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Further Inequalities for the Weighted Numerical Radius of Operators

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Abstract: This paper deals with the so-called A-numerical radius associated with a positive (semi-definite) bounded linear operator A acting on a complex Hilbert space $\mathcal{H}$. Several new inequalities involving this concept are established. In particular, we prove several estimates for 2×2 operator matrices whose entries are A-bounded operators. Some of the obtained results cover and extend well-known recent results due to Bani-Domi and Kittaneh. In addition, several improvements of the generalized Kittaneh estimates are obtained. The inequalities given by Feki in his work represent a generalization of the inequalities given by Kittaneh. Some refinements of the inequalities due to Feki are also presented.

Keywords: positive operator; A-numerical radius; 2×2-operator matrix; A-adjoint operator

MSC: 46C05; 47B65; 47A05; 47A12

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1. Introduction

Along this work $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ denotes a complex Hilbert space with associated norm $\| \cdot \|$. Let $\mathbb{B}(\mathcal{H})$ be the algebra of all bounded linear operators acting on $\mathcal{H}$. An operator $T \in \mathbb{B}(\mathcal{H})$ is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x \rangle \geq 0$ for all $x \in \mathcal{H}$. For every operator $T \in \mathbb{B}(\mathcal{H})$, its range is denoted by $\mathcal{R}(T)$, its null space by $\mathcal{N}(T)$, and its adjoint by T^*. By $\overline{\mathcal{R}(T)}$, we mean the closure of $\mathcal{R}(T)$ with respect to the norm topology of $\mathcal{H}$. Throughout this paper, we retain the notation A for a nonzero positive operator on $\mathcal{H}$ which clearly induces the following positive semidefinite sesquilinear form

$$\langle \cdot, \cdot \rangle_A : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{C}, \ (x, y) \longmapsto \langle x, y \rangle_A := \langle Ax, y \rangle.$$

The seminorm on $\mathcal{H}$ induced by $\langle \cdot, \cdot \rangle_A$ is given by $\|x\|_A = \sqrt{\langle x, x \rangle_A}$, for every $x \in \mathcal{H}$. We remark that $\| \cdot \|_A$ is a norm on $\mathcal{H}$ if and only if A is an injective operator, i.e., $\mathcal{N}(A) = \{0_{\mathcal{H}}\}$. In addition, the semi-Hilbert space $(\mathcal{H}, \| \cdot \|_A)$ is complete if, and only if, $\mathcal{R}(A)$ is closed in $\mathcal{H}$. Next, when we use an operator, it means that it is an operator in $\mathbb{B}(\mathcal{H})$. For recent contributions related to operators acting on the A-weighted space $(\mathcal{H}, \| \cdot \|_A)$, the readers may consult [1–3]. Before we proceed further, we recall that $\langle \cdot, \cdot \rangle_A$ induces on the quotient $\mathcal{H}/\mathcal{N}(A)$ an inner product which is not complete unless $\mathcal{R}(A)$ is closed in $\mathcal{H}$. On the other hand, it was proved in [4] (see also [5]) that the completion of $\mathcal{H}/\mathcal{N}(A)$ is isometrically isomorphic to the Hilbert space $\mathcal{R}(A^{1/2})$ endowed with the following inner product

$$\langle A^{1/2}x, A^{1/2}y\rangle_{\mathbf{R}(A^{1/2})} := \langle P_{\overline{\mathcal{R}(A)}}x, P_{\overline{\mathcal{R}(A)}}y\rangle, \quad \forall\, x, y \in \mathcal{H}, \tag{1}$$

where $P_{\overline{\mathcal{R}(A)}}$ stands for the orthogonal projection onto $\overline{\mathcal{R}(A)}$. Notice that the Hilbert space $\left(\mathcal{R}(A^{1/2}), \langle\cdot,\cdot\rangle_{\mathbf{R}(A^{1/2})}\right)$ will be simply denoted by $\mathbf{R}(A^{1/2})$. Let us emphasize that $\mathcal{R}(A)$ is dense in $\mathbf{R}(A^{1/2})$ (see [6]). More results related involving the Hilbert space $\mathbf{R}(A^{1/2})$ can be found in [6] and the references therein. An application of (1) gives

$$\langle Ax, Ay\rangle_{\mathbf{R}(A^{1/2})} = \langle x, y\rangle_A, \quad \forall\, x, y \in \mathcal{H}. \tag{2}$$

The numerical range and the numerical radius of $T \in \mathbb{B}(\mathcal{H})$ are defined by $W(T) = \{\langle Tx, x\rangle\,;\; x \in \mathcal{H}, \|x\| = 1\}$, and $\omega(T) = \sup\{|\xi|\,;\; \xi \in W(T)\}$, respectively. It is well known that the numerical radius of Hilbert space operators plays an important role in various fields of operator theory and matrix analysis (cf. [7–10]). Recently, several generalizations for the concept of $\omega(\cdot)$ have been introduced (cf. [11–13]). One of these generalizations is the so-called A-numerical radius of an operator $T \in \mathbb{B}(\mathcal{H})$, which was firstly defined by Saddi in [14] as

$$\omega_A(T) = \sup\left\{\left|\langle Tx, x\rangle_A\right|\,;\; x \in \mathcal{H},\, \|x\|_A = 1\right\}. \tag{3}$$

For an account of the recent results related to the A-numerical radius, we refer the reader to [15–19] and the references therein. If $\mathbb{T} = T_{ij}$ is a 2×2-operator matrix with $T_{ij} \in \mathbb{B}(\mathcal{H})$ for all $i, j \in \{1, 2\}$, then (3) can be written as:

$$\omega_{\mathbb{A}}(\mathbb{T}) = \sup\left\{\left|\langle \mathbb{T}\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}}\right|\,;\; \mathbf{x} \in \mathcal{H} \oplus \mathcal{H},\, \|x\|_{\mathbb{A}} = 1\right\}.$$

In addition, Zamani defined in [20] the notion of A-Crawford number of an operator T as follows:

$$c_A(T) = \inf\left\{|\langle Tx, x\rangle_A|\,;\; x \in \mathcal{H},\, \|x\|_A = 1\right\}.$$

Zamani used this notion in [20] in order to derive some improvements of inequalities related to $\omega_A(\cdot)$.

Before continuing, let us recall from [21] the concept of A-adjoint operator. For $T \in \mathbb{B}(\mathcal{H})$, an operator $S \in \mathbb{B}(\mathcal{H})$ is said to be an A-adjoint operator of T if $\langle Tx, y\rangle_A = \langle x, Sy\rangle_A$ for all $x, y \in \mathcal{H}$; that is, S is the solution in $\mathbb{B}(\mathcal{H})$ of the equation $AX = T^*A$. This kind of operator equations may be investigated by using the well-known Douglas theorem [22]. Briefly, this theorem says that equation $TX = S$ has a solution $X \in \mathbb{B}(\mathcal{H})$ if and only if $\mathcal{R}(S) \subseteq \mathcal{R}(T)$. This, in turn, equivalent to the existence of some positive constant λ such that $\|S^*x\| \leq \lambda\|T^*x\|$ for all $x \in \mathcal{H}$. Furthermore, among its many solutions, there is only one, denoted by Q, which satisfies $\mathcal{R}(Q) \subseteq \overline{\mathcal{R}(T^*)}$. Such Q is called the reduced solution of $TX = S$. Let $\mathbb{B}_A(\mathcal{H})$ and $\mathbb{B}_{A^{1/2}}(\mathcal{H})$ denote the sets of all operators that admit A-adjoints and $A^{1/2}$-adjoints, respectively. An application of Douglas theorem shows that

$$\mathbb{B}_A(\mathcal{H}) = \left\{T \in \mathbb{B}(\mathcal{H})\,;\; \mathcal{R}(T^*A) \subseteq \mathcal{R}(A)\right\},$$

and

$$\mathbb{B}_{A^{1/2}}(\mathcal{H}) = \left\{T \in \mathbb{B}(\mathcal{H})\,;\; \exists\, c > 0\,;\; \|Tx\|_A \leq c\|x\|_A,\; \forall x \in \mathcal{H}\right\}.$$

We remark that $\mathbb{B}_A(\mathcal{H})$ and $\mathbb{B}_{A^{1/2}}(\mathcal{H})$ are two subalgebras of $\mathbb{B}(\mathcal{H})$ which are neither closed nor dense in $\mathbb{B}(\mathcal{H})$. It is easy to see that the following property is satisfied: $\mathbb{B}_A(\mathcal{H}) \subseteq \mathbb{B}_{A^{1/2}}(\mathcal{H})$ (see [18]). The operators from $\mathbb{B}_{A^{1/2}}(\mathcal{H})$ are called A-bounded. If $T \in \mathbb{B}_A(\mathcal{H})$, then the "reduced" solution of the equation $AX = T^*A$ is a distinguished A-adjoint operator of T, which will be denoted by $T^{\sharp_A}$. Observe that $T^{\sharp_A} = A^{\dagger}T^*A$. Here, $A^{\dagger}$ denotes the Moore–Penrose inverse of the operator A. Notice that if $T \in \mathbb{B}_A(\mathcal{H})$, then $T^{\sharp_A} \in \mathbb{B}_A(\mathcal{H})$, $(T^{\sharp_A})^{\sharp_A} = P_{\overline{\mathcal{R}(A)}} T P_{\overline{\mathcal{R}(A)}}$ and $\left((T^{\sharp_A})^{\sharp_A}\right)^{\sharp_A} = T^{\sharp_A}$. Further results involving the operator $T^{\sharp_A}$ and the theory of the Moore–Penrose inverse of Hilbert space operators can be found in [2,21,23] and the references therein. We equip $\mathbb{B}_{A^{1/2}}(\mathcal{H})$ with the following seminorm

$$\|T\|_A := \sup_{\substack{x \in \overline{\mathcal{R}(A)},\\ x \neq 0}} \frac{\|Tx\|_A}{\|x\|_A} = \sup\left\{\|Tx\|_A\,;\ x \in \mathcal{H}, \|x\|_A = 1\right\} < +\infty \tag{4}$$

