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A More Flexible Asymmetric Exponential Modification of the Laplace Distribution with Applications for Chemical Concentration and Environment Data

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Abstract: In this work, a new family of distributions based on the Laplace distribution is introduced. We define this new family by its stochastic representation as the sum of two independent random variables, one with a Laplace distribution and the other with an exponential distribution. Using a Monte Carlo simulation study, the statistical performance of the estimators obtained by the moments and maximum likelihood methods were empirically evaluated. We studied the coverage probabilities and mean length of the confidence intervals of the corresponding parameters based on the asymptotic normality of these estimators. This simulation study reported a good statistical performance of these estimators. Fits were made to three real data sets with the new distribution, two related to chemical concentrations and one to the environment, comparing it with three similar distributions given in the literature. We have used information criteria for the selection of models. These results showed that the exponentially modified Laplace model can be an alternative distribution to model skewed data with high kurtosis. The new approach is a contribution to the tools of statisticians and various professionals interested in modeling data with high kurtosis.

Keywords: exponentially modified Laplace distribution; moments; skewness and kurtosis coefficients

MSC: 62P12

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1. Introduction

There are several investigations that use the Laplace distribution to model data from certain fields based on an empirical fit using goodness-of-fit techniques. For example, in environmental problems, the Laplace distribution is used to analyze (or model) random variables that determine maximum pollution values and describe times of high pollution. In mining, the Laplace distribution is used to analyze the mineral content in soil samples [1,2].

However, not all data related to these types of problems have a symmetric behavior. For this reason, other distributions have been proposed that are capable of better modeling this type of data. In this sense, Agu and Onwukwe [3] presented the modified Laplace distribution (ML), Grushka [4] presented the exponentially modified Gaussian distribution (EMG) and Reyes et al. [5] presented the exponentially modified logistic distribution ($EMLOG$). One of the advantages of these new probability distributions obtained through mixtures is that the obtained distributions generally have longer tails than the base distribution, thus giving rise to better fits for empirical frequency distributions, [4,5].

Our research is based on the theory of probability distributions and based on the process of mixtures of probability distributions, it proposes a new parametric probability distribution using the Laplace distribution. The new distribution depends on three parameters and is obtained by adding two independent random variables: one with a Laplace distribution and the other with an exponential distribution. This distribution can be used as an alternative to some existing distributions. The density function of the new distribution is obtained using the stochastic representation $Y = \sigma(X + V) + \mu$ where X and V are independent random variables, such that X is standard Laplace distribution and V is exponentially distributed with parameter λ, where μ is the location parameter, σ is the scale parameter, and λ is the skewness parameter. This document is organized as follows: Section 2, in order to make this work self-contained, presents the probability distributions of the Laplace, exponential, modified Laplace, exponentially modified Gaussian, and exponentially modified logistic distributions with some characteristics of these that will be useful later. In Section 3, the exponentially modified Laplace probability distribution is constructed, obtaining the density and the main characteristics of the distribution. In Section 4, the methods of moments and maximum likelihood are presented to estimate the parameters of the distribution. A simulation study for the theoretical validation of the model is also presented. Section 5 shows a comparative analysis and a discussion of the results obtained by fitting the different data sets with the modified Laplace (ML), exponentially modified Gaussian (EMG), and exponentially modified logistic distributions ($EMLOG$) and the proposed exponentially modified Laplace distribution (EML). Finally, in Section 6, conclusions are drawn from the work.

2. Preliminaries

The classical Laplace distribution (also known as Laplace's first law) is a probability distribution, given by the density function

$$f(x;\theta,s) \quad = \quad \frac{1}{2s}e^{-\frac{|x-\theta|}{s}}\,,\ x\in\mathbb{R}$$

where $-\infty < \theta < \infty$ and $s > 0$ are the location and scale parameters, respectively (Johnson et al. [6]), and we will denote it as $X \sim L(\theta,s)$. When the location parameter is equal to zero and the scale parameter is equal to one, then the standard Laplace distribution function is obtained, denoted by $L(0,1)$. The nth moment for a random variable $X \sim L(0,1)$, is given by:

$$E(X^n) \quad = \quad \frac{1}{2}n!\{1+(-1)^n\}\ n=1,2,\ldots \tag{1}$$

The continuous random variable, say X, is said to have an exponential distribution if it has the following probability density function:

$$f(x;\lambda)=\begin{cases}\lambda\, e^{-\lambda x} & si \quad x>0\\ 0 & si \quad x\le 0\end{cases}$$

where λ is called the rate of the distribution and will be represented as $X \sim exp(\lambda)$. The nth moment for a random variable $X \sim exp(\lambda)$ is given by the following expression:

$$E(X^n) \quad = \quad \frac{n!}{\lambda^n},\ n=1,2,\ldots \tag{2}$$

Agu and Onwukwe [3] presented the modified Laplace distribution whose density function is given by

$$f_X(x)=\begin{cases}\frac{\lambda}{2\sigma}\left(\frac{1}{2}e^{\frac{x-\mu}{\sigma}}\right)^{\lambda-1}e^{\frac{x-\mu}{\sigma}} & , \quad x\le\mu\\[2ex] \frac{\lambda}{2\sigma}\left(1-\frac{1}{2}e^{-\frac{x-\mu}{\sigma}}\right)^{\lambda-1}e^{-\frac{x-\mu}{\sigma}} & , \quad x>\mu\end{cases}$$

$x\in\mathbb{R}$, which is denoted by $X \sim ML(\mu,\sigma,\lambda)$.

The pdf of a random variable with an exponentially modified Gaussian distribution EMG (Grushka [4]) is given by:

$$f_Y(y;\mu,\sigma,\lambda)=\frac{\lambda}{2}e^{-\frac{\lambda}{2}(2y-2\mu-\lambda\sigma^2)}erfc\left(\frac{2\mu+\lambda\sigma^2-y}{\sqrt{2\sigma^2}}\right),\quad x\in\mathbb{R}$$

and is denoted as $Y \sim EMG(\mu, \sigma, \lambda)$, where $erfc(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2}\, dt$.

A random variable X has a logistic distribution with location parameter $\alpha \in \mathbb{R}$ and scale parameter $\beta > 0$ if its density function is:

$$f_X(x; \alpha, \beta) = \frac{e^{-(x-\alpha)/\beta}}{\beta\left(1 + e^{-(x-\alpha)/\beta}\right)^2}, \quad x \in \mathbb{R}$$

which is denoted as $X \sim LOG(\alpha, \beta)$. When the location parameter is 0 and the scale parameter is 1, then the standard logistic distribution function is obtained.

Reyes et al. [5], using the methodology given by [4], introduces the exponentially modified logistic distribution by the following stochastic representation:

$$Y = Z + T,$$

where $Z \sim LOG(\alpha, \beta)$ and $T \sim exp(1/\beta)$ are random independent variables and are denoted by $Y \sim EMLOG(\alpha, \beta)$, transforming this into a more flexible distribution in terms of working with data that have high kurtosis. Its function is given by:

$$f_Y(y|\alpha, \beta) = \frac{1}{\beta^2} e^{\frac{y-\alpha}{\beta}} \int_0^\infty e^{-\frac{2w}{\beta}} \left[1 + e^{\frac{y-w-\alpha}{\beta}}\right]^{-2} dw, \quad -\infty < y < \infty$$

and we denote as $Y \sim EMLOG(\alpha, \beta)$.

3. Exponentially Modified Laplace Distribution

In this section, the exponentially modified Laplace distribution (EML) is presented using the Grushka methodology [4], considering the location and scale parameters. This distribution is obtained by substituting the normal distribution for the standard Laplace distribution in the stochastic representation. The flexibility of this new distribution allows better capture of outliers. We will start by deriving its density function.

3.1. Density Function

The exponentially modified Laplace distribution admits the following stochastic representation as

$$Y = \sigma(X + V) + \mu, \tag{3}$$

where X and V are independent random variables such that $X \sim L(0, 1)$ and $V \sim \exp(\lambda)$, where μ is the location parameter, σ is the scale parameter, and λ is the skewness parameter, so we say that Y follows an exponentially modified Laplace distribution and is denoted by $Y \sim EML(\mu, \sigma, \lambda)$.

