mathematics

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General Master Theorems of Integrals with Applications

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Abstract: Many formulas of improper integrals are shown every day and need to be solved in different areas of science and engineering. Some of them can be solved, and others require approximate solutions or computer software. The main purpose of this research is to present new fundamental theorems of improper integrals that generate new formulas and tables of integrals. We present six main theorems with associated remarks that can be viewed as generalizations of Cauchy's results and I.S. Gradshteyn integral tables. Applications to difficult problems are presented that cannot be solved with the usual techniques of residue or contour theorems. The solutions of these applications can be obtained directly, depending on the proposed theorems with an appropriate choice of functions and parameters.

Keywords: improper integrals; power series; analytic function; Cauchy residue theorem; Ramanujan's principal theorem; integral equation

MSC: 44A20; 40G99

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Citation: Abu-Ghuwaleh, M.; Saadeh, R.; Qazza, A. General Master Theorems of Integrals with Applications. *Mathematics* **2022**, *10*, 3547. https://doi.org/10.3390/math10193547

Academic Editor: Youssef Raffoul

Received: 2 August 2022
Accepted: 20 September 2022
Published: 28 September 2022

1. Introduction

In recent decades, many improper integrals have emerged in various fields of science, physics, and engineering [1–5], and these integrals are very important when dealing with mathematical applications. Therefore, many mathematicians try to discover new theorems and techniques to calculate them. The importance of these integrals arose because of their application in applied mathematical physics, electrical engineering, etc. [6–11]. Some of these integrations can be solved directly, and others require long and hard calculations. Some of these integrals cannot be solved manually and need computer software such as Mathematica and Maple to be solved. In addition, sometimes numerical methods can be used to solve some improper integrals that cannot be solved using previous methods [12–18].

The evaluation of improper integrals is a process that does not depend on any specific rules or techniques that can be applied directly. Many methods and theorems have been introduced and implemented by mathematicians and researchers to present a closed expression for indefinite integrals, such as the technique of double integrals, series methods, residual theorem, calculus under the integral sign, and other methods that are used to exactly or approximately solve improper complex integrals (see [19–23]).

In recent years, many researchers have investigated new theorems to compute improper integrals. The first was the residue theorem, which was introduced by Cauchy in 1826, and it is considered as one of the most powerful tools in computing improper and contour integrals. Many other researchers have studied improper integrals such as Ramanujan who presented Ramanujan's master theorem [24–26], which gives expressions for the Mellin transform of any continuous analytic function in terms of its Taylor expansion. The study of the application of such integrals has continued and appeared in solving integral equations, integral transforms, fractional calculus, and differential equations as well as other applications that include the procedure of computing integrals (see [27–30]).

The proposed results in this work are applicable to solving and generating some families of improper integrals and integral transforms. The main goal of this work is to simplify the procedure of computing improper integrals that might take a long time and effort to solve or that cannot be solved manually. The outcomes of this study can be generalized and stated in tables to compute some difficult integrals directly without the need to find contours or factorize, etc. We simply choose the suitable functions and generate a large number of integrals.

In this research, we introduce new theorems about improper integrals with proof. Each theorem can generate new formulas of improper integrals that cannot be handled by conventional methods or that require time and effort to obtain results. The proposed theorems present the solutions of improper integrals directly in a simple finite sum that depends on the target problem. The motivation of this work is to generate new problems involving improper integrals and their solutions that can be used in various physical and engineering applications. The theorems obtained can be implemented to produce integration tables, which can be used to help researchers calculate difficult problems that may arise during their research or to study new approximation methods for solving improper integrals. They may check the accuracy of their answers using these tables.

This article is organized as follows. We present some basic definitions and theorems essential to our work in Section 2. Six main theorems are presented in Section 3 with some related results. Some remarks and applications are presented in Section 4. Finally, the conclusion of our research is presented in Section 5.

2. Preliminaries

To understand our new theorems, we introduce some basic definitions and theorems that are needed in our work.

Definition 1. [8] Let f be an analytic function in an open set Ω, and D is a disc centered at z_0; whose closure is contained in Ω, then f has a power series expansion at z_0.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{1}$$

Definition 2. [9] Let f be a real analytic function that is infinitely differentiable, such that the Taylor series at any point x_0 in the domain is

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n, \tag{2}$$

that converges to $f(x)$ in a neighborhood of x_0 pointwise.

Definition 3. [8] The Cauchy principal value of a finite integral of a function f about a point c, with $a \leq c \leq b$, is given by

$$PV \int_a^b f(x)dx = \lim_{\varepsilon \to 0^+} \left(\int_a^{c-\varepsilon} f(x)dx + \int_{c+\varepsilon}^b f(x)dx \right). \tag{3}$$

Lemma 1. De Moivre's factorization formula [11], which can be considered a more general formula expressible in a modern form than Cotes–Newton factorization, is obtained by

$$x^{2n} - 2\cos(\varphi)x^n + 1 = \prod_{k=0}^{n-1} \left(x^2 - 2\cos\left(\frac{2k\pi + \varphi}{n}\right) x + 1 \right), \tag{4}$$

where $n \in \mathbb{N}$.

Lemma 2. [11,12] The partial fractions of $\frac{x^{m-1}}{x^{2n}-2\cos(\varphi)x^n+1}$, where m and n are positive integers, $m-1<2n,\ \varphi>0$, and $x>0$ are of the form

$$\frac{x^{m-1}}{x^{2n}-2\cos(\varphi)x^n+1}=\sum_{s=0}^{n-1}\frac{x\left(\frac{\sin\left(m\ \left(\frac{2s\pi+\varphi}{n}\right)-\varphi\right)}{n\sin(\varphi)}\right)-\frac{\sin\left((m-1)\ \left(\frac{2s\pi+\varphi}{n}\right)-\varphi\right)}{n\sin(\varphi)}}{x^2-2x\cos\left(\frac{2s\pi+\varphi}{n}\right)+1}. \tag{5}$$

Putting $\varphi=\frac{\pi}{2}$ in Lemma 2, we obtain the following result.

Corollary 1. Let m and n be positive integers, where $m-1<2n$, then we obtain

$$\frac{x^{m-1}}{1+x^{2n}}=\frac{1}{n}\sum_{k=1}^{n}\frac{\cos\left(\frac{(2k-1)(m-1)\pi}{2n}\right)-x\cos\left((2k-1)m\frac{\pi}{2n}\right)}{x^2-2x\ \cos\left(\frac{(2k-1)\pi}{2n}\right)+1}. \tag{6}$$

Lemma 3. Let m and n be positive integers, where $m-1<2n$ and $x>0$. Then, the partial fractions of $\frac{x^{m-1}}{1-x^{2n}}$ has the form

$$\frac{x^{m-1}}{1-x^{2n}}=\frac{1}{n}\left(\frac{1}{1-x^2}+\sum_{k=1}^{n-1}\frac{\cos\left(\frac{k(m-1)\pi}{n}\right)-x\cos\left(km\frac{\pi}{n}\right)}{x^2-2x\cos\left(\frac{k\pi}{n}\right)+1}\right). \tag{7}$$

Proof of Lemma 3. Let $x^{2n}=1=e^{2ik\pi}$. Then, using de Moivre's theorem, we obtain

$$x=e^{\frac{ik\pi}{n}}=a+ib=\cos\left(\frac{k\pi}{n}\right)+i\sin\left(\frac{k\pi}{n}\right),\text{ for } k=0,1,2,\dots,(n-1).$$

$(x+1)(x+1)$ are factors of $1=x^{2n}$, and the other factors are given as products of conjugate factors. Therefore, we obtain

$$\prod_{k=1}^{n-1}(x-(a+ib))(x-(a-ib))=\prod_{k=1}^{n-1}\left(x^2-2ax+a^2+b^2\right)$$

$$\therefore x^{2n}-1=(x^2-1)\prod_{k=1}^{n-1}\left(x^2-2ax+a^2+b^2\right),$$

$$x^{2n}-1=(x^2-1)\prod_{k=1}^{n-1}(x^2-2x\cos\left(k\frac{\pi}{n}\right)+1)\ ,$$

$$1-x^{2n}=\left(1-x^2\right)\prod_{k=1}^{n-1}\left(x^2-2x\cos\left(k\frac{\pi}{n}\right)+1\right). \tag{8}$$

We use Equation (8) to find the partial fractions as follows

$$\frac{x^{m-1}}{1-x^{2n}}=\sum_{s=1}^{n-1}\frac{D}{1-x^2}+\frac{A_s+B_sx}{(x^2-2x\cos\left(s\frac{\pi}{n}\right)+1)}. \tag{9}$$

Let $\omega = \omega(s) = s\frac{\pi}{n}$ in Equation (9). Then, we obtain

$$\frac{x^{m-1}}{1-x^{2n}} = \sum_{s=1}^{n-1} \frac{D}{(1-x^2)} + \frac{A_s + B_s x}{(x^2 - 2x\cos(\omega) + 1)}. \tag{10}$$

We use the Euler approach in [2] to find D, A_s, and B_s to obtain

$$D = \frac{1}{n}, B_s = \frac{\sin\left(m\,\omega - \frac{\pi}{2}\right)}{n}, \text{ and } A_s = -\frac{\sin\left((m-1)\,\omega - \frac{\pi}{2}\right)}{n}.$$

Substituting the values of D, A_s, and B_s in (10), we obtain

$$\frac{x^{m-1}}{1-x^{2n}} = \frac{1}{n}\left(\frac{1}{1-x^2} + \sum_{s=1}^{n-1} \frac{\cos\left(\frac{s(m-1)\pi}{n}\right) - x\cos\left(sm\frac{\pi}{n}\right)}{x^2 - 2x\,\cos\left(\frac{s\pi}{n}\right) + 1}\right). \tag{11}$$

The proof is completed. □

3. Fundamental Theorems

In this section, we introduce six new theorems for solving improper integrals. These theorems can be used to generate new integrals and solve difficult applications. These theories are considered as generalizations of the master theorems found in [31]. The difficulty of generalizing these results lies in the partial fractions mentioned in Lemmas 1–3, in addition to the idea of merging the theorems used in the previous research with these lemmas and finding the integrals in Appendix A.