(see [18] and the references therein). Let us emphasize here that it may happen that $\|T\|_A = +\infty$ for some $T \in \mathbb{B}(\mathcal{H})$ (see [18]). It is pertinent to point out that $\|T\|_A = 0$ if, and only if, $AT = 0$. It can be observed that for $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, $\|T\|_A = 0$ if and only if $AT = 0$. Moreover, it is not difficult to see that for $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, it holds that $\|Tx\|_A \leq \|T\|_A\|x\|_A$ for all $x \in \mathcal{H}$. This yields that

$$\|TS\|_A \leq \|T\|_A\|S\|_A, \quad \forall\, T, S \in \mathbb{B}_{A^{1/2}}(\mathcal{H}). \tag{5}$$

An operator $T \in \mathbb{B}(\mathcal{H})$ is called A-selfadjoint if $AT = T^*A$. Moreover, it was proved in [18] that if T is A-selfadjoint, then

$$\|T\|_A = \omega_A(T). \tag{6}$$

In addition, Baklouti et al. showed in [24] that for an A-selfadjoint operator T, it holds

$$\|T^n\|_A = \|T\|_A^n, \quad \forall\, n \in \mathbb{N}^*, \tag{7}$$

where $\mathbb{N}^*$ denotes the set all positive integers. Furthermore, if $AT \geq 0$, then T is called A-positive and we write $T \geq_A 0$. Baklouti et al. [15] obtained the following A-numerical radius inequality for $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$:

$$\frac{1}{2}\|T\|_A \leq \omega_A(T) \leq \|T\|_A. \tag{8}$$

The first inequality in (8) becomes an equality if $AT^2 = 0$ and the second inequality becomes an equality if T is A-selfadjoint (see [18]). Several authors improved recently the inequalities (8) (see, e.g., [16,25] and the references therein). In particular, the second author of this paper proved in [26] Theorem 2.5 that

$$\frac{1}{4}\|T^{\sharp_A}T + TT^{\sharp_A}\|_A \leq \omega_A^2(T) \leq \frac{1}{2}\|T^{\sharp_A}T + TT^{\sharp_A}\|_A, \quad \forall\, T \in \mathbb{B}_A(\mathcal{H}). \tag{9}$$

Obviously, for $A = I$, we obtain the well-known inequalities due to Kittaneh (see [27] Theorem 1).

The main objective of the present paper is to present a few new $\mathbb{A}$-numerical radius inequalities for 2×2 operator matrices. In Theorem 1, we obtain a bound for the $\mathbb{A}$-numerical radius for the 2×2 operator matrix. By particularization, we deduce an improvement of the second inequality (9). Another bound for $\mathbb{A}$-numerical radius for the 2×2 operator matrix is given in Theorem 2. Next, we present an improvement of the Cauchy–Schwarz inequality type using the inner product $\langle \cdot, \cdot \rangle_{\mathbb{A}}$. This result is used to find a new bound for $\mathbb{A}$-numerical radius of operator matrix $\mathbb{S}^{\sharp_{\mathbb{A}}}\mathbb{T}$. Applying the Bohr inequality, we deduce another new bound for the $\mathbb{A}$-numerical radius for the 2×2 operator matrix. In addition to these, we aim to establish an alternative and easy proof of the generalized Kittaneh inequalities (9). In addition, several improvements of the first inequality in (9) are established.

2. Main Results

To establish our first main result in the present work, we require the following two lemmas.

Lemma 1 ([28])**.** *Let $P, Q, R, S \in \mathbb{B}_A(\mathcal{H})$. Then, the following assertions hold*

(i) $\omega_{\mathbb{A}}\left[\begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix}\right] = \max\{\omega_A(P), \omega_A(S)\}$.

(ii) $\left\|\begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix}\right\|_{\mathbb{A}} = \max\{\|P\|_A, \|S\|_A\}$.

(iii) $\begin{pmatrix} P & Q \\ R & S \end{pmatrix}^{\sharp_{\mathbb{A}}} = \begin{pmatrix} P^{\sharp_A} & R^{\sharp_A} \\ Q^{\sharp_A} & S^{\sharp_A} \end{pmatrix}$.

Lemma 2 ([14])**.** *Let $x, y, e \in \mathcal{H}$ be such that $\|e\|_A = 1$. Then*

$$2|\langle x, e\rangle_A \langle e, y\rangle_A| \leq |\langle x, y\rangle_A| + \|x\|_A \|y\|_A. \tag{10}$$

Now, we can prove the following result, which generalizes Theorem 2.1 in [29].

Theorem 1. *Let $P, Q, R, S \in \mathbb{B}_A(\mathcal{H})$. Then,*

$$\omega_{\mathbb{A}}^2\left[\begin{pmatrix} P & Q \\ R & S \end{pmatrix}\right] \leq \max\left\{\omega_A^2(P), \omega_A^2(S)\right\} + \omega_{\mathbb{A}}^2\left[\begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix}\right] + \omega_{\mathbb{A}}^2\left[\begin{pmatrix} 0 & QS \\ RP & 0 \end{pmatrix}\right]$$
$$+ \frac{1}{2}\max\left\{\|P^{\sharp_A}P + QQ^{\sharp_A}\|_A, \|S^{\sharp_A}S + RR^{\sharp_A}\|_A\right\}.$$

Proof. Let $\mathbf{x} \in \mathcal{H} \oplus \mathcal{H}$ be such that $\|\mathbf{x}\|_{\mathbb{A}} = 1$. One has

$$\begin{aligned}
\left|\left\langle \begin{pmatrix} P & Q \\ R & S \end{pmatrix}\mathbf{x}, \mathbf{x}\right\rangle_{\mathbb{A}}\right|^2 &= \left|\left\langle \begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix}\mathbf{x}, \mathbf{x}\right\rangle_{\mathbb{A}} + \left\langle \begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix}\mathbf{x}, \mathbf{x}\right\rangle_{\mathbb{A}}\right|^2 \\
&\leq \left(\left|\left\langle \begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix}\mathbf{x}, \mathbf{x}\right\rangle_{\mathbb{A}}\right| + \left|\left\langle \begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix}\mathbf{x}, \mathbf{x}\right\rangle_{\mathbb{A}}\right|\right)^2 \\
&= \left|\left\langle \begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix}\mathbf{x}, \mathbf{x}\right\rangle_{\mathbb{A}}\right|^2 + \left|\left\langle \begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix}\mathbf{x}, \mathbf{x}\right\rangle_{\mathbb{A}}\right|^2 \\
&\quad + 2\left|\left\langle \begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix}\mathbf{x}, \mathbf{x}\right\rangle_{\mathbb{A}}\right| \left|\left\langle \begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix}\mathbf{x}, \mathbf{x}\right\rangle_{\mathbb{A}}\right| \\
&\leq \omega_{\mathbb{A}}^2\left[\begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix}\right] + \omega_{\mathbb{A}}^2\left[\begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix}\right] \\
&\quad + 2\left|\left\langle \begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix}\mathbf{x}, \mathbf{x}\right\rangle_{\mathbb{A}}\right| \left|\left\langle \begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix}\mathbf{x}, \mathbf{x}\right\rangle_{\mathbb{A}}\right| \\
&= \max\left\{\omega_A^2(P), \omega_A^2(S)\right\} + \omega_{\mathbb{A}}^2\left[\begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix}\right] \\
&\quad + 2\left|\left\langle \begin{pmatrix} (P^{\sharp_A})^{\sharp_A} & 0 \\ 0 & (S^{\sharp_A})^{\sharp_A} \end{pmatrix}\mathbf{x}, \mathbf{x}\right\rangle_{\mathbb{A}}\right| \left|\left\langle \mathbf{x}, \begin{pmatrix} 0 & R^{\sharp_A} \\ Q^{\sharp_A} & 0 \end{pmatrix}\mathbf{x}\right\rangle_{\mathbb{A}}\right|,
\end{aligned}$$

where the last equality follows by using Lemma 1 (i) and (iii). So, by applying (10) together with the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
&\left|\left\langle\begin{pmatrix} P & Q \\ R & S \end{pmatrix}\mathbf{x},\mathbf{x}\right\rangle_{\mathbb{A}}\right|^2 \\
&\le \max\left\{\omega_A^2(P),\omega_A^2(S)\right\} + \left|\left\langle\begin{pmatrix} (P^{\sharp_A})^{\sharp_A} & 0 \\ 0 & (S^{\sharp_A})^{\sharp_A} \end{pmatrix}\mathbf{x},\begin{pmatrix} 0 & R^{\sharp_A} \\ Q^{\sharp_A} & 0 \end{pmatrix}\mathbf{x}\right\rangle_{\mathbb{A}}\right| \\
&+ \omega_{\mathbb{A}}^2\left[\begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix}\right] + \left\|\begin{pmatrix} (P^{\sharp_A})^{\sharp_A} & 0 \\ 0 & (S^{\sharp_A})^{\sharp_A} \end{pmatrix}\mathbf{x}\right\|_{\mathbb{A}}\left\|\begin{pmatrix} 0 & R^{\sharp_A} \\ Q^{\sharp_A} & 0 \end{pmatrix}\mathbf{x}\right\|_{\mathbb{A}} \\
&\le \max\left\{\omega_A^2(P),\omega_A^2(S)\right\} + \left|\left\langle\begin{pmatrix} 0 & (Q^{\sharp_A})^{\sharp_A} \\ (R^{\sharp_A})^{\sharp_A} & 0 \end{pmatrix}\begin{pmatrix} (P^{\sharp_A})^{\sharp_A} & 0 \\ 0 & (S^{\sharp_A})^{\sharp_A} \end{pmatrix}\mathbf{x},\mathbf{x}\right\rangle_{\mathbb{A}}\right| \\
&+ \omega_{\mathbb{A}}^2\left[\begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix}\right] + \left\|\begin{pmatrix} (P^{\sharp_A})^{\sharp_A} & 0 \\ 0 & (S^{\sharp_A})^{\sharp_A} \end{pmatrix}\mathbf{x}\right\|_{\mathbb{A}}\left\|\begin{pmatrix} 0 & R^{\sharp_A} \\ Q^{\sharp_A} & 0 \end{pmatrix}\mathbf{x}\right\|_{\mathbb{A}} \\
&= \max\left\{\omega_A^2(P),\omega_A^2(S)\right\} + \omega_{\mathbb{A}}^2\left[\begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix}\right] + \left|\left\langle\begin{pmatrix} 0 & (Q^{\sharp_A})^{\sharp_A}(S^{\sharp_A})^{\sharp_A} \\ (R^{\sharp_A})^{\sharp_A}(P^{\sharp_A})^{\sharp_A} & 0 \end{pmatrix}\mathbf{x},\mathbf{x}\right\rangle_{\mathbb{A}}\right| \\
&+ \sqrt{\left\langle\begin{pmatrix} P^{\sharp_A} & 0 \\ 0 & S^{\sharp_A} \end{pmatrix}\begin{pmatrix} (P^{\sharp_A})^{\sharp_A} & 0 \\ 0 & (S^{\sharp_A})^{\sharp_A} \end{pmatrix}\mathbf{x},\mathbf{x}\right\rangle_{\mathbb{A}}\left\langle\begin{pmatrix} 0 & (Q^{\sharp_A})^{\sharp_A} \\ (R^{\sharp_A})^{\sharp_A} & 0 \end{pmatrix}\begin{pmatrix} 0 & R^{\sharp_A} \\ Q^{\sharp_A} & 0 \end{pmatrix}\mathbf{x},\mathbf{x}\right\rangle_{\mathbb{A}}},
\end{aligned}$$