Proposition 1. *Let Y be a random variable such that $Y \sim EML(\mu, \sigma, \lambda)$. Then, its probability density function (pdf) is given by*

$$f_Y(y; \mu, \sigma, \lambda) = \begin{cases} \frac{\lambda}{2\sigma(\lambda-1)}\left[e^{-\frac{y-\mu}{\sigma}} - \left(\frac{2}{\lambda+1}\right)e^{-\lambda\left(\frac{y-\mu}{\sigma}\right)}\right] & , \quad y > \mu,\ \lambda \neq 1 \\ \left[\frac{2\left(\frac{y-\mu}{\sigma}\right)+1}{4\sigma}\right]e^{-\frac{y-\mu}{\sigma}} & , \quad y > \mu,\ \lambda = 1 \\ \frac{\lambda}{2\sigma(\lambda+1)}e^{\frac{y-\mu}{\sigma}} & , \quad y < \mu \end{cases} \tag{4}$$

Proof. Using the stochastic representation in (3), we have

$$X \sim L(0,1) \quad \Rightarrow \quad f_X(x) = \frac{1}{2}e^{-|x|}, \, -\infty < x < \infty,$$
$$V \sim exp(\lambda) \quad \Rightarrow \quad f_V(v) = \lambda e^{-\lambda v}, \, v > 0$$

and the Jacobian transformation approach, it follows that:

$$\left.\begin{array}{l} Y = \sigma(X+V)+\mu \\ W = V \end{array}\right\} \Rightarrow \begin{array}{l} X = \frac{Y-\mu}{\sigma} - W \\ V = W \end{array} \Rightarrow J = \begin{vmatrix} \frac{\partial x}{\partial y} & \frac{\partial x}{\partial w} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial w} \end{vmatrix} = \begin{vmatrix} \frac{1}{\sigma} & -1 \\ 0 & 1 \end{vmatrix} = \frac{1}{\sigma}.$$

Then,

$$\begin{aligned} f_{Y,W}(y,w) &= |J| f_{X,V}\left(\frac{y-\mu}{\sigma} - w, w\right) \\ f_{Y,W}(y,w) &= \frac{1}{\sigma} f_X\left(\frac{y-\mu}{\sigma} - w\right) f_V(w) \\ f_Y(y) &= \int_0^\infty \frac{1}{\sigma} f_X\left(\frac{y-\mu}{\sigma} - w\right) f_V(w)\, dw \\ f_Y(y) &= \frac{\lambda}{2\sigma} \int_0^\infty e^{-\lambda w} e^{-\left|\frac{y-\mu}{\sigma} - w\right|}\, dw, -\infty < y < \infty, \end{aligned}$$

solving the integral, for $\lambda \neq 1$ and $\lambda = 1$, the result (4) is obtained. □

Proposition 2. *If* $Y \sim EML(\mu, \sigma, \lambda)$ *and* $\lambda \to \infty$, *then* $Y \sim L(\mu, \sigma)$.

Proof. If $\lambda \to \infty$ in the density function given in (4), the result is obtained. □

Figure 1 graphically illustrates the behavior of the density function of the exponentially modified Laplace distribution and the standard Laplace for different values of λ (upper), it is observed that as the parameter λ decreases, the tails become heavier. On the other hand, on the lower portion of the figure, the densities of the standard Laplace, modified Laplace, and exponentially modified Laplace distributions are plotted, in which greater flexibility is observed in the *EML* model.

Proposition 3. *Let* Y *be a random variable such that* $Y \sim EML(\mu, \sigma, \lambda)$, *then its cdf is given by*

$$F_Y(t;\mu,\sigma,\lambda) = \begin{cases} \frac{\lambda}{2(\lambda+1)} + \frac{\lambda}{2(\lambda-1)}\left[1 - e^{-\frac{t-\mu}{\sigma}} - \frac{2}{\lambda(\lambda+1)}\left(1 - e^{-\lambda\left(\frac{t-\mu}{\sigma}\right)}\right)\right] & , \quad t > \mu, \lambda \neq 1 \\ \frac{1}{4}\left[4 - 3e^{-\frac{t-\mu}{\sigma}} - \frac{2(t-\mu)}{\sigma}e^{-\frac{t-\mu}{\sigma}}\right] & , \quad t > \mu, \lambda = 1 \\ \frac{\lambda}{2(\lambda+1)}e^{\frac{t-\mu}{\sigma}} & , \quad t < \mu. \end{cases} \tag{5}$$

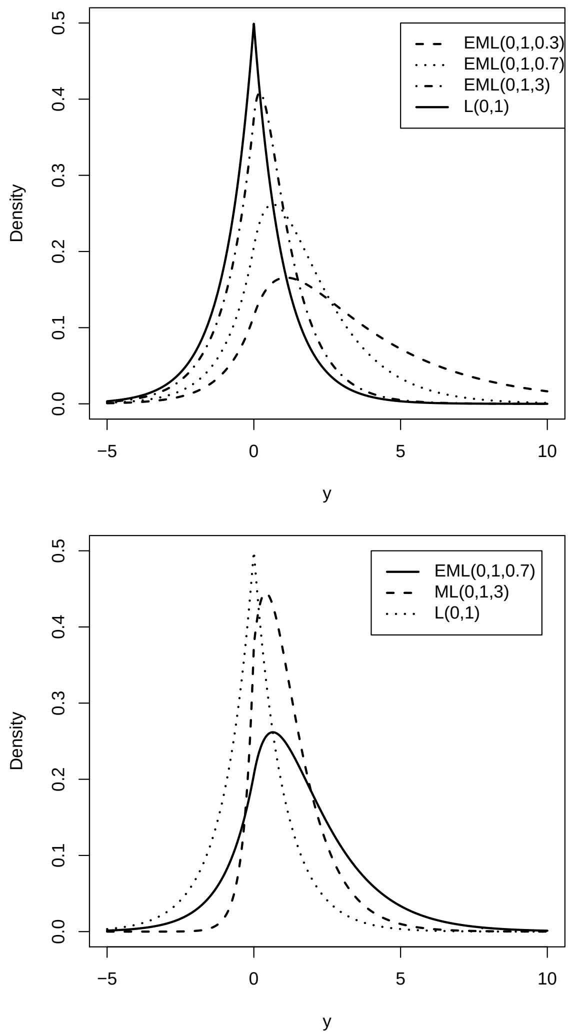

Figure 1. Graphical comparison of EML distributions with L for different values of λ (**upper**) and with ML and L (**lower**).

Proof. Using the definition of cdf, we have

$$F_Y(t;\mu,\sigma,\lambda) \quad = \quad \int_{-\infty}^{t} \frac{\lambda}{2\sigma} \int_{0}^{\infty} e^{-\lambda w} e^{-\left|\frac{y-\mu}{\sigma}-w\right|}\, dw dy,\ -\infty < t < \infty,$$

solving the integral for, $\lambda \neq 1$ and $\lambda = 1$, the result (5) is obtained. □

Corollary 1. *Let Y be a random variable such that $Y \sim EML(\mu,\sigma,\lambda)$. Then, the reliability function defined as $R(y) = P(Y > y) = 1 - F_Y(y)$, $y > 0$ is given by*

$$R(y) \quad = \quad \begin{cases} 1 - \frac{\lambda}{2(\lambda+1)} - \frac{\lambda}{2(\lambda-1)}\left[1 - e^{-\frac{t-\mu}{\sigma}} - \frac{2}{\lambda(\lambda+1)}\left(1 - e^{-\lambda\left(\frac{t-\mu}{\sigma}\right)}\right)\right] & , \quad t > \mu,\ \lambda \neq 1 \\ 1 - \frac{1}{4}\left[4 - 3e^{-\frac{t-\mu}{\sigma}} - \frac{2(t-\mu)}{\sigma} e^{-\frac{t-\mu}{\sigma}}\right] & , \quad t > \mu,\ \lambda = 1 \\ 1 - \frac{\lambda}{2(\lambda+1)} e^{\frac{t-\mu}{\sigma}} & , \quad t < \mu. \end{cases} \tag{6}$$

Proof. Using the reliability function definition $R(y)$ and (5), the result is directly obtained. □

Through Figure 2, we graphically illustrate the behavior of the cumulative distribution function (cdf) for the exponentially modified Laplace distribution. Compared to the standard Laplace distribution, it reflects a slower growth, implying a greater capture of outlier data.