To achieve our goal, we need to present some results concerning analytic functions. For more details, see [7–9].

Let $f(z)$ be an analytic function around α, then, according to Taylor's series where α, β, and θ denote positive or negative real quantities, we obtain

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(\alpha)}{k!}(z-\alpha)^k,$$

$$f\left(\alpha + \beta e^{\theta x}\right) = \sum_{k=0}^{\infty} \frac{f^{(k)}(\alpha)}{k!}\beta^k e^{\theta k x}. \tag{12}$$

Using Euler formulas

$$e^{i\theta x} + e^{-i\theta x} = 2\cos(\theta x),$$
$$e^{i\theta x} - e^{-i\theta x} = 2i\sin(\theta x).$$

we can gain

$$\begin{aligned}\tfrac{1}{2}\left(f\left(\alpha + \beta e^{i\theta x}\right) + f\left(\alpha + \beta e^{-i\theta x}\right)\right) &= \tfrac{1}{2}\sum_{k=0}^{\infty} \tfrac{f^{(k)}(\alpha)}{k!}\beta^k \left(e^{i\theta k x} + e^{-i\theta k x}\right) \\ &= \sum_{k=0}^{\infty} \tfrac{f^{(k)}(\alpha)}{k!}\beta^k \cos(k\theta x) \\ &= f(\alpha) + f'(\alpha)\beta\cos(\theta x) + \tfrac{f''(\alpha)}{2!}\beta^2\cos(2\theta x) + \ldots.\end{aligned} \tag{13}$$

Similarly,

$$\begin{aligned}&\tfrac{1}{2i}\left(f\left(\alpha + \beta e^{i\theta x}\right) - f\left(\alpha + \beta e^{-i\theta x}\right)\right) \\ &\quad = \tfrac{1}{2}\sum_{k=0}^{\infty} \tfrac{f^{(k)}(\alpha)}{k!}\beta^k \left(e^{i\theta k x} - e^{-i\theta k x}\right) \\ &\quad = f'(\alpha)\beta\sin(\theta x) + \tfrac{f''(\alpha)}{2!}\beta^2\sin(2\theta x) + \ldots = \sum_{k=1}^{\infty} \tfrac{f^{(k)}(\alpha)}{k!}\beta^k \sin(k\theta x).\end{aligned} \tag{14}$$

In the following arguments, we present new theorems to solve improper integrals that can be used in solving various applications. It is worth mentioning here that these theorems can be considered as generalizations of Cauchy's results [3].

Theorem 1. Let $f(z)$ be an analytic function around α, where $\alpha \in \mathbb{R}$. Then, we obtain the following result

$$\int_0^\infty \frac{x^{m-1}\left(f\left(\alpha+\beta e^{i\theta x}\right)+f\left(\alpha+\beta e^{-i\theta x}\right)\right)}{(1+x^{2n})^r}dx = \frac{(-1)^{r-1}}{\Gamma(r)}\frac{\partial^{r-1}}{\partial u^{r-1}}\left((u)^{\frac{r-2n}{2n}}\frac{\pi}{n}\sum_{s=1}^{n}\left(\frac{1}{2}\sin(m\omega)(\psi+\phi)+\frac{1}{2i}\cos(m\omega)(\psi-\phi)\right)\right)\bigg|_{u=1},$$

where $\theta > 0$, m is odd, $n, m \in \mathbb{N}$, $0 < m < 2n$, $r \in \mathbb{R}$, $\omega = \omega(s) = \frac{(2s-1)\pi}{2n}$, $\psi = \psi(\omega) = f\left(\alpha + \beta e^{-\theta\sqrt[2n]{u}\sin(\omega)+i\theta\sqrt[2n]{u}\cos(\omega)}\right)$, and $\phi = \phi(\omega) = f\left(\alpha + \beta e^{-\theta\sqrt[2n]{u}\sin(\omega)-i\theta\sqrt[2n]{u}\cos(\omega)}\right)$.

Proof of Theorem 1. To obtain our result, we need the following fact.

Consider

$$j = \int_0^\infty \frac{x^{m-1}\left(f\left(\alpha+\beta e^{i\theta x}\right)+f\left(\alpha+\beta e^{-i\theta x}\right)\right)}{x^{2n}+u}dx.$$

Differentiating j with respect to u, $(r-1)$ times, we obtain the following

$$\frac{\partial^{r-1}j}{\partial u^{r-1}} = \Gamma(r)(-1)^{r-1}\int_0^\infty \frac{x^{m-1}\left(f\left(\alpha+\beta e^{i\theta x}\right)+f\left(\alpha+\beta e^{-i\theta x}\right)\right)}{(x^{2n}+u)^r}dx. \quad (15)$$

After simple computations and substituting $u = 1$ in Equation (15), we obtain

$$\begin{aligned}\frac{(-1)^{r-1}}{\Gamma(r)}\frac{\partial^{r-1}j}{\partial u^{r-1}}\bigg|_{u=1} &= \left(\int_0^\infty \frac{x^{m-1}\left(f\left(\alpha+\beta e^{i\theta x}\right)+f\left(\alpha+\beta e^{-i\theta x}\right)\right)}{(u+x^{2n})^r}dx\right)\bigg|_{u=1} \\ &= \int_0^\infty \frac{x^{m-1}\left(f\left(\alpha+\beta e^{i\theta x}\right)+f\left(\alpha+\beta e^{-i\theta x}\right)\right)}{(1+x^{2n})^r}dx = I.\end{aligned}$$

Now,

$$\begin{aligned}I &= \int_0^\infty \frac{x^{m-1}\left(f\left(\alpha+\beta e^{i\theta x}\right)+f\left(\alpha+\beta e^{-i\theta x}\right)\right)}{(1+x^{2n})^r}dx \\ &= \frac{(-1)^{r-1}}{\Gamma(r)}\frac{\partial^{r-1}}{\partial u^{r-1}}\left(\int_0^\infty \frac{x^{m-1}\left(f\left(\alpha+\beta e^{i\theta x}\right)+f\left(\alpha+\beta e^{-i\theta x}\right)\right)}{x^{2n}+u}dx\right)\bigg|_{u=1}.\end{aligned} \quad (16)$$

Letting $\frac{x}{\sqrt[2n]{u}} = y \to dx = \sqrt[2n]{u}\,dy$ in Equation (16), we obtain

$$I = \frac{(-1)^{r-1}}{\Gamma(r)}\frac{\partial^{r-1}}{\partial u^{r-1}}\left(\int_0^\infty \frac{(u)^{\frac{m-2n}{2n}}\, y^{m-1}\left(f\left(\alpha+\beta e^{i\theta y\sqrt[2n]{u}}\right)+f\left(\alpha+\beta e^{-i\theta y\sqrt[2n]{u}}\right)\right)}{y^{2n}+1}dy\right)\bigg|_{u=1}. \quad (17)$$

Now, since f is an analytic around α, we obtain

$$f\left(\alpha+\beta e^{i\theta y\sqrt[2n]{u}}\right)+f\left(\alpha+\beta e^{-i\theta y\sqrt[2n]{u}}\right) = 2\sum_{k=0}^{\infty}\frac{f^{(k)}(\alpha)\beta^k}{k!}\cos\left(k\theta y\sqrt[2n]{u}\right). \quad (18)$$

Substituting Equation (18) into Equation (17), we obtain

$$I=\frac{(-1)^{r-1}}{\Gamma(r)}\frac{\partial^{r-1}}{\partial u^{r-1}}\left(\int_0^\infty \frac{(u)^{\frac{m-2n}{2n}}\ y^{m-1}\left(2\sum_{k=0}^{\infty}\frac{f^{(k)}(\alpha)\beta^k}{k!}\cos\left(k\theta y\sqrt[2n]{u}\right)\right)}{y^{2n}+1}dy\right)\Bigg|_{u=1}. \quad (19)$$

Therefore, by changing the order of the summation and the improper integral and using Fubini's theorem, Equation (19) becomes

$$I=\frac{(-1)^{r-1}}{\Gamma(r)}\frac{\partial^{r-1}}{\partial u^{r-1}}\left(2\ (u)^{\frac{m-2n}{2n}}\sum_{k=0}^{\infty}\frac{f^{(k)}(\alpha)\beta^k}{k!}\int_0^\infty\frac{y^{m-1}\left(\cos\left(k\theta y\sqrt[2n]{u}\right)\right)}{(y^{2n}+1)}dy\right)\Bigg|_{u=1}. \quad (20)$$

Now, using Equation (1) on Appendix A, we obtain

$$I=\frac{(-1)^{r-1}}{\Gamma(r)}\frac{\partial^{r-1}}{\partial u^{r-1}}\left(\frac{\pi}{n}(u)^{\frac{m-2n}{2n}}\sum_{k=0}^{\infty}\frac{f^{(k)}(\alpha)\beta^k}{k!}\mu_k\right)\Bigg|_{u=1}, \quad (21)$$

where

$$\mu_k=\sum_{s=1}^{n}e^{-k\theta\sqrt[2n]{u}\sin\left(\frac{(2s-1)\pi}{2n}\right)}\sin\left(\left(\frac{(2s-1)\ m\ \pi}{2n}\right)+k\theta\sqrt[2n]{u}\cos\left(\frac{(2s-1)\pi}{2n}\right)\right). \quad (22)$$

To simplify the calculations, let $\omega=\omega(s)=\frac{(2s-1)\ \pi}{2n}$ in Equation (22).