where the last equality follows by using Lemma 1 (iii). Furthermore, by taking again Lemma 1 (iii) into consideration and the fact that $\omega_{\mathbb{A}}(\mathbb{X}) = \omega_{\mathbb{A}}(\mathbb{X}^{\sharp_{\mathbb{A}}})$ for all 2×2 operator matrix $\mathbb{X}\in\mathbb{B}_{\mathbb{A}}(\mathcal{H}\oplus\mathcal{H})$, we see that

$$\begin{aligned}
&\left|\left\langle\begin{pmatrix} P & Q \\ R & S \end{pmatrix}\mathbf{x},\mathbf{x}\right\rangle_{\mathbb{A}}\right|^2 \\
&\le \max\left\{\omega_A^2(P),\omega_A^2(S)\right\} + \omega_{\mathbb{A}}^2\left[\begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix}\right] + \omega_{\mathbb{A}}\left[\begin{pmatrix} 0 & (S^{\sharp_A}Q^{\sharp_A})^{\sharp_A} \\ (P^{\sharp_A}R^{\sharp_A})^{\sharp_A} & 0 \end{pmatrix}\right] \\
&+ \sqrt{\left\langle\begin{pmatrix} (P^{\sharp_A}P)^{\sharp_A} & 0 \\ 0 & (S^{\sharp_A}S)^{\sharp_A} \end{pmatrix}\mathbf{x},\mathbf{x}\right\rangle_{\mathbb{A}}\left\langle\begin{pmatrix} (QQ^{\sharp_A})^{\sharp_A} & 0 \\ 0 & (RR^{\sharp_A})^{\sharp_A} \end{pmatrix}\mathbf{x},\mathbf{x}\right\rangle_{\mathbb{A}}} \\
&= \max\left\{\omega_A^2(P),\omega_A^2(S)\right\} + \omega_{\mathbb{A}}^2\left[\begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix}\right] + \omega_{\mathbb{A}}\left[\begin{pmatrix} 0 & (RP)^{\sharp_A} \\ (QS)^{\sharp_A} & 0 \end{pmatrix}^{\sharp_{\mathbb{A}}}\right] \\
&+ \sqrt{\left\langle\begin{pmatrix} P^{\sharp_A}P & 0 \\ 0 & S^{\sharp_A}S \end{pmatrix}^{\sharp_{\mathbb{A}}}\mathbf{x},\mathbf{x}\right\rangle_{\mathbb{A}}\left\langle\begin{pmatrix} QQ^{\sharp_A} & 0 \\ 0 & RR^{\sharp_A} \end{pmatrix}^{\sharp_{\mathbb{A}}}\mathbf{x},\mathbf{x}\right\rangle_{\mathbb{A}}} \\
&= \max\left\{\omega_A^2(P),\omega_A^2(S)\right\} + \omega_{\mathbb{A}}^2\left[\begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix}\right] + \omega_{\mathbb{A}}\left[\begin{pmatrix} 0 & QS \\ RP & 0 \end{pmatrix}\right] \\
&+ \sqrt{\left\langle\begin{pmatrix} P^{\sharp_A}P & 0 \\ 0 & S^{\sharp_A}S \end{pmatrix}\mathbf{x},\mathbf{x}\right\rangle_{\mathbb{A}}\left\langle\begin{pmatrix} QQ^{\sharp_A} & 0 \\ 0 & RR^{\sharp_A} \end{pmatrix}\mathbf{x},\mathbf{x}\right\rangle_{\mathbb{A}}}.
\end{aligned}$$

By applying the arithmetic–geometric mean inequality, we obtain

$$\begin{aligned}
\left|\left\langle\begin{pmatrix} P & Q \\ R & S \end{pmatrix}\mathbf{x},\mathbf{x}\right\rangle_{\mathbb{A}}\right|^2 &\le \max\left\{\omega_A^2(P),\omega_A^2(S)\right\} + \omega_{\mathbb{A}}^2\left[\begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix}\right] + \omega_{\mathbb{A}}\left[\begin{pmatrix} 0 & QS \\ RP & 0 \end{pmatrix}\right] \\
&+ \frac{1}{2}\left(\left\langle\begin{pmatrix} P^{\sharp_A}P & 0 \\ 0 & S^{\sharp_A}S \end{pmatrix}\mathbf{x},\mathbf{x}\right\rangle_{\mathbb{A}} + \left\langle\begin{pmatrix} QQ^{\sharp_A} & 0 \\ 0 & RR^{\sharp_A} \end{pmatrix}\mathbf{x},\mathbf{x}\right\rangle_{\mathbb{A}}.\right)
\end{aligned}$$

This implies that

$$\begin{aligned}
\left|\left\langle\begin{pmatrix} P & Q \\ R & S \end{pmatrix}\mathbf{x},\mathbf{x}\right\rangle_{\mathbb{A}}\right|^2 &\le \frac{1}{2}\left\langle\begin{pmatrix} P^{\sharp_A}P + QQ^{\sharp_A} & 0 \\ 0 & S^{\sharp_A}S + RR^{\sharp_A} \end{pmatrix}\mathbf{x},\mathbf{x}\right\rangle_{\mathbb{A}} \\
&+ \max\left\{\omega_A^2(P),\omega_A^2(S)\right\} + \omega_{\mathbb{A}}^2\left[\begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix}\right] + \omega_{\mathbb{A}}\left[\begin{pmatrix} 0 & QS \\ RP & 0 \end{pmatrix}\right].
\end{aligned}$$

Furthermore, it can be verified that $\begin{pmatrix} P^{\sharp_A}P + QQ^{\sharp_A} & 0 \\ 0 & S^{\sharp_A}S + RR^{\sharp_A} \end{pmatrix}$ is $\mathbb{A}$-positive. So, by applying the Cauchy–Schwarz inequality, we obtain

$$\left|\left\langle \begin{pmatrix} P & Q \\ R & S \end{pmatrix}\mathbf{x}, \mathbf{x} \right\rangle_{\mathbb{A}}\right|^2 \leq \frac{1}{2}\left\| \begin{pmatrix} P^{\sharp_A}P + QQ^{\sharp_A} & 0 \\ 0 & S^{\sharp_A}S + RR^{\sharp_A} \end{pmatrix} \right\|_{\mathbb{A}} + \max\left\{\omega_A^2(P), \omega_A^2(S)\right\} + \omega_{\mathbb{A}}^2\left[\begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix}\right] + \omega_{\mathbb{A}}\left[\begin{pmatrix} 0 & QS \\ RP & 0 \end{pmatrix}\right].$$

Thus, an application of Lemma 1 (ii) gives

$$\left|\left\langle \begin{pmatrix} P & Q \\ R & S \end{pmatrix}\mathbf{x}, \mathbf{x} \right\rangle_{\mathbb{A}}\right|^2 \leq \frac{1}{2}\max\left\{\|P^{\sharp_A}P + QQ^{\sharp_A}\|_A, \|S^{\sharp_A}S + RR^{\sharp_A}\|_A\right\} + \max\left\{\omega_A^2(P), \omega_A^2(S)\right\} + \omega_{\mathbb{A}}^2\left[\begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix}\right] + \omega_{\mathbb{A}}\left[\begin{pmatrix} 0 & QS \\ RP & 0 \end{pmatrix}\right].$$

So, by taking the supremum over $\mathbf{x} \in \mathcal{H} \oplus \mathcal{H}$ with $\|\mathbf{x}\|_{\mathbb{A}} = 1$ in the last inequality, we obtain the desired result. □

By letting $P = Q = R = S = T$, we obtain the following corollary which considerably improves the second inequality in (9) and was already proved by the present author in [26]. Notice that this corollary is also stated by Bhunia et al. in [30] when A is an injective linear operator.

Corollary 1. *Let $T \in \mathbb{B}_A(\mathcal{H})$. Then*

$$\omega_A(T) \leq \frac{1}{2}\sqrt{\left\|TT^{\sharp_A} + T^{\sharp_A}T\right\|_A + 2\omega_A(T^2)}.$$

In order to prove our next result which generalizes Theorem 2.1 in [29], we need the following two lemmas. The first one follows immediately by using Lemma 3.1 in [29] and the second is recently proved in [31].