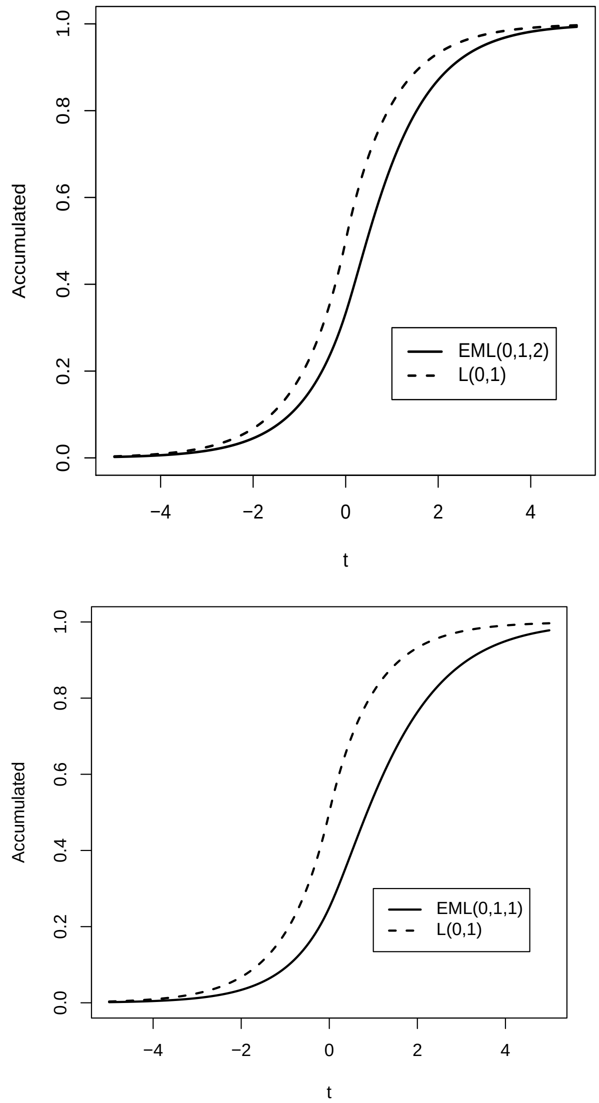

Figure 2. Comparison of the cdf of the *EML* distribution (solid line) for $\lambda = 2$ (**upper**) and $\lambda = 1$ (**lower**) with the cdf of the distribution L (dashed line).

3.2. Reliability Function Comparison of ML, EMLOG, EMG, and EML Distributions

The reliability function of a random variable Y indicates the probability that a variable exceeds the value of y. In this section, using Table 1, for a fixed value of $\lambda = 0.7$, we make a brief comparison where it is observed that the tails of the *EML* distribution are heavier than those of the *ML*, *EMLOG*, and *EMG* distributions.

Table 1. Reliability function comparison for distributions *ML*, *EMLOG*, *EMG*, and *EML*.

Distribution	$P(Y > 2)$	$P(Y > 2.5)$	$P(Y > 3)$	$P(Y > 3.5)$	$P(Y > 4)$	$P(Y > 4.5)$	$P(Y > 5)$
ML	0.2444	0.1543	0.0958	0.0530	0.0361	0.0220	0.0134
EMLOG	0.2878	0.2116	0.1517	0.1065	0.0735	0.0501	0.0377
EMG	0.3073	0.2202	0.1561	0.1102	0.0775	0.0547	0.0385
EML	0.3256	0.2449	0.1820	0.1339	0.0978	0.0711	0.0515

Likewise, observing the graphical illustration represented in Figure 3, it can be seen that the tails of the *EML* distribution are heavier than those of the *ML*, *EMLOG*, and *EMG* distributions.

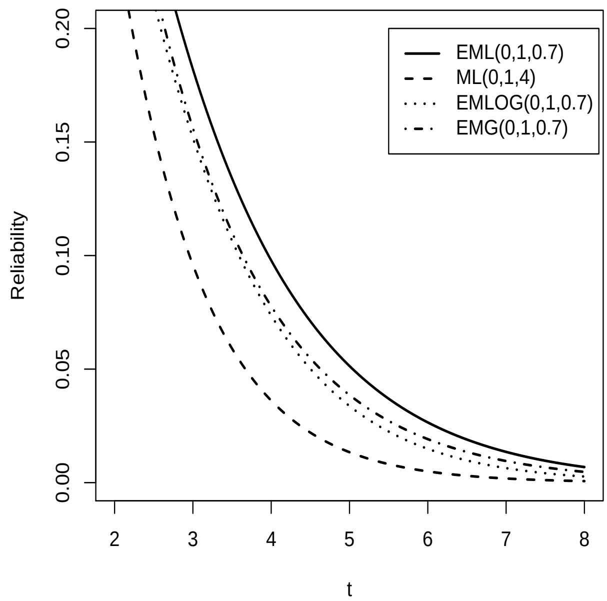

Figure 3. Comparison of the reliability function of the *EML* distribution (solid line) for $\lambda = 0.7$ with the reliability function of the *ML*, *EMLOG*, and *EMG* distributions (dashed line, dotted line, dash-dotted line).

3.3. Moments

The following proposition presents us with a formula that, with the use of numerical techniques, allows us to calculate the rth moment of an exponentially modified Laplace distribution.

Proposition 4. *If $Y \sim EML(\mu, \sigma, \lambda)$, the rth moment of Y is given by:*

$$\mu_r = E[Y^r] = \sum_{j=0}^{r} \binom{r}{j} \sigma^j \mu^{r-j} \left[\sum_{k=0}^{j} \binom{j}{k} \frac{k!\{1 + (-1)^k)\}(j-k)!}{2\lambda^{j-k}} \right]$$

Proof. Using the stochastic representation given in (3), applying the binomial theorem and the moments of the standard Laplace and exponential distributions given in (1) and (2), respectively, the result is obtained. □

Corollary 2. *Let $Y \sim EML(\mu, \sigma, \lambda)$, then*

$$\begin{aligned}
\mu_1 &= \frac{\sigma}{\lambda} + \mu \\
\mu_2 &= 2\sigma^2\left(1 + \frac{1}{\lambda^2}\right) + \frac{2\sigma\mu}{\lambda} + \mu^2 \\
\mu_3 &= \frac{6\sigma^3}{\lambda}\left(1 + \frac{1}{\lambda^2}\right) + 6\sigma^2\mu\left(1 + \frac{1}{\lambda^2}\right) + \frac{3\sigma\mu^2}{\lambda} + \mu^3 \\
\mu_4 &= 24\sigma^4\left(1 + \frac{1}{\lambda^2} + \frac{1}{\lambda^4}\right) + \frac{24\sigma^3\mu}{\lambda}\left(1 + \frac{1}{\lambda^2}\right) + \frac{12\sigma^2\mu^2}{\lambda}\left(1 + \frac{1}{\lambda^2}\right) + \frac{4\sigma\mu^3}{\lambda} + \mu^4
\end{aligned}$$

Proof. Using Proposition 4 with $r = 1, 2, 3, 4$ we obtain the results. □

Corollary 3. *Let $Y \sim EML(\mu, \sigma, \lambda)$. Then, the mean and variance are given, respectively, by*

$$\begin{aligned} E(Y) &= \mu + \frac{\sigma}{\lambda} \\ Var(Y) &= \sigma^2 \left(2 + \frac{1}{\lambda^2}\right) \end{aligned}$$

Proof. Using μ_1 and μ_2 obtained in Corollary 2, and substituting in $V(Y) = \mu_2 - (\mu_1)^2$, we obtain the results. □

Corollary 4. *Let $Y \sim EML(\mu, \sigma, \lambda)$, then the asymmetry and kurtosis coefficient of Y is given by*

$$\begin{aligned} \sqrt{\beta_1} &= \frac{2}{(2\lambda^2 + 1)^{\frac{3}{2}}} \\ \beta_2 &= \frac{24\lambda^4 + 12\lambda^2 + 9}{(2\lambda^2 + 1)^2} \end{aligned}$$

Proof. Using the standardized skewness and kurtosis coefficients of Y, the result is reached. □

Figure 4 shows that the kurtosis coefficient for the distribution (EML) takes values in the interval $[5, 9]$, decreasing for values of λ between $[0, 1]$ and increasing for values greater than one.

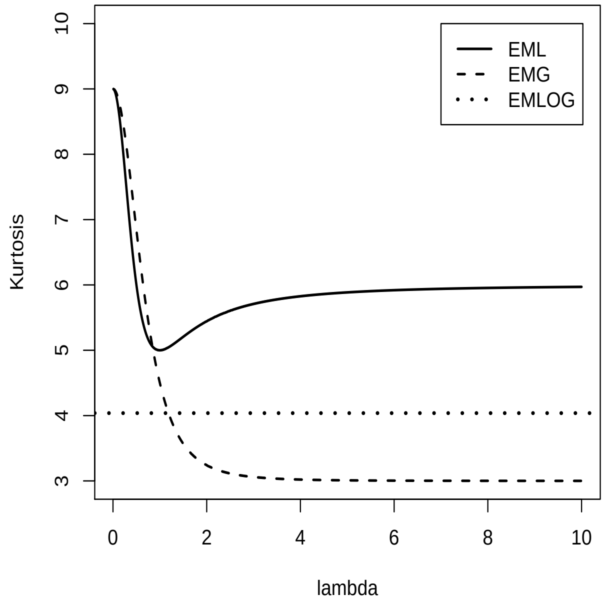

Figure 4. Graphical comparison of the kurtosis coefficient between the exponentially modified Laplace distribution (solid line), the exponentially modified Gaussian distribution (dashed line), and the exponentially modified logistic distribution (dotted line).