To write Equation (21) in terms of the original function, we must rewrite Equation (22) in the exponential form as follows

$$\begin{aligned} &e^{-k\theta\sqrt[2n]{u}\sin(\omega)}\sin\left((m\omega)+k\theta\sqrt[2n]{u}\cos(\omega)\right)\\ &=\tfrac{1}{2}\sin(m\omega)\left(e^{-k\theta\sqrt[2n]{u}\sin(\omega)+ik\theta\sqrt[2n]{u}\cos(\omega)}+e^{-k\theta\sqrt[2n]{u}\sin(\omega)-ik\theta\sqrt[2n]{u}\cos(\omega)}\right)\\ &+\tfrac{1}{2i}\cos(m\omega)\left(e^{-k\theta\sqrt[2n]{u}\sin(\omega)+ik\theta\sqrt[2n]{u}\cos(\omega)}-e^{-k\theta\sqrt[2n]{u}\sin(\omega)-ik\theta\sqrt[2n]{u}\cos(\omega)}\right).\end{aligned} \quad (23)$$

Therefore, substituting Equation (23) into Equation (22), Equation (21) becomes

$$I=\frac{(-1)^{r-1}}{\Gamma(r)}\frac{\partial^{r-1}}{\partial u^{r-1}}\left((u)^{\frac{m-2n}{2n}}\frac{\pi}{n}\sum_{k=0}^{\infty}\frac{f^{(k)}(\alpha)\beta^k}{k!}\sum_{s=1}^{n}\left(\tfrac{1}{2}\sin(m\omega)(A+B)+\tfrac{1}{2i}\cos(m\omega)(A-B)\right)\right)\Bigg|_{u=1}, \quad (24)$$

where

$$A=e^{-k\theta\sqrt[2n]{u}\sin(\omega)+ik\theta\sqrt[2n]{u}\cos(\omega)},\ \text{and}\ B=e^{-k\theta\sqrt[2n]{u}\sin(\omega)-ik\theta\sqrt[2n]{u}\cos(\omega)}. \quad (25)$$

Now, using the fact in Equation (12), Equation (25) becomes

$$I=\frac{(-1)^{r-1}}{\Gamma(r)}\frac{\partial^{r-1}}{\partial u^{r-1}}\left((u)^{\frac{m-2n}{2n}}\frac{\pi}{n}\sum_{s=1}^{n}\left(\tfrac{1}{2}\sin(m\omega)(\psi(\omega)+\phi(\omega))+\tfrac{1}{2i}\cos(m\omega)(\psi(\omega)-\phi(\omega))\right)\right)\Bigg|_{u=1},$$

where $\omega=\omega(s)=\frac{(2s-1)\pi}{2n}$, $\psi=\psi(\omega)=f\left(\alpha+\beta e^{-\theta\sqrt[2n]{u}\sin(\omega)+i\theta\sqrt[2n]{u}\cos(\omega)}\right)$, and $\phi=\phi(\omega)=f\left(\alpha+\beta e^{-\theta\sqrt[2n]{u}\sin(\omega)-i\theta\sqrt[2n]{u}\cos(\omega)}\right)$.

This completes the proof. □

Theorem 2. Let $f(z)$ be an analytic function around α, where $\alpha\in\mathbb{R}$. Then, we obtain the following result

$$I = \int_0^\infty \frac{x^{m-1}\left(f\left(\alpha+\beta e^{i\theta x}\right)-f\left(\alpha+\beta e^{-i\theta x}\right)\right)}{i\,(1+x^{2n})^r}dx$$
$$= \frac{(-1)^{r-1}}{\Gamma(r)}\frac{\partial^{r-1}}{\partial u^{r-1}}\left((u)^{\frac{m-2n}{2n}}\left(\frac{-\pi}{n}\right)\sum_{s=1}^{n}\left(\frac{\cos(m\omega)}{2}(\psi+\phi-2f(\alpha))+\frac{\sin(m\omega)}{2i}(\psi-\phi)\right)\right)\Bigg|_{u=1}, \tag{26}$$

where m is even, $n \in \mathbb{N}$, $0 < m < 2n$, $\theta > 0$, $r \in \mathbb{R}$, $\omega = \omega(s) = \frac{(2s-1)\pi}{2n}$, $\psi = \psi(\omega) = f\left(\alpha+\beta e^{-\theta\sqrt[2n]{u}\sin(\omega)+i\theta\sqrt[2n]{u}\cos(\omega)}\right)$, and $\phi = \phi(\omega) = f\left(\alpha+\beta e^{-\theta\sqrt[2n]{u}\sin(\omega)-i\theta\sqrt[2n]{u}\cos(\omega)}\right)$.

The proof of Theorem 2 can be obtained by similar arguments to Theorem 1 and using Fact (2) in Appendix A.

Putting $m = 1$ in the left-hand side of Theorem 2, with similar arguments to the proof of Theorem 1 and using Fact (3) in Appendix A, we can obtain the following result.

Corollary 2. Let $f(z)$ be an analytic function around α, where $\alpha \in \mathbb{R}$. Then, we obtain the following result

$$I = \int_0^\infty \frac{f\left(\alpha+\beta e^{i\theta x}\right)-f\left(\alpha+\beta e^{-i\theta x}\right)}{i\,x\,(1+x^{2n})^r}dx$$
$$= \frac{(-1)^{r-1}}{\Gamma(r)}\frac{\partial^{r-1}}{\partial u^{r-1}}\left(\frac{\pi}{n}(u)^{-1}\sum_{s=1}^{n}\left(f(\alpha+\beta)-\frac{1}{2}(\psi+\phi)\right)\right)\Bigg|_{u=1}, \tag{27}$$

where $n \in \mathbb{N}$, $\theta > 0$, $r \in \mathbb{R}$, $\omega = \omega(s) = \frac{(2s-1)\pi}{2n}$, $\psi = \psi(s) = f\left(\alpha+\beta \mathrm{e}^{(\mathrm{i}\theta\sqrt[2n]{u}\cos(\omega)-\theta\sqrt[2n]{u}\sin(\omega))}\right)$, and $\phi = \phi(s) = f\left(\alpha+\beta \mathrm{e}^{(-\mathrm{i}\theta\sqrt[2n]{u}\cos(\omega)-\theta\sqrt[2n]{u}\sin(\omega))}\right)$.

Theorem 3. Let $f(z)$ be an analytic function around α, where $\alpha \in \mathbb{R}$. Then, we obtain the following result

$$PV\int_0^\infty \frac{x^{m-1}\left(f\left(\alpha+\beta e^{i\theta x}\right)+f\left(\alpha+\beta e^{-i\theta x}\right)\right)}{(1-x^{2n})^r}dx$$
$$= \frac{(-1)^{r-1}}{\Gamma(r)}\frac{\partial^{r-1}}{\partial u^{r-1}}\left((u)^{\frac{m-2n}{2n}}\frac{\pi}{n}\left(\frac{1}{2i}(\eta-\vartheta)+\sum_{s=1}^{n-1}\left(\frac{1}{2}\sin(m\omega)(\psi+\phi)+\frac{1}{2i}\cos(m\omega)(\psi-\phi)\right)\right)\right)\Bigg|_{u=1}, \tag{28}$$

where m is odd, $n \in \mathbb{N}$, $0 < m < 2n$, $\theta > 0$, $r \in \mathbb{R}$, $\omega = \omega(s) = \frac{s\pi}{n}$, $\eta = f\left(\alpha+\beta e^{i\theta\sqrt[2n]{u}}\right)$, $\vartheta = f\left(\alpha+\beta e^{-i\theta\sqrt[2n]{u}}\right)$, $\psi = \psi(\omega) = f\left(\alpha+\beta e^{-\theta\sqrt[2n]{u}\sin(\omega)+i\theta\sqrt[2n]{u}\cos(\omega)}\right)$ and $\phi = \phi(\omega) = f\left(\alpha+\beta e^{-\theta\sqrt[2n]{u}\sin(\omega)-i\theta\sqrt[2n]{u}\cos(\omega)}\right)$.

Proof of Theorem 3. Let

$$I = PV\int_0^\infty \frac{x^{m-1}\left(f\left(\alpha+\beta e^{i\theta x}\right)+f\left(\alpha+\beta e^{-i\theta x}\right)\right)}{(1-x^{2n})^r}dx$$
$$= \frac{(-1)^{r-1}}{\Gamma(r)}\frac{\partial^{r-1}}{\partial u^{r-1}}\left(PV\int_0^\infty \frac{x^{m-1}\left(f\left(\alpha+\beta e^{i\theta x}\right)+f\left(\alpha+\beta e^{-i\theta x}\right)\right)}{u-x^{2n}}dx\right)\Bigg|_{u=1}. \tag{29}$$

Let $\frac{x}{\sqrt[2n]{u}} = y$ in Equation (29), then $dx = \sqrt[2n]{u}\,dy$. Therefore, Equation (29) becomes

$$I = \frac{(-1)^{r-1}}{\Gamma(r)}\frac{\partial^{r-1}}{\partial u^{r-1}}\left(PV\int_0^\infty \frac{(u)^{\frac{m-2n}{2n}}\,y^{r-1}\left(f\left(\alpha+\beta e^{i\theta y\sqrt[2n]{u}}\right)+f\left(\alpha+\beta e^{-i\theta y\sqrt[2n]{u}}\right)\right)}{1-y^{2n}}dy\right)\Bigg|_{u=1}.$$

Now, since f is an analytic function around α, and using the fact in Equation (13) in Equation (29), we obtain

$$I = \frac{(-1)^{r-1}}{\Gamma(r)} \frac{\partial^{r-1}}{\partial u^{r-1}} \left(PV \int_0^\infty \frac{(u)^{\frac{m-2n}{2n}} y^{m-1} \left(2 \sum_{k=0}^{\infty} \frac{f^{(k)}(\alpha)\beta^k}{k!} \cos\left(k\theta y \sqrt[2n]{u}\right) \right)}{1 - y^{2n}} dy \right) \Bigg|_{u=1}. \quad (30)$$