Lemma 3. *Let $x, y, e \in \mathcal{H}$ be such that $\|e\|_A = 1$. Then*

$$4|\langle x, e\rangle_A \langle e, y\rangle_A|^2 \leq 3\|x\|_A^2\|y\|_A^2 + \|x\|_A\|y\|_A\,|\langle x, y\rangle_A|.$$

Lemma 4. *Let $T \in \mathbb{B}(\mathcal{H})$ be such that $T \geq_A 0$. Then, for every $x \in \mathcal{H}$ with $\|x\|_A = 1$ we have*

$$\langle Tx, x\rangle_A^n \leq \langle T^n x, x\rangle_A, \quad \forall n \in \mathbb{N}^*. \tag{11}$$

Now, we are ready to prove the following theorem.

Theorem 2. *Let $\mathbb{T} = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$ be such that $P, Q, R, S \in \mathbb{B}_A(\mathcal{H})$. Then,*

$$\omega_{\mathbb{A}}^4(\mathbb{T}) \leq \max\{\omega_A(QR), \omega_A(RQ)\}\max\left\{\|R^{\sharp_A}R + QQ^{\sharp_A}\|_A, \|Q^{\sharp_A}Q + RR^{\sharp_A}\|_A\right\} + 8\max\left\{\omega_A^4(P), \omega_A^4(S)\right\} + 3\max\{\mu, \nu\},$$

where $\mu = \|(R^{\sharp_A}R)^2 + (QQ^{\sharp_A})^2\|_A$ and $\nu = \|(Q^{\sharp_A}Q)^2 + (RR^{\sharp_A})^2\|_A$.

Proof. Let $\mathbf{x} \in \mathcal{H} \oplus \mathcal{H}$ be such that $\|\mathbf{x}\|_{\mathbb{A}} = 1$. One has

$$\begin{aligned}
|\langle \mathbb{T}\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}}|^4 &= \left|\left\langle \begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix} \mathbf{x}, \mathbf{x} \right\rangle_{\mathbb{A}} + \left\langle \begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix} \mathbf{x}, \mathbf{x} \right\rangle_{\mathbb{A}}\right|^4 \\
&\leq \left(\left|\left\langle \begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix} \mathbf{x}, \mathbf{x} \right\rangle_{\mathbb{A}}\right| + \left|\left\langle \begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix} \mathbf{x}, \mathbf{x} \right\rangle_{\mathbb{A}}\right| \right)^4 \\
&= \left(\frac{2\left|\left\langle \begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix} \mathbf{x}, \mathbf{x} \right\rangle_{\mathbb{A}}\right| + 2\left|\left\langle \begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix} \mathbf{x}, \mathbf{x} \right\rangle_{\mathbb{A}}\right|}{2} \right)^4 \\
&\leq 8\left(\left|\left\langle \begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix} \mathbf{x}, \mathbf{x} \right\rangle_{\mathbb{A}}\right|^4 + \left|\left\langle \begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix} \mathbf{x}, \mathbf{x} \right\rangle_{\mathbb{A}}\right|^4 \right),
\end{aligned}$$

where the last inequality follows by applying the convexity of the function $f(t) = t^4$ with $t \geq 0$. This implies, by taking Lemma 1 (i) into consideration, that

$$\begin{aligned}
|\langle \mathbb{T}\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}}|^4 &\leq 8\omega_{\mathbb{A}}^4\left[\begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix}\right] + 8\left|\left\langle \begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix} \mathbf{x}, \mathbf{x} \right\rangle_{\mathbb{A}}\right|^4 \\
&= 8\max\{\omega_A^4(P), \omega_A^4(S)\} + 8\left|\left\langle \begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix} \mathbf{x}, \mathbf{x} \right\rangle_{\mathbb{A}}\right|^4.
\end{aligned}$$

On the other hand, let $\mathbb{S} = \begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix}$. By using Lemma 3, we obtain

$$\begin{aligned}
&|\langle \mathbb{T}\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}}|^4 \\
&\leq 8\max\{\omega_A^4(P), \omega_A^4(S)\} + 8\left|\langle \mathbb{S}\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}} \langle \mathbf{x}, \mathbb{S}^{\sharp_{\mathbb{A}}}\mathbf{x}\rangle_{\mathbb{A}}\right|^2 \\
&\leq 8\max\{\omega_A^4(P), \omega_A^4(S)\} + 2\Big(3\|\mathbb{S}\mathbf{x}\|_{\mathbb{A}}^2 \|\mathbb{S}^{\sharp_{\mathbb{A}}}\mathbf{x}\|_{\mathbb{A}}^2 + \|\mathbb{S}\mathbf{x}\|_{\mathbb{A}} \|\mathbb{S}^{\sharp_{\mathbb{A}}}\mathbf{x}\|_{\mathbb{A}} \, |\langle \mathbb{S}\mathbf{x}, \mathbb{S}^{\sharp_{\mathbb{A}}}\mathbf{x}\rangle_{\mathbb{A}}|\Big) \\
&= 8\max\{\omega_A^4(P), \omega_A^4(S)\} + 6\langle \mathbb{S}^{\sharp_{\mathbb{A}}}\mathbb{S}\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}} \langle \mathbb{S}\mathbb{S}^{\sharp_{\mathbb{A}}}\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}} \\
&\qquad + 2\sqrt{\langle \mathbb{S}^{\sharp_{\mathbb{A}}}\mathbb{S}\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}} \langle \mathbb{S}\mathbb{S}^{\sharp_{\mathbb{A}}}\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}}} \, |\langle \mathbb{S}^2\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}}| \\
&\leq 8\max\{\omega_A^4(P), \omega_A^4(S)\} + 3\Big(\langle \mathbb{S}^{\sharp_{\mathbb{A}}}\mathbb{S}\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}}^2 + \langle \mathbb{S}\mathbb{S}^{\sharp_{\mathbb{A}}}\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}}^2\Big) \\
&\qquad + \langle (\mathbb{S}^{\sharp_{\mathbb{A}}}\mathbb{S} + \mathbb{S}\mathbb{S}^{\sharp_{\mathbb{A}}})\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}} |\langle \mathbb{S}^2\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}}|,
\end{aligned}$$

where the last inequality follows by applying the arithmetic–geometric mean inequality. Now, since $\mathbb{S}^{\sharp_{\mathbb{A}}}\mathbb{S}$ and $\mathbb{S}\mathbb{S}^{\sharp_{\mathbb{A}}}$ are $\mathbb{A}$-positive, then an application of Lemma 4 with $n = 2$ gives

$$\begin{aligned}
|\langle \mathbb{T}\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}}|^4 &\leq 8\max\{\omega_A^4(P), \omega_A^4(S)\} + 3\big\langle [(\mathbb{S}^{\sharp_{\mathbb{A}}}\mathbb{S})^2 + (\mathbb{S}\mathbb{S}^{\sharp_{\mathbb{A}}})^2]\mathbf{x}, \mathbf{x}\big\rangle_{\mathbb{A}} \\
&\qquad + \langle (\mathbb{S}^{\sharp_{\mathbb{A}}}\mathbb{S} + \mathbb{S}\mathbb{S}^{\sharp_{\mathbb{A}}})\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}} |\langle \mathbb{S}^2\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}}| \\
&\leq 8\max\{\omega_A^4(P), \omega_A^4(S)\} + 3\|(\mathbb{S}^{\sharp_{\mathbb{A}}}\mathbb{S})^2 + (\mathbb{S}\mathbb{S}^{\sharp_{\mathbb{A}}})^2\|_{\mathbb{A}} \\
&\qquad + \left\|\mathbb{S}^{\sharp_{\mathbb{A}}}\mathbb{S} + \mathbb{S}\mathbb{S}^{\sharp_{\mathbb{A}}}\right\|_{\mathbb{A}} \omega_{\mathbb{A}}\left(\mathbb{S}^2\right).
\end{aligned}$$

A short calculation reveals that

$$\mathbb{S}^{\sharp_{\mathbb{A}}}\mathbb{S} + \mathbb{S}\mathbb{S}^{\sharp_{\mathbb{A}}} = \begin{pmatrix} R^{\sharp_A}R + QQ^{\sharp_A} & 0 \\ 0 & Q^{\sharp_A}Q + RR^{\sharp_A} \end{pmatrix}, \quad \mathbb{S}^2 = \begin{pmatrix} QR & 0 \\ 0 & RQ \end{pmatrix}$$

and

$$(\mathbb{S}^{\sharp_{\mathbb{A}}}\mathbb{S})^2 + (\mathbb{S}\mathbb{S}^{\sharp_{\mathbb{A}}})^2 = \begin{pmatrix} (R^{\sharp_A}R)^2 + (QQ^{\sharp_A})^2 & 0 \\ 0 & (Q^{\sharp_A}Q)^2 + (RR^{\sharp_A})^2 \end{pmatrix}.$$

Hence, by applying Lemma 1 (i) and (ii), we infer that

$$\left\|\mathbb{S}^{\sharp_{\mathbb{A}}}\mathbb{S} + \mathbb{S}\mathbb{S}^{\sharp_{\mathbb{A}}}\right\|_{\mathbb{A}} = \max\left\{\|R^{\sharp_A}R + QQ^{\sharp_A}\|_A, \|Q^{\sharp_A}Q + RR^{\sharp_A}\|_A\right\},$$

$$\omega_{\mathbb{A}}\left(\mathbb{S}^2\right) = \max\{\omega_A(QR), \omega_A(RQ)\} \text{ and } \|(\mathbb{S}^{\sharp_{\mathbb{A}}}\mathbb{S})^2 + (\mathbb{S}\mathbb{S}^{\sharp_{\mathbb{A}}})^2\|_{\mathbb{A}} = \max\{\mu, \nu\},$$

where $\mu = \|(R^{\sharp_A}R)^2 + (QQ^{\sharp_A})^2\|_A$ and $\nu = \|(Q^{\sharp_A}Q)^2 + (RR^{\sharp_A})^2\|_A$. So, we obtain

$$\begin{aligned}\left|\langle\mathbb{T}\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}}\right|^4 \le{}& \max\{\omega_A(QR), \omega_A(RQ)\}\max\left\{\|R^{\sharp_A}R + QQ^{\sharp_A}\|_A, \|Q^{\sharp_A}Q + RR^{\sharp_A}\|_A\right\} \\ &+ 8\max\left\{\omega_A^4(P), \omega_A^4(S)\right\} + 3\max\{\mu, \nu\},\end{aligned}$$

