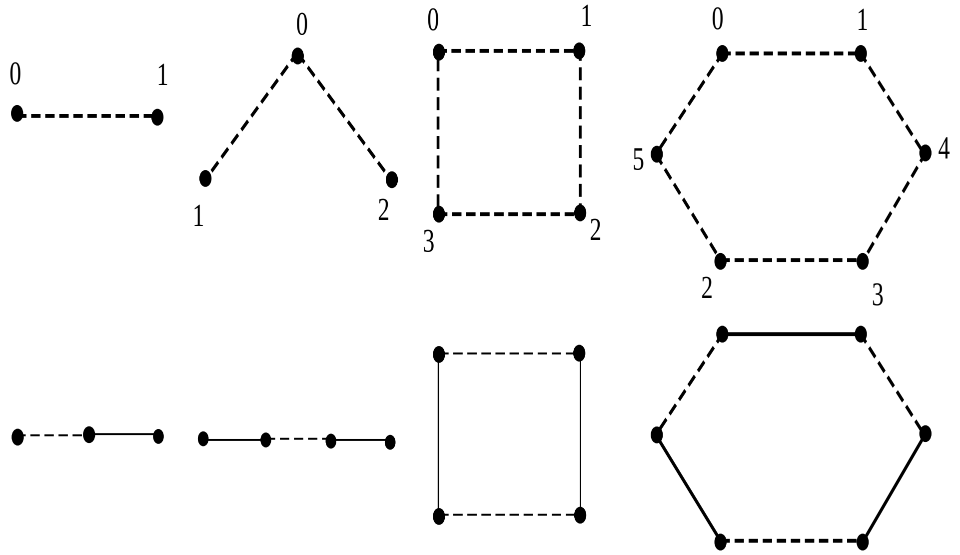

Figure 8. Showing $\Sigma_2^\wedge$, $\Sigma_3^\wedge$, $\Sigma_4^\wedge$ and $\Sigma_6^\wedge$ and its $\times-$line signed root graphs.

2.4. $\mathcal{C}$-Consistency of $\Sigma_n^\wedge$

Lemma 5. *For any prime p, $p \neq 2$ and $n = p^\alpha$, the $d^-(2)$ and $d^-(4)$ in $\Sigma_n^\wedge$ is odd.*

Proof. Given a $\Sigma_n^\wedge = (G_n, \sigma^\wedge)$, where $n = p^\alpha$ and p is an odd prime. Since n is odd; 2, $4 \in U_n$. It is obvious that $d^-(2)$ and $d^-(4)$ in $\Sigma_n^\wedge$ appear only when 2 and 4 are adjacent to kp, where k is some positive integer. Now, $(2+4) + cp \neq kp$; positive integers c and k. Additionally, 2 and 4 are connected to all the multiples of p, which are $p^{\alpha-1}$. Therefore $d^-(2)$ $(d^-(4)) = p^{\alpha-1}$ is odd, hence the lemma. □

Theorem 10 ([25]). *Let a, b and m be integers with m positive. The linear congruence $ax \equiv b \pmod{m}$ is soluble if and only if $(a,m)|b$. If x_0 is a solution, there are exactly (a,m) incongruent solutions given by $\{x_0 + tm/(a,m)\}$, where $t = 0, 1, \ldots, (a,m) - 1$.*

Corollary 1. *If $(a,m) = 1$ then the congruence $ax \equiv b \pmod{m}$ has exactly one incongruent solution.*

Lemma 6. *In addition, signed Cayley graph $\Sigma_n^\wedge = (G_n, \sigma^\wedge)$, if $n = p_1^{a_1} p_2^{a_2}$, where p_1 and p_2 are two distinct odd primes, then $d^-(2)(d^-(4))$ = odd.*

Proof. Given that $n = p_1^{a_1} p_2^{a_2}$ in $\Sigma_n^\wedge$, p_1 and p_2 are distinct odd primes. As n is odd, $2 \in U_n$. Now, the negative degree of 2 of $\Sigma_n^\wedge$ appears only when 2 is adjacent with the multiples of p_1 and p_2. Let $A_i = \{cp_i; c$ certain positive integers$, i = 1, 2\}$. Then,

$$|A_1| = p_1^{a_1-1} p_2^{a_2}$$
$$|A_2| = p_1^{a_1} p_2^{a_2-1}$$

and

$$|A_1 \cap A_2| = p_1^{a_1-1} p_2^{a_2-1}$$

Thus, using the inclusion–exclusion principle

$$|A_1 \cup A_2| = p_1^{a_1-1} p_2^{a_2} + p_1^{a_1} p_2^{a_2-1} - p_1^{a_1-1} p_2^{a_2-1}$$