Therefore, by changing the order of the summation and the improper integral using Fubini's theorem, we obtain

$$I = \frac{(-1)^{r-1}}{\Gamma(r)} \frac{\partial^{r-1}}{\partial u^{r-1}} \left(2\,(u)^{\frac{m-2n}{2n}} \sum_{k=0}^{\infty} \frac{f^{(k)}(\alpha)\beta^k}{k!} PV \int_0^\infty \frac{y^{m-1}\left(\cos\left(k\theta y \sqrt[2n]{u}\right)\right)}{(1 - y^{2n})} dy \right) \Bigg|_{u=1}. \quad (31)$$

Using Fact (4) in Appendix A, we obtain

$$I = \frac{(-1)^{r-1}}{\Gamma(r)} \frac{\partial^{r-1}}{\partial u^{r-1}} \left(2\,(u)^{\frac{m-2n}{2n}} \frac{\pi}{2n} \sum_{k=0}^{\infty} \frac{f^{(k)}(\alpha)\beta^k}{k!} \mu_k \right) \Bigg|_{u=1}, \quad (32)$$

where

$$\mu_k = \sin\left(k\theta \sqrt[2n]{u}\right) + \sum_{s=1}^{n-1} e^{-k\theta \sqrt[2n]{u} \sin\left(\frac{s\pi}{n}\right)} \sin\left(\left(\frac{sr\pi}{n}\right) + k\theta \sqrt[2n]{u} \cos\left(\frac{s\pi}{n}\right)\right). \quad (33)$$

To simplify the calculations, let $\omega = \omega(s) = \frac{s\pi}{n}$ in Equation (33), and to rewrite the answer in a closed form of the original function, Equation (33) should be rewritten in the exponential form as

$$\begin{aligned} & e^{-k\theta \sqrt[2n]{u} \sin(\omega)} \sin\left((r\omega) + k\theta \sqrt[2n]{u} \cos(\omega)\right) \\ = & \tfrac{1}{2} \sin(m\omega) \left(e^{-k\theta \sqrt[2n]{u} \sin(\omega) + ik\theta \sqrt[2n]{u} \cos(\omega)} + e^{-k\theta \sqrt[2n]{u} \sin(\omega) - ik\theta \sqrt[2n]{u} \cos(\omega)} \right) \\ & + \frac{\cos(m\omega)}{2i} \left(e^{-k\theta \sqrt[2n]{u} \sin(\omega) + ik\theta \sqrt[2n]{u} \cos(\omega)} - e^{-k\theta \sqrt[2n]{u} \sin(\omega) - ik\theta \sqrt[2n]{u} \cos(\omega)} \right). \end{aligned} \quad (34)$$

Substituting Equation (34) into Equation (33) with some computations, Equation (32) becomes

$$I = \frac{(-1)^{r-1}}{\Gamma(r)} \frac{\partial^{r-1}}{\partial u^{r-1}} \left((u)^{\frac{m-2n}{2n}} \frac{\pi}{n} \sum_{k=0}^{\infty} \frac{f^{(k)}(\alpha)\beta^k}{k!} \sin\left(k\theta \sqrt[2n]{u}\right) + \sum_{s=1}^{n-1} \left(\frac{\sin(m\omega)}{2}(A+B) + \frac{\cos(m\omega)}{2i}(A-B) \right) \right) \Bigg|_{u=1}, \quad (35)$$

where $A = A(\omega) = e^{-k\theta \sqrt[2n]{u} \sin(\omega) + ik\theta \sqrt[2n]{u} \cos(\omega)}$, $B = B(\omega) = e^{-k\theta \sqrt[2n]{u} \sin(\omega) - ik\theta \sqrt[2n]{u} \cos(\omega)}$.

Using the fact in Equation (12), we obtain

$$I = \frac{(-1)^{r-1}}{\Gamma(r)} \frac{\partial^{r-1}}{\partial u^{r-1}} \left((u)^{\frac{m-2n}{2n}} \frac{\pi}{n} \left(\frac{1}{2i}(\eta - \vartheta) + \sum_{s=1}^{n-1} \left(\frac{1}{2} \sin(m\omega)(\psi + \phi) + \frac{1}{2i} \cos(m\omega)(\psi - \phi) \right) \right) \right) \Bigg|_{u=1},$$

where $\omega = \omega(s) = \frac{s\pi}{n}$, $\eta = f\left(\alpha + \beta e^{i\theta \sqrt[2n]{u}}\right)$, $\vartheta = f\left(\alpha + \beta e^{-i\theta \sqrt[2n]{u}}\right)$, $\psi = \psi(\omega) = f\left(\alpha + \beta e^{-\theta \sqrt[2n]{u} \sin(\omega) + i\theta \sqrt[2n]{u} \cos(\omega)}\right)$ and $\phi = \phi(\omega) = f\left(\alpha + \beta e^{-\theta \sqrt[2n]{u} \sin(\omega) - i\theta \sqrt[2n]{u} \cos(\omega)}\right)$.

This completes the proof. □

Theorem 4. Let $f(z)$ be an analytic function around α, where $\alpha \in \mathbb{R}$. Then, we obtain the following result

$$
\begin{aligned}
I &= PV\int_0^{\infty} \frac{x^{m-1}\left(f\left(\alpha+\beta e^{i\theta x}\right)-f\left(\alpha+\beta e^{-i\theta x}\right)\right)}{i(1-x^{2n})^{r}}dx \\
&= \frac{(-1)^{r-1}}{\Gamma(r)}\frac{\partial^{r-1}}{\partial u^{r-1}}\left(\frac{-\pi}{2n(u)^{\frac{2n-m}{2n}}}\left((\eta+\vartheta-2f(\alpha))+\sum_{s=1}^{n-1}\left(\cos(m\omega)(\psi+\phi-2f(\alpha))+\frac{\sin(m\omega)}{i}(\psi-\phi)\right)\right)\right)\Bigg|_{u=1},
\end{aligned} \tag{36}
$$

where m is even, $n \in \mathbb{N}$, $0 < m < 2n$, $\theta > 0$ and $r \in \mathbb{R}$, $\omega = \omega(s) = \frac{s\pi}{n}$, $\eta = f\left(\alpha+\beta e^{i\theta \sqrt[2n]{u}}\right)$, $\vartheta = f\left(\alpha+\beta e^{-i\theta \sqrt[2n]{u}}\right)$, $\psi = \psi(\omega) = f\left(\alpha+\beta e^{-\theta \sqrt[2n]{u}\sin(\omega)+i\theta \sqrt[2n]{u}\cos(\omega)}\right)$, and $\phi = \phi(\omega) = f\left(\alpha+\beta e^{-\theta \sqrt[2n]{u}\sin(\omega)-i\theta \sqrt[2n]{u}\cos(\omega)}\right)$.

The proof of Theorem 4 can be obtained by similar arguments to Theorem 3 and using Fact (5) in Appendix A.

Putting $m = 0$ in the left-hand side of Theorem 4, with similar arguments to the proof of Theorem 3 and using Fact (6) in Appendix A, we can obtain the following result.

Corollary 3. Let $f(z)$ be an analytic function around α, where $\alpha \in \mathbb{R}$. Then we obtain the following result

$$
\begin{aligned}
I &= PV\int_0^{\infty} \frac{f\left(\alpha+\beta e^{i\theta x}\right)-f\left(\alpha+\beta e^{-i\theta x}\right)}{ix(1-x^{2n})^{r}}dx \\
&= \frac{(-1)^{r-1}}{\Gamma(r)}\frac{\partial^{r-1}}{\partial u^{r-1}}\left(\frac{-\pi}{un}\left(\frac{1}{2}(\eta+\vartheta-2f(\alpha))-n(f(\alpha+\beta)-f(\alpha))+\frac{1}{2}\sum_{s=1}^{n-1}(\psi+\phi-2f(\alpha))\right)\right)\Bigg|_{u=1},
\end{aligned} \tag{37}
$$

where $n \in \mathbb{N}$, $\theta > 0$, $r \in \mathbb{R}$, $\omega = \omega(s) = \frac{s\pi}{n}$, $\eta = f\left(\alpha+\beta e^{i\theta \sqrt[2n]{u}}\right)$, $\vartheta = f\left(\alpha+\beta e^{-i\theta \sqrt[2n]{u}}\right)$, $\psi = \psi(\omega) = f\left(\alpha+\beta e^{-\theta \sqrt[2n]{u}\sin(\omega)+i\theta \sqrt[2n]{u}\cos(\omega)}\right)$, and $\phi = \phi(\omega) = f\left(\alpha+\beta e^{-\theta \sqrt[2n]{u}\sin(\omega)-i\theta \sqrt[2n]{u}\cos(\omega)}\right)$.

Theorem 5. Let $f(z)$ be an analytic function around α, where $\alpha \in \mathbb{R}$. Then, we obtain the following result

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \frac{x^{m-1}\left(f\left(\alpha+\beta e^{i\theta x}\right)+f\left(\alpha+\beta e^{-i\theta x}\right)\right)}{x^{2n}-2\cos(\varphi)x^{n}+1}dx \\
&\qquad = \frac{2\pi}{n\sin(\varphi)}\sum_{s=0}^{n-1}\left(\frac{\cos(\varphi-m\omega)}{2}(\psi+\phi)+\frac{\sin(\varphi-m\omega)}{2i}(\psi-\phi)\right),
\end{aligned} \tag{38}
$$

where $\theta > 0$, $|\varphi| < \pi$, $\varphi \neq 0$, $0 < m < 2n$, m is odd, $n \in \mathbb{N}$, $\omega = \omega(s) = \frac{2s\pi+\varphi}{n}$, $\psi = \psi(\omega) = f\left(\alpha+\beta e^{(i\,\theta\cos\omega-\theta\sin(\omega))}\right)$ and $\phi = \phi(\omega) = f\left(\alpha+\beta e^{(-i\,\theta\cos\omega-\theta\sin(\omega))}\right)$.