4. Estimation

4.1. Moment Estimators

The following proposition shows analytic expressions for the moment estimators of μ, σ, and λ for the exponentially modified Laplace distribution (EML).

Proposition 5. *Let $y_1, y_2, \ldots, y_n$ be a random sample from the distribution of random variable $Y \sim EML(\mu, \sigma, \lambda)$, so that the moment estimators for $\theta = (\mu, \sigma, \lambda)$ are obtained by solving the following numerical equation for μ:*

$$\mu^3 - 8\overline{y}\mu^2 + 15\overline{y}^2\mu - 6\overline{y}^3 - 3\overline{y}s^2 + \overline{y^3} = 0,$$

later, the moment estimator for σ is obtained by substituting the moment estimator for μ ($\widehat{\mu}_M$), in the following equation:

$$\widehat{\sigma}_M = \sqrt{\frac{\overline{y^2} - 2\overline{y}(\overline{y} - \widehat{\mu}_M) - \widehat{\mu}_M^2}{2}}$$

and finally, the estimator of moments for λ is obtained:

$$\widehat{\lambda}_M = \frac{\widehat{\sigma}_M}{\overline{y} - \widehat{\mu}_M}$$

where $\overline{y}$, $\overline{y^2}$, $\overline{y^3}$, and s^2 are the sample moments, and sample variance, respectively.

Proof. Equating the first three population moments to the sample moments, we obtain:

$$\begin{aligned}
\overline{y} &= \frac{\sigma}{\lambda} + \mu \\
\overline{y^2} &= 2s^2 - 2\sigma^2 + \left(\frac{\sigma}{\lambda} + \overline{y}\right)\mu \\
\overline{y^3} &= 6s^2\left(\frac{\sigma}{\lambda} + \mu\right) + \left(\frac{\sigma}{\lambda} + \overline{y}\right)\mu^2,
\end{aligned}$$

solving the system, we arrive at the result. □

4.2. Likelihood Function

Consider a random sample of size n, $y_1, \ldots, y_n$, from the distribution $EML(\mu, \sigma, \lambda)$. So, the log-likelihood function for $\theta = (\mu, \sigma, \lambda)^T$ can be expressed as

$$\ell(\theta) = n \log \lambda - n \log 2 - n \log \sigma + \sum_{i=1}^{n} \log G(y_i, \theta), \tag{7}$$

where $G(y_i, \theta) = \int_0^\infty e^{-\lambda w} e^{-\left|\frac{y_i - \mu}{\sigma} - w\right|} dw$.

Maximum likelihood estimators (MLEs) were acquired maximizing the likelihood function given in (7). Since there is no analytical solution, we used the iterative numerical method "BFGS", created by Byrd et al. [7]. The "BFGS" method is a limited-memory quasi-Newton method for approximating the Hessian matrix of the target distribution. This method allows us to numerically obtain the maximum likelihood estimates of the parameters of a distribution and their respective standard errors derived from the Hessian matrix.

4.3. Simulation Study

We used the Monte Carlo method to generate random numbers from the distribution $EML(\mu, \sigma, \lambda)$. The results obtained are a sequence of n random numbers that are stored inside an array that we call $n-$vector. For this, we used 1000 samples of size 50, 100, 200 and 500, obtaining the estimates of the parameters by means of the moment and maximum likelihood methods. In addition, we analyze the standard deviation, average length of the confidence intervals, and the empirical coverage, for the parameters of the distribution, based on a 95% confidence level.

To develop the algorithm (Algorithm 1) we will use the following notation:

1. n: The length of the $n-$vector.
2. Y: A random variable with the distribution EML.
3. $f_Y(y)$: The PDF of EML.
4. $L1$: Number of samples of size n.
5. μ, σ, λ: Parameters.

Algorithm 1: Monte Carlo algorithm to generate random numbers from the $EML(\mu, \sigma, \lambda)$ distribution

1. Start
   Input: $f_Y(y), L1, n, \mu, \sigma, \lambda$.
   Output: $n-$vector.
2. Generate a random variable $X \sim L(0, 1)$.
3. Generate a random variable $V \sim exp(\lambda)$.
4. Compute $Y = X + V$.
5. Since $Y \sim EML(\mu, \sigma, \lambda)$, append y to n-vector.
6. Repeat steps 2–5 for each sample of size n obtained.
7. For each estimate, the 95% confidence interval is obtained and the length calculated. Additionally, the number of intervals containing the value of each parameter is counted. By obtaining the average of these 1000 values, the value ali and the empirical coverage c are obtained.
8. end.

Table 2 contains the values of the estimates of the parameters, standard deviation, average interval length, and empirical coverage, based on a 95% confidence interval from simulations obtained by the method of moments for 1000 generated samples of size n = 50, 100, 200, and 500 from the population with distribution $EML(\mu, \sigma, \lambda)$. These estimates were obtained by solving the system of equations given in Proposition 5. Similarly, Table 3 shows the results of the simulation studies, illustrating the behavior of the MLEs. For each sample generated, MLEs are calculated numerically using the Newton–Raphson [8] procedure. In both tables, it can be seen that the simulations carried out by these methods show that the average estimates of the parameters are close to the proposed values. Additionally, the standard deviation and the average length of the interval decrease as the sample size increases. This is an expected result, since the ME and MLE are asymptotically consistent. On the other hand, the empirical coverage is adequate since it is close to 95%.

Table 2. ME simulation of 1000 iterations of the model $EML(\mu, \sigma, \lambda)$.

n	μ	σ	λ	$\widetilde{\mu}$	$sd(\widetilde{\mu})$	$ali(\widetilde{\mu})$	$c(\widetilde{\mu})$	$\widetilde{\sigma}$	$sd(\widetilde{\sigma})$	$ali(\widetilde{\sigma})$	$c(\widetilde{\sigma})$	$\widetilde{\lambda}$	$sd(\widetilde{\lambda})$	$ali(\widetilde{\lambda})$	$c(\widetilde{\lambda})$
50	0	1	0.3	0.0341	0.1189	0.4661	93.9	1.0341	0.1189	0.4661	93.9	0.3341	0.1189	0.4661	93.9
100	0	1	0.3	0.0123	0.0650	0.2548	95.2	1.0123	0.0650	0.2548	95.2	0.3123	0.0650	0.2548	95.2
200	0	1	0.3	0.0066	0.0411	0.1612	94.7	1.0066	0.0411	0.1612	94.7	0.3066	0.0411	0.1612	94.7
500	0	1	0.3	0.0036	0.0245	0.0959	94.6	1.0036	0.0245	0.0959	94.6	0.3036	0.0245	0.0959	94.6
50	0	1	0.7	−0.1249	0.2589	1.0150	97.4	0.8751	0.2589	1.0150	97.4	0.5751	0.2589	1.0150	97.4
100	0	1	0.7	−0.1162	0.2293	0.8989	96.1	0.8838	0.2293	0.8989	96.1	0.5838	0.2293	0.8989	96.1
200	0	1	0.7	−0.0785	0.1901	0.7451	91.0	0.9215	0.1901	0.7451	91.0	0.6215	0.1901	0.7451	91.0
500	0	1	0.7	−0.0434	0.1540	0.6038	93.8	0.9566	0.1540	0.6038	93.8	0.6566	0.1540	0.6038	93.8
50	0	1	1	−0.1006	0.3208	1.2576	92.7	0.8994	0.3208	1.2576	92.7	0.8994	0.3208	1.2576	92.7
100	0	1	1	−0.0399	0.2174	0.8522	96.7	0.9601	0.2174	0.8522	96.7	0.9601	0.2174	0.8522	96.7
200	0	1	1	−0.0149	0.1373	0.5381	97.3	0.9851	0.1373	0.5381	97.3	0.9851	0.1373	0.5381	97.3
500	0	1	1	−0.0038	0.0760	0.2978	93.7	0.9962	0.0760	0.2978	93.7	0.9962	0.0760	0.2978	93.7
50	0	1	1.2	−0.0525	0.2984	1.1698	96.6	0.9475	0.2984	1.1698	96.6	1.1475	0.2984	1.1698	96.6
100	0	1	1.2	−0.0114	0.1827	0.7161	98.1	0.9886	0.1827	0.7161	98.1	1.1886	0.1827	0.7161	98.1
200	0	1	1.2	0.0004	0.1118	0.4383	96.0	1.0004	0.1118	0.4383	96.0	1.2004	0.1118	0.4383	96.0
500	0	1	1.2	−0.0024	0.0637	0.2499	94.3	0.9976	0.0637	0.2499	94.3	1.1976	0.0637	0.2499	94.3
50	−1	2	0.3	−0.9902	0.0641	0.2512	94.7	2.0098	0.0641	0.2512	94.7	0.3098	0.0641	0.2512	94.7
100	−1	2	0.3	−0.9923	0.0462	0.1810	94.9	2.0077	0.0462	0.1810	94.9	0.3077	0.0462	0.1810	94.9
200	−1	2	0.3	−0.9958	0.0301	0.1181	94.7	2.0042	0.0301	0.1181	94.7	0.3042	0.0301	0.1181	94.7
500	−1	2	0.3	−0.9987	0.0185	0.0723	94.8	2.0013	0.0185	0.0723	94.8	0.3013	0.0185	0.0723	94.8

sd corresponds to the standard deviation, *ali* (average length of interval) is the average length of the confidence interval, and *c* the empirical coverage of the respective ME of the parameters, based on a 95% confidence interval.