This proves the desired by letting the supremum over $\mathbf{x} \in \mathcal{H} \oplus \mathcal{H}$ be such that $\|\mathbf{x}\|_{\mathbb{A}} = 1$ in the last inequality. □

Next, we present a result which is an improvement of the inequality of Cauchy–Schwarz type,

$$|\langle\mathbf{x}, \mathbf{y}\rangle_{\mathbb{A}}| \le \|\mathbf{x}\|_{\mathbb{A}}\|\mathbf{y}\|_{\mathbb{A}}, \tag{12}$$

where $\mathbf{x}, \mathbf{y} \in \mathcal{H} \oplus \mathcal{H}$, similar to a result of [32]; thus:

Lemma 5. *Let $\lambda \in [0,1]$. Then*

$$\left|\langle\mathbf{x}, \mathbf{y}\rangle_{\mathbb{A}}\right|^2 \le \lambda\|\mathbf{x}\|_{\mathbb{A}}^2\|\mathbf{y}\|_{\mathbb{A}}^2 + (1-\lambda)\left|\langle\mathbf{x}, \mathbf{y}\rangle_{\mathbb{A}}\right|\|\mathbf{x}\|_{\mathbb{A}}\|\mathbf{y}\|_{\mathbb{A}} \le \|\mathbf{x}\|_{\mathbb{A}}^2\|\mathbf{y}\|_{\mathbb{A}}^2, \tag{13}$$

for any $\mathbf{x}, \mathbf{y} \in \mathcal{H} \oplus \mathcal{H}$.

Proof. Using inequality (12), we deduce that

$$\left|\langle\mathbf{x}, \mathbf{y}\rangle_{\mathbb{A}}\right| \le \lambda\|\mathbf{x}\|_{\mathbb{A}}\|\mathbf{y}\|_{\mathbb{A}} + (1-\lambda)\left|\langle\mathbf{x}, \mathbf{y}\rangle_{\mathbb{A}}\right| \le \|\mathbf{x}\|_{\mathbb{A}}\|\mathbf{y}\|_{\mathbb{A}},$$

for any $\mathbf{x}, \mathbf{y} \in \mathcal{H} \oplus \mathcal{H}$ and $\lambda \in [0,1]$.

Multiplying by $\|\mathbf{x}\|_{\mathbb{A}}\|\mathbf{y}\|_{\mathbb{A}}$ in the above inequality, we have that

$$\left|\langle\mathbf{x}, \mathbf{y}\rangle_{\mathbb{A}}\right|\|\mathbf{x}\|_{\mathbb{A}}\|\mathbf{y}\|_{\mathbb{A}} \le \lambda\|\mathbf{x}\|_{\mathbb{A}}^2\|\mathbf{y}\|_{\mathbb{A}}^2 + (1-\lambda)\left|\langle\mathbf{x}, \mathbf{y}\rangle_{\mathbb{A}}\right|\|\mathbf{x}\|_{\mathbb{A}}\|\mathbf{y}\|_{\mathbb{A}} \le \|\mathbf{x}\|_{\mathbb{A}}^2\|\mathbf{y}\|_{\mathbb{A}}^2,$$

and using again inequality (12), we deduce that the inequality of the statement is true. □

Remark 1. *Inequality (13) can be written as:*

$$\left|\langle\mathbf{x}, \mathbf{y}\rangle_{\mathbb{A}}\right| \le \sqrt{\lambda\|\mathbf{x}\|_{\mathbb{A}}^2\|\mathbf{y}\|_{\mathbb{A}}^2 + (1-\lambda)\left|\langle\mathbf{x}, \mathbf{y}\rangle_{\mathbb{A}}\right|\|\mathbf{x}\|_{\mathbb{A}}\|\mathbf{y}\|_{\mathbb{A}}} \le \|\mathbf{x}\|_{\mathbb{A}}\|\mathbf{y}\|_{\mathbb{A}},$$

for any $\mathbf{x}, \mathbf{y} \in \mathcal{H} \oplus \mathcal{H}$ and $\lambda \in [0,1]$.

Theorem 3. *Let $\mathbb{T}, \mathbb{S} \in \mathbb{B}_{\mathbb{A}}(\mathcal{H} \oplus \mathcal{H})$ and $\lambda \in [0,1]$. Then, the inequality*

$$\omega_{\mathbb{A}}^2(\mathbb{S}^{\sharp_{\mathbb{A}}}\mathbb{T}) \le \frac{\lambda}{2}\left\|(\mathbb{T}^{\sharp_{\mathbb{A}}}\mathbb{T})^2 + (\mathbb{S}^{\sharp_{\mathbb{A}}}\mathbb{S})^2\right\|_{\mathbb{A}} + \frac{1-\lambda}{2}\omega_{\mathbb{A}}(\mathbb{S}^{\sharp_{\mathbb{A}}}\mathbb{T})\left\|\mathbb{T}^{\sharp_{\mathbb{A}}}\mathbb{T} + \mathbb{S}^{\sharp_{\mathbb{A}}}\mathbb{S}\right\|_{\mathbb{A}} \tag{14}$$

holds.

Proof. We take the first inequality from Lemma 5:

$$\left|\langle\mathbf{x}, \mathbf{y}\rangle_{\mathbb{A}}\right|^2 \le \lambda\|\mathbf{x}\|_{\mathbb{A}}^2\|\mathbf{y}\|_{\mathbb{A}}^2 + (1-\lambda)\left|\langle\mathbf{x}, \mathbf{y}\rangle_{\mathbb{A}}\right|\|\mathbf{x}\|_{\mathbb{A}}\|\mathbf{y}\|_{\mathbb{A}},$$

for any $\mathbf{x}, \mathbf{y} \in \mathcal{H} \oplus \mathcal{H}$ and $\lambda \in [0,1]$. Because we need to apply the inequality Hölder–McCarthy for positive operators, it is easy to see that the operators $\mathbb{T}^{\sharp_{\mathbb{A}}}\mathbb{T}$ and $\mathbb{S}^{\sharp_{\mathbb{A}}}\mathbb{S}$ are positive. Now, we replace $\mathbf{x}$ and $\mathbf{y}$ by $\mathbb{T}\mathbf{x}$ and $\mathbb{S}\mathbf{x}$, in the above inequality, and we assume that $\|\mathbf{x}\|_{\mathbb{A}} = 1$; then, we obtain

$$\begin{aligned}
&\left|\langle \mathbb{S}^{\sharp_{\mathbb{A}}}\mathbb{T}\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}}\right|^2 = \left|\langle \mathbb{T}\mathbf{x}, \mathbb{S}\mathbf{x}\rangle_{\mathbb{A}}\right|^2 \\
&\leq \lambda \|\mathbb{T}\mathbf{x}\|_{\mathbb{A}}^2 \|\mathbb{S}\mathbf{x}\|_{\mathbb{A}}^2 + (1-\lambda)\left|\langle \mathbb{S}^{\sharp_{\mathbb{A}}}\mathbb{T}\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}}\right| \|\mathbb{T}\mathbf{x}\|_{\mathbb{A}} \|\mathbb{S}\mathbf{x}\|_{\mathbb{A}} \\
&\leq \lambda \langle \mathbb{T}\mathbf{x}, \mathbb{T}\mathbf{x}\rangle_{\mathbb{A}} \langle \mathbb{S}\mathbf{x}, \mathbb{S}\mathbf{x}\rangle_{\mathbb{A}} + (1-\lambda)\left|\langle \mathbb{S}^{\sharp_{\mathbb{A}}}\mathbb{T}\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}}\right| \sqrt{\langle \mathbb{T}\mathbf{x}, \mathbb{T}\mathbf{x}\rangle_{\mathbb{A}} \langle \mathbb{S}\mathbf{x}, \mathbb{S}\mathbf{x}\rangle}_{\mathbb{A}} \\
&= \lambda \langle \mathbb{T}^{\sharp_{\mathbb{A}}}\mathbb{T}\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}} \langle \mathbb{S}^{\sharp_{\mathbb{A}}}\mathbb{S}\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}} + (1-\lambda)\left|\langle \mathbb{S}^{\sharp_{\mathbb{A}}}\mathbb{T}\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}}\right| \sqrt{\langle \mathbb{T}^{\sharp_{\mathbb{A}}}\mathbb{T}\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}} \langle \mathbb{S}^{\sharp_{\mathbb{A}}}\mathbb{S}\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}}} \\
&\leq \frac{\lambda}{4}\left(\langle \mathbb{T}^{\sharp_{\mathbb{A}}}\mathbb{T}\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}} + \langle \mathbb{S}^{\sharp_{\mathbb{A}}}\mathbb{S}\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}}\right)^2 + \frac{1-\lambda}{2}\left|\langle \mathbb{S}^{\sharp_{\mathbb{A}}}\mathbb{T}\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}}\right|\left(\langle \mathbb{T}^{\sharp_{\mathbb{A}}}\mathbb{T}\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}} + \langle \mathbb{S}^{\sharp_{\mathbb{A}}}\mathbb{S}\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}}\right) \\
&\leq \frac{\lambda}{2}\left(\langle \mathbb{T}^{\sharp_{\mathbb{A}}}\mathbb{T}\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}}^2 + \langle \mathbb{S}^{\sharp_{\mathbb{A}}}\mathbb{S}\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}}^2\right) + \frac{1-\lambda}{2}\left|\langle \mathbb{S}^{\sharp_{\mathbb{A}}}\mathbb{T}\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}}\right| \langle (\mathbb{T}^{\sharp_{\mathbb{A}}}\mathbb{T} + \mathbb{S}^{\sharp_{\mathbb{A}}}\mathbb{S})\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}} \\
&\leq \frac{\lambda}{2}\left(\left(\mathbb{T}^{\sharp_{\mathbb{A}}}\mathbb{T}\right)^2\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}} + \langle \left(\mathbb{S}^{\sharp_{\mathbb{A}}}\mathbb{S}\right)^2\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}}\right) + \frac{1-\lambda}{2}\left|\langle \mathbb{S}^{\sharp_{\mathbb{A}}}\mathbb{T}\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}}\right| \langle (\mathbb{T}^{\sharp_{\mathbb{A}}}\mathbb{T} + \mathbb{S}^{\sharp_{\mathbb{A}}}\mathbb{S})\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}}.
\end{aligned}$$