Since $cp_1(p_2) + 2 = p_2(p_1)$, for certain positive integers c and so, $cp_1(p_2)2 \notin E(\Sigma_n^\wedge)$ for those c. Thus, according to Theorem 10, we have

$$p_1 x \equiv -2 \pmod{p_2} \tag{2}$$

and

$$p_2 y \equiv -2 \pmod{p_1} \tag{3}$$

Due to Corollary 1, we have an incongruent solution x_0 (say), which is unique for Equation (2). So, for Equation (2) where $p_1 x + 2 < n$, we have:

$$x_0 + 0(p_2), x_0 + 1(p_2), x_0 + 2(p_2), \ldots, x_0 + (p_1^{a_1-1} p_2^{a_2-1} - 1)(p_2) \tag{4}$$

Thus, Equation (2) has $p_1^{a_1-1} p_2^{a_2-1}$ total solutions. Similarly, the total solutions of Equation (3) are $p_1^{a_1-1} p_2^{a_2-1}$. Hence,

$$\begin{aligned} d^-(2) &= p_1^{a_1-1} p_2^{a_2} + p_1^{a_1} p_2^{a_2-1} - p_1^{a_1-1} p_2^{a_2-1} - p_1^{a_1-1} p_2^{a_2-1} - p_1^{a_1-1} p_2^{a_2-1} \\ &= p_1^{a_1-1} p_2^{a_2-1} (p_1 + p_2 - 3) \end{aligned}$$

p_1 and p_2 are odd primes, which implies $d^-(2)$ is odd. The proof for $d^-(4)$ is analogous. □

Lemma 7. *In $\Sigma_n^\wedge = (G_n, \sigma^\wedge)$, if $n = 3^{a_1} 5^{a_2}$, then $d^-(7)$ = odd.*

Proof. This is easy to prove using the same logic as mentioned in Lemma 6. □

Theorem 11. *Let n have at most two distinct odd prime factors, then $\Sigma_n^\wedge$ is $\mathcal{C}$ consistent if—and only if—n is even or 3.*

Proof. ***Necessity***: Let n have, at most, two distinct prime factors and let $\Sigma_n^\wedge$ be $\mathcal{C}$ consistent. If possible, let n be odd but not 3.

Case (a): Let $n \equiv 1 \pmod 3$ or $n \equiv 2 \pmod 3$. As n is odd, $2 \in U_n$. Clearly, 0 is adjacent with 1, 2, $n-1$ in $\Sigma_n^\wedge$. Since, $n - 1 + 2 = 1 \in U_n$, $n-1$ and 2 are connected in $\Sigma_n^\wedge$. Since, 3 is not a factor of n, $3 \in U_n$. Now, $2 + 1 = 3 \in U_n$. Hence, 2 and 1 are adjacent in $\Sigma_n^\wedge$. Now, the cycles $Z_1 = (0, 1, 2, 0)$, $Z_2 = (0, 2, n-1, 0)$ have a common chord with end vertices 0 and 2. By Lemma 6,

$$\mu_\sigma(2) = -$$

Since the vertex $0 \notin U_n$, $d(0) = d^-(0) = \phi(n)$ = even. It follows,

$$\mu_\sigma(0) = +.$$

Now, if either Z_1 or Z_2 is not a $\mathcal{C}$-consistent cycle, a contradiction. Thus, Z_1 and Z_2 both cycle are $\mathcal{C}$-consistent. The common chord with end vertices zero and two are oppositely marked, in contradiction with (Theorem 2, [26]).

Case (b): Let $n \equiv 0 \pmod 3$. Then, either $n = 3^{a_1}$ or $n = 3^{a_1} \times p_2^{a_2}$. First, suppose $p_2 \neq 5$. Since, n is odd, $2, 4 \in U_n$. According to Lemma 2, $n-2 \in U_n$. Clearly, 0 is adjacent to 1, 4 and $n-2$ in $\Sigma_n^\wedge$. Since, $n-2+4 = n+2 = 2 \in U_n$, $n-2$ is adjacent to 4 in $\Sigma_n^\wedge$. Now, for cycle $Z_1 = (0,1,4,0)$, $Z_2 = (0,4,n-2,0)$; Z_1, Z_2 have a common chord with end vertices 0 and 4. According to Lemma 6,

$$\mu_\sigma(4) = -$$

Since the vertex $0 \notin U_n$, $d(0) = d^-(0) = \phi(n) =$ even. It follows,

$$\mu_\sigma(0) = +.$$

Now, if either Z_1 or Z_2 is a cycle which is not $\mathcal{C}$ consistent, a contradiction. Therefore, Z_1 and Z_2 are the cycles which are $\mathcal{C}$-consistent. However, there is a chord whose end vertices 0 and 4 have opposite marking. Here again, we find a contradiction to the (Theorem 2, [26]).

Now, suppose $p_2 = 5$. In this case, we consider two cycles $Z_1 = (0,1,7,0)$ and $Z_2 = (0,7,10,13,0)$ in $\Sigma_n^\wedge$. For cycles Z_1, Z_2 have a common chord with end vertices 0 and 7, according to Lemma 7,

$$\mu_\sigma(7) = -$$

Since the vertex $0 \notin U_n$, $d(0) = d^-(0) = \phi(n) =$ even. It follows that

$$\mu_\sigma(0) = +.$$

Now, if either Z_1 or Z_2 is a cycle which is not $\mathcal{C}$ consistent, this is a contradiction. Therefore, Z_1 and Z_2 are the cycles which are $\mathcal{C}$ consistent. However, the end vertices 0 and 7 have the opposite marking. Here, we have a contradiction to the (Theorem 2, [26]). Hence, n is either even or $n = 3$.