Table 3. MLE simulation of 1000 iterations of the model $EML(\mu, \sigma, \lambda)$.

n	μ	σ	λ	$\widehat{\mu}$	$sd(\widehat{\mu})$	$ali(\widehat{\mu})$	$c(\widehat{\mu})$	$\widehat{\sigma}$	$sd(\widehat{\sigma})$	$ali(\widehat{\sigma})$	$c(\widehat{\sigma})$	$\widehat{\lambda}$	$sd(\widehat{\lambda})$	$ali(\widehat{\lambda})$	$c(\widehat{\lambda})$
50	0	1	0.3	0.0470	0.5226	2.0486	93.6	0.9327	0.3885	1.5229	94.2	0.3175	0.2096	0.8216	96.2
100	0	1	0.3	0.0326	0.3439	1.3480	94.2	0.9822	0.2534	0.9933	94.8	0.3081	0.1139	0.4463	94.9
200	0	1	0.3	0.0093	0.2300	0.9015	94.5	0.9937	0.1746	0.6843	95.4	0.3035	0.0706	0.2769	95.1
500	0	1	0.3	−0.0061	0.1408	0.5521	95.0	0.9956	0.1087	0.4260	95.3	0.2999	0.0444	0.1740	94.6
50	0	1	0.7	−0.0023	0.3834	1.5031	94.5	0.9491	0.2611	1.0237	95.6	0.7974	0.5617	2.2017	94.8
100	0	1	0.7	0.0255	0.2872	1.1257	94.1	0.9692	0.1913	0.7498	94.6	0.7607	0.3987	1.5630	95.7
200	0	1	0.7	0.0261	0.2011	0.7882	94.9	1.0004	0.1396	0.5471	95.9	0.7527	0.2750	1.0779	96.1
500	0	1	0.7	0.0084	0.1211	0.4747	95.2	0.9968	0.0829	0.3249	94.9	0.7130	0.1229	0.4816	94.8
50	0	1	1.0	−0.0149	0.3755	1.4718	95.4	0.9182	0.2369	0.9286	93.7	1.2052	1.0383	4.0701	93.8
100	0	1	1.0	0.0176	0.2805	1.0997	94.7	0.9654	0.1697	0.6653	95.0	1.1874	0.8215	3.2204	93.7
200	0	1	1.0	0.0248	0.2150	0.8428	94.6	0.9898	0.1266	0.4962	94.3	1.1528	0.6274	2.4595	94.8
500	0	1	1.0	0.0106	0.1361	0.5337	94.7	0.9924	0.0842	0.3300	94.7	1.0473	0.3208	1.2574	96.5
50	0	1	1.2	−0.0644	0.3430	1.3444	94.8	0.8979	0.2139	0.8385	92.1	1.2577	0.9488	3.7195	95.5
100	0	1	1.2	−0.0013	0.2758	1.0812	94.6	0.9568	0.1629	0.6384	93.5	1.3821	0.9002	3.5287	93.1
200	0	1	1.2	0.0090	0.2123	0.8322	94.9	0.9849	0.1200	0.4704	95.1	1.3675	0.7209	2.8259	94.2
500	0	1	1.2	0.0123	0.1373	0.5383	94.5	0.9969	0.0792	0.3104	94.6	1.2831	0.4290	1.6819	95.5
50	−1	2	0.3	−0.8821	0.9899	3.8805	95.0	1.8777	0.7352	2.8819	93.8	0.3135	0.2027	0.7946	97.3
100	−1	2	0.3	−0.9601	0.6781	2.6582	96.1	1.9483	0.5054	1.9813	94.6	0.3071	0.1150	0.4507	95.5
200	−1	2	0.3	−0.9681	0.4653	1.8239	95.0	1.9917	0.3324	1.3030	94.9	0.3061	0.0702	0.2754	94.5
500	−1	2	0.3	−0.9883	0.2807	1.1004	94.6	1.9960	0.2197	0.8612	94.7	0.3021	0.0448	0.1755	94.9

sd corresponds to the standard deviation, *ali* (average length of interval) is the average length of the confidence interval, and *c* the empirical coverage of the respective EMV of the parameters, based on a 95% confidence interval.

5. Three Illustrative Examples with a Real Data Set

In this section, three applications are presented in which the parameter estimators are obtained based on the maximum likelihood method (MLE) for (μ, σ and λ) through of the fitted models ML, $EMLOG$, EMG, and EML to a set of real data. The numerical illustrations below are intended to show that the EML model is an alternative to unimodal data modeling in different areas.

5.1. Illustrative Example 1

In our first illustration, the data set corresponds to the nickel content in soil samples analyzed at the Department of Mining (Department of Mines) of the University of Atacama, Chile, (see Appendix A, Table A1). Table 4 presents summary statistics for the data set of nickel content in soil samples, where γ_1 and γ_2 are the skewness and kurtosis coefficients of the sample, respectively. The moment estimators for these data are given by: $\widehat{\theta}_M = (\widehat{\mu}_M, \widehat{\sigma}_M, \widehat{\lambda}_M) = (6.7497\,,\, 5.0626\,,\, 0.3412)$.

Table 4. Summary Statistics for the Nickel Concentration Data Set.

n	$\overline{y}$	s_y	γ_1	γ_2
85	21.3372	16.6391	2.3559	11.1917

Table 5 shows the maximum likelihood estimates and the standard deviations for the ML, $EMLOG$, EMG, and EML models. In addition, we report the values of the Akaike [9] (AIC), Bayesian information criteria [10] (BIC), Akaike information criterion consistent [11] (CAIC), and Hannan—Quinn information criterion [12] (HQIC). On the other hand, Figure 5 shows the histogram with estimated pdf. This indicates that the EML model fits the data better than the ML, $EMLOG$, and EMG models. This result is supported by Figure 6 based on theoretical versus empirical (QQ) quantile plots.

Table 5. Maximum likelihood estimators for *ML*, *EMLOG*, *EMG*, and *EML* models for the soil nickel concentration data set, with their corresponding standard deviations in parentheses and comparison criteria AIC, BIC, CAIC, and HQIC.

Parameter Estimates	*ML*	*EMLOG*	*EMG*	*EML*
$\widehat{\mu}$	11.0020 (0.0657)	18.9149 (1.4048)	10.0433 (0.0771)	7.020 (0.0462)
$\widehat{\sigma}$	11.6843 (1.21645)	7.6833 (0.7231)	9.0165 (0.0766)	5.1452 (0.0258)
$\widehat{\lambda}$	2.2279 (0.2418)		0.7810 (0.0610)	0.3733 (0.0443)
AIC	687.022	699.207	685.0531	682.053
BIC	694.385	704.092	692.381	689.381
CAIC	695.548	705.092	693.381	690.381
HQIC	690.168	701.172	688.001	685.001

Figure 5 presents the histogram of the data with adjustment of the modified Laplace, exponentially modified Laplace, exponentially modified Gaussian, and exponentially modified logistic (upper) distributions, fitted with the values of the maximum likelihood estimators of their parameters. Notice that the fitted exponentially modified Laplace distribution has heavier tails and a magnification of the upper tails of the soil nickel concentration data (lower).