So, we obtain

$$\begin{aligned}
&\left|\langle \mathbb{S}^{\sharp_{\mathbb{A}}}\mathbb{T}\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}}\right|^2 \\
&\leq \frac{\lambda}{2}\langle\left(\left(\mathbb{T}^{\sharp_{\mathbb{A}}}\mathbb{T}\right)^2 + \left(\mathbb{S}^{\sharp_{\mathbb{A}}}\mathbb{S}\right)^2\right)\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}} + \frac{1-\lambda}{2}\left|\langle \mathbb{S}^{\sharp_{\mathbb{A}}}\mathbb{T}\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}}\right| \langle (\mathbb{T}^{\sharp_{\mathbb{A}}}\mathbb{T} + \mathbb{S}^{\sharp_{\mathbb{A}}}\mathbb{S})\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}}.
\end{aligned}$$

Taking the supremum over $\mathbf{x} \in \mathcal{H} \oplus \mathcal{H}$ with $\|\mathbf{x}\|_{\mathbb{A}} = 1$ in the above inequality, we obtain the inequality of the statement. □

Remark 2. *Through various particular cases of λ in Theorem 3, we obtain some results, thus: for $\lambda = 1$ in the inequality of (14), we deduce inequality*

$$\omega_{\mathbb{A}}^2(\mathbb{S}^{\sharp_{\mathbb{A}}}\mathbb{T}) \leq \frac{1}{2}\left\|(\mathbb{T}^{\sharp_{\mathbb{A}}}\mathbb{T})^2 + (\mathbb{S}^{\sharp_{\mathbb{A}}}\mathbb{S})^2\right\|_{\mathbb{A}}$$

and for $\lambda = 0$, we find inequality

$$\omega_{\mathbb{A}}(\mathbb{S}^{\sharp_{\mathbb{A}}}\mathbb{T}) \leq \frac{1}{2}\left\|\mathbb{T}^{\sharp_{\mathbb{A}}}\mathbb{T} + \mathbb{S}^{\sharp_{\mathbb{A}}}\mathbb{S}\right\|_{\mathbb{A}}.$$

Corollary 2. *Let $T, S \in \mathbb{B}_A(\mathcal{H})$ and $\lambda \in [0,1]$. Then, the inequality*

$$\omega_A^2(S^{\sharp_A}T) \leq \frac{\lambda}{2}\left\|(T^{\sharp_A}T)^2 + (S^{\sharp_A}S)^2\right\|_A + \frac{1-\lambda}{2}\omega_A(S^{\sharp_A}T)\left\|T^{\sharp_A}T + S^{\sharp_A}S\right\|_A$$

holds.

Proof. Let $\mathbb{S} = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix}$ and $\mathbb{T} = \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}$ $\left(\text{or } \mathbb{T} = \begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix}\right)$. By using Lemma 1 (iii), we obtain

$$\mathbb{S}^{\sharp_{\mathbb{A}}}\mathbb{S} = \begin{pmatrix} S^{\sharp_A}S & 0 \\ 0 & S^{\sharp_A}S \end{pmatrix}, \mathbb{T}^{\sharp_{\mathbb{A}}}\mathbb{T} = \begin{pmatrix} T^{\sharp_A}T & 0 \\ 0 & T^{\sharp_A}T \end{pmatrix} \text{ and } \mathbb{S}^{\sharp_{\mathbb{A}}}\mathbb{T} = \begin{pmatrix} S^{\sharp_A}T & 0 \\ 0 & S^{\sharp_A}T \end{pmatrix}.$$

Therefore, we obtain $\omega_{\mathbb{A}}\left(\mathbb{S}^{\sharp_{\mathbb{A}}}\mathbb{T}\right) = \omega_A\left(S^{\sharp_A}T\right)$.

Applying relation (ii) from Lemma 1, we find

$$\|\mathbb{T}^{\sharp_{\mathbb{A}}}\mathbb{T} + \mathbb{S}^{\sharp_{\mathbb{A}}}\mathbb{S}\|_{\mathbb{A}} = \|T^{\sharp_A}T + S^{\sharp_A}S\|_A.$$

Using Theorem 3 and the above results, we deduce the inequality of the statement. □

Theorem 4. *Let $P,Q,R,S \in \mathbb{B}_A(\mathcal{H})$. Then,*

$$\omega_{\mathbb{A}}^2\left[\begin{pmatrix} P & Q \\ R & S \end{pmatrix}\right] \leq p\max\left\{\omega_A^2(P),\omega_A^2(S)\right\} + \frac{q}{2}\max\{\omega_A(QR),\omega_A(RQ)\}$$
$$+\frac{q}{4}\max\{\|QQ^{\sharp_A}+R^{\sharp_A}R\|_A,\|RR^{\sharp_A}+Q^{\sharp_A}Q\|_A\},$$

where $p,q>1$, with $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Let $\mathbf{x} \in \mathcal{H}\oplus\mathcal{H}$ be such that $\|\mathbf{x}\|_{\mathbb{A}}=1$. We use the classical Bohr inequality [33]

$$|a+b|^2 \leq p|a|^2+q|b|^2,$$

where $p,q>1$, with $\frac{1}{p}+\frac{1}{q}=1$ and $a,b\in\mathbb{C}$.

Therefore, we have

$$\begin{aligned}\left|\left\langle\begin{pmatrix} P & Q \\ R & S \end{pmatrix}\mathbf{x},\mathbf{x}\right\rangle_{\mathbb{A}}\right|^2 &= \left|\left\langle\begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix}\mathbf{x},\mathbf{x}\right\rangle_{\mathbb{A}}+\left\langle\begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix}\mathbf{x},\mathbf{x}\right\rangle_{\mathbb{A}}\right|^2 \\ &\leq p\left|\left\langle\begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix}\mathbf{x},\mathbf{x}\right\rangle_{\mathbb{A}}\right|^2+q\left|\left\langle\begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix}\mathbf{x},\mathbf{x}\right\rangle_{\mathbb{A}}\right|^2 \\ &\leq p\omega_{\mathbb{A}}^2\left[\begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix}\right]+q\omega_{\mathbb{A}}^2\left[\begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix}\right] \\ &= p\max\left\{\omega_A^2(P),\omega_A^2(S)\right\}+q\omega_{\mathbb{A}}^2\left[\begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix}\right].\end{aligned}$$

However, we have the inequality given by Xu et al. in [28]:

$$\omega_{\mathbb{A}}^2\left[\begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix}\right] \leq \frac{1}{4}\max\{\|QQ^{\sharp_A}+R^{\sharp_A}R\|_A,\|RR^{\sharp_A}+Q^{\sharp_A}Q\|_A\}$$
$$+\frac{1}{2}\max\{\omega_A(QR),\omega_A(RQ)\},$$

for all $Q,R\in\mathbb{B}_A(\mathcal{H})$.

Therefore, we proved the inequality of the statement. □

Our next goal consists of deriving an alternative and easy proof of the generalized Kittaneh inequalities (9). In all that follows, for any arbitrary operator $T\in\mathbb{B}_A(\mathcal{H})$, we write $\Re_A(T) := \frac{T+T^{\sharp_A}}{2}$ and $\Im_A(T) := \frac{T-T^{\sharp_A}}{2i}$. In order to provide the alternative proof of (9), we require the following two lemmas.

Lemma 6 ([25])**.** *Let $T\in\mathbb{B}(\mathcal{H})$ be an A-selfadjoint operator. Then, $T^{2n}\geq_A 0$ for all $n\in\mathbb{N}^*$.*

Lemma 7 ([20,25])**.** *Let $T\in\mathbb{B}_A(\mathcal{H})$. Then*

$$\omega_A(T)=\sup_{\theta\in\mathbb{R}}\left\|\Re_A(e^{i\theta}T)\right\|_A=\sup_{\theta\in\mathbb{R}}\left\|\Im_A(e^{i\theta}T)\right\|_A. \tag{15}$$

Now, we are ready to derive our proof in the next result.