Proof of Theorem 5. Let

$$
I = \int_{-\infty}^{\infty} \frac{x^{m-1}\left(f\left(\alpha+\beta e^{i\theta x}\right)+f\left(\alpha+\beta e^{-i\theta x}\right)\right)}{x^{2n}-2\cos(\varphi)x^{n}+1}dx.
$$

Now, since f is an analytic function around α, we use the fact in Equation (13), and by interchanging the order of the summation and the improper integral, we obtain

$$
I = 2\sum_{k=0}^{\infty}\frac{f^{k}(\alpha)\beta^{k}}{k!}\int_{-\infty}^{\infty}\frac{x^{m-1}(\cos(k\theta x))}{x^{2n}-2\cos(\varphi)x^{n}+1}dx.
$$

Now, by using Equation (7) in Appendix A, we obtain

$$I = \frac{2\pi}{n\sin(\varphi)} \sum_{k=0}^{\infty} \frac{f^k(\alpha)\beta^k}{k!} \sum_{s=0}^{n-1} e^{-\theta k \sin\left(\frac{2s\pi+\varphi}{n}\right)} \cos\left(m\left(\frac{2s\pi+\varphi}{n}\right) + \theta k \cos\left(\frac{2s\pi+\varphi}{n}\right) - \varphi\right). \tag{39}$$

To simplify the calculations, let $\omega = \omega(s) = \frac{2s\pi+\varphi}{n}$ in Equation (39).

To write the answer in a closed form of the original function, the internal sum in Equation (39) should be rewritten in the exponential form

$$e^{-\theta k \sin(\omega)} \cos(m\omega + \theta k \cos\omega - \varphi) = \frac{A}{2}\cos(\varphi - m\omega)\frac{B}{2i}\sin(\varphi - m\omega). \tag{40}$$

where $A = A(\omega) = e^{k(i\,\theta\cos\omega - \theta\sin(\omega))} + e^{k(-i\,\theta\cos\omega - \theta\sin(\omega))}$, and $B = B(\omega) = \frac{1}{2i}\left(e^{k(i\,\theta\cos\omega - \theta\sin(\omega))} - e^{k(-i\,\theta\cos\omega - \theta\sin(\omega))}\right)$.

Now, substituting Equation (40) into Equation (39), we obtain

$$I = \frac{2\pi}{n\sin(\varphi)} \sum_{k=0}^{\infty} \frac{f^k(\alpha)\beta^k}{k!} \sum_{s=0}^{n-1} \frac{A}{2}\cos(\varphi - m\omega)\frac{B}{2i}\sin(\varphi - m\omega). \tag{41}$$

Now, using the fact in Equation (12), Equation (41) becomes

$$I = \frac{2\pi}{n\sin(\varphi)} \sum_{s=0}^{n-1} \left(\frac{\cos(\varphi - m\omega)}{2}(\psi + \phi) + \frac{\sin(\varphi - m\omega)}{2i}(\psi - \phi)\right),$$

where $\omega = \omega(s) = \frac{2s\pi+\varphi}{n}$, $\psi = \psi(\omega) = f\left(\alpha + \beta e^{(i\,\theta\cos\omega - \theta\sin(\omega))}\right)$, and $\phi = \phi(\omega) = f\left(\alpha + \beta e^{(-i\,\theta\cos\omega - \theta\sin(\omega))}\right)$. Hence, the proof is completed.

Theorem 6. Let $f(z)$ be an analytic function around α, where $\alpha \in \mathbb{R}$. Then, we obtain the following result

$$\int_{-\infty}^{\infty} \frac{x^{m-1}\left(f\left(\alpha+\beta e^{i\theta x}\right) - f\left(\alpha+\beta e^{-i\theta x}\right)\right)}{i(x^{2n} - 2\cos(\varphi)x^n + 1)} dx = \frac{\pi}{n\sin(\varphi)} \sum_{s=0}^{n-1} \left(\frac{\cos(\varphi - m\omega)}{i}(\psi - \phi) - \sin(\varphi - m\omega)(\psi + \phi - 2f(\alpha))\right), \tag{42}$$

where $\theta > 0$, $|\varphi| < \pi$, $\varphi \neq 0$,$0 < m < 2n$, m is even, $n \in \mathbb{N}$, $\omega = \omega(s) = \frac{2s\pi+\varphi}{n}$, $\psi = \psi(\omega) = f\left(\alpha + \beta e^{(i\,\theta\cos\omega - \theta\sin(\omega))}\right)$, and $\phi = \phi(\omega) = f\left(\alpha + \beta e^{(-i\,\theta\cos\omega - \theta\sin(\omega))}\right)$.

The proof of Theorem 6 can be obtained by similar arguments to Theorem 5 and using the fact (8) in Appendix A.

Putting $m = 0$ in the left-hand side of Theorem 6, with similar arguments to the proof of Theorem 5 and using the fact (9) in Appendix A, we obtain the following result.

Corollary 4. Let $f(z)$ be an analytic function around α, where $\alpha \in \mathbb{R}$. Then, we obtain the following result

$$\int_{-\infty}^{\infty} \frac{f\left(\alpha+\beta e^{i\theta x}\right) - f\left(\alpha+\beta e^{-i\theta x}\right)}{ix(x^{2n} - 2\cos(\varphi)x^n + 1)} dx = \frac{2\pi}{n\sin(\varphi)} \sum_{s=0}^{n-1} \left(\sin(\varphi)(f(\alpha+\beta) - f(\alpha)) + \frac{\cos(\varphi)}{2i}(\psi - \phi) - \frac{\sin\varphi}{2}(\psi + \phi - 2f(\alpha))\right), \tag{43}$$

where $\theta > 0$, $|\varphi| < \pi$, $\varphi \neq 0$ and $n \in \mathbb{N}$, $\omega = \omega(s) = \frac{2s\pi+\varphi}{n}$, $\psi = \psi(\omega) = f\left(\alpha + \beta e^{(i\,\theta\cos\omega - \theta\sin(\omega))}\right)$, and $\phi = \phi(\omega) = f\left(\alpha + \beta e^{(-i\,\theta\cos\omega - \theta\sin(\omega))}\right)$.

4. Applications and Examples

In this section, we present remarks, applications, and comparisons of the proposed theorems. We also show that simple cases of our master theorems are generalizations of some of Cauchy's results from his memoirs [4,5].

4.1. Remarks on Theorems

In this section, we introduce some remarks on improper integrals and comparisons with Cauchy's results. These remarks are illustrated in the following Table 1.

4.2. Generating Improper Integrals

In this section, we present the technique of generating an infinite number of integrals using the theorems by only choosing the function $f(z)$ and finding the real or imaginary part. It is worth mentioning that many of these integrals with particular cases appear in [32–35] when solving some applications referred to in finding Green's function, integral representations of the Mittag–Leffler function on the positive real axis, wave motion in elastic solids, and implementing Fourier cosine and Fourier sine transforms.

To demonstrate the idea, we show some general examples that are applied to Theorems (1) and (2) as follows.

1. Setting $f(z) = z^v$, $v \in \mathbb{R}^+$
   - Using Theorem (1) and setting $\alpha = 0$, $\beta = 1$, we obtain

$$f\left(e^{i\theta x}\right) + f\left(e^{-i\theta x}\right) = e^{i\theta v x} + e^{-i\theta v x} = 2\cos(\theta\, v x).$$

Therefore,

$$\int_0^\infty \frac{x^{m-1}(2\cos(\theta\ v\ x))}{(1+x^{2n})^r}dx$$
$$= \frac{(-1)^{r-1}}{\Gamma(r)}\frac{\partial^{r-1}}{\partial u^{r-1}}\left((u)^{\frac{r-2n}{2n}}\frac{\pi}{n}\sum_{s=1}^{n}\left(\sin(m\omega)\left(e^{-\theta v\sqrt[2n]{u}\sin(\omega)}\cos\left(\theta v\sqrt[2n]{u}\cos(\omega)\right)\right)\right.\right.$$
$$\left.\left.\left.+\cos(m\omega)\left(e^{-\theta v\sqrt[2n]{u}\sin(\omega)}\sin\left(\theta v\sqrt[2n]{u}\cos(\omega)\right)\right)\right)\right)\right|_{u=1},$$

where $\theta > 0$, m is odd, $n, m \in \mathbb{N}$, $0 < m < 2n$, $r \in \mathbb{R}$.

- Setting $v = 1$, the obtained integral is a Fourier cosine transform [33] of the function $f(t) = \frac{t^{m-1}}{(1+t^{2n})^r}$. where $\theta > 0$, m is odd, $n, m \in \mathbb{N}$, $0 < m < 2n$, $r \in \mathbb{R}$.
- This can be used as an integral representation of the Mittag–Leffler function on the positive real axis (see [32]).
- Using Theorem (2) and setting $\alpha = 0$, $\beta = 1$, we obtain

$$\frac{1}{i}\left(f\left(e^{i\theta x}\right) - f\left(e^{-i\theta x}\right)\right) = \frac{1}{i}\left(e^{i\theta v x} + e^{-i\theta v x}\right) = 2\sin(\theta\, v x).$$

Therefore,

$$\int_0^\infty \frac{x^{m-1}\ (2\sin(\theta v x))}{(1+x^{2n})^r}dx$$
$$= \frac{(-1)^{r-1}}{\Gamma(r)}\frac{\partial^{r-1}}{\partial u^{r-1}}\left((u)^{\frac{r-2n}{2n}}\frac{\pi}{n}\sum_{s=1}^{n}\left(\cos(m\omega)\left(e^{-\theta v\sqrt[2n]{u}\sin(\omega)}\cos\left(\theta v\sqrt[2n]{u}\cos(\omega)\right)\right)\right.\right.$$
$$\left.\left.\left.+\sin(m\omega)\left(e^{-\theta v\sqrt[2n]{u}\sin(\omega)}\sin\left(\theta v\sqrt[2n]{u}\cos(\omega)\right)\right)\right)\right)\right|_{u=1},$$

where $\theta > 0$, m is even, $n, m \in \mathbb{N}$, $0 < m < 2n$, $r \in \mathbb{R}$.