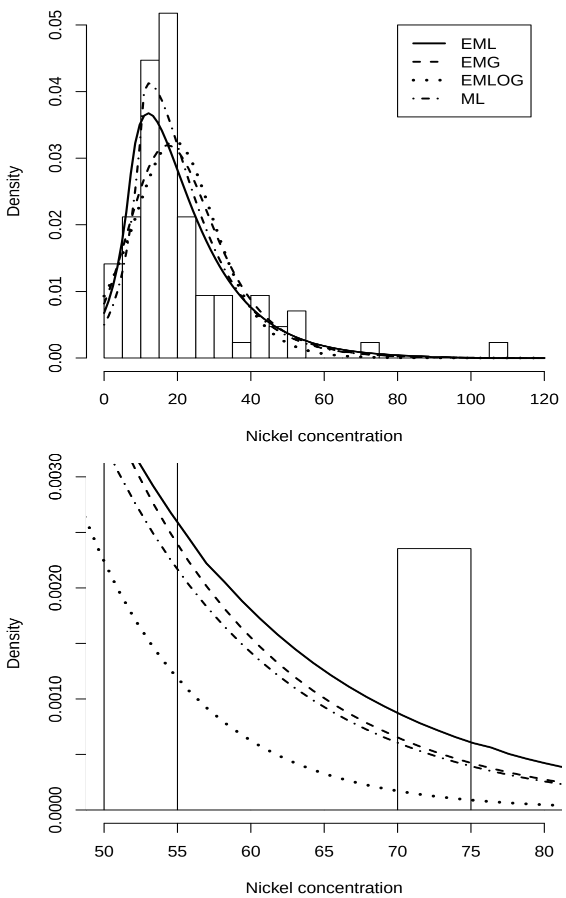

Figure 5. Histogram (**upper**) and tail (**lower**) for nickel concentration data set. Overlaid on top is the density *EML* with parameters estimated via MLE (solid line), exponentially modified Gaussian density with parameters estimated via MLE (dotted line), exponentially modified logistic (dashed line), and modified Laplace (dash-dotted line).

On the other hand, Figure 6 shows the QQ plot of the fitted models. From these results, it can be seen that the exponentially modified Laplace distribution provided a better fit than the other distributions in consideration.

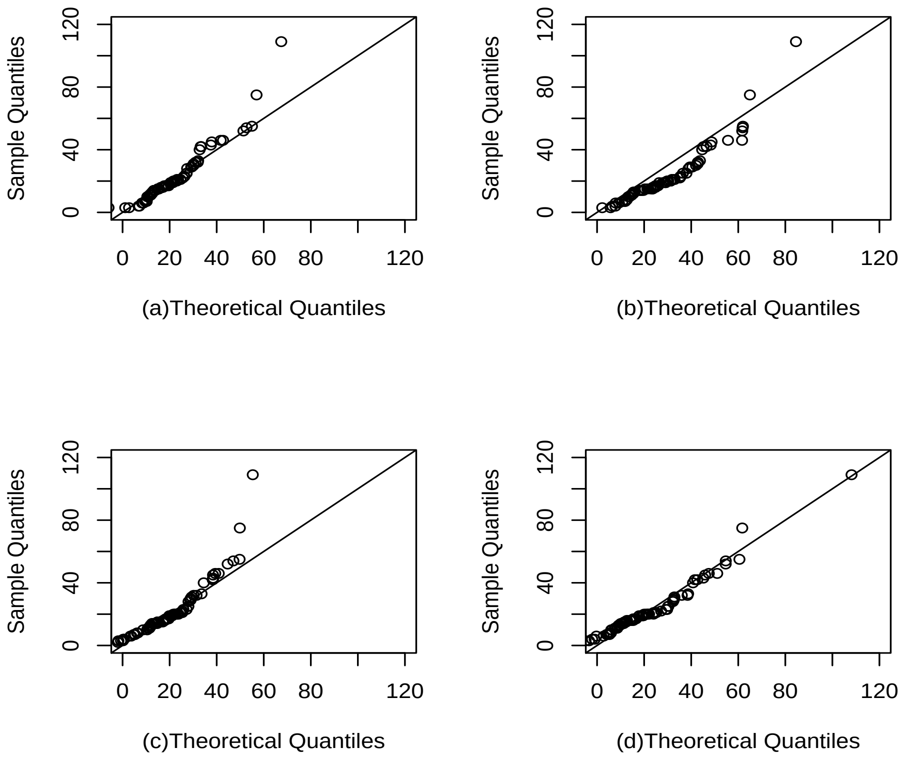

Figure 6. QQ plot for nickel concentration data set. The modified Laplace density (**a**), exponentially modified logistic density (**b**), exponentially modified Gaussian density (**c**), and exponentially modified Laplace density (**d**).

5.2. Illustrative Example 2

The second illustration is related to the neodymium content in soil samples analyzed at the Department of Mining (Department of Mines) of the University of Atacama, Chile (see Appendix A, Table A2). Table 6 presents summary statistics for the data set of the neodymium content in soil samples, where γ_1 and γ_2 are the skewness and kurtosis coefficients of the sample, respectively. The moment estimators for these data are given by: $\widehat{\theta}_M = (\widehat{\mu}_M, \widehat{\sigma}_M, \widehat{\lambda}_M) = (4.2094,\ 10.3030,\ 0.5868)$.

Table 6. Summary statistics for neodymium concentration data.

n	$\overline{y}$	s_y	γ_1	γ_2
86	35.1032	34.3307	3.8847	17.3951

The modified Laplace, exponentially modified logistic, exponentially modified Gaussian, and exponentially modified Laplace distributions were fitted to the data set. Table 7 shows the maximum likelihood estimates of the parameters, with the corresponding standard deviations (*sd*) in parentheses, for the three mentioned distributions. The adjustment criteria (AIC, BIC, CAIC, and HQIC) indicate that the data fit better to the exponentially modified Laplace model, because they present a smaller or lower value.

Figure 7 shows the histogram plots and a magnification of the upper tails of the soil neodymium concentration data with the modified Laplace, exponentially modified logistic, exponentially modified Gaussian, exponentially modified Laplace, and distributions fitted with the maximum likelihood estimators of its parameters where the fit of outliers is best observed. In addition, Figure 8 shows the QQ plot of the fitted models, observing that the proposed model achieves a better capture of extreme values.

Table 7. Maximum likelihood estimators for models *ML*, *EMLOG*, *EMG*, and *EML* for the neodymium concentration data set in the soil, with their corresponding standard deviations in parentheses and comparison criteria AIC, BIC, CAIC, and HQIC.

Parameter Estimates	***ML***	***EMLOG***	***EMG***	***EML***
$\widehat{\mu}$	13.0001 (0.2804)	29.0578 (2.1736)	15.3836 (2.8609)	10.44577 (0.0388)
$\widehat{\sigma}$	18.9313 (1.8033)	12.4762 (1.1937)	17.9653 (1.0407)	6.8136 (0.0091)
$\widehat{\lambda}$	2.9883 (0.3412)		0.9147 (0.1363)	0.2831 (0.0339)
AIC	768.088	802.523	792.496	763.294
BIC	775.451	807.432	799.859	770.567
CAIC	776.451	808.432	800.859	771.657
HQIC	771.051	804.499	795.459	766.257

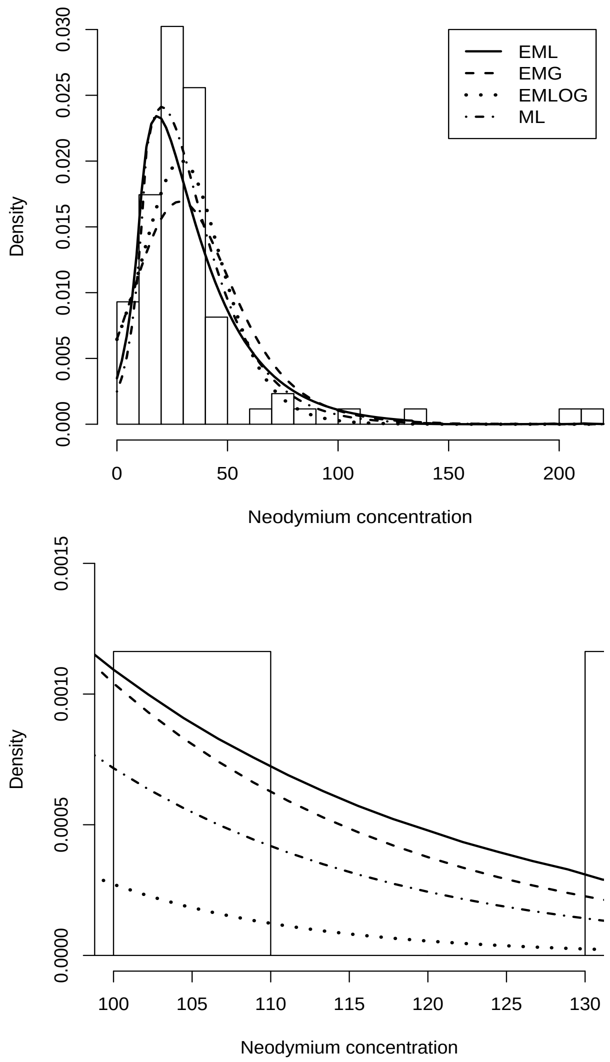

Figure 7. Histogram (**upper**) and tail (**lower**) for the neodymium concentration data set. The first graph shows the densities of the exponentially modified Laplace (solid line), Gaussian modified exponentially (dashed line), exponentially modified logistic (dotted line), and modified Laplace (dash-dotted line) distributions, with their parameters estimated by MLE.