Theorem 5 ([26])**.** *Let $T\in\mathbb{B}_A(\mathcal{H})$. Then,*

$$\frac{1}{4}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A \leq \omega_A^2(T) \leq \frac{1}{2}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A.$$

Proof. Let $\theta \in \mathbb{R}$. By making simple computations, we see that

$$\left[\Re_A(e^{i\theta}T)\right]^2 + \left[\Im_A(e^{i\theta}T)\right]^2 = \frac{1}{2}\left(TT^{\sharp_A} + T^{\sharp_A}T\right). \tag{16}$$

Since $\Im_A(e^{i\theta}T)$ is an A-selfadjoint operator, then by Lemma 6, we deduce that $[\Im_A(e^{i\theta}T)]^2 \geq_A 0$. So, in view of (16), we infer that

$$\frac{1}{2}\left(TT^{\sharp_A} + T^{\sharp_A}T\right) - \left[\Re_A(e^{i\theta}T)\right]^2 = \left[\Im_A(e^{i\theta}T)\right]^2 \geq_A 0.$$

This implies that

$$\langle \left[\Re_A(e^{i\theta}T)\right]^2 x, x\rangle_A \leq \frac{1}{2}\langle \left(TT^{\sharp_A} + T^{\sharp_A}T\right)x, x\rangle_A, \tag{17}$$

for all $x \in \mathcal{H}$. Moreover, since $\Re_A(e^{i\theta}T)$ is an A-selfadjoint operator, then by taking into account (17), it can be seen that

$$\left\| \left[\Re_A(e^{i\theta}T)\right]x \right\|_A^2 \leq \frac{1}{2}\langle \left(TT^{\sharp_A} + T^{\sharp_A}T\right)x, x\rangle_A, \tag{18}$$

for all $x \in \mathcal{H}$. Furthermore, clearly, we have $TT^{\sharp_A} + T^{\sharp_A}T \geq_A 0$. So, by taking the supremum over all $x \in \mathcal{H}$ with $\|x\|_A = 1$ in (18) and then using (4) and (6), we obtain

$$\left\|\Re_A(e^{i\theta}T)\right\|_A^2 \leq \frac{1}{2}\|TT^{\sharp_A} + T^{\sharp_A}T\|_A.$$

Therefore, by taking the supremum over all $\theta \in \mathbb{R}$ in the above inequality and then applying Lemma 7, we obtain

$$\omega_A^2(T) \leq \frac{1}{2}\|TT^{\sharp_A} + T^{\sharp_A}T\|_A. \tag{19}$$

On the other hand, by taking $\theta = 0$ in (16), we obtain

$$\frac{1}{2}\left(TT^{\sharp_A} + T^{\sharp_A}T\right) = [\Re_A(T)]^2 + [\Im_A(T)]^2.$$

This implies that

$$\begin{aligned} \frac{1}{2}\left\|\left(TT^{\sharp_A} + T^{\sharp_A}T\right)\right\|_A &= \left\|[\Re_A(T)]^2 + [\Im_A(T)]^2\right\|_A \\ &\leq \left\|[\Re_A(T)]^2\right\|_A + \left\|[\Im_A(T)]^2\right\|_A \\ &\leq 2\omega_A^2(T) \quad \text{(by Lemma 7 and (5))}. \end{aligned}$$

Hence, we obtain

$$\frac{1}{4}\left\|\left(TT^{\sharp_A} + T^{\sharp_A}T\right)\right\|_A \leq \omega_A^2(T). \tag{20}$$

Hence, the required result follows by combining (20) together with (19). □

The following lemma plays a crucial rule in proving our next result.

Lemma 8. *Let $T, S \in \mathbb{B}(\mathcal{H})$ be A-positive operators. Then,*

$$\|T + S\|_A \leq \max\{\|T\|_A, \|S\|_A\} + \sqrt{\|TS\|_A}.$$

To prove Lemma 8, we need the following two results.

Lemma 9 ([5,6])**.** *Let $T \in \mathbb{B}(\mathcal{H})$. Then $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ if and only if there exists a unique $\widetilde{T} \in \mathbb{B}(\mathbf{R}(A^{1/2}))$ such that $Z_A T = \widetilde{T} Z_A$. Here, $Z_A : \mathcal{H} \to \mathbf{R}(A^{1/2})$ is defined by $Z_A x = Ax$. Furthermore, the following properties hold*

(i) $\|T\|_A = \|\widetilde{T}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))}$ *for every $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$.*

(ii) $\widetilde{T} + \widetilde{S} = \widetilde{T+S}$ *and* $\widetilde{T}\widetilde{S} = \widetilde{TS}$ *for every $T, S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$.*

Lemma 10 ([34])**.** *Let $X, Y \in \mathbb{B}(\mathcal{H})$ be such that $X \geq 0$ and $Y \geq 0$. Then*

$$\|X + Y\| \leq \max\{\|X\|, \|Y\|\} + \sqrt{\|XY\|}. \tag{21}$$

Now, we are in a position to prove Lemma 8.

Proof of Lemma 8. Notice first that since T and S are A-positive, then clearly T and S are A-selfadjoint. This implies that $T, S \in \mathbb{B}_A(\mathcal{H}) \subseteq \mathbb{B}_{A^{1/2}}(\mathcal{H})$. Thus, by Lemma 9, there exist two unique operators $\widetilde{T}$ and $\widetilde{S}$ in $\mathbb{B}(\mathbf{R}(A^{1/2}))$ such that $Z_A T = \widetilde{T} Z_A$ and $Z_A S = \widetilde{S} Z_A$. Furthermore, since $T \geq_A 0$, then $\langle ATx, x\rangle \geq 0$ for all $x \in \mathcal{H}$. So, by taking (2) into consideration, we see that

$$\langle Tx, x\rangle_A = \langle ATx, Ax\rangle_{\mathbf{R}(A^{1/2})} = \langle \widetilde{T}Ax, Ax\rangle_{\mathbf{R}(A^{1/2})} \geq 0, \quad \forall\, x \in \mathcal{H}.$$

On the other hand, the density of $\mathcal{R}(A)$ in $\mathbf{R}(A^{1/2})$ yields that

$$\langle \widetilde{T}A^{1/2}x, A^{1/2}x\rangle_{\mathbf{R}(A^{1/2})} \geq 0, \;\; \forall\, x \in \mathcal{H}.$$

Therefore, the operator $\widetilde{T}$ is positive on the Hilbert space $\mathbf{R}(A^{1/2})$. By using similar arguments, one may prove that $\widetilde{S}$ is also positive on $\mathbf{R}(A^{1/2})$. So, by applying (21) together with Lemma 9, we observe that

$$\begin{aligned}
\|T + S\|_A &= \|\widetilde{T+S}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} \\
&= \|\widetilde{T} + \widetilde{S}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} \\
&\leq \max\Big\{\|\widetilde{T}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))}, \|\widetilde{S}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))}\Big\} + \sqrt{\|\widetilde{T}\widetilde{S}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))}} \\
&= \max\{\|T\|_A, \|S\|_A\} + \sqrt{\|\widetilde{TS}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))}} \\
&= \max\{\|T\|_A, \|S\|_A\} + \sqrt{\|TS\|_A}.
\end{aligned}$$

This proves the desired result. □

Now, we are able to establish the next result which provides a refinement of the first inequality in (9). The inspiration for our investigation comes from [35].

Theorem 6. *Let $T \in \mathbb{B}_A(\mathcal{H})$. Then*

$$\begin{aligned}
\frac{1}{4}\Big\|TT^{\sharp_A} + T^{\sharp_A}T\Big\|_A &\leq \frac{1}{4}\Big(\max\{\|\Re_A(T) + \Im_A(T)\|_A, \|\Re_A(T) - \Im_A(T)\|_A\}\Big)^2 \\
&\qquad + \frac{1}{4}\|\Re_A(T) + \Im_A(T)\|_A \|\Re_A(T) - \Im_A(T)\|_A \\
&\leq \omega_A^2(T).
\end{aligned}$$

Proof. Notice first that a short calculation reveals that

$$T^{\sharp_A}T + TT^{\sharp_A} = \Big(\Re_A(T) + \Im_A(T)\Big)^2 + \Big(\Re_A(T) - \Im_A(T)\Big)^2. \tag{22}$$

Moreover, one may immediately check that the operators $\Re_A(T)$ and $\Im_A(T)$ are A-selfadjoint. Thus, by Lemma 6, we deduce that $\big(\Re_A(T)+\Im_A(T)\big)^2 \geq_A 0$ and $\big(\Re_A(T)-\Im_A(T)\big)^2 \geq_A 0$. Therefore, an application of (22) together with Lemma (8) ensures that

$$\begin{aligned}
\frac{1}{4}\|T^{\sharp_A}T + TT^{\sharp_A}\|_A &= \frac{1}{4}\left\|\Big(\Re_A(T)+\Im_A(T)\Big)^2 + \Big(\Re_A(T)-\Im_A(T)\Big)^2\right\|_A \\
&\leq \frac{1}{4}\max\left\{\left\|\Big(\Re_A(T)+\Im_A(T)\Big)^2\right\|_A, \left\|\Big(\Re_A(T)-\Im_A(T)\Big)^2\right\|_A\right\} \\
&\quad + \frac{1}{4}\sqrt{\left\|\Big(\Re_A(T)+\Im_A(T)\Big)^2\Big(\Re_A(T)-\Im_A(T)\Big)^2\right\|_A} \\
&\leq \frac{1}{4}\max\left\{\left\|\Big(\Re_A(T)+\Im_A(T)\Big)^2\right\|_A, \left\|\Big(\Re_A(T)-\Im_A(T)\Big)^2\right\|_A\right\} \\
&\quad + \frac{1}{4}\|\Re_A(T)+\Im_A(T)\|_A\|\Re_A(T)-\Im_A(T)\|_A,
\end{aligned}$$

where the last inequality follows by using (5). In addition, since the operators $\Re_A(T)+\Im_A(T)$ and $\Re_A(T)-\Im_A(T)$ are A-selfadjoint, then by applying (7), we obtain

$$\begin{aligned}
\frac{1}{4}\left\|TT^{\sharp_A} + T^{\sharp_A}T\right\|_A &\leq \frac{1}{4}\Big(\max\{\|\Re_A(T)+\Im_A(T)\|_A, \|\Re_A(T)-\Im_A(T)\|_A\}\Big)^2 \\
&\quad + \frac{1}{4}\|\Re_A(T)+\Im_A(T)\|_A\|\Re_A(T)-\Im_A(T)\|_A. \qquad (23)
\end{aligned}$$

On the other hand, let $x \in \mathcal{H}$ be such that $\|x\|_A = 1$. Clearly, T can be decomposed as $T = \Re_A(T) + i\Im_A(T)$. Notice that $\Re_A(T)$ and $\Im_A(T)$ are A-selfadjoint operators. This implies that $\langle \Re_A(T)x, x\rangle_A$ and $\langle \Im_A(T)x, x\rangle_A$ are real numbers. Furthermore, we see that

$$\begin{aligned}
|\langle Tx, x\rangle_A|^2 &= \langle \Re_A(T)x, x\rangle_A^2 + \langle \Im_A(T)x, x\rangle_A^2 \\
&= \frac{1}{2}\Big(\langle \Re_A(T)x, x\rangle_A + \langle \Im_A(T)x, x\rangle_A\Big)^2 + \frac{1}{2}\Big(\langle \Re_A(T)x, x\rangle_A - \langle \Im_A(T)x, x\rangle_A\Big)^2 \\
&= \frac{1}{2}\left|\langle\Big(\Re_A(T)+\Im_A(T)\Big)x, x\rangle_A\right|^2 + \frac{1}{2}\left|\langle\Big(\Re_A(T)-\Im_A(T)\Big)x, x\rangle_A\right|^2 \\
&\geq \frac{1}{2}c_A^2\Big(\Re_A(T)+\Im_A(T)\Big) + \frac{1}{2}\left|\langle\Big(\Re_A(T)-\Im_A(T)\Big)x, x\rangle_A\right|^2.
\end{aligned}$$