Sufficiency: Let n be even. According to Lemma 3, $\Sigma_n^\wedge$ is all negative. Additionally, according to Theorem 13, $d(v) = d^-(v)$ = even $\forall\, v \in V(\Sigma_n^\wedge)$. So, according to canonical marking $\mu_\sigma(v) = + \;\forall\, v \in V(\Sigma_n^\wedge)$. So when n is even, $\Sigma_n^\wedge$ is trivially $\mathcal{C}$ consistent. If $n = 3$, then G_3 is a path, which is trivially $\mathcal{C}$- consistent, hence the result. □

3. Balance in Certain Derived Signed Graphs of $\Sigma_n^\wedge$

Theorem 12. *$\eta(\Sigma_n^\wedge)$ is balanced if—and only if—n is 3 or even.*

Proof. Let $\eta(\Sigma_n^\wedge)$ be balanced. If possible, n is odd but not 3, and p is the smallest prime factor of n. Since $n-2+1 = n-1 \in U_n$, $n-2$ and 1 are connected in $\Sigma_n^\wedge$. $p+1 \in U_n$ implies that p and 1 are connected in $\Sigma_n^\wedge$. Additionally, as n is odd, $2 \in U_n$ and $n-2 \in U_n$. according to Lemma 2. Since, $n-2+p = n+(p-2) = p-2 \in U_n$, $(n-2)p \in E(\Sigma_n^\wedge)$. Now, for the cycle $Z = (1,p,n-2,1)$ in $\Sigma_n^\wedge$ we have a one positive edge $1(n-2)$ and two negative edges $1p$ and $p(n-2)$ in Z. However, in $\eta(\Sigma_n^\wedge)$, there is a cycle $Z' = (1,p,n-2,1)$ with one negative edge $1(n-2)$ and two positive edges $1p$ and $p(n-2)$. Thus, we have a negative cycle that contradicts the given condition. Therefore, the only possibility is that n is 3 or even.

Conversely, let n be even. $\Sigma_n^\wedge$, according to Lemma 3 is an all-negative signed graph. So $\eta(\Sigma_n^\wedge)$ is balanced and is all positive. $\eta(\Sigma_n^\wedge)$ for $n = 3$ is a tree which is trivially balanced, hence the converse. □

We present the following theorem for the degree of the vertices of G_n (see [20]).

Theorem 13 ([20]). *Let m be any vertex of the unitary addition Cayley graph G_n. Then,*

$$d(m) = \begin{cases} \phi(n) & \text{if } n \text{ is even,} \\ \phi(n) & \text{if } n \text{ is odd and } (m,n) \neq 1, \\ \phi(n) - 1 & \text{if } n \text{ is odd and } (m,n) = 1. \end{cases}$$

Additionally, for a signed graph S, the balance property of $L(S)$ is discussed in ([27], Theorem 4).

Theorem 14. *For an additional signed Cayley graph $\Sigma_n^{\wedge} = (G_n, \sigma^{\wedge})$, its line signed graph $L(\Sigma_n^{\wedge})$ is balanced if—and only if—$n \in \{2,3,4,6\}$.*

Proof. Let $L(\Sigma_n^{\wedge})$ be balanced and $n \neq 2,3,4$ and 6. Now, according to Theorem 13, $d(0) = d^-(0) = \phi(n)$ = even, which implies $d(0) = d^-(0) = \phi(n) \geq 4$. This shows that condition *ii* (of Theorem 4, [27]) is not satisfied for $\Sigma_n^{\wedge}$. This is a contradiction. Hence, $n \in \{2,3,4,6\}$. The converse part is easy to prove. □

For a signed graph S, the balance property of $C_E(S)$ is discussed in ([9], Theorem 13).

Theorem 15. *For an additional signed Cayley graph $\Sigma_n^{\wedge} = (G_n, \sigma^{\wedge})$, its common-edge signed graph $C_E(\Sigma_n^{\wedge})$ is balanced if—and only if—$n \in \{3,4,6\}$.*

Proof. Let $n \notin \{3,4,6\}$. It is clear that $0 \notin U_n$. Now, by Theorem 13, $d(0) = d^-(0) = \phi(n)$ = even, which implies $d(0) = d^-(0) = \phi(n) \geq 4$. This shows that condition *ii* (of Theorem 13, [9]) is not satisfied for $\Sigma_n^{\wedge}$. Thus, $C_E(\Sigma_n^{\wedge})$ is not balanced, which is a contradiction. Hence, $n \in \{3,4,6\}$. The converse part is easy to prove. □

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