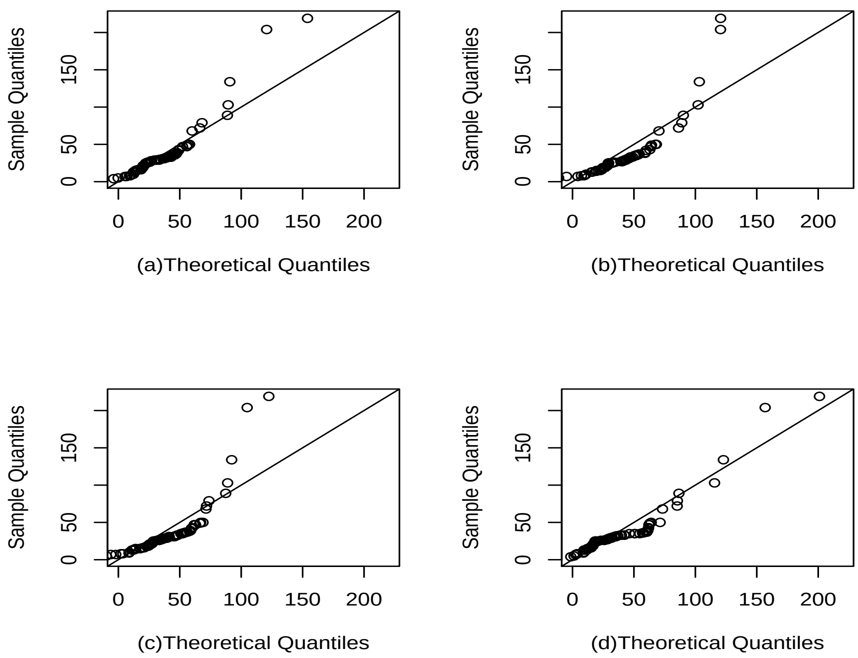

Figure 8. QQ plot for the neodymium concentration data set. The modified Laplace density (**a**), exponentially modified logistic density (**b**), exponentially modified Gaussian density (**c**), and exponentially modified Laplace density (**d**).

5.3. Illustrative Example 3

In this application, we used daily nitrogen concentration data obtained by chromatography [13]. Data are given in the Appendix A (Tabla A3). Table 8 presents summary statistics for the nitrogen concentration data set, where γ_1 and γ_2 are the sample skewness and kurtosis coefficients, respectively. Moment estimators for these data are given by: $\widehat{\theta}_M = (\widehat{\mu}_M, \widehat{\sigma}_M, \widehat{\lambda}_M) = (0.0965, 1.0965, 2.0965)$. Table 9 shows the maximum likelihood estimates for the parameters with their corresponding standard deviations (*sd*) in parentheses for the modified Laplace, exponentially modified logistic, exponentially modified Gaussian and exponentially modified Laplace distributions. The fit criteria used, AIC, BIC, CAIC and HQIC, indicate that the exponentially modified Laplace model fits the data better.

Table 8. Summary statistics for nitrogen concentration data.

n	$\overline{y}$	s_y	γ_1	γ_2
367	0.6189	0.0078	-1.3205	12.4692

Figure 9 shows the histogram plots and a magnification of the lower tails of the nitrogen concentration data with the modified Laplace, exponentially modified logistic, exponentially modified Gaussian, and exponentially modified Laplace distributions fitted with the maximum likelihood estimators of its parameters where the fit of outliers is best observed. In addition, Figure 10 shows the QQ plot of the fitted models, observing that the proposed model achieves a better capture of extreme values.

Table 9. Comparison of the maximum likelihood estimators for nitrogen concentration data between the *ML*, *EMLOG*, *EMG*, and *EML* distributions with their corresponding standard deviations in parentheses and comparison criteria AIC, BIC, CAIC and HQIC.

Parameter Estimates	*ML*	*EMLOG*	*EMG*	*EML*
$\widehat{\mu}$	0.6165 (0.0007)	0.6192 (0.0003)	0.6132 (0.0005)	0.6147 (0.0005)
$\widehat{\sigma}$	0.0065 (0.0003)	0.0041 (0.0001)	0.0064 (0.0002)	0.0045 (0.0002)
$\widehat{\lambda}$	1.5045 (0.1761)		1.0049 (0.0969)	0.9616 (0.1368)
AIC	−2549.257	−2062.155	−2465.067	−2560.045
BIC	−2530.541	−2054.345	−2453.351	−2548.329
CAIC	−2536.541	−2053.344	−2452.351	−2547.329
HQIC	−2544.602	−2059.052	−2460.412	−2555.390

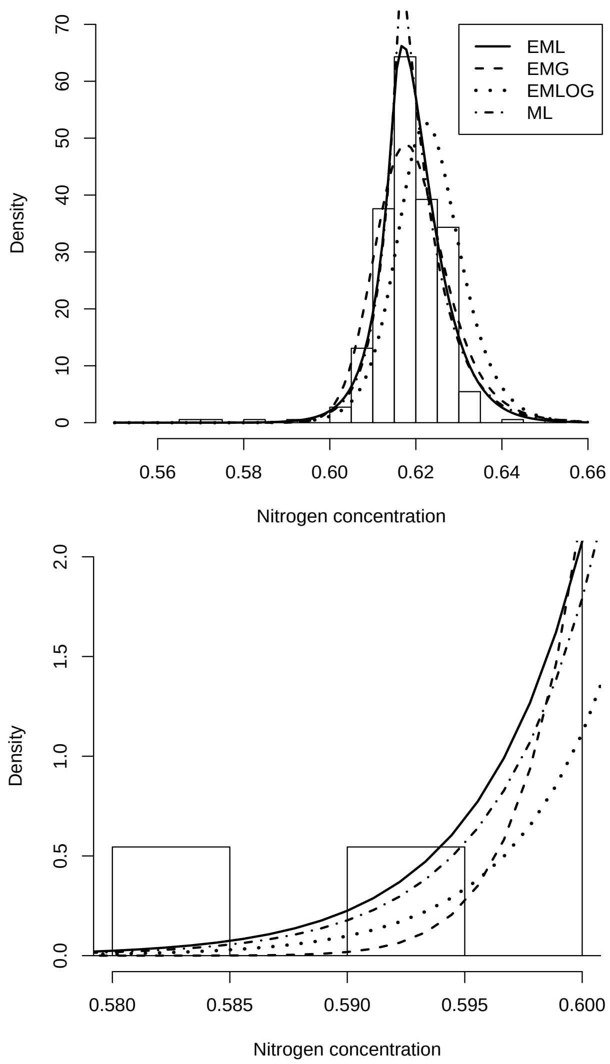

Figure 9. Histogram (**upper**) and tail (**lower**) for nitrogen concentration data set. The first graph shows the densities of exponentially modified Laplace (solid line), Gaussian modified exponentially (dashed line), exponentially modified logistic (dotted line) and modified Laplace (dash-dotted line) distributions, with their parameters estimated by MLE.

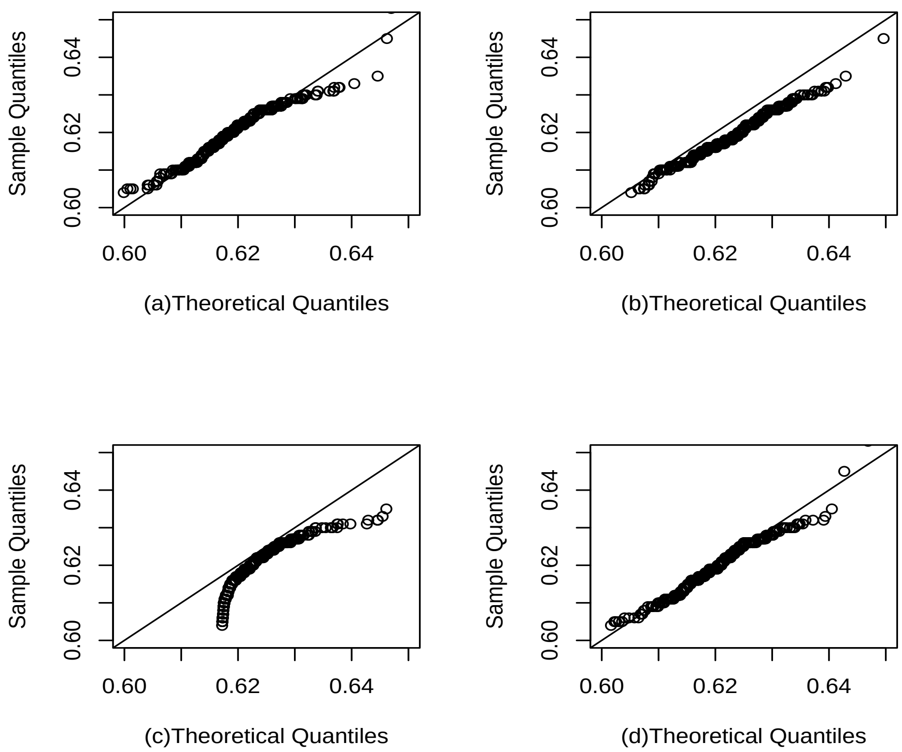

Figure 10. QQ plot for Nitrogen concentration data set. The density *ML* (**a**), *EMLOG* (**b**), *EMG* density (**c**), and *EML* (**d**).