Table 1. Remarks on improper integrals, where $\theta > 0$.

	Conditions	Theorem/ Corollary	$g(x)$	$\int_0^\infty g(x)dx$	Remarks
1	$\alpha = 0, \beta = 1$, and $r, m, n = 1$	Theorem 1	$\frac{f\left(e^{i\theta x}\right)+f\left(e^{-i\theta x}\right)}{1+x^2}$	$\pi f\left(e^{-\theta}\right)$	This is identical to Cauchy's theorem obtained in [4] (P.62, formula 8) and in [13] (3.037 Theorem 1).
2	$\alpha = 0, \beta = 1$, and $r, n = 1,\ m = 2$	Theorem 2	$\frac{x\left(f\left(e^{i\theta x}\right)-f\left(e^{-i\theta x}\right)\right)}{i\left(1+x^2\right)}$	$\pi\left(f\left(e^{-\theta}\right) - f(0)\right)$	Cauchy made a mistake in this result (see [4]) (P.62, formula 8). He corrected his result in his next memoir (see [5,6]).
3	$\alpha = 0,\ r,\ n = 1$	Corollary 1	$\frac{f\left(\beta e^{i\theta x}\right)-f\left(\beta e^{-i\theta x}\right)}{ix\left(1+x^2\right)}$	$\pi\left(f(\beta) - f\left(\beta e^{-\theta}\right)\right)$	Cauchy also made a mistake in this result (see [4] (P.62, formula 10)). This result appears in [13] (3.037 Theorem 4).
4	$r = 1,\ n = 1$	Theorem 5	$\frac{f\left(\alpha+\beta e^{i\theta x}\right)+f\left(\alpha+\beta e^{-i\theta x}\right)}{x^2-2\cos(\varphi)x+1}$	$\frac{\pi}{\sin(\varphi)}\left(f\left(\alpha + \beta e^{(i\theta\cos\varphi - \theta\sin(\varphi))}\right) + f\left(\alpha + \beta e^{(-i\,\theta\cos\varphi - \theta\sin(\varphi))}\right)\right)$	$\lvert\varphi\rvert < \pi,\ \varphi \neq 0$
5	$r = 2,\ n = 1$	Theorem 5	$\frac{x\ \left(f\left(\alpha+\beta e^{i\theta x}\right)-f\left(\alpha+\beta e^{-i\theta x}\right)\right)}{i(x^2-2\cos(\varphi)x+1)}$	$\frac{\pi}{\sin(\varphi)}\left(\sin\varphi(\psi + \phi - 2f(\alpha)) + \frac{\cos(\varphi)}{i}(\psi - \phi)\right)$	$\psi(\varphi) = f\left(\alpha + \beta e^{(i\theta\cos\varphi - \theta\sin\varphi)}\right)$ $\phi(\varphi) = f\left(\alpha + \beta e^{(-i\theta\cos\varphi - \theta\sin\varphi)}\right)$
6	$n = 1$	Theorem 6	$\frac{f\left(\alpha+\beta e^{i\theta x}\right)-f\left(\alpha+\beta e^{-i\theta x}\right)}{ix(x^2-2\cos(\varphi)x+1)}$	$\frac{2\pi}{\sin(\varphi)}\left(\sin(\varphi)(f(\alpha + \beta) - f(\alpha)) + \frac{\cos(\varphi)}{2i}(\psi - \phi) - \frac{\sin\varphi}{2}(\psi + \phi - 2f(\alpha))\right)$	$\psi = f\left(\alpha + \beta e^{(i\,\theta\cos\varphi - \theta\sin\varphi)}\right)$ $\phi = f\left(\alpha + \beta e^{(-i\theta\cos\varphi - \theta\sin\varphi)}\right)$

- Setting $v = 1$, the obtained integral is a Fourier sine transform [33,34] of the function $f(t) = \frac{t^{m-1}}{(1+t^{2n})^r}$. where $\theta > 0$, m is even, $n, m \in \mathbb{N}$, $0 < m < 2n$, $r \in \mathbb{R}$.
- This can be used as an integral representation of the Mittag–Leffler function on the positive real axis (see [32]).

2. Setting $f(z) = e^z$
   - Using Theorem (1), we obtain

$$f\left(\alpha+\beta e^{i\theta x}\right)+f\left(\alpha+\beta e^{-i\theta x}\right)=e^{\alpha+\beta e^{i\theta x}}+e^{\alpha+\beta e^{-i\theta x}}=2e^{\alpha+\beta\cos(\theta x)}\cos(\beta\sin(\theta x))$$

Therefore,

$$\begin{aligned}&\int_0^\infty \frac{x^{m-1}\left(2e^{\alpha+\beta\cos(\theta x)}\cos(\beta\sin(\theta x))\right)}{(1+x^{2n})^r}dx\\&=\frac{(-1)^{r-1}}{\Gamma(r)}\frac{\partial^{r-1}}{\partial u^{r-1}}\left((u)^{\frac{r-2n}{2n}}\frac{\pi}{n}\left(\sum_{s=1}^{n}\left(\frac{1}{2}\sin(m\omega)\left(e^{\alpha+\beta e^{-\theta\sqrt[2n]{u}\sin(\omega)+i\theta\sqrt[2n]{u}\cos(\omega)}}+e^{\alpha+\beta e^{-\theta\sqrt[2n]{u}\sin(\omega)-i\theta\sqrt[2n]{u}\cos(\omega)}}\right)\right)\right)\right.\\&\left.+\sum_{s=1}^{n}\frac{1}{2i}\sin(m\omega)\left(e^{\alpha+\beta e^{-\theta\sqrt[2n]{u}\sin(\omega)+i\theta\sqrt[2n]{u}\cos(\omega)}}-e^{\alpha+\beta e^{-\theta\sqrt[2n]{u}\sin(\omega)-i\theta\sqrt[2n]{u}\cos(\omega)}}\right)\right)\Bigg|_{u=1}\end{aligned}$$

where $\theta > 0$, m is odd, $n, m \in \mathbb{N}, 0 < m < 2n, r \in \mathbb{R}$.

- Using Theorem (2), we obtain

$$\begin{aligned}\frac{1}{i}\left(f\left(\alpha+\beta e^{i\theta x}\right)-f\left(\alpha+\beta e^{-i\theta x}\right)\right)&=\frac{1}{i}\left(e^{\alpha+\beta e^{i\theta x}}-e^{\alpha+\beta e^{-i\theta x}}\right)\\&=2e^{\alpha+\beta\cos(\theta x)}\sin(\beta\sin(\theta x))\end{aligned}$$

Thus,

$$\begin{aligned}&\int_0^\infty \frac{2x^{m-1}\left(e^{\alpha+\beta\cos(\theta x)}\sin(\beta\sin(\theta x))\right)}{(1+x^{2n})^r}dx\\&=\frac{(-1)^{r-1}}{\Gamma(r)}\frac{\partial^{r-1}}{\partial u^{r-1}}\left((u)^{\frac{r-2n}{2n}}\frac{\pi}{n}\left(\sum_{s=1}^{n}\left(\frac{1}{2}\cos(m\omega)\left(e^{\alpha+\beta e^{-\theta\sqrt[2n]{u}\sin(\omega)+i\theta\sqrt[2n]{u}\cos(\omega)}}+e^{\alpha+\beta e^{-\theta\sqrt[2n]{u}\sin(\omega)-i\theta\sqrt[2n]{u}\cos(\omega)}}-2e^{\alpha}\right)\right)\right)\right.\\&\left.+\sum_{s=1}^{n}\frac{1}{2i}\sin(m\omega)\left(e^{\alpha+\beta e^{-\theta\sqrt[2n]{u}\sin(\omega)+i\theta\sqrt[2n]{u}\cos(\omega)}}-e^{\alpha+\beta e^{-\theta\sqrt[2n]{u}\sin(\omega)-i\theta\sqrt[2n]{u}\cos(\omega)}}\right)\right)\Bigg|_{u=1}\end{aligned}$$

where $\theta > 0$, m is even, $n, m \in \mathbb{N}$, $0 < m < 2n$, $r \in \mathbb{R}$.

- Setting $\alpha = 0$, $\beta = 1, n = 1, r = 1, \theta = 1$, $m = 2$, we obtain

$$\int_0^\infty \frac{2xe^{\cos(x)}\sin(\sin(x))}{x^2+1}dx=\pi\left(e^{\frac{1}{e}}-1\right).$$

where this example was discussed in [6], and the author solved this example using the residue theorem and elucidated the analytical aspects of this example, knowing that Cauchy had put this question and made a mistake in his first attempt to solve it. However, by using the theorems, this question is solved using very simple steps, as mentioned above.

3. Setting $f(z) = \text{sinh}z$
   - Using Theorem (1), we obtain

$$\begin{aligned}f\left(\alpha+\beta e^{i\theta x}\right)+f\left(\alpha+\beta e^{-i\theta x}\right)&=\sinh\left(\alpha+\beta e^{i\theta x}\right)+\sinh\left(\alpha+\beta e^{-i\theta x}\right)\\&=2\cos(\beta\sin(\theta x))\sinh(\alpha+\beta\cos(\theta x)).\end{aligned}$$

Thus,

$$\int_0^\infty \frac{2x^{m-1}\cos(\beta\sin(\theta x))\sinh(\alpha+\beta\cos(\theta x))}{(x^{2n}+1)^r}dx$$
$$= \frac{(-1)^r}{\Gamma(r)}\frac{\partial^{r-1}}{\partial u^{r-1}}\left((u)^{\frac{r-2n}{2n}}\frac{\pi}{n}\left(\sum_{s=1}^{n}\left(\frac{\sin(m\omega)}{2}\left(\sinh\left(\alpha+\beta e^{-\theta\sqrt[2n]{u}\sin(\omega)+i\theta\sqrt[2n]{u}\cos(\omega)}\right)\right)\right)\right)\right.$$
$$\left.+\sum_{s=1}^{n}\frac{\cos(m\omega)}{2i}\left(\sinh\left(\alpha+\beta e^{-\theta\sqrt[2n]{u}\sin(\omega)-i\theta\sqrt[2n]{u}\cos(\omega)}\right)\right)\right)\Bigg|_{u=1},$$

where $\theta > 0$, m is odd, $n, m \in \mathbb{N}$, $0 < m < 2n$, $r \in \mathbb{R}$.