This implies, by taking the supremum over all $x \in \mathcal{H}$ with $\|x\|_A = 1$ in the last inequality, that

$$\frac{1}{2}c_A^2\Big(\Re_A(T)+\Im_A(T)\Big) + \frac{1}{2}\omega_A\Big(\Re_A(T)-\Im_A(T)\Big) \leq \omega_A^2(T).$$

In addition, since $\Re_A(T)-\Im_A(T)$ is A-selfadjoint, then an application of (6) gives

$$\frac{1}{2}c_A^2\Big(\Re_A(T)+\Im_A(T)\Big) + \frac{1}{2}\|\Re_A(T)-\Im_A(T)\|_A^2 \leq \omega_A^2(T). \qquad (24)$$

Similarly, it can be proved that

$$\frac{1}{2}c_A^2\Big(\Re_A(T)-\Im_A(T)\Big) + \frac{1}{2}\|\Re_A(T)+\Im_A(T)\|^2 \leq \omega_A^2(T). \qquad (25)$$

Combing (24) together with (25) gives

$$\frac{1}{2}\max\left\{\|\Re_A(T)-\Im_A(T)\|_A^2, \|\Re_A(T)-\Im_A(T)\|_A^2\right\} \leq \omega_A^2(T). \qquad (26)$$

Hence, by taking (23) into consideration and then using (26), we observe that

$$\begin{aligned}\frac{1}{4}\left\|TT^{\sharp_A}+T^{\sharp_A}T\right\|_A &\le \frac{1}{4}\Big(\max\{\|\Re_A(T)+\Im_A(T)\|_A,\|\Re_A(T)-\Im_A(T)\|_A\}\Big)^2\\ &\quad+\frac{1}{4}\|\Re_A(T)+\Im_A(T)\|_A\|\Re_A(T)-\Im_A(T)\|_A\\ &\le \frac{1}{2}\omega_A^2(T)+\frac{1}{4}\|\Re_A(T)+\Im_A(T)\|_A\|\Re_A(T)-\Im_A(T)\|_A\\ &\le \frac{1}{2}\omega_A^2(T)+\frac{1}{4}\big[\sqrt{2}\omega_A(T)\big]^2=\omega_A^2(T).\end{aligned}$$

This finishes the proof of our result. □

By using the inequalities (24) and (25), we derive in the next theorem another improvement of the first inequality in Theorem 5: $\frac{1}{4}\left\|TT^{\sharp_A}+T^{\sharp_A}T\right\|_A \le \omega_A^2(T)$.

Theorem 7. *Let $T\in\mathbb{B}_A(\mathcal{H})$. Then*

$$\begin{aligned}\frac{1}{4}\left\|TT^{\sharp_A}+T^{\sharp_A}T\right\|_A &\le \frac{1}{4}\|\Re_A(T)+\Im_A(T)\|_A^2+\frac{1}{4}\|\Re_A(T)-\Im_A(T)\|_A^2\\ &\quad+\frac{1}{4}c_A^2(\Re_A(T)+\Im_A(T))+\frac{1}{4}c_A^2(\Re_A(T)-\Im_A(T))\\ &\le \omega_A^2(T).\end{aligned}$$

Proof. By applying (22), we see that

$$\begin{aligned}\frac{1}{4}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A &= \frac{1}{4}\left\|\Big(\Re_A(T)+\Im_A(T)\Big)^2+\Big(\Re_A(T)-\Im_A(T)\Big)^2\right\|_A\\ &\le \frac{1}{4}\left\|\Big(\Re_A(T)+\Im_A(T)\Big)^2\right\|_A+\left\|\Big(\Re_A(T)-\Im_A(T)\Big)^2\right\|_A\\ &\le \frac{1}{4}\|\Re_A(T)+\Im_A(T)\|_A^2+\frac{1}{4}\|\Re_A(T)-\Im_A(T)\|_A^2 \quad \text{(by (5))}.\end{aligned}$$

This implies, through (5), that

$$\frac{1}{4}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A \le \frac{1}{4}\|\Re_A(T)+\Im_A(T)\|_A^2+\frac{1}{4}\|\Re_A(T)-\Im_A(T)\|_A^2. \tag{27}$$

On the other hand, it follows from the inequalities (24) and (25) that

$$\begin{aligned}&\frac{1}{4}\|\Re_A(T)+\Im_A(T)\|_A^2+\frac{1}{4}\|\Re_A(T)-\Im_A(T)\|_A^2\\ &+\frac{1}{4}c_A^2\Big(\Re_A(T)+\Im_A(T)\Big)+\frac{1}{4}c_A^2\Big(\Re_A(T)-\Im_A(T)\Big)\le\omega_A^2(T).\end{aligned} \tag{28}$$

Combining (27) together with (28) yields the desired result. □

Our next result provides also another refinement of the first inequality in Theorem 5.

Theorem 8. *Let $T\in\mathbb{B}_A(\mathcal{H})$. Then*

$$\begin{aligned}\frac{1}{4}\left\|TT^{\sharp_A}+T^{\sharp_A}T\right\|_A &\le \frac{1}{2}\max\Big\{\|\Re_A(T)-\Im_A(T)\|_A^2,\|\Re_A(T)+\Im_A(T)\|_A^2\Big\}\\ &\le \max\{\gamma_A(T),\delta_A(T)\}\\ &\le \omega_A^2(T),\end{aligned}$$

where

$$\gamma_A(T) = \frac{1}{2}c^2(\Re_A(T) + \Im_A(T)) + \frac{1}{2}\|\Re_A(T) - \Im_A(T)\|_A^2$$

$$\text{and} \quad \delta_A(T) = \frac{1}{2}c^2(\Re_A(T) - \Im_A(T)) + \frac{1}{2}\|\Re_A(T) + \Im_A(T)\|_A^2.$$

Proof. Notice first that the third inequality in Theorem 8 follows immediately by applying the inequalities (24) and (25). Moreover, the second inequality holds immediately. So, it remains to prove the first inequality. We recall the following elementary equality

$$\max\{a, b\} = \frac{1}{2}\Big(a + b + |a - b|\Big), \quad \forall\, x, y \in \mathbb{R}. \tag{29}$$

An application of (29) shows that

$$\begin{aligned}
&\frac{\max\{\|\Re_A(T) + \Im_A(T)\|_A^2, \|\Re_A(T) - \Im_A(T)\|_A^2\}}{2} \\
&= \frac{\|\Re_A(T) + \Im_A(T)\|_A^2 + \|\Re_A(T) - \Im_A(T)\|_A^2}{4} \\
&\quad + \frac{\Big|\|\Re_A(T) + \Im_A(T)\|_A^2 - \|\Re_A(T) - \Im_A(T)\|_A^2\Big|}{4} \\
&\geq \frac{\Big\|\Big(\Re_A(T) + \Im_A(T)\Big)^2 + \Big(\Re_A(T) - \Im_A(T)\Big)^2\Big\|_A}{4} \\
&\quad + \frac{\Big|\|\Re_A(T) + \Im_A(T)\|_A^2 - \|\Re_A(T) - \Im_A(T)\|_A^2\Big|}{4} \\
&= \frac{\|TT^{\sharp_A} + T^{\sharp_A}T\|_A}{4} + \frac{\Big|\|\Re_A(T) + \Im_A(T)\|_A^2 - \|\Re_A(T) - \Im_A(T)\|_A^2\Big|}{4},
\end{aligned}$$

where the last equality follows from (22). This immediately proves the first inequality in Theorem 8. Hence, the proof is complete. □

Remark 3. *The inequalities from Theorem 5 given by Feki in [26] represent a generalization of the inequalities given by Kittaneh [27]. In Theorems 6 and 7, we present some refinements of the inequalities due to Feki.*

3. Conclusions

The main objective of the present paper is to present a few new $\mathbb{A}$-numerical radius inequalities for 2×2 operator matrices. In Theorem 1, we obtain a bound for the $\mathbb{A}$-numerical radius for the 2×2 operator matrix. We use an inequality of Buzano type (see Lemma 2) to estimate the $\mathbb{A}$-numerical radius of an operator $\mathbb{T} = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$, where $P, Q, R, S \in \mathbb{B}_A(\mathcal{H})$. By particularization, we deduce an improvement of the second inequality (9). Another bound for $\mathbb{A}$-numerical radius for the 2×2 operator matrix is given in Theorem 2. Next, we present an improvement of the Cauchy–Schwarzv inequality type using the inner product $\langle \cdot, \cdot \rangle_{\mathbb{A}}$. This result is used to find a new bound for the $\mathbb{A}$-numerical radius of operator matrix $\mathbb{S}^{\sharp_\mathbb{A}}\mathbb{T}$. Applying the Bohr inequality, we deduce another new bound for the $\mathbb{A}$-numerical radius for the 2×2 operator matrix. In addition to these, we aim to establish an alternative and easy proof of the generalized Kittaneh inequalities (9). We also give a lemma which plays a crucial rule in proving a result concerning to norm $\| \cdot \|_A$ (see Lemma 8). Finally, we establish some improvements of the well-known inequalities due to Kittaneh (see [27] Theorem 1) and generalized by Feki in [26]. In the future, we will study better estimates of the $\mathbb{A}$-numerical radius for the 2×2 operator matrix and we will study new inequalities involving the Berezin norm and Berezin number of bounded linear operators in Hilbert and semi-Hilbert space. We can also define the A-Berezin norm and number.

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