6. Conclusions

In this paper, a new and more flexible distribution, called the exponentially modified Laplace distribution, has been proposed. We estimate the parameters of the model by the moment and maximum likelihood methods. Likewise, we apply information criteria to select the models and evaluate the goodness of fit of the new distribution compared to other similar distributions in the current literature. We performed a Monte Carlo simulation study to empirically assess the statistical performance of the estimates obtained. In addition, we study the standard deviations, the mean length of the confidence intervals, and the empirical coverage based on 95% confidence intervals. This simulation study reported a good statistical performance of these estimates. Three illustrations were made using data related to the chemical and environmental concentrations, comparing them with three similar distributions presented in the literature. The analyses reported a good performance of the new distribution compared to similar distributions, providing evidence that the proposed model is a good alternative for modeling skewed and high-kurtosis data. These results reported that the exponentially modified Laplace model can be an alternative to analyze this type of data. The new approach is a contribution to the tools of statisticians and various professionals interested in data modeling. From these applications, we have obtained useful information that can be used by professionals and users of statistics. A limitation of the proposed distribution is the loss of goodness of fit for data sets whose sample kurtosis is less than five. Some topics for future research based on this new distribution are related to the study of multivariate procedures, quantile regression, spatial methods, temporal methods, partial least squares, principal components, etc.

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Data Availability Statement: The analyzed data are available in the Appendix A of the article.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

Table A1. Nickel Data.

2	3	3	3	4	4	6	6	7	7	7	8	8	10	10	11	11	11
12	12	13	13	14	14	14	14	14	14	14	15	15	15	15	15	16	16
16	16	16	16	17	17	17	17	17	19	19	19	19	19	20	20	20	20
20	20	21	21	21	21	22	23	23	25	25	28	29	29	30	31	32	32
33	40	42	42	43	45	46	46	52	54	55	75	109					

Table A2. Neodymium Data.

47	26	29	22	33	16	7	13	4	31	27	13	36	8	42	15	5	29
25	29	36	18	16	50	18	28	16	29	10	31	7	15	32	33	35	31
72	89	37	43	29	35	14	25	21	8	26	49	47	19	14	33	35	21
25	30	15	27	27	9	26	33	13	204	33	38	25	22	35	31	39	24
50	103	28	219	134	68	25	37	21	26	36	32	79	19				

Table A3. Nitrogen Data.

0.607	0.605	0.606	0.606	0.609	0.631	0.617	0.626
0.610	0.611	0.610	0.606	0.610	0.612	0.614	0.613
0.614	0.614	0.615	0.616	0.616	0.616	0.616	0.615
0.616	0.616	0.616	0.618	0.617	0.617	0.617	0.617
0.617	0.617	0.617	0.619	0.619	0.618	0.618	0.622
0.619	0.620	0.620	0.619	0.617	0.616	0.614	0.617
0.611	0.611	0.612	0.611	0.612	0.612	0.612	0.613
0.610	0.612	0.613	0.614	0.613	0.612	0.610	0.609
0.613	0.612	0.616	0.612	0.611	0.611	0.613	0.609
0.612	0.612	0.612	0.605	0.604	0.615	0.620	0.622
0.617	0.619	0.621	0.622	0.630	0.626	0.616	0.617
0.621	0.623	0.625	0.626	0.624	0.618	0.618	0.618
0.621	0.623	0.625	0.626	0.624	0.618	0.618	0.618
0.622	0.623	0.623	0.608	0.624	0.620	0.619	0.615
0.611	0.615	0.612	0.620	0.623	0.627	0.628	0.625
0.627	0.628	0.626	0.627	0.626	0.625	0.625	0.625
0.624	0.626	0.627	0.626	0.628	0.626	0.619	0.618
0.627	0.626	0.626	0.627	0.626	0.626	0.628	0.629
0.627	0.627	0.627	0.627	0.625	0.625	0.629	0.623
0.619	0.573	0.565	0.585	0.595	0.608	0.614	0.614
0.612	0.615	0.616	0.617	0.615	0.615	0.615	0.614
0.610	0.610	0.611	0.611	0.611	0.612	0.610	0.609
0.611	0.614	0.617	0.617	0.620	0.622	0.619	0.618
0.619	0.622	0.618	0.619	0.620	0.619	0.620	0.621
0.617	0.620	0.621	0.623	0.626	0.627	0.626	0.626
0.627	0.626	0.628	0.626	0.624	0.624	0.621	0.620
0.621	0.619	0.621	0.626	0.627	0.624	0.622	0.622
0.622	0.622	0.622	0.625	0.622	0.621	0.618	0.616
0.621	0.619	0.623	0.626	0.625	0.624	0.619	0.620
0.630	0.629	0.630	0.631	0.632	0.624	0.625	0.628
0.623	0.628	0.626	0.629	0.628	0.630	0.618	0.607
0.631	0.630	0.629	0.630	0.629	0.631	0.632	0.633
0.625	0.619	0.619	0.653	0.624	0.622	0.645	0.619

Table A3. *Cont.*

0.619	0.622	0.622	0.618	0.620	0.620	0.619	0.619
0.620	0.619	0.618	0.620	0.620	0.621	0.618	0.614
0.617	0.616	0.616	0.616	0.615	0.616	0.617	0.616
0.615	0.617	0.616	0.614	0.616	0.617	0.616	0.617
0.618	0.618	0.619	0.622	0.622	0.623	0.622	

References

1. Oosterbaan, R.J. Chapter 6, Frequency and Regression Analysis. In *Drainage Principles and Applications, Publication 16*; En Ritzema, H.P., Ed.; International Institute for Land Reclamation and Improvement (ILRI): Wageningen, The Netherlands, 1994; pp. 175–224. ISBN 90-70754-33-9.
2. Burke, E.J.; Perry, R.H.; Brown, S.J. An extreme value analysis of UK drought and projections of change in the future. *J. Hydrol.* **2010**, *388*, 131. [CrossRef]
3. Agu, F.I.; Onwukwe, E. Modified Laplace Distribution, Its Statistical Properties and Applications Asian. *J. Probab. Stat.* **2019**, *4*, 49277. [CrossRef]
4. Grushka, E. Characterization of Exponentially Modified Gaussian Peaks in Chromatography. *Anal. Chem.* **1972**, *44*, 1733–1738.
5. Reyes, J.; Venega, O.; Gómez, H.W. Exponentially Modified Logistic Distribution with Application to Mining and Nutrition. *Appl. Math. Inf. Sci.* **2018**, *12*, 1–8. [CrossRef] [PubMed]
6. Johnson, N.L.; Kotz, S.; Balakrishnan, N. *Continuous Univariate Distributions*, 2nd ed.; Wiley: New York, NY, USA, 1995. [CrossRef]
7. Byrd, R.H.; Lu, P.; Nocedal, J.; Zhu, C. A Limited Memory Algorithm for Bound Constrained Optimization. *SIAM J. Sci. Comput.* **1995**, *16*, 1190–1208.
8. Tjalling, J.Y. Historical development of the Newton-Raphson method. *SIAM Rev.* **1995**, *37*, 531–551. [CrossRef]
9. Akaike, H. A new look at the statistical model identification. *IEEE Trans. Autom. Control* **1974**, *19*, 716–723.
10. Schwarz, G. Estimating the dimension of a model. *Ann. Stat.* **1978**, *6*, 461–464. [CrossRef]
11. Bozdogan, H. Model selection and Akaike's information criterion (AIC): The general theory and its analytical extensions. *Psychometrika* **1987**, *52*, 345–370. [CrossRef]
12. Hannan, E.J.; Quinn, B.G. The Determination of the order of an autoregression. *J. R. Stat. Soc. Ser. B* **1979**, *41*, 190–195. [CrossRef]
13. Torres de Young, S. *Introduccion a la Cromatografia*; Universidad Nacional de Colombia: Palmira, Colombia, 1994; 134p. ISBNS 9581701192/9789581701193. [CrossRef]