- Using Theorem (2), we obtain

$$\frac{1}{i}\left(f\left(\alpha+\beta e^{i\theta x}\right)-f\left(\alpha+\beta e^{-i\theta x}\right)\right)=\frac{1}{i}\left(\sinh\left(\alpha+\beta e^{i\theta x}\right)-\sinh\left(\alpha+\beta e^{-i\theta x}\right)\right)$$
$$=2\sin(\beta\sin(\theta x))\cosh(\alpha+\beta\cos(\theta x)).$$

Thus,

$$\int_0^\infty \frac{2x^{m-1}\sin(\beta\sin(\theta x))\cosh(\alpha+\beta\cos(\theta x))}{(1+x^{2n})^r}dx$$
$$= \frac{(-1)^r}{\Gamma(r)}\frac{\partial^{r-1}}{\partial u^{r-1}}\left((u)^{\frac{r-2n}{2n}}\frac{\pi}{n}\left(\sum_{s=1}^{n}\left(\frac{\cos(m\omega)}{2}\left(\sinh\left(\alpha+\beta e^{-\theta\sqrt[2n]{u}\sin(\omega)+i\theta\sqrt[2n]{u}\cos(\omega)}\right)\right)\right)\right)\right.$$
$$\left.+\sum_{s=1}^{n}\frac{\sin(m\omega)}{2i}\left(\sinh\left(\alpha+\beta e^{-\theta\sqrt[2n]{u}\sin(\omega)-i\theta\sqrt[2n]{u}\cos(\omega)}\right)\right)\right)\Bigg|_{u=1},$$

where $\theta > 0$, m is even, $n, m \in \mathbb{N}$, $0 < m < 2n$, $r \in \mathbb{R}$.

4. Setting $f(z) = \cos(e^z)$
   - Using Theorem 1, we obtain

$$f\left(\alpha+\beta e^{i\theta x}\right)+f\left(\alpha+\beta e^{-i\theta x}\right)=\cos\left(e^{\alpha+\beta e^{i\theta x}}\right)+\cos\left(e^{\alpha+\beta e^{-i\theta x}}\right)$$
$$=2\cos\left(e^{\alpha+\beta\cos(\theta x)}\cos(\beta\sin(\theta x))\right)\cosh\left(\sin(\beta\sin(\theta x))e^{\alpha+\beta\cos(\theta x)}\right).$$

Thus,

$$\int_0^\infty \frac{2x^{m-1}\cos\left(e^{\alpha+\beta\cos(\theta x)}\cos(\beta\sin(\theta x))\right)\cosh\left(\sin(\beta\sin(\theta x))e^{\alpha+\beta\cos(\theta x)}\right)}{(1+x^{2n})^r}dx$$
$$= \frac{(-1)^r}{\Gamma(r)}\frac{\partial^{r-1}}{\partial u^{r-1}}\left((u)^{\frac{r-2n}{2n}}\frac{\pi}{n}\left(\sum_{s=1}^{n}\frac{\sin(m\omega)}{2}\left(\cos\left(e^{(\alpha+\beta e^{-\theta\sqrt[2n]{u}\sin(\omega)+i\theta\sqrt[2n]{u}\cos(\omega)})}\right)\right)\right)\right.$$
$$\left.+\sum_{s=1}^{n}\frac{\cos(m\omega)}{2i}\left(\cos\left(e^{\alpha+\beta e^{-\theta\sqrt[2n]{u}\sin(\omega)-i\theta\sqrt[2n]{u}\cos(\omega)}}\right)\right)\right)\Bigg|_{u=1}.$$

5. Setting $f(z) = \ln(1+z)$
   - Using Theorem 1, we obtain

$$f\left(1+\alpha+\beta e^{i\theta x}\right)+f\left(1+\alpha+\beta e^{-i\theta x}\right)$$
$$=\ln\left(1+\alpha+\beta e^{i\theta x}\right)+\ln\left(1+\alpha+\beta e^{-i\theta x}\right)$$
$$=\ln\left((\alpha+1)^2+\beta^2+2(\alpha+1)\beta\cos(\theta x)\right).$$

Thus,

$$\int_0^\infty \frac{x^{m-1}\ \ln\left((\alpha+1)^2+\beta^2+2(\alpha+1)\beta\cos(\theta x)\right)}{(x^{2n}+1)^r}dx$$
$$= \frac{(-1)^r}{\Gamma(r)}\frac{\partial^{r-1}}{\partial u^{r-1}}\left((u)^{\frac{r-2n}{2n}}\frac{\pi}{n}\left(\sum_{s=1}^{n}\frac{\sin(m\omega)}{2}\left(\ln\left(1+\alpha+\beta e^{-\theta\sqrt[2n]{u}\sin(\omega)+i\theta\sqrt[2n]{u}\cos(\omega)}\right)\right)\right)\right.$$
$$\left.+\sum_{s=1}^{n}\frac{\cos(m\omega)}{2i}\left(\ln\left(1+\alpha+\beta e^{-\theta\sqrt[2n]{u}\sin(\omega)-i\theta\sqrt[2n]{u}\cos(\omega)}\right)\right)\right)\Bigg|_{u=1}.$$

- Setting $\alpha = 0$, $\beta = 1$, we obtain

$$f\left(e^{i\theta x}\right) + f\left(e^{-i\theta x}\right) = \ln\left(1 + e^{i\theta x}\right) + \ln\left(1 + e^{-i\theta x}\right) = 2\ln\left|2\cos\left(\frac{\theta x}{2}\right)\right|$$

Thus,

$$\int_0^\infty \frac{2x^{m-1}\ln\left|2\cos\left(\frac{\theta x}{2}\right)\right|}{(x^{2n}+1)^r}dx$$
$$= \frac{(-1)^r}{\Gamma(r)}\frac{\partial^{r-1}}{\partial u^{r-1}}\left((u)^{\frac{r-2n}{2n}}\frac{\pi}{n}\left(\sum_{s=1}^{n}\left(\frac{\sin(m\omega)}{2}\left(\ln\left(1 + e^{-\theta\sqrt[2n]{u}\sin(\omega) + i\theta\sqrt[2n]{u}\cos(\omega)}\right)\right)\right)\right)\right.$$
$$\left.+\sum_{s=1}^{n}\frac{\cos(m\omega)}{2i}\left(\ln\left(1 + e^{-\theta\sqrt[2n]{u}\sin(\omega) - i\theta\sqrt[2n]{u}\cos(\omega)}\right)\right)\right)\Bigg|_{u=1}.$$

4.3. Solving Improper Integrals

In this section, we introduce examples of some complicated integrals that cannot be easily solved using familiar methods or that may take effort and time to be solved.

We show that using the new results in this article, the solution can be directly determined; it is worth noting that Mathematica and Maple could not solve similar examples.

Example 1. Evaluate the following integral

$$PV\int_0^\infty \frac{x^3\tan(\pi x)}{1 + x^6}dx.$$

Solution. Using Theorem 1, let $\alpha = 0$, $\beta = 1$, $r = 3$, $m = 1$, $n = 3$, and setting $f(z) = \ln(1+z)$.

Thus, we obtain

$$f\left(e^{i\theta x}\right) + f\left(e^{-i\theta x}\right) = \ln\left(1 + e^{i\theta x}\right) + \ln\left(1 + e^{-i\theta x}\right) = \ln(2\cos(\theta x) + 2) = 2\ln\left|2\cos\left(\tfrac{\theta x}{2}\right)\right|,$$
$$I(\theta) = PV\int_0^\infty \frac{2x^2\left(\ln\left|2\cos\left(\frac{\theta x}{2}\right)\right|\right)}{1+x^6}dx = \frac{\pi}{6}\sum_{s=1}^{3}\left(\left(\sin\left(\frac{(2s-1)\pi}{2}\right)\right)(\psi + \phi)\right),$$

where

$$\psi = \ln\left(1 + \left(e^{-\theta\sin\left(\frac{(2s-1)\pi}{6}\right) + i\theta\cos\left(\frac{(2s-1)\pi}{6}\right)}\right)\right),$$
$$\phi = \ln\left(1 + \left(e^{-\theta\sin\left(\frac{(2s-1)\pi}{6}\right) - i\theta\cos\left(\frac{(2s-1)\pi}{6}\right)}\right)\right)$$
$$= \frac{\pi}{6}\left(-2\ln\left(1 + \mathrm{e}^{-\theta}\right) + 2\ln\left(1 + \mathrm{e}^{-\frac{\theta}{2} - \frac{1}{2}\mathrm{i}\sqrt{3}\theta}\right)\right.$$
$$\left. + 2\ln\left(1 + \mathrm{e}^{-\frac{\theta}{2} + \frac{1}{2}\mathrm{i}\sqrt{3}\theta}\right)\right)$$

Taking the derivative for $I(\theta)$ with respect to θ, we obtain

$$\frac{\partial I}{\partial\theta} = PV\int_0^\infty \frac{-x^3\tan\left(\frac{\theta x}{2}\right)}{1 + x^6}dx.$$

Therefore,

$$PV\int_0^\infty \frac{x^3\tan(\pi x)}{1+x^6}dx$$
$$= \frac{-\pi}{6}\left(\frac{2e^{-\theta}}{1+e^{-\theta}} + \frac{2(-\frac{1}{2}-\frac{i\sqrt{3}}{2})e^{-\frac{\theta}{2}-\frac{1}{2}i\sqrt{3}\theta}}{1+e^{-\frac{\theta}{2}-\frac{1}{2}i\sqrt{3}\theta}} + \frac{2(-\frac{1}{2}+\frac{i\sqrt{3}}{2})e^{-\frac{\theta}{2}+\frac{1}{2}i\sqrt{3}\theta}}{1+e^{-\frac{\theta}{2}+\frac{1}{2}i\sqrt{3}\theta}}\right)\Bigg|_{\theta=2\pi}$$
$$= \frac{\pi}{6}\left(\frac{\sqrt{3}\sin\left(\frac{\sqrt{3}\theta}{2}\right)+\cos\left(\frac{\sqrt{3}\theta}{2}\right)\tanh\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\sqrt{3}\theta}{2}\right)+\cosh\left(\frac{\theta}{2}\right)}\right)\Bigg|_{\theta=2\pi}$$
$$= \frac{\pi\left(\sqrt{3}\sin\left(\sqrt{3}\pi\right)+\cos\left(\sqrt{3}\pi\right)\tanh(\pi)\right)}{6\left(\cos\left(\sqrt{3}\pi\right)+\cosh(\pi)\right)}.$$

Example 2. Evaluate the following integral

$$\int_0^\infty \frac{x\sin(\theta x)}{(1+x^{2n})(1+2c\ \cos(\theta x)+c^2)}dx,$$

where $n\in\mathbb{N}$, $\theta>0$, and $c>1$.

Solution. Using Theorem 2, taking $\alpha=0,\ \beta=1$, and letting $f(z)=\frac{1}{1+ce^z}$, we obtain

$$\frac{1}{i}\left(f\left(e^{i\theta x}\right)-f\left(e^{-i\theta x}\right)\right)=\frac{1}{i}\left(\frac{1}{1+ce^{i\theta x}}-\frac{1}{1+ce^{-i\theta x}}\right)=\frac{-2c\ \sin(x\theta)}{1+c^2+2c\cos(x\theta)}.$$

Therefore,

$$\int_0^\infty \frac{-2cx\ \sin(\theta x)}{(1+x^{2n})(1+2c\ \cos(\theta x)+c^2)}dx$$
$$= \frac{-\pi}{n}\sum_{s=1}^{n}\left(\frac{1}{2}(\cos(2\omega))\left(\psi+\phi-2\left(\frac{1}{1+c}\right)\right)+\frac{1}{2i}(\sin(2\omega))(\psi-\phi)\right),$$

where

$$\psi=\psi(\omega)=\frac{1}{1+ce^{e^{-\theta\sqrt[2n]{u}\sin(\omega)+i\theta y\sqrt[2n]{u}\cos(\omega)}}},$$
$$\phi=\phi(\omega)=\frac{1}{1+ce^{-\theta\sqrt[2n]{u}\sin(\omega)-i\theta\sqrt[2n]{u}\cos(\omega)}},$$
$$\omega=\omega(s)=\frac{(2s-1)\pi}{2n}.$$

$$\therefore\int_0^\infty \frac{x\sin(\theta x)}{(1+x^{2n})(1+2c\ \cos(\theta x)+c^2)}dx$$
$$= \frac{\pi}{2nc}\sum_{s=1}^{n}\left(\frac{1}{2}(\cos(2\omega))\left(\psi+\phi-2\left(\frac{1}{1+c}\right)\right)+\frac{1}{2i}(\sin(2\omega))(\psi-\phi)\right)$$

where

$$\psi=\psi(\omega)=\frac{1}{1+ce^{e^{-\theta\sqrt[2n]{u}\sin(\omega)+i\theta y\sqrt[2n]{u}\cos(\omega)}}},$$
$$\phi=\phi(\omega)=\frac{1}{1+ce^{-\theta\sqrt[2n]{u}\sin(\omega)-i\theta\sqrt[2n]{u}\cos(\omega)}},$$
$$\omega=\omega(s)=\frac{(2s-1)\pi}{2n}.$$

5. Conclusions

The main purpose of this work is to generate new formulas of improper integrals and implement them in solving problems. In this article, we introduced new master theorems of improper integrals. Tables were established to present and generate new formulas of improper integrals. Comparisons with previous results were made and introduced in tables. Finally, various applications on difficult problems were presented and solved using the theorems.

In the future, we will use these new results to solve ordinary differential equations and integral equations.

Author Contributions: Formal analysis, M.A.-G., R.S. and A.Q.; investigation, A.Q., R.S. and M.A.-G., data curation, M.A.-G., R.S. and A.Q.; methodology, A.Q., R.S. and M.A.-G.; writing—original draft, A.Q., R.S. and M.A.-G.; project administration, A.Q., R.S. and M.A.-G.; resources, R.S., M.A.-G. and A.Q.; writing—review and editing, R.S., M.A.-G. and A.Q. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Acknowledgments: The authors express their gratitude to the dear referees, who wish to remain anonymous and the editor for their helpful suggestions, which improved the final version of this paper.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

Table A1. Formulas of improper integrals.

			Conditions
1.	$\int_0^\infty \frac{x^{m-1}\cos(\theta x)}{x^{2n}+1}dx$	$\frac{\pi}{2n}\sum_{k=1}^{n} e^{-\theta\sin\left(\frac{(2k-1)\pi}{2n}\right)}\sin\left(\left(\frac{(2k-1)m\pi}{2n}\right)+\theta\cos\left(\frac{(2k-1)\pi}{2n}\right)\right)$	$\theta > 0$, m be odd, $n \in \mathbb{N}$, and $0 < m < 2n$
2.	$\int_0^\infty \frac{x^{m}\sin(\theta x)}{x^{2n}+1}dx$	$\frac{-\pi}{2n}\sum_{k=1}^{n} e^{-\theta\sin\left(\frac{(2k-1)\pi}{2n}\right)}\cos\left(\left(\frac{(2k-1)\,(m+1)\pi}{2n}\right)+\theta\cos\left(\frac{(2k-1)\pi}{2n}\right)\right)$	m is odd, $\theta > 0$, $n \in \mathbb{N}$, and $-1 < m < 2n-1$
3.	$\int_0^\infty \frac{\sin(yx)}{x\left(x^{2n}+1\right)}dx$	$\frac{\pi}{2n}\sum_{k=1}^{n} 1-\mathrm{e}^{-y\sin\left(\frac{(2k-1)\pi}{2n}\right)}\cos\left(y\cos\left(\frac{(2k-1)\pi}{2n}\right)\right)$	$n \in \mathbb{N}$, $y > 0$
4.	$PV\int_0^\infty \frac{x^{m-1}\cos(\theta x)}{1-x^{2n}}dx$	$\frac{\pi}{2n}\left(\sin(\theta)+\sum_{k=1}^{n-1} e^{-\theta\sin\left(\frac{k\pi}{n}\right)}\sin\left(\left(\frac{km\pi}{n}\right)+\theta\cos\left(\frac{k\pi}{n}\right)\right)\right)$	$\theta > 0$, m be odd, $n \in \mathbb{N}$, and $0 < m < 2n$
5.	$PV\int_0^\infty \frac{x^{m}\sin(\theta x)}{1-x^{2n}}dx$	$\frac{-\pi}{2n}\left(\cos(\theta)+\sum_{k=1}^{n-1} e^{-\theta\sin\left(\frac{k\pi}{n}\right)}\cos\left(\left(\frac{k\pi(m+1)}{n}\right)+\theta\cos\left(\frac{k\pi}{n}\right)\right)\right)$	m be odd, $\theta > 0$, and $-1 < m < 2n-1$
6.	$PV\int_0^\infty \frac{\sin(yx)}{x\left(1-x^{2n}\right)}dx$	$\frac{\pi}{2n}\left(1-\cos(y)+\sum_{k=1}^{n-1} 1-\mathrm{e}^{-y\sin\left(\frac{k\pi}{n}\right)}\cos\left(y\cos\left(\frac{k\pi}{n}\right)\right)\right)$	$y > 0$
7.	$\int_{-\infty}^\infty \frac{x^{m-1}\cos(\theta x)dx}{x^{2n}-2\cos(\varphi)x^n+1}$	$\frac{\pi}{n\sin(\varphi)}\sum_{s=0}^{n-1} e^{-\theta\sin\left(\frac{2s\pi+\varphi}{n}\right)}\cos\left(m\frac{2s\pi+\varphi}{n}+\left(\theta\cos\left(\frac{2s\pi+\varphi}{n}\right)\right)-\varphi\right)$	$\theta > 0$, $\lvert\varphi\rvert < \pi$, $\varphi \neq 0$, $m < 2n+1$, $n \in \mathbb{N}$, and m is odd
8.	$\int_{-\infty}^\infty \frac{x^{m}\sin(\theta x)dx}{x^{2n}-2\cos(\varphi)x^n+1}$	$\frac{-\pi}{n\sin(\varphi)}\sum_{s=0}^{n-1} e^{-\theta\sin\left(\frac{2s\pi+\varphi}{n}\right)}\left(\sin\left(\varphi-(m+1)\left(\frac{2s\pi+\varphi}{n}\right)-\left(\theta\cos\left(\frac{2s\pi+\varphi}{n}\right)\right)\right)\right)$	$\theta > 0$, $\lvert\varphi\rvert < \pi$, $\varphi \neq 0$, $m < 2n+1$, m is odd, and $n \in \mathbb{N}$
9.	$\int_{-\infty}^\infty \frac{\sin(xy)}{x\left(x^{2n}-2\cos(\varphi)x^n+1\right)}dx$	$\frac{\pi}{n\sin(\varphi)}\sum_{s=0}^{n-1}\sin(\varphi)-\mathrm{e}^{-y\sin\left(\frac{2\pi s+\varphi}{n}\right)}\sin\left(\varphi-y\cos\left(\frac{2\pi s+\varphi}{n}\right)\right)$	$y > 0